Asymptotic stability in probability for Stochastic Boolean Networks ⋆

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Abstract

In this paper, a new class of Boolean networks, called Stochastic Boolean Networks, is presented. These systems combine some features of the classical deterministic Boolean networks (the state variables admit two operation levels, either 0 or 1) and of Probabilistic Boolean Networks (at each time instant the transition map is selected through a random process), enriching the set of admissible dynamical behaviors, thanks to the set–valued nature of the transition map. Necessary and sufficient Lyapunov conditions are given to guarantee global asymptotic stability (resp., global asymptotic stability in probability) of a given set for a deterministic Boolean network with set–valued transition map (resp., for a Stochastic Boolean Network). A constructive procedure to compute a Lyapunov function (resp., stochastic Lyapunov function) relative to a given set for a deterministic Boolean network with set–valued transition map (resp., Stochastic Boolean Network) is reported.

1 Introduction

The study of the relation between the expression of a gene and the synthesis of a particular biochemical product is one of the most challenging problems in modern molecular biology (Perdew et al. 2014). In the literature, different frameworks have been proposed to model and analyze this complex relationship, such as: cluster analysis (Eisen et al. 1998), Bayesian networks (Friedman et al. 2000, Yu et al. 2004), information–theoretic approaches (Margolin et al. 2006), and Ordinary Differential Equations (Bansal et al. 2006). Among these analytical models, Boolean networks are receiving growing interest (Grieb et al. 2015, Kaushik & Sahi 2015).

A Boolean network is a discrete–time nonlinear system described by variables with binary operation levels (Kauffman 1969). At each time instant, the state of the system is updated by using a logic function of the current variables. In fact, each gene can have two states: 1, when it is expressed, and 0, when it is not. Similarly, each biochemical product can have two states, 0 or 1, depending on its presence above or below a certain concentration threshold, respectively. This kind of structure can capture the behavior of complex regulatory networks (Albert & Barabási 2000, Harris et al. 2002). In the literature, many different approaches have been proposed to characterize the dynamical behavior of this class of systems. For instance, in Cheng & Qi (2010), a mathematical framework has been proposed to convert a Boolean network into a classical discrete–time, time–invariant system, and it is shown that, by analyzing the transition matrix of such a system, one can identify some features of the Boolean network such as: the number of fixed points, the number of cycles of given length, the transient period for all points to enter the set of attractors, and the basin of attraction for each attractor. On the other hand, in Hinkelmann et al. (2011), an algebraic geometry approach has been proposed to identify attractors. Algebraic geometry techniques have been used also to compute Darboux polynomials (Menini & Tornambe 2013*a*) and to design observers for Boolean networks (Menini & Tornambe 2013*b*). Even if these systems have been first used to model biological relationships, they are receiving most attention also in other fields such as: financial markets (Caetano & Yoneyama 2015), electronics (Rosin 2015), and industrial networks (Easton et al. 2008).

One of the most important limitations of classical Boolean networks is their determinism (somehow mitigated in Thomas (1973) by the introduction of Boolean networks with asynchronous updates). In Shmulevich et al. (2002*b*), Probabilistic Boolean Networks (briefly,

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PBN) have been introduced; they share the appealing structure of Deterministic Boolean Networks, but are also able to cope with uncertainty both in the data and in the model selection. Namely, a PBN is a discrete–time system that shares the structure of a classical Boolean network (i.e., the state variables admit two operation levels), but the transition from a state to another one is governed by a random process. In fact, a PBN involves a set of possible Boolean maps for each state variable and, at each update time, the process of choosing a certain map rather than another is governed by a random process (for further details and the formal definition of a PBN, see Section 2.2). The interest in these systems arises from the advent of gene expression microarrays that yield quantitative and semi-quantitative data on the cell status in a specific condition and time (Bansal et al. 2007). However, many times, the available data are not sufficient to estimate all the parameters that are present in the system (e.g., when the number of variables involved in the process is higher than the number of available measures, or when some essential variables are unmeasurable). In these cases, it may be preferable to have a probabilistic description of the process being analyzed.

In this paper, a new class of Boolean networks, called Stochastic Boolean Networks (briefly, SBN), is presented. This kind of system admits state variables with binary operation levels as classical Boolean networks and the transition from a state to the following one is governed by a stochastic process as in PBNs. The difference between these systems and PBNs is that, at each time instant and for each outcome of the random process, the map from the current state to the subsequent one needs not be single–valued, but can be set–valued. The advantage of this feature is that, when the number of possible states is too large for precise estimation or when some essential variables are either not measurable or unknown, it is not necessary to restrict the number of considered values to an essential set that defines a function. In fact, a whole branch of behaviors can be encoded by a single SBN. Moreover, the structure of SBNs allows to cope with biological dynamical models having non–unique solutions (Conte et al. 2004, Kaitala & Heino 1996, Kaitala et al. 2000, Upadhyay 2003). Two motivating examples are given in Section 2.

2 Notation and Preliminaries

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and real numbers, respectively. Given $k \in \mathbb{Z}$, let $\mathbb{Z}_{\geqslant k} := \{z \in \mathbb{Z} : z \geqslant k\}$, $\mathbb{R}_{\geqslant k} := \{r \in \mathbb{R} : r \geqslant k\}$, and $\mathbb{Z}_{<k} := \{z \in \mathbb{Z}_{\geqslant 0} : z < k\}$. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$, denoted $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}_\infty$, denoted $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and it is unbounded. Let $(\mathbb{K}, \mathbb{d})$ be a metric space. Since $\mathbb{d}$ is a metric for $\mathbb{K}$, the concept of convergence is well defined. Namely, a sequence $\{x^\nu\}_{\nu=0}^\infty$ is said to converge to x, denoted $x^\nu \to x$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $\nu \geqslant N$ implies $\mathbb{d}(x^\nu, x) \leqslant \varepsilon$. A set–valued mapping $S : \mathbb{K} \rightrightarrows \mathbb{K}$ is a left–total relation assigning to each element $x \in \mathbb{K}$ a set $S(x) \subset \mathbb{K}$. A set–valued mapping $S : \mathbb{K} \rightrightarrows \mathbb{K}$ is *outer semicontinuous* at $\bar{x} \in \mathbb{K}$ if $\limsup_{x \to \bar{x}} S(x) \subset S(\bar{x})$, where $\limsup_{x\to\bar{x}} S(x) := \{y \in \mathbb{K} : \exists x^\nu \to \bar{x}, \exists y^\nu \to y, \text{ with } y^\nu \in S(x^\nu)\}$. A mapping $S : \mathbb{K} \rightrightarrows \mathbb{K}$ is locally bounded if, for each bounded set $K \subset \mathbb{K}$, $S(K) := \bigcup_{x\in K} S(x)$ is bounded. A mapping $S : \mathbb{K}_1 \rightrightarrows \mathbb{K}_2$ is measurable if, for every open set $\mathcal{O} \subset \mathbb{K}_2$, the set $S^{-1}(\mathcal{O}) := \{y \in \mathbb{K}_1 : S(y) \cap \mathcal{O} \neq \emptyset\}$ is measurable. Given $\mathcal{A} \subset \mathbb{K}$, a continuous function $\varrho : \mathbb{K} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{PD}(\mathcal{A})$, denoted $\varrho \in \mathcal{PD}(\mathcal{A})$, if $\varrho(x) = 0$, for all $x \in \mathcal{A}$ and $\varrho(x) > 0$, for all $x \in \mathbb{K} \setminus \mathcal{A}$. Given a finite set $\Psi \subset \mathbb{K}$, the symbol $\mathcal{P}(\Psi)$ denotes the *power set* of Ψ, i.e., the set of all the subsets of Ψ. The symbols $\neg$, $\vee$, $\wedge$, and $\oplus$ represent the entry wise logical "not", "or", "and", and "exclusive or" operators, respectively. The symbol $(\cdot)^+$ denotes the next value.

2.1 The Galois field $\mathbb{F}_2$

Let $\mathbb{F}_2 := \{0, 1\}$ denote the *Galois field of order 2* (Lidl & Niederreiter 1994). The set of all the n–dimensional vectors whose entries are in $\mathbb{F}_2$ is denoted $\mathbb{F}_2^n$. Note that each vector in $\mathbb{F}_2^n$ is essentially an n–bit digital number $[\, x_1 \; x_2 \; \cdots \; x_n \,]^\top$, whose decimal equivalent is given by $\pi_n : \mathbb{F}_2^n \to \mathbb{Z}_{\geqslant 0}$, $\pi_n(x) = \sum_{i=1}^n 2^{i-1}\psi^{-1}(x_i)$, where $\psi^{-1} : \mathbb{F}_2 \to \{0, 1\} \subset \mathbb{Z}$ maps each $x \in \mathbb{F}_2$ to the corresponding integer value in $\{0, 1\} \subset \mathbb{Z}$. In the following, π_n^{-1} denotes the inverse map of π_n. Let a point $y \in \mathbb{F}_2^n$ be given. For each $x \in \mathbb{F}_2^n$, the *distance between x and y* is $\mathbb{d}(x, y) : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{Z}_{\geqslant 0}$, $\mathbb{d}(x, y) := \sum_{i=1}^n \psi^{-1}(x_i \oplus y_i)$, where $x_i, y_i \in \mathbb{F}_2$, because $x, y \in \mathbb{F}_2^n$. The distance $\mathbb{d}$ is usually known in coding theory as *Hamming distance* (Hamming 1950), when applied to strings of equal length. On the other hand, letting $\mathcal{A} \subset \mathbb{F}_2^n$, the *distance between x and $\mathcal{A}$* is $|x|_\mathcal{A} := \min_{y\in\mathcal{A}} \mathbb{d}(x, y)$. The following lemma, whose proof is well known (Bourbaki 1998), states that the function $\mathbb{d}$ is a metric on $\mathbb{F}_2^n$ and hence the definitions given at the beginning of this section apply to such a field, when the distance $\mathbb{d}$ is used as a metric.

