SCATTERING FOR THE L^2 SUPERCRITICAL POINT NLS

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Abstract. We consider the 1D nonlinear Schrödinger equation with focusing point nonlinearity. "Point" means that the pure-power nonlinearity has an inhomogeneous potential and the potential is the delta function supported at the origin. This equation is used to model a Kerr-type medium with a narrow strip in the optic fibre. There are several mathematical studies on this equation and the local/global existence of solution, blow-up occurrence and blow-up profile have been investigated. In this paper we focus on the asymptotic behavior of the global solution, i.e, we show that the global solution scatters as $t \to \pm\infty$ in the L^2 supercritical case. The main argument we use is due to Kenig-Merle, but it is required to make use of an appropriate function space (not Strichartz space) according to the smoothing properties of the associated integral equation.

1. Introduction

In this paper, we address a theoretical study on a model, proposed in [16], that describes a wave propagation in a 1D linear medium containing a narrow strip of nonlinear material, where the nonlinear strip is assumed to be much smaller than the typical wavelength. Considering such nonlinear strip may allow to model a wave propagation in nanodevices, in particular the authors in [13] consider some nonlinear quasi periodic super lattices and investigate an interplay between the nonlinearity and the quasi periodicity. Such a strip is described as an impurity, i.e. a delta measure in the nonlinearity of nonlinear Schrödinger equation. For applications in nanodevices, it should be important to study NLS with a quasi periodic location of delta measures, but in this paper, as a first step, we will treat the Schrödinger equation which has

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only one impurity in the nonlinearity:

$$\begin{cases} i\partial_t\psi + \partial_x^2\psi + K(x)|\psi|^{p-1}\psi = 0, & t \in \mathbb{R},\ x \in \mathbb{R} \\ \psi(x,0) = \psi_0(x) \end{cases} \tag{1.1}$$

where $p > 1$, and $K = \delta$, δ is the Dirac mass at $x = 0$. This singularity in the nonlinearity is interpreted as the linear Schrödinger equation:

$$i\partial_t\psi + \partial_x^2\psi = 0, \qquad t \in \mathbb{R}, \quad x \neq 0$$

together with the jump condition at $x = 0$

$$\begin{aligned} &\psi(0,t) := \psi(0-,t) = \psi(0+,t) \\ &\partial_x\psi(0+,t) - \partial_x\psi(0-,t) = -|\psi(0,t)|^{p-1}\psi(0,t). \end{aligned}$$

Remark that this equation (1.1) also appears as a limiting case of nonlinear Schrödinger equation with a concentrated nonlinearity (see [7]).

In [3, 11], it was proved that the equation (1.1) is locally well-posed for any $\psi_0 \in H^1(\mathbb{R})$ for $p > 1$, and Equation (1.1) has two conservative quantities: the mass

$$M(\psi) = \int |\psi|^2$$

and the energy

$$E(\psi) = \frac{1}{2}\int |\partial_x\psi|^2 - \frac{1}{p+1}|\psi(0)|^{p+1}.$$

The mass condition for the global existence/blow-up, further an analysis of the blow-up profile were established in [11, 12]. Furthermore, the problem of asymptotic stability of the standing waves of equation (1.1) has been treated in [5] and [14].

As far as we know, the asymptotic behavior, in particular, the scattering of the solution is not known for (1.1). For the standard NLS, i.e. $K \equiv 1$, in one dimensional case, such a result in H^1 was firstly established in [17]. This topic has been very active these decades thanks to a breakthrough result by Kenig-Merle [15]. Our proof therefore essentially will be based on Kenig-Merle [15], and some results after [15], for example [10]. However, it is required to make use of an appropriate function space (not Strichartz space) according to the smoothing properties of the associated integral equation to (1.1).

Higher-dimensional models with a generalization of the delta potential have been introduced in [2] and in [6] for the three and two-dimensional setting, respectively. While, at a qualitative level, the model in dimension three behaves like that in dimension one, the two-dimensional setting displays some uncommon features still to be understood (for the analysis of the blow-up, see [1]).