Lemma 1 *The pair $(\mathbb{F}_2^n, \mathbb{d})$ constitutes a metric space.*

Since the pair $(\mathbb{F}_2^n, \mathbb{d})$ is a metric space, it is possible to define the *open ball of radius $r > 0$ about $x \in \mathbb{F}_2^n$* as $\mathbb{B}(x, r) = \{y \in \mathbb{F}_2^n : \mathbb{d}(y, x) < r\}$. A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *open* if, for every $x \in \mathcal{A}$, $\exists r > 0$ such that $\mathbb{B}(x, r) \subset \mathcal{A}$. A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *closed* if $\mathbb{F}_2^n \setminus \mathcal{A}$ is open. For any set $\mathcal{A}$ and $\varepsilon > 0$, let $\mathcal{A} + \mathbb{B}(0, \varepsilon) = \{x \in \mathbb{F}_2^n : |x|_\mathcal{A} < \varepsilon\}$. Next lemma characterizes the topology of the metric space $(\mathbb{F}_2^n, \mathbb{d})$.

Lemma 2 *Each set $\mathcal{A} \subset \mathbb{F}_2^n$ is both open and closed.*

Proof. By Lemma 1, $(\mathbb{F}_2^n, \mathbb{d})$ constitutes a metric space. Hence, the open ball $\mathbb{B}(x, r)$ is well defined. Consider the set $\mathcal{A}_i = \{\bar{x}\}$, with $\bar{x} \in \mathbb{F}_2^n$. The set $\mathcal{A}_i$ is open because $\mathbb{B}(\bar{x}, 1) \subset A$. Hence, since every set $\mathcal{A} \subset \mathbb{F}_2^n$ is such that $\mathcal{A} = \bigcup_{i \in I} A_i$, for some finite I, $\mathcal{A}$ is open (Bourbaki 1998, § 2.6, § 2.7). Consider now $\mathcal{B}_i := \mathbb{F}_2^n \setminus \mathcal{A}_i$. The set $\mathcal{B}_i$ is closed, because it is the complement of an open set. Since every set $\mathcal{A}_i = \bigcap_{i \in I} \mathcal{B}_i$ for some finite I, the set A_i is closed. Therefore, since each set $\mathcal{A} = \bigcup_{i \in I} A_i$, for some finite I, and the union of finitely many closed sets is closed, the set $\mathcal{A}$ is closed. $\square$

2.2 Classes of Boolean Networks

A map $g : \mathbb{F}_2^n \to \mathbb{F}_2^\ell$ is called *Boolean* and can be defined by assigning to each of the 2^n elements of $\mathbb{F}_2^n$ one of the 2^ℓ elements of $\mathbb{F}_2^\ell$. A *Deterministic Boolean Network* (briefly, *DBN*) is a discrete–time system of the form

$$x^+ = g(x) \tag{1}$$

where $x \in \mathbb{F}_2^n$ and $g : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is a Boolean map.

In order to deal with non–unique solutions (see the subsequent Examples 1 and 2), the concept of DBN can be extended through the notion of Boolean network with set–valued transition map, written formally as

$$x^+ \in G(x), \tag{2}$$

with $G : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$ having nonempty values for every $x \in \mathbb{F}_2^n$.

The following lemma states that the number of dynamical behaviors modeled by a Boolean network is bounded.

Lemma 3 *Let $M := 2^n$ and $N := (2^M - 1)^M$. There exist M^M different $g : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and N different $G : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$ such that $G(x)$ is nonempty for each $x \in \mathbb{F}_2^n$.*

Proof. The first part of the proof follows trivially by Cheng & Qi (2010). On the other hand, a set–valued mapping $G : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$ maps each $x \in \mathbb{F}_2^n$ into $G(x) \subset \mathbb{F}_2^n$. The statement follows by the fact that the number of points in $\mathbb{F}_2^n$ equals M and that the number of nonempty sets $G(x) \subset \mathbb{F}_2^n$ equals $2^M - 1$. $\square$

A *Probabilistic Boolean Network* (briefly, PBN) consists of a set of Boolean vector maps $\{f_1, f_2, \ldots, f_K\}$, $f_\ell : \mathbb{F}_2^n \to \mathbb{F}_2^n$, $\ell = 1, \ldots, K$, governing the state transitions of the discrete–time system (Shmulevich et al. 2002*a*). At each time instant, a random decision (with probability q) is made on whether to switch the network function to the next transition, where the probability q is a parameter of the PBN. If a decision is made to switch the network function, then a new function among $f_1, f_2, \ldots, f_K$ is chosen, with c_ℓ being the probability of choosing the function f_ℓ, $\ell = 1, \ldots, K$. Namely, each f_ℓ determines a DBN, and the PBN dynamics are the same ones of a DBN, until a random decision (with probability q) is taken. Hence, a new function f_ℓ is chosen among $f_1, \ldots, f_K$, with c_ℓ being the probability of choosing the function f_ℓ, and the Boolean network is updated as the DBN defined by such f_ℓ (Shmulevich et al. 2002*b*).

Let $M := 2^n$ and $N := (2^M - 1)^M$. A *Stochastic Boolean Network* (briefly, *SBN*) is a stochastic discrete–time system written formally as

$$x^+ \in G(x, w), \quad \mu(\cdot), \tag{3}$$

where $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$ is the transition map, $G(x, w) \neq \emptyset$, for each $(x, w) \in \mathbb{F}_2^n \times \mathbb{Z}_{<N}$. The distribution μ is derived from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $\boldsymbol{w} := \{\boldsymbol{w}_j\}_{j=0}^{\infty}$ of independent, identically distributed (i.i.d.) input random variables $\boldsymbol{w}_j : \Omega \to \mathbb{Z}_{<N}$, $j \in \mathbb{Z}_{\geqslant 0}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Note that, by Lemma 3, there is not loss of generality in considering random variables $\boldsymbol{w}_j : \Omega \to \mathbb{Z}_{<N}$, because the set of all the possible SBNs has cardinality N. Hence, for each $A \in \mathcal{P}(\mathbb{Z}_{<N})$, $\boldsymbol{w}_j^{-1}(A) := \{\omega \in \Omega : \boldsymbol{w}_j(\omega) \in A\} \in \mathcal{F}$. Since, for every $A \in \mathcal{P}(\mathbb{Z}_{<N})$, $\mathbb{P}(\omega \in \Omega : \boldsymbol{w}_j(\omega) \in A)$ is independent of j, the distribution $\mu : \mathcal{P}(\mathbb{Z}_{<N}) \to [0, 1]$ is defined as $\mu(A) := \mathbb{P}(\omega \in \Omega : \boldsymbol{w}_j(\omega) \in A)$ (Fristedt & Gray 1997, Sec. 2.1 and 11.1). Let $\mathcal{F}_j$ denote the collection of sets $\{\omega \in \Omega : (\boldsymbol{w}_0(\omega), \ldots, \boldsymbol{w}_j(\omega)) \in A\}$, $A \in \mathcal{P}((\mathbb{Z}_{<N})^{j+1})$, which are all the sub–$\sigma$–fields of $\mathcal{F}$ that form the minimal filtration of $\boldsymbol{w}$ (Fristedt & Gray 1997, Sec. 11.3, Def. 4).

The next two examples motivate the interest in Boolean networks with set–valued transition map.

Example 1 *Consider the reaction system depicted in Figure 1, which includes the main reactions of glycolysis and some adjacent reactions occurring, for example, in liver cells (Heinrich & Schuster 2012).*

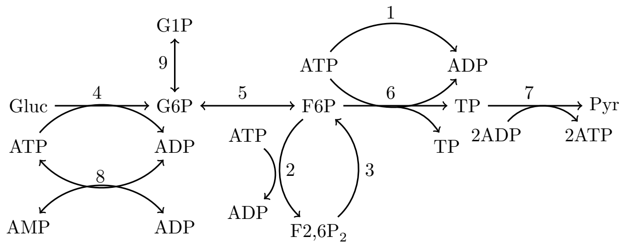


Figure 1. Scheme of the main reactions of glycolysis and some adjacent reactions (Heinrich & Schuster 2012).

In Heinrich & Schuster (2012, Sec. 3.2.2) such a chemical system has been characterized through the analysis of the corresponding stoichiometric matrix. It turns out that, in this reaction system, there are multiple behaviors that can occur and hence the main reactions of glycolysis cannot be modeled by a DBN that admits unique solutions. However, a SBN, that admits non–unique solutions, can be used to model this chemical reaction. Note *that, in the reaction system depicted in Figure 1, the metabolite ATP appears multiple times, to take into account its different sources. A SBN modeling such a reaction system can be obtained by considering, instead, a single source of ATP and assuming that each chemical reaction consumes all its sources. Namely, by inspecting the chemical network depicted in Figure 1 and letting* $x_1 = \mathrm{Gluc}$, $x_2 = \mathrm{G1P}$, $x_3 = \mathrm{F6P}$, $x_4 = \mathrm{G6P}$, $x_5 = \mathrm{F2,6P_2}$, $x_6 = \mathrm{TP}$, $x_7 = \mathrm{Pyr}$, $x_8 = \mathrm{ATP}$, $x_9 = \mathrm{ADP}$, $x_{10} = \mathrm{AMP}$, *a SBN modeling such a reaction system is*

$$
\begin{aligned}
x_1 &\in \{\neg x_8 \wedge x_1\},\\
\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} &\in \left\{ \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}, \begin{bmatrix} 0 \\ x_4 \vee x_5 \end{bmatrix} \right\},\\
\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} &\in \begin{bmatrix} (x_1 \wedge x_8) \vee x_2 \\ 0 \end{bmatrix} \vee \left\{ \begin{bmatrix} x_3 \\ 0 \end{bmatrix}, \begin{bmatrix} x_3 \wedge \neg x_8 \\ x_3 \wedge x_8 \end{bmatrix} \right\},\\
x_6 &\in \{x_3 \wedge x_8\},\\
x_7 &\in \{(x_9 \wedge x_6) \vee x_7\},\\
x_8 &\in \{x_9\},\\
x_9 &\in \{x_8 \vee x_{10}\},\\
x_{10} &\in \{x_9\},
\end{aligned}
$$

with the understanding that $x_i = 1$, *if the corresponding chemical species is present above a certain concentration threshold, and* $x_i = 0$, *otherwise,* $i = 1, \ldots, 10$. △

Example 2 *Typical Boolean models (as, for instance, the ones given in Albert & Othmer (2003), Kauffman et al. (2003)) assume synchronous updates of the Boolean state of the system. Namely, at predetermined time instants, all the nodes exchange information about their current state with all the other nodes and update their state according to such information. The assumption underlying these models is that the time scales of all the involved processes are similar. In reality the time scales of transcription, translation, and degradation can vary widely from gene to gene and can be anywhere from minutes to hours (Chaves et al. 2005). Hence, Boolean models allowing asynchronous updates are receiving increasing interest (Albert et al. 2008, Ghysen & Thomas 2003, Thomas 1973). The framework of SBN presented here allows to deal with Boolean networks with asynchronous updates. Namely, let* $\tau = [\,\tau_1 \ \cdots \ \tau_h\,]^\top \in \mathbb{F}_2^h$, *and consider the* Boolean counter *discrete–time system*

$$\tau^+ = \psi_h(\tau),$$

where $\psi_h : \mathbb{F}_2^h \to \mathbb{F}_2^h$, $\psi_h(\tau) = [\,\psi_{1,h}(\tau) \ \cdots \ \psi_{h,h}(\tau)\,]^\top$, $\psi_{h,h}(\tau) = \tau_h \oplus 1$, $\psi_{i,h}(\tau) = \tau_i \oplus (\tau_h \wedge \cdots \wedge \tau_{i+1})$, $i = 1, \ldots, h-1$. *We denote such a systems as "counter" because, by construction,* $\pi_h(\tau^+) = \pi_h(\tau) + 1$, *for each* $\tau \in \mathbb{F}_2^h$ *such that* $\pi_h(\tau) < 2^h - 1$, *and* $\pi_h(\tau^+) = 0$ *if* $\pi_h(\tau) = 2^h - 1$, *for each* $h \in \mathbb{Z}_{>0}$. *Hence, consider the SBN with state* $[\,x^\top \ t^\top \ q^\top\,]^\top$, $x \in \mathbb{F}_2^n$, $t = [\,t_1^\top \ \cdots \ t_n^\top\,]^\top$, $t_i \in \mathbb{F}_2^{h_i}$, $h_i \in \mathbb{Z}_{>0}$ $i = 1, \ldots, n$, , $q = [\,q_1 \ \cdots \ q_n\,]^\top \in \mathbb{F}_2^n$,

$$
\begin{aligned}
x_i^+ &\in \{(q_i \wedge g_i(x)) \vee (\neg q_i \wedge x_i)\}, && i = 1, \ldots, n,\\
t_i^+ &\in \{\neg q_i \wedge \psi_{h_i}(t_i)\}, && i = 1, \ldots, n,\\
q_i^+ &\in \begin{cases} \{0,1\}, & \text{if } \pi_{h_i}(t_i) < T_i,\\ \{1\} & \text{otherwise,} \end{cases} && i = 1, \ldots, n,
\end{aligned} \tag{4}
$$

where $T_i \in \mathbb{Z}_{\geqslant 0}$ *is a fixed constant imposing a maximum dwell–time for the jumps of the state* x_i *(that is, at most* T_i *amount of time passes between two consecutive intervals on which there are no jumps of the state* x_i*),* $i = 1, \ldots, n$, *and* $g : \mathbb{F}_2^n \to \mathbb{F}_2^n$ *is a Boolean function. The SBN* (4) *is able to encode the structure of Boolean network with asynchronous updates (Bertsekas & Tsitsiklis 1989, Chaves et al. 2005). As a matter of fact, the updates of the states of the SBN* (4) *need not be synchronous, but each state* x_i *may or may not be updated at time* $j+1$, *according to the value of the decision variable* $q_i \in \mathbb{F}_2$ *at time* j. *In fact, if* $q_i = 1$, *then the state* x_i *is updated, otherwise, if* $q_i = 0$, x_i *is not updated.* △

The following proposition states that every mapping $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$ satisfies the Stochastic Hybrid Basic Conditions (Teel 2013, Ass. 1.1–3), establishing, with the subsequent Assumption 1, existence of random solutions.

Proposition 1 *Every set–valued mapping* $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$ *is locally bounded and, for each* $w \in \mathbb{Z}_{<N}$, $x \mapsto G(x, w)$ *is outer semicontinuous.*

Proof. Every set–valued mapping $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$ is globally bounded, because $\mathbb{d}(g, 0) < 2^n$, for all $g \in G(x, w)$ and $(x, w) \in \mathbb{F}_2^n \times \mathbb{Z}_{<N}$ (Rockafellar & Wets 2009, Def. 5.14). Moreover, for each $w \in \mathbb{Z}_{<N}$, $G_w : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$, $G_w(x) = G(x, w)$, is such that $\operatorname{graph} G_w := \{(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : y \in G_w(x)\} \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$, whence, by Lemma 2, $\operatorname{graph} G_w$ is closed. Therefore, for each $w \in \mathbb{Z}_{<N}$, the mapping $x \mapsto G(x, w)$ is outer semicontinuous (Goebel et al. 2012, Lem. 5.10). □

3 Stability of Boolean networks with set–valued jump map

In this section, we characterize the stability properties of $\mathcal{A} \subset \mathbb{F}_2^n$ for Boolean networks of the form (2).

A sequence $\boldsymbol{x}$ is a *solution to* (2) starting at x, denoted $\boldsymbol{x} \in \mathcal{S}(x)$, if $\boldsymbol{x}_{j+1} \in G(\boldsymbol{x}_j)$, and $\boldsymbol{x}_0 = x$. It is worth noticing that system (2) with this concept of solution is essentially a finite automaton whose alphabet is composed by singletons and whose final–state set is empty.

A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *globally stable for* (2) if there exists a function $\alpha \in \mathcal{K}_\infty$ (Khalil 1996, Def 4.2) such that, letting $\boldsymbol{x} = \{\boldsymbol{x}_j\}_{j=0}^\infty$, $\boldsymbol{x} \in \mathcal{S}(x)$, $x \in \mathbb{F}_2^n$, $|\boldsymbol{x}_j|_\mathcal{A} \leqslant \alpha(|x|_\mathcal{A})$, $\forall j \in \mathbb{Z}_{\geqslant 0}$. A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *attractive for* (2) if, for each $\varepsilon > 0$ and $R > 0$ and $\boldsymbol{x} = \{\boldsymbol{x}_j\}_{j=0}^\infty$, $\boldsymbol{x} \in \mathcal{S}(x)$, $|x|_\mathcal{A} \leqslant R$, there exists $T > 0$ such that $|\boldsymbol{x}_j|_\mathcal{A} \leqslant \varepsilon$, for every $j \in \mathbb{Z}_{\geqslant T}$. The set $\mathcal{A}$ is *globally asymptotically stable* (briefly, *GAS*) *for* (2) if it is both stable and attractive for (2) (Goebel et al. 2012, Def 3.6).

Define the *inverse mapping* $G^{-1} : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$, $G^{-1}(x) = \{y \in \mathbb{F}_2^n : x \in G(y)\}$. The following two lemmas show that the global stability of $\mathcal{A} \subset \mathbb{F}_2^n$ for (2) is equivalent to the strong forward invariance of $\mathcal{A}$ for (2) (Goebel et al. 2012, Def. 6.25), and that a set $\mathcal{A}$ is GAS if and only if it strongly forward invariant and the solutions to (2) converges to $\mathcal{A}$ in finite time.

Lemma 4 *Let the system* (2) *and* $\mathcal{A} \subset \mathbb{F}_2^n$ *be given. The set* $\mathcal{A}$ *is stable for* (2) *if and only if*

$$G(\mathcal{A}) := \bigcup_{a \in \mathcal{A}} G(a) \subset \mathcal{A}. \tag{5}$$

Proof. If (5) holds, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that, letting $\boldsymbol{x} \in \mathcal{S}(x)$, if $|x|_\mathcal{A} < \delta$, then $|\boldsymbol{x}_j|_\mathcal{A} < \varepsilon$. As a matter of fact, it is enough to choose $\delta = 1$, for every $\varepsilon > 0$, to satisfy such an inequality. Hence, by Khalil (1996, Lem. 4.5), the function $\alpha \in \mathcal{K}_\infty$, $\alpha(s) = 2^n s$ is such that, $|\boldsymbol{x}_j|_\mathcal{A} \leqslant \alpha(|x|_\mathcal{A})$, $\forall j \in \mathbb{Z}_{\geqslant 0}$.

On the other hand, assume that $\mathcal{A}$ is stable and that (5) does not hold. Hence, for some $x \in \mathcal{A}$, there exists $y \in G(x)$ such that $y \notin \mathcal{A}$. Since $y \notin \mathcal{A}$, the distance $|y|_\mathcal{A} > 0$. This contradicts the hypothesis that $\mathcal{A}$ is stable. As a matter of fact, there exists $\boldsymbol{x} \in \mathcal{S}(x)$, $\boldsymbol{x} := \{\boldsymbol{x}_j\}_{j=1}^\infty$, such that $\boldsymbol{x}_1 = y$, whence $|\boldsymbol{x}_1|_\mathcal{A} > \alpha(|x|_\mathcal{A}) = 0$. $\square$

Lemma 4 essentially states that, in this context, the concepts of global stability, stability and forward invariance of $\mathcal{A} \subset \mathbb{F}_2^n$ for (2) are the same. In the following lemma, we provide a method to verify whether $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS.

Lemma 5 *Let the system* (2) *and* $\mathcal{A} \subset \mathbb{F}_2^n$ *be given. The set* $\mathcal{A}$ *is GAS if and only if it is globally stable for* (2) *and there exists* $T \in \mathbb{Z}_{\geqslant 0}$ *such that* $|\boldsymbol{x}_T|_\mathcal{A} = 0$*, for any* $\boldsymbol{x} := \{\boldsymbol{x}_j\}_{j=0}^\infty \in \mathcal{S}(x)$, $x \in \mathbb{F}_2^n$.

Proof. If $\exists T \in \mathbb{Z}_{\geqslant 0}$ such that, for each $x \in \mathbb{F}_2^n$, $\boldsymbol{x} \in \mathcal{S}(x)$, $|\boldsymbol{x}_T| = 0$ and the set is stable, then $|\boldsymbol{x}_j|_\mathcal{A} = 0$, $\forall j \in \mathbb{Z}_{\geqslant T}$, and thus $\mathcal{A}$ is GAS.

On the other hand, if $\mathcal{A}$ is GAS, then $\mathcal{A}$ is strongly forward invariant and globally stable (Lemma 4). Moreover, by the definition of global asymptotic stability of $\mathcal{A}$ for $R = 2^n$ and $\varepsilon = 1$, $\exists T \in \mathbb{Z}_{\geqslant 0}$ such that, for any $x \in \mathbb{F}_2^n$ and $\boldsymbol{x} \in \mathcal{S}(x)$, $|\boldsymbol{x}_T| = 0$. $\square$

The set $\mathcal{A}$ is *the smallest GAS set for* (2) if, letting $\mathcal{Q}$ be any GAS set for (2), $\mathcal{A} \subset \mathcal{Q}$. Since the set $\mathbb{F}_2$ is GAS and the intersection of two nonempty GAS sets is GAS and nonempty, then, for each Boolean network (2), there exists a unique, nonempty smallest GAS set.

The function $V : \mathbb{F}_2^n \to \mathbb{R}_{\geqslant 0}$ is a *Lyapunov function for* (2) (Goebel et al. 2012, Def. 3.16 and Thm. 3.18) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\varrho \in \mathcal{PD}(\mathcal{A})$ such that, $\forall x \in \mathbb{F}_2^n$,

$$\alpha_1(|x|_\mathcal{A}) \leqslant V(x) \leqslant \alpha_2(|x|_\mathcal{A}), \tag{6a}$$
$$V(g) - V(x) \leqslant -\varrho(x), \quad \forall g \in G(x). \tag{6b}$$

The following theorem gives necessary and sufficient conditions for global asymptotic stability of the set $\mathcal{A}$ for (2), together with a constructive procedure to build a Lyapunov function V and $\varrho \in \mathcal{PD}(\mathcal{A})$ for this systems.

Theorem 1 *Let* $\mathcal{A} \subset \mathbb{F}_2^n$ *and the system* (2) *be given. The set* $\mathcal{A}$ *is GAS for* (2) *if and only if there exists a Lyapunov function* V *for* (2).

Proof. If $\mathcal{A} = \mathbb{F}_2^n$, the proof is trivial. Hence, assume that $\mathcal{A} \neq \mathbb{F}_2^n$ is GAS for (2). Hence, by Lemma 5, there exists $T \in \mathbb{Z}_{\geqslant 0}$ such that, for any $\boldsymbol{x} \in \mathcal{S}(x)$, $x \in \mathbb{F}_2^n$, $|\boldsymbol{x}_T| = 0$, and $G(\mathcal{A}) \subset \mathcal{A}$. Thus, consider the set $\mathcal{I}_1 = G^{-1}(\mathcal{A}) \setminus \mathcal{A}$. Since $\mathcal{A}$ is attractive and $G(\mathcal{A}) \subset \mathcal{A}$, the set $\mathcal{I}_1$ is nonempty. Note that, since $\mathcal{A}$ is attractive, for any $\boldsymbol{x} \in \mathcal{S}(x)$, $x \notin \mathcal{A}$, there exists $T_1 \in \mathbb{Z}_{\geqslant 0}$ (depending on x), such that $x_{T_1} \in \mathcal{I}_1$. Thus, define $\mathcal{I}_\nu = G^{-1}(\mathcal{I}_{\nu-1})$, $\nu = 2, \ldots, 2^n - 1$, and

$$\Theta_\nu = I_\nu \cap \left(\mathbb{F}_2^n \setminus \left(\bigcup_{h=\nu+1}^{2^n-1} \mathcal{I}_h \cup \mathcal{A} \right) \right)$$

$\nu = 1, \ldots, 2^n - 1$. Since the set $\mathcal{A}$ is attractive, whence $(G^{-1})^h(\mathcal{A}) := G^{-1} \circ G^{-1} \circ \cdots \circ G^{-1}(\mathcal{A}) = \mathbb{F}_2^n$, for sufficiently large h, $\bigcup_{\nu=1}^{2^n} \Theta_\nu \cup \mathcal{A} = \mathbb{F}_2^n$. Hence, since $\Theta_{\nu_1} \cap \Theta_{\nu_2} = \emptyset$, for $\nu_1 \neq \nu_2$, and $\Theta_\nu \cap \mathcal{A} = \emptyset$, $\nu = 1, \ldots, 2^n$, the sets Θ_ν, $\nu = 1, \ldots, 2^n$, and $\mathcal{A}$, form a *partition* of $\mathbb{F}_2^n$. Moreover, each set Θ_ν is such that $G(\Theta_\nu) \subset \bigcup_{h=1}^{\nu-1} \Theta_h \cup \mathcal{A}$ (namely, by construction, each set Θ_ν is the set of all the points in $\mathbb{F}_2^n$ that reaches $\mathcal{A}$ in at most ν time instants). Note that if $\mathcal{A}$ is GAS, it is reached by all the points in $\mathbb{F}_2^n$ at most in $2^n - 1$ steps. Therefore, a Lyapunov function for (2), satisfying (6), is given by $V : \mathbb{F}_2^n \to \mathbb{Z}_{\geqslant 0}$,

$$V(x) = \begin{cases} 0, & \text{if } x \in \mathcal{A}, \\ \nu, & \text{if } x \in \Theta_\nu,\ \nu = 1, \ldots, 2^n - 1. \end{cases}$$

which is well defined over $\mathbb{F}_2^n$, because Θ_ν and $\mathcal{A}$ form a partition of $\mathbb{F}_2^n$. As a matter of fact, V satisfies (6a), with $\alpha_1 : r \mapsto 2^{-n} r$ and $\alpha_2 : r \mapsto 2^n r$. Moreover, V satisfies (6b), with $\varrho(x) = 1 - \mathbb{I}_\mathcal{A}(x)$, where $\mathbb{I}_\mathcal{A}(\cdot)$ is the *indicator function of* $\mathcal{A}$ (Fristedt & Gray 1997, Chap. 2, Def. 8), because, for each $x \in \Theta_\nu$, $G(x) \subset \bigcup_{h=1}^{\nu-1} \Theta_h \cup \mathcal{A}$, whence $V(g) \leqslant V(x) - 1$, $\forall g \in G(x)$, $\forall x \notin \mathcal{A}$. Note that, by

Menini et al. (2017, Thm. 3), there exists a polynomial $\upsilon : \mathbb{F}_2^n \to \mathbb{Z}_{\geqslant 0}$, such that $\upsilon(x) = V(x)$, for all $x \in \mathbb{F}_2^n$.

Sufficiency follows from Goebel et al. (2012). $\square$

Remark 1 *The technique given in the proof of Theorem 1 allows to determine whether a given set $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS for* (2)*. In fact, by computing $\Theta_1, \ldots, \Theta_p, \Theta_{p+1}$ as in such a proof, where $p+1 \leqslant 2^n$ is such that $\Theta_p \neq \emptyset$ and $\Theta_{p+1} = \emptyset$, one has that $\mathcal{A}$ is GAS for* (2) *if and only if $G(\mathcal{A}) \subset \mathcal{A}$ and $\mathcal{A}, \Theta_1, ..., \Theta_p$ partition $\mathbb{F}_2^n$.* $\diamond$

Remark 2 *In order to determine if the set $\mathcal{A}$ is GAS for* (2) *either the statement of Lemma 5 or the procedure given in the proof of Theorem 1 can be used. The complexity of these two methods is, in the worst case, $\frac{1}{3}2^{n-1}(4^n - 1)$ and $2^{n-1}(2^n - 1)$, respectively. Therefore, the latter is more effective.* $\diamond$

Remark 3 *The constructive procedure given in the proof of Theorem 1 to build a Lyapunov function for* (2) *with respect to $\mathcal{A} \subset \mathbb{F}_2^n$ can be interpreted in a graph theoretic framework. Namely, consider the directed graph (W, E), where $W := \{0, \ldots, 2^n - 1\}$ is the* vertex set *and $E := \{(x, y) \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, 2^n - 1\} : \pi_n^{-1}(y) \in G(\pi_n^{-1}(x))\}$ is the* edge set*. In the following, we refer to the graph (W, E) as* transition graph*. A* strongly connected component *(briefly,* SCC*) of (W, E) is a maximal set of vertices $C \subset W$ such that, for every pair of vertices $u, v \in C$, there exists a* path *from u to v and a path from v to u. In Cormen (2009, Sec. 22.5) a procedure is given to compute all the SCCs components of a given graph. A SCC is said to be* terminal *if no other SCC can be reached from it. The* condensation digraph *of a (W, E), denoted by $\mathcal{C}((W, E))$, is defined as follows: the nodes of $\mathcal{C}((W, E))$ are the strongly connected components of (W, E), and there exists a directed edge in $\mathcal{C}((W, E))$ from node H_1 to node H_2 if and only if there exists a directed edge in (W, E) from a node of H_1 to a node of H_2 (Bollobás 2013). By construction, the graph $\mathcal{C}((W, E))$ is acyclic. The set of all the terminal SCCs can be obtained by applying the topological sort algorithm (Cormen 2009, Sec. 22.4) to $\mathcal{C}((W, E))$. By Lemma 5, it can be easily argued that the union of all the terminal SCCs of the transition graph corresponds to the unique smallest GAS set $\mathcal{A}$ for* (2)*, provided that the graph $(W, E) \setminus (\mathcal{A}, \{(x, y) \in E : x \in \mathcal{A} \vee y \in \mathcal{A}\})$ is acyclic. The constructive procedure given in the proof of Theorem 1 to build a Lyapunov function for* (2) *is related to such a procedure. As a matter of fact, the method given in the proof of Theorem 1 to construct the sets $\mathcal{I}_\nu$ and Θ_ν, $\nu = 1, \ldots, 2^n - 1$, translates in the Boolean framework of this paper the* DFS(G) *algorithm given in Cormen (2009, Sec. 22.3) that is employed in the* TOPOLOGICAL–SORT(G) *algorithm given in Cormen (2009, Sec. 22.4) to carry out the topological sort of a given directed acyclic graph.* $\diamond$

The advantage of having at one's disposal a Lyapunov characterization of global asymptotic stability of Boolean networks with set–valued transition map relies on the concept of Control Lyapunov Function (briefly, CLF). Namely, let a *Boolean Control Network* (i.e., a DBN where the transition map $g : \mathbb{F}_2^n \times \mathbb{F}_2^m \to \mathbb{F}_2^n$ depends on an input variable $u \in \mathbb{F}_2^m$) of the form

$$x^+ = g(x, u), \tag{7}$$

be given. Hence, following the idea of the proof of Theorem 1, a feedback $u \in \kappa(x)$ such that the set $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS for the closed loop system can be obtained by determining $V : \mathbb{F}_2^n \to \mathbb{R}_{\geqslant 0}$ such that (6a) holds and

$$\min_u \{V(g(x, u)) - V(x)\} \leqslant \mathbb{I}_{\mathcal{A}}(x) - 1, \quad \forall x \in \mathbb{F}_2^n.$$

If such a function exists, then $u \in \arg\min_u \{V(g(x, u)) - V(x)\}$ is such that the set $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS for the closed loop system, otherwise there does not exist a feedback policy $u \in \kappa(x)$ such that the set $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS for the closed loop system. The possibly non–unique value of the function $\arg\min(\cdot)$ highlights once more the interest in Boolean networks with set–valued transition map.

4 Random solutions and stability properties of Stochastic Boolean Networks

Consider the SBN (3). Let $\mathcal{S}_{c,m}(x)$ denote the set of maximal random solutions to (3) starting at x that are causal, measurable functions of the inputs. That is, $\boldsymbol{\phi} \in \mathcal{S}_{c,m}(x)$, if $\boldsymbol{\phi}$ comprises a sequence of measurable functions $\boldsymbol{\phi}_j : \operatorname{dom} \boldsymbol{\phi}_j \to \mathbb{F}_2^n$, with $\operatorname{dom} \boldsymbol{\phi}_j \subset (\mathbb{Z}_{<N})^j$, $j \in \mathbb{Z}_{\geqslant 0}$ and $\boldsymbol{\phi}_0 = x$, such that $\boldsymbol{\phi}_{j+1}(w_0, \ldots, w_j) \in G(\boldsymbol{\phi}_j(w_0, \ldots, w_{j-1}), w_j)$, for all $j \in \mathbb{Z}_{\geqslant 0}$ and all $(w_0, \ldots, w_j) \in \operatorname{dom} \boldsymbol{\phi}_{j+1}$

A *random process $\boldsymbol{x}$ from $x \in \mathbb{F}_2^n$* is a sequence of random variables $\boldsymbol{x}_j : \Omega \to \mathbb{F}_2^n$, $j \in \mathbb{Z}_{\geqslant 0}$, with $\boldsymbol{x}_0 = x$, for all $\omega \in \Omega$. A random process $\boldsymbol{x}$ is *adapted to the natural filtration of $\boldsymbol{w}$* if $\boldsymbol{x}_{j+1}$ is $\mathcal{F}_j$–measurable for each $j \in \mathbb{Z}_{\geqslant 0}$. A random process $\boldsymbol{x}$ from $x \in \mathbb{F}_2^n$ that is adapted to the natural filtration of $\boldsymbol{w}$ (i.e.,, $\boldsymbol{x}_{j+1}^{-1}(F) \in \mathcal{F}_j$, for each $F \in \mathcal{P}(\mathbb{F}_2^n)$) is a *random solution to* (3) if $\boldsymbol{x}_{j+1}(\omega) \in G(\boldsymbol{x}_j(\omega), \boldsymbol{w}_j(\omega))$, for all $\omega \in \operatorname{dom} \boldsymbol{x}_{j+1}$ and $j \in \mathbb{Z}_{\geqslant 0}$ (Subbaraman & Teel 2013). A random solution $\boldsymbol{x}$ from $x \in \mathbb{F}_2^n$ is said to be *maximal*, denoted $\boldsymbol{x} \in \mathcal{S}_r(x)$, if it cannot be extended, i.e., there does not exist another random solution $\boldsymbol{y}$ from x such that $\operatorname{dom} \boldsymbol{x}_j \subset \operatorname{dom} \boldsymbol{y}_j$ for all $j \in \mathbb{Z}_{\geqslant 0}$, $\boldsymbol{y}_j(\omega) = \boldsymbol{x}_j(\omega)$, for all $\omega \in \operatorname{dom} \boldsymbol{x}_i$ and all $j \in \mathbb{Z}_{\geqslant 0}$, and $\operatorname{dom} \boldsymbol{x}_j \neq \operatorname{dom} \boldsymbol{y}_j$ for some $j \in \mathbb{Z}_{\geqslant 0}$. Define graph $(\boldsymbol{x}(\omega)) := \bigcup_{j \in \mathbb{Z}_{\geqslant 0}} \{j\} \times \boldsymbol{x}_j(\omega)$.

A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *Lyapunov stable in probability for* (3) if, for each $\varepsilon > 0$ and $\varrho > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &|x|_{\mathcal{A}} < \delta, \boldsymbol{x} \in \mathcal{S}_r(x) \implies \\ &\quad \mathbb{P}(\operatorname{graph}(\boldsymbol{x}) \subset (\mathbb{Z}_{\geqslant 0} \times (\mathcal{A} + \mathbb{B}(0, \varepsilon)))) \geqslant 1 - \varrho. \end{aligned} \tag{8}$$

A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *Lagrange stable in probability for* (3) if for each $\delta > 0$ and $\varrho > 0$, there exists $\varepsilon > 0$ such that (8) holds. The set $\mathcal{A}$ is *globally stable in probability for* (3) if it is both Lyapunov stable and Lagrange stable in probability for (3). The set $\mathcal{A}$ is *attractive in probability for* (3) if for each $\varepsilon > 0$, $\varrho > 0$, there exists $\tau > 0$ such that, for any $x \in \mathbb{F}_2^n$ and $\boldsymbol{x} \in \mathcal{S}_r(x)$,

$$\mathbb{P}((\mathrm{graph}\,(\boldsymbol{x}) \cap (\mathbb{Z}_{\geqslant \tau} \times \mathbb{F}_2^n)) \subset (\mathbb{Z}_{\geqslant \tau} \times (\mathcal{A} + \mathbb{B}(0, \varepsilon)))) \geqslant 1 - \varrho. \quad (9)$$

A set $\mathcal{A} \subset \mathbb{F}_2^n$ is *globally asymptotically stable* (briefly, *GAS*) *in probability for* (3) if it is both globally stable and attractive in probability for (3). Informally, a set $\mathcal{A} \subset \mathbb{F}_2^n$ is GAS if the probability that all the solutions starting nearby $\mathcal{A}$ stay nearby $\mathcal{A}$ is greater that $1 - \rho$, for each $\rho > 0$, while $\mathcal{A}$ is GAS if it is stable and the probability that all the solutions reach the set $\mathcal{A}$ tends to 1 as time goes to infinity.

The following lemma shows that a set $\mathcal{A} \in \mathbb{F}_2^n$ is stable if and only if it is *strongly forward invariant in probability for* (3), i.e., if it is such that the subsequent (10) holds.

Lemma 6 *Let the system* (3) *and* $\mathcal{A} \subset \mathbb{F}_2^n$ *be given. The set* $\mathcal{A}$ *is globally stable in probability for* (3) *if and only if*

$$x \in \mathcal{A}, \boldsymbol{x} \in \mathcal{S}_r(x) \implies \mathbb{P}(\mathrm{graph}\,(\boldsymbol{x}) \subset (\mathbb{Z}_{\geqslant 0} \times \mathcal{A})) = 1. \quad (10)$$

Proof. Every set $\mathcal{A} \subset \mathbb{F}_2^n$ is clearly Lagrange stable, because $|x|_{\mathcal{A}} \leqslant 2^n$, for each $x \in \mathbb{F}_2^n$. Hence, if (10) holds, then (8) is satisfied with $\delta = 1$. Thus, since $\mathcal{A}$ is both Lyapunov and Lagrange stable in probability for (3), $\mathcal{A}$ is globally stable in probability for (3).

Assume now that $\mathcal{A}$ is globally stable in probability for (3). Consider that if (8) holds for some $\delta > 0$, it holds also for $\delta = 1$. As a matter of fact, one has $|x|_{\mathcal{A}} < \delta$, for some $\delta \in (0, 1)$, if and only if $x \in \mathcal{A}$, and $x \in \mathcal{A}$ if and only if $|x|_{\mathcal{A}} < 1$. Hence, since $\mathcal{A} + \mathbb{B}(0, \varepsilon) = \mathcal{A}$ and (8) holds for all $\varrho > 0$, the set $\mathcal{A}$ is such that (10) holds. $\square$

Lemma 6 essentially states that the set $\mathcal{A} \subset \mathbb{F}_2^n$ is globally stable in probability for (3) if and only if the probability that all the solutions starting in $\mathcal{A}$ stay in $\mathcal{A}$ equals 1. The proof of the following lemma follows from (9).

Lemma 7 *Let the system* (3) *and* $\mathcal{A} \subset \mathbb{F}_2^n$ *be given. The set* $\mathcal{A}$ *is attractive in probability for* (3) *if and only if*

$$\lim_{\tau \to \infty} \mathbb{P}((\mathrm{graph}\,(\boldsymbol{x}) \cap (\mathbb{Z}_{\geqslant \tau} \times \mathbb{F}_2^n)) \subset (\mathbb{Z}_{\geqslant \tau} \times A)) = 1. \quad (11)$$

The following assumption guarantees that the integrals appearing in the study of (3) are well defined (Subbaraman & Teel 2013) and existence of random solutions to (3) (Teel et al. 2014, Prop. 1).

Assumption 1 *(Stochastic Basic Condition) The mapping* $w \mapsto \mathrm{graph}\,(G(\cdot, w))$ *is measurable, where* $\mathrm{graph}\,(G(\cdot, w)) = \{(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : y \in G(x, w)\}$.

Note that, since, for each $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$, the domain of $w \mapsto \mathrm{graph}\,(G(\cdot, w))$ is countable (and hence measurable), Assumption 1 holds for each SBN (Teel et al. 2014). The following fact states that our formalism allows to encode the dynamical behavior of PBNs.

Fact 1 *Let a PBN be given, let* $\tilde{N} := 2^{n2^n}$. *There exists* $g : \mathbb{F}_2^n \times \mathbb{Z}_{<\tilde{N}} \to \mathbb{F}_2^n$ *such that*

$$x^+ = g(x, w), \quad \mu(\cdot), \quad (12)$$

is a discrete–time representation of the PBN, where the probability measure $\mu(\cdot)$ *depends on the data of the PBN.*

Proof. In Datta et al. (2003), Pal et al. (2005), an expression for the transition probability from state ${}^i\bar{x} \in \mathbb{F}_2^n$ to state ${}^k\bar{x} \in \mathbb{F}_2^n$, $i, k \in \{0, 1, \ldots, 2^n - 1\}$ is derived from the data of a PBN. Note that, since $\mathbb{F}_2^n$ is composed by 2^n different elements, the transition probability from any state ${}^i\bar{x} \in \mathbb{F}_2^n$ to any state ${}^k\bar{x} \in \mathbb{F}_2^n$ can be encoded into a $2^n \times 2^n$ matrix P (Pal et al. 2005). Namely, by ordering the points in $\mathbb{F}_2^n$ so that ${}^i\bar{x} < {}^k\bar{x}$ if and only if $\pi_n({}^i\bar{x}) < \pi_n({}^k\bar{x})$, $i, k \in \{0, 1, \ldots, 2^n - 1\}$, (which is a total, well ordering on the points in $\mathbb{F}_2^n$), the transition probability from ${}^i\bar{x}$ to ${}^k\bar{x}$, $i, k \in \{0, 1, \ldots, 2^n - 1\}$, for the PBN constitutes the $(i+1, k+1)$–th entry $P_{i+1,k+1}$ of the matrix P. By Lemma 3, there exist $\tilde{N} := 2^{n2^n}$ transition maps updating the Boolean network state, at each time $j \in \mathbb{Z}_{\geqslant 0}$. The probability of selecting one of these $\tilde{N}$ transition maps can be easily obtained by P (e.g., the probability of choosing the map $x \mapsto x$ is given by $\prod_{i=1}^{2^n} P_{i,i}$). Moreover, the probability distribution is independent of the time and of the previous selections. By this reasoning, for each PBN, there exists $g : \mathbb{F}_2^n \times \mathbb{Z}_{<\tilde{N}} \to \mathbb{F}_2^n$ such that (12) is a discrete–time representation of the PBN, where $\boldsymbol{w} = \{\boldsymbol{w}_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. input random variables. $\square$

The following example, which employs the tools used in the proof of Fact 1, shows how to define a SBN encoding the dynamical behavior of a given PBN.

Example 3 *Consider the PBN given in Shmulevich et al. (2002b, Ex. 1) consisting of three genes (i.e.,* $x \in \mathbb{F}_2^3$*). On the basis of the truth table given in (Shmulevich et al. 2002b, p. 267), define the following Boolean maps for all* $x \in \mathbb{F}_2^3$, $\hat{g}_1^{(1)}(x) = \hat{g}_1^{(2)}(x) = f_1^{(1)}(x) = (x_2 \wedge x_3) \oplus x_2 \oplus x_3$, $\hat{g}_1^{(3)}(x) = \hat{g}_1^{(4)}(x) = f_1^{(2)}(x) = (x_1 \wedge x_2 \wedge x_3) \oplus x_2 \oplus x_3$,

$\hat{g}_2^{(1)}(x) = \hat{g}_2^{(2)}(x) = \hat{g}_2^{(3)}(x) = \hat{g}_2^{(4)}(x) = f_2^{(1)}(x) = (x_1 \wedge x_2 \wedge x_3) \oplus (x_1 \wedge x_3) \oplus x_1 \oplus x_2 \oplus x_3$, $\hat{g}_3^{(1)}(x) = \hat{g}_3^{(3)}(x) = f_3^{(1)}(x) = (x_1 \wedge x_2) \oplus (x_1 \wedge x_3) \oplus (x_2 \wedge x_3)$, *and* $\hat{g}_3^{(2)}(x) = g_3^{(4)}(x) = f_3^{(2)}(x) = x_1 \wedge x_2 \wedge x_3$. *Hence, letting* $\hat{g}^{(i)} = [\,\hat{g}_1^{(i)}\ \hat{g}_2^{(i)}\ g_3^{(i)}\,]^\top$, $\hat{g}^{(i)} : \mathbb{F}_2^3 \to \mathbb{F}_2^3$, *the probability* P_i *that the state of the PBN is updated according to the Boolean map* $g^{(i)}$ *is given by Shmulevich et al. (2002b, (7)),* $i = 1, \ldots, 4$. *Hence, a SBN encoding the probabilistic behavior of such a PBN is given by* (12), *with* $g : \mathbb{F}_2^3 \times \mathbb{Z}_{<4} \to \mathbb{F}_2^3$, $g(x, w) = \hat{g}^{(w)}(x)$, $w : \Omega \to \mathbb{Z}_{<4}$, $\mu(\{w\}) = P_{w+1}$, *for all* $(x, w) \in \mathbb{F}_2^3 \times \mathbb{Z}_{<4}$. $\triangle$

Example 3 highlights the interest arising from the analysis of SBNs, showing that these systems can encode the dynamical behavior of PBNs. Moreover, SBNs enrich the set of dynamical behaviors that can be modeled by PBNs, thanks to the set–valued transition map G.

As already done for non–stochastic Boolean networks with set–valued transition map, we can frame global stability of the SBN (3) in terms of a nonnegative function.

Proposition 2 *Let* $\mathcal{A} \subset \mathbb{F}_2^n$ *and the system* (3) *be given. Then,* $\mathcal{A}$ *is globally stable for* (3) *if and only if there exists* $V : \mathbb{F}_2^n \to \mathbb{R}_{\geqslant 0}$ *such that there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha_1(|x|_{\mathcal{A}}) \leqslant V(x) \leqslant \alpha_2(|x|_{\mathcal{A}}), \tag{13a}$$

$$\sum_{w=0}^{N-1} \sup_{g \in G(x,w)} V(g)\mu(\{w\}) = 0, \quad \forall x \in \mathcal{A}. \tag{13b}$$

Proof. By Lemma 6, the set $\mathcal{A}$ is stable for (3) if and only if (10) hold, i.e. if and only if (Teel 2013)

$$x \in \mathcal{A}, \boldsymbol{x} \in \mathcal{S}_r(x) \implies \mathbb{P}(\boldsymbol{x}_j \in \mathcal{A}, j \in \mathbb{Z}_{\geqslant 0}) = 1. \tag{14}$$

Note that, if V is such that (13a) holds, then $V(x) = 0$, for all $x \in \mathcal{A}$, and $V(x) > 0$, for all $x \in \mathbb{F}_2^n \setminus \mathcal{A}$. Assume that (14) holds but (13b) does not. Thus, there exists $w \in \mathbb{Z}_{<N}$ such that $\mu(\{w\}) \neq 0$ and $\sup_{g \in G(x,w)} V(g) \neq 0$, for some $x \in \mathcal{A}$, i.e., for such a w, letting $M : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$, $M(x) = G(x, w)$, $M(\mathcal{A}) \cap (\mathbb{F}_2^n \setminus \mathcal{A}) \neq \emptyset$. Hence, (14) does not hold, leading to a contradiction.

On the other hand, if (13b) holds, then there exists no $w \in \mathbb{Z}_{<N}$, with $\mu(\{w\}) \neq 0$, such that $\sup_{g \in G(x,w)} V(g) > 0$, for some $x \in \mathcal{A}$, i.e., for all w such that $\mu(\{w\}) \neq 0$, $G(x, w) \subset \mathcal{A}$, for any $x \in \mathcal{A}$. Therefore, (14) holds. $\square$

A function $V : \mathbb{F}_2^n \to \mathbb{R}_{\geqslant 0}$ is a *stochastic Lyapunov function relative to* $\mathcal{A} \subset \mathbb{F}_2^n$ *for* (3) if exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\varrho \in \mathcal{PD}(\mathcal{A})$, such that, for all $x \in \mathbb{F}_2^n$, (13a) holds and

$$\sum_{v=0}^{N-1} \sup_{g \in G(x,v)} V(g)\mu(v) \leqslant V(x) - \varrho(x). \tag{15}$$

The following theorem follows from Teel et al. (2014, Thm. 1) and from the fact that all the maps $w \mapsto \operatorname{graph}(G(\cdot, w))$ have measurable domains.

Theorem 2 *The set* $\mathcal{A} \subset \mathbb{F}_2^n$ *is GAS in probability for* (3) *if and only if there exists a stochastic Lyapunov function relative to* $\mathcal{A}$ *for* (3).

In the following, a constructive procedure, similar to the one given in Possieri & Teel (2016), is given to compute a stochastic Lyapunov function. Recall the definition of the map $\pi_n : \mathbb{F}_2^n \to \mathbb{Z}_{\geqslant 0}$, $\pi_n(x) = \sum_{i=1}^n 2^{i-1}\psi^{-1}(x_i)$. Let $\mathcal{N} := \{w \in \mathbb{Z}_{<N} : \mu(\{w\}) \neq 0\} = \{w_1, \ldots, w_H\}$, and let $G_w : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$ be the set–valued mappings defined as $G_w(x) = G(x, w)$, for each $x \in \mathbb{F}_2^n$, for all $w \in \mathcal{N}$. For each $i \in \{0, \ldots, 2^n - 1\}$, $j \in \{1, \ldots, H\}$, let $\Lambda_{i,j} := \{y \in \mathbb{Z}_{\geqslant 0} : \exists z \in G(\pi_n^{-1}(i), w_j)$ such that $y = \pi_n(z)\}$. Define vectors $v = [\,v_0 \cdots v_{2^n-1}\,]^\top$, $\rho = [\,\rho_0 \cdots \rho_{2^n-1}\,]^\top$ of unknowns, and the constant vector $\gamma = [\,\mu(\{w_1\}) \cdots \mu(\{w_H\})\,]^\top \in \mathbb{R}^H$. Let S be a matrix of unknowns, whose (i, j)–th entry is defined as

$$S_{ij} := \max_{k \in \Lambda_{i-1,j}} v_k, \quad i = 1, \ldots, 2^n, \quad j = 1, \ldots, H.$$

Consider the following problem:

$$\begin{aligned} &S\gamma + \rho \leqslant v, \\ &v_i = 0,\ \rho_i = 0, \quad \forall i \text{ such that } \pi_n^{-1}(i) \in \mathcal{A}, \\ &v_i > 0,\ \rho_i > 0, \quad \forall i \text{ such that } \pi_n^{-1}(i) \notin \mathcal{A}, \end{aligned} \tag{16}$$

The following theorem shows that there exists a stochastic Lyapunov function for (3) if and only if there exists a solution to the problem given in (16).

Theorem 3 *Let the SBN* (3) *be given. The set* $\mathcal{A} \subset \mathbb{F}_2^n$ *is GAS in probability for* (3) *if and only if there exists a solution to the problem given in* (16).

Proof. By Theorem 2, $\mathcal{A}$ is GAS in probability for (3) if and only if there exists a stochastic Lyapunov function relative to $\mathcal{A}$ for (3). If there exists $[\,v_0 \cdots v_{2^n-1}\,]^\top \in \mathbb{R}^{2^n}$ that solves (16), then let $V : \mathbb{F}_2^n \to \mathbb{R}_{\geqslant 0}$,

$$V(x) = v_{\pi_n(x)}, \quad x \in \mathbb{F}_2^n.$$

Such a function is such that (13a) holds, because $V(x) = 0$, $\forall x \in \mathcal{A}$ and $V(x) > 0$, $\forall x \notin \mathcal{A}$, whence there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (13a) holds. Moreover, the function V is such that (15) holds, with $\varrho(x) = \rho_{\pi_n(x)}$, for all $x \in \mathbb{F}_2^n$, and ϱ is such that $\varrho \in \mathcal{PD}(\mathcal{A})$, because $\varrho(x) = 0$ for all $x \in \mathcal{A}$ and $\varrho(x) > 0$ for all $x \notin \mathcal{A}$.

On the other hand, assume that there exists a function V such that (13a) and (15) hold, but there exists no solution to the problem given in (16). Thus, let $v_i = V(\pi_n^{-1}(i))$, $\rho_i = \varrho(\pi_n^{-1}(i))$, $i = 0, \ldots, 2^n - 1$. Since (15) holds, the real numbers v_i are such that $\sum_{j=1}^{H} \max_{k \in \Lambda_{i-1,j}} \{v_k\} \hat{\mu}(w_j) \leqslant v_i - \rho_i$, $i = 0, \ldots, 2^n - 1$. Moreover, since the function V is such that (13a) holds, $v_i = 0$, $\forall i$ such that $\pi_n^{-1}(i) \in \mathcal{A}$, $v_i > 0$, $\forall i$ such that $\pi_n^{-1}(i) \notin \mathcal{A}$. Furthermore, since $\varrho \in \mathcal{PD}(\mathcal{A})$, $\rho_i = 0$, $\forall i$ such that $\pi_n^{-1}(i) \in \mathcal{A}$, $\rho_i > 0$, $\forall i$ such that $\pi_n^{-1}(i) \notin \mathcal{A}$. Therefore, vectors $v = [\, v_0 \;\cdots\; v_{2^n-1} \,]^\top$ and $\rho = [\, \rho_0 \;\cdots\; \rho_{2^n-1} \,]^\top$ are a solution to (16). By this reasoning, if there exists a function V such that (13a) and (15) hold, then there exists a solution to the problem given in (16). $\square$

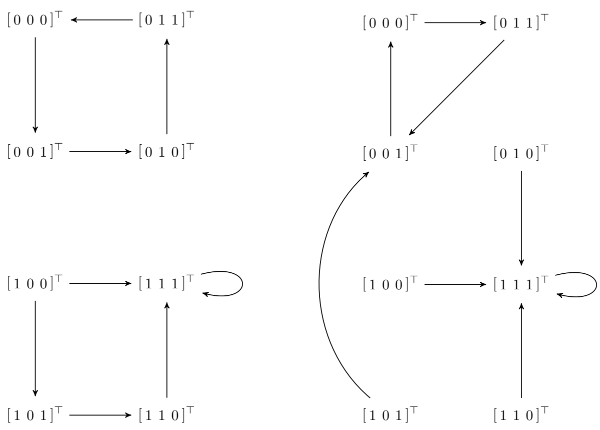
Remark 4 *A solution to the problem given in* (16) *can be obtained by computing the feasible region of a finite set of linear programming problems (Luenberger 1973), by considering all the possible inequality relation between the elements belonging to each entry* S_{ij} *of the matrix* S. *As a matter of fact, assuming that, for each* $i \in \{0, \ldots, 2^n - 1\}$, $j \in \{1, \ldots, H\}$, *a certain* $v_{k_{ij}}$, $k_{ij} \in \Lambda_{i,j}$, *is greater than or equal to* $v_{q_{ij}}$, *for all* $q_{ij} \in \Lambda_{i,j}$, *corresponds to add a finite set of linear constraint to the problem given in* (16). *Note that, with these assumptions, the problem given in* (16) *is a linear programming problem. Hence, by considering all the permutations of these assumptions, a solution to the problem given in* (16) *can be actually computed. Note that, even if each linear programming problem can be solved in polynomial time (Karmarkar 1984), in the worst case, the computational complexity of the problem given in* (16) *is* $O((2^n - 1)!)$. *As a matter of fact, in the worst case, all the orderings of the variables* v_i, $i = 0, \ldots, 2^n - 1$, *have to be taken into consideration. However, in many cases of practical interest, such a complexity is substantially lower, because only a small subset of the possible orderings of the variables* v_i, $i = 0, \ldots, 2^n - 1$, *has to be taken into account.* $\diamond$

The following example shows how the solution to the problem given in (16) can be used to compute a stochastic Lyapunov function relative to a set $\mathcal{A} \subset \mathbb{F}_2^n$ for (3).

Example 4 *Consider the SBN depicted in Figure 2, with* $G : \mathbb{F}_2^3 \times \mathbb{F}_2 \rightrightarrows \mathbb{F}_2^3$, $\mu(\{0\}) = 2/3$, *and* $\mu(\{1\}) = 1/3$.

Clearly, the set $\mathcal{A} = \{[\,1\;1\;1\,]^\top\}$ *is GAS in probability for the SBN. Define the vectors* $v = [\, v_0 \;\cdots\; v_7 \,]^\top$, $\rho = [\, \rho_0 \;\cdots\; \rho_7 \,]^\top$. *Hence, the vector* γ *is defined as*

$$\gamma = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \text{ while } S = \begin{bmatrix} v_1 & v_3 \\ v_2 & v_0 \\ v_3 & v_7 \\ v_0 & v_1 \\ \max\{v_5, v_7\} & v_7 \\ v_6 & v_1 \\ v_7 & v_7 \\ v_7 & v_7 \end{bmatrix}.$$



(a) Map $G(x, 0)$. (b) Map $G(x, 1)$.

Figure 2. SBN with three states.

By considering that $\pi_n^{-1}(7) = [\,1\;1\;1\,]^\top \in \mathcal{A}$, $v_7 = 0$, *whence, since* $v_i \geqslant 0$, $i = 0, \ldots, 7$, $\max\{v_5, v_7\} = v_5$. *A solution to the problem given in* (16) *is*

$$v = [\, 48 \;\, 43 \;\, 35 \;\, 49 \;\, 15 \;\, 19 \;\, 2 \;\, 0 \,]^\top,$$
$$\rho = [\, 1 \;\, 1 \;\, 1 \;\, 1 \;\, 1 \;\, 1 \;\, 1 \;\, 0 \,]^\top.$$

Therefore, by Menini et al. (2017) and Theorem 3,

$$V(x) = -25\psi^{-1}(x_2)\,\psi^{-1}(x_3)\,\psi^{-1}(x_1) + 9\psi^{-1}(x_3)\,\psi^{-1}(x_1) - 33\psi^{-1}(x_1) - 13\psi^{-1}(x_2) + 19\psi^{-1}(x_2)\,\psi^{-1}(x_3) - 5\psi^{-1}(x_3) + 48$$

is such that (15) *holds, with* $\varrho(x) = 1 - \mathbb{I}_{\mathcal{A}}$. *Moreover,* V *is such that* (13a) *holds, with* $\alpha_1(s) = \frac{1}{2}(s)$, $\alpha_2(s) = 48s$. *Thus,* V *is a stochastic Lyapunov function for the SBN.*

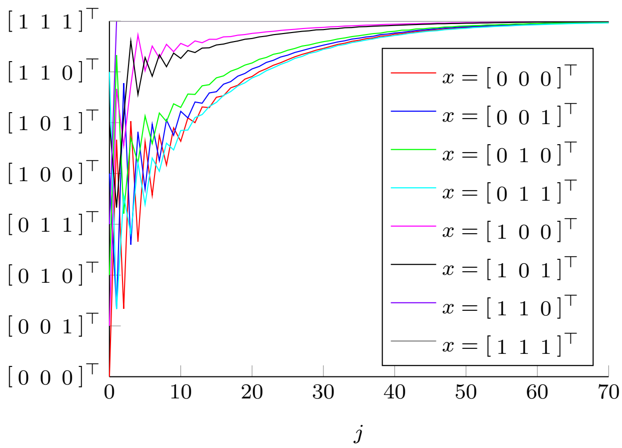


Figure 3. Numerical simulations of the solution to the SBN.

Numerical simulations have been carried out to analyze the dynamical behavior of the SBN depicted in Figure 2.

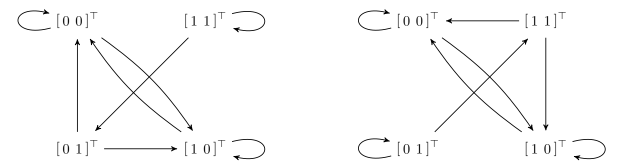
Namely, 10^4 simulation have been carried out from each initial condition $x \in \mathbb{F}_2^3$. Figure 3 depicts the results of such simulations, showing the expectation $\mathbb{E}(\boldsymbol{x})$. $\triangle$

Note that the existence of a stochastic Lyapunov function relative to a set $\mathcal{A} \subset \mathbb{F}_2^n$ guarantees the asymptotic stability in probability of such a set with respect to causal perturbations (Grammatico et al. 2013). The following (negative) example, recalling Grammatico et al. (2013, Ex. 1), shows that the existence of a stochastic Lyapunov function for system (3) does not imply asymptotic stability in presence of "non causal" selections.

Example 5 *Consider the SBN with state $x \in \mathbb{F}_2^2$*

$$x_1^+ \in \{0, 1\}, \tag{17a}$$
$$x_2^+ = (x_1 \oplus w)x_2, \tag{17b}$$

and assume that the random input $w \in \mathbb{F}_2$ has probability measure $\mu(\cdot)$ such that $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$. The transition map of the SBN (17) is depicted in Figure 4.



(a) Transition map $G(x, 0)$. (b) Transition map $G(x, 1)$.

Figure 4. Transition map of the system (17).

Define the set $\mathcal{A} := \{[\,0\ 0\,]^\top, [\,1\ 0\,]^\top\}$, the vectors $v := [\,v_0 \cdots v_3\,]^\top$, $\rho := [\,\rho_0 \cdots \rho_3\,]^\top$, $\gamma := [\,1/2\ 1/2\,]^\top$, and

$$S := \begin{bmatrix} \max\{v_0,v_2\} & \max\{v_0,v_2\} \\ \max\{v_0,v_2\} & \max\{v_1,v_3\} \\ \max\{v_0,v_2\} & \max\{v_0,v_2\} \\ \max\{v_1,v_3\} & \max\{v_0,v_2\} \end{bmatrix}.$$

A solution to the problem given in (16) *is* $v = [\,0\ 7\ 0\ 6\,]^\top$, $\rho = [\,0\ 1\ 0\ 1\,]^\top$. *Therefore, the function*

$$V(x) := 7\psi^{-1}(x_2) - \psi^{-1}(x_1 x_2)$$

is a stochastic Lyapunov function relative to $\mathcal{A}$ for (17). *Hence, by Menini et al. (2017, Thm. 3) and Theorem 3, the set $\mathcal{A}$ is GAS in probability for* (17). *However, by admitting "non causal" solutions, one has that the set $\mathcal{A}$ is not GAS in probability for* (17). *In fact, the "non causal" solution $x_1(\omega) := w(\omega) \oplus 1$ is such that $x_2^+ = x_2$, whence the set $\mathcal{A}$ is not GAS in probability. We refer to the solution $x_1(\omega) := w(\omega) + 1$ as* non causal *because x_1 depends on the current value of w.* $\triangle$

Remark 5 *In Example 1, a Boolean network with set–valued transition map modeling the main reactions of glycolysis is given. The set–valued transition map $G : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$ of such an example has been obtained by inspection of the network diagram corresponding to the stoichiometric matrix of such a reaction. However, a wholly probabilistic procedure can be employed to obtain the transition map $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$ and the distribution function $\mu(\cdot)$ from biological data. Namely, following the idea of Busetto & Lygeros (2014), let $\mathbb{G}$ be the set of all the set–valued transition maps $G : \mathbb{F}_2^n \rightrightarrows \mathbb{F}_2^n$. By Lemma 3, the cardinality of the set $\mathbb{G}$ is finite and equals $(2^{2^n} - 1)^{2^n}$. Assume that the modeler can measure the readout vector $y \in \mathbb{F}_2^r$, that consists of realizations of the random variable $\mathcal{Y}_t$, $y_j \sim H(\mathcal{Y}_j|x_j)$, where H is an experiment–dependent and task specific measurement distribution function. Hence, let $D := \{y_j\}_{j=1}^T$, $T \in \mathbb{Z}_{>0}$, be the set of measures available to reconstruct the transition map $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$. The modeler starts with a* prior *distribution function $p(\mathcal{G})$, $\mathcal{G} \in \mathbb{G}$ (possibly uniform, if the modeler has no previous knowledge on the system). Thus, by employing Bayes' Theorem (Jaynes 2003), the prior distribution $p(\mathcal{G})$ can be updated as*

$$p(\mathcal{G}|D) = \frac{p(D|\mathcal{G})p(\mathcal{G})}{p(D)} \tag{18}$$

where $p(D) = \sum_{\mathcal{G} \in \mathbb{G}} p(D|\mathcal{G})p(\mathcal{G})$. Note that, in (18), *the likelihood function $p(D|\mathcal{G})$ relates the models to the data and is specified by the known distribution H. Hence, by enumerating the set–valued mappings in $\mathbb{G}$ so that $\mathbb{G} = \{\mathcal{G}_0, \cdots, \mathcal{G}_{N-1}\}$, a SBN modeling the observed data in a wholly probabilistic framework is given by* (3), *where $G : \mathbb{F}_2^n \times \mathbb{Z}_{<N} \rightrightarrows \mathbb{F}_2^n$, $G(x, w) = \mathcal{G}_w(x)$, for all $(x, w) \in \mathbb{F}_2^n \times \mathbb{Z}_{<N}$, and $\mu(\{w\}) = p(\mathcal{G}_w|D)$, $w = 0, \ldots, N-1$. Even if the cardinality of $\mathbb{G}$ may be very large, usually the modeler is interested in a much smaller subclass of models, that are considered interesting on the basis of previous experiments or domain experience. Thus, the method proposed here to build SBNs from real data is tractable in many cases of practical interest.* $\diamond$

5 Conclusions

In this paper a new class of Boolean networks, called Stochastic Boolean Networks is presented. In these systems the state variables admit two operation levels (either 0 or 1), and the transition map, at each time instant, is selected through a random process. Hence, these systems enrich the set of admissible dynamical behaviors already encoded in the classical definition of DBNs and of PBNs, thanks to the set–valued nature of the transition map. The interest in SBN arises from the observation of non–unique dynamical behaviors of certain biological systems. Lyapunov conditions for the stability of these systems, together with constructive procedures to build Lyapunov functions, are given both for deterministic Boolean networks with set–valued transition map, and for SBNs. A procedure to identify SBN from real experimental data is proposed.

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