We remark that the model of a NLS with a standard power nonlinearity and a linear point interaction has been studied in [4].

Notation. If I is an interval of $\mathbb{R}$, and $1 \leq r \leq \infty$, then L^r_I is the space of strongly Lebesgue measurable, complex-valued functions v from I into $\mathbb{C}$ satisfying $\|v\|_{L^r_I} := \int_I |v(t)|^r dt < +\infty$ if $r < +\infty$, when $r = +\infty$, $\|v\|_{L^\infty_I} := \sup_{t\in I} |v(t)| < +\infty$. The space $C^0_I E$ denotes the space of continuous functions on I with values in a Banach space E.

For $s \in \mathbb{R}$, we define the Sobolev space

$$H^s = \{v \in \mathcal{S}'(\mathbb{R}),\ \|v\|_{H^s} := \|(1 + |\xi|^2)^{\frac{s}{2}}\widehat{v}(\xi)\|_{L^2_{\mathbb{R}}} < +\infty\},$$

and the homogeneous Sobolev space

$$\dot{H}^s = \{v \in \mathcal{S}'(\mathbb{R}),\ \|v\|_{\dot{H}^s} := \||\xi|^s\widehat{v}(\xi)\|_{L^2_{\mathbb{R}}} < +\infty\},$$

where $\widehat{f}$ is the Fourier transform of the function f. Thus, $H^0 = \dot{H}^0 = L^2_{\mathbb{R}}$, and this will be simply denoted as L^2. Sometimes we put an index t or x like $\dot{H}^s_t$ or $\dot{H}^s_x$ to enlighten which variable concerns. For $\alpha \in \mathbb{R}$, $|\nabla|^\alpha$ denotes the Fourier multiplier with symbol $|\xi|^\alpha$. For $s \geq 0$, define $v \in H^s_I$ if, when $v(x)$ is extended to $\tilde{v}(x)$ on $\mathbb{R}$ by setting $\tilde{v}(x) = 0$ for $x \notin I$, then $\tilde{v} \in H^s$; in this case we set $\|v\|_{H^s_I} = \|\tilde{v}\|_{H^s}$. Finally, χ_I denotes the characteristic function for the interval $I \subset \mathbb{R}$.

The equation (1.1) has a scaling invariance: if $\psi(x, t)$ is a solution to (1.1) then $\lambda^{\frac{1}{p-1}}\psi(\lambda x, \lambda^2 t)$, $\lambda > 0$ is also. The scale-invariant Sobolev space for (1.1) is $\dot{H}^{\sigma_c}$ with

$$\sigma_c = \frac{1}{2} - \frac{1}{p-1},$$

thus, for (1.1), $p = 3$ is the L^2 critical setting. If $p > 3$, then $0 < \sigma_c < \frac{1}{2}$ and

$$\frac{1}{4} < \frac{2\sigma_c + 1}{4} < \frac{1}{2}, \quad -\frac{1}{4} < \frac{2\sigma_c - 1}{4} < 0.$$

We take q and $\tilde{q}$ to be given by

$$\frac{1}{q} = \frac{1}{2} - \frac{2\sigma_c + 1}{4}, \quad \frac{1}{2} = \frac{1}{\tilde{q}} - \frac{1 - 2\sigma_c}{4},$$

and from the definition of σ_c, we find that

$$q = 2(p-1)\,, \qquad \tilde{q} = \frac{2(p-1)}{p}.$$

In the remainder of the paper, once $p > 3$ is selected, we will take σ_c, q and $\tilde{q}$ to have the corresponding values as defined above.

Recall that by Sobolev embedding, one has

$$\|\psi\|_{L^q_{\mathbb{R}}} \lesssim \|\psi\|_{\dot{H}^{\frac{2\sigma_c+1}{4}}}, \qquad \|f\|_{\dot{H}^{\frac{2\sigma_c-1}{4}}} \lesssim \|f\|_{L^{\tilde{q}}_{\mathbb{R}}}.$$

More generally than the above case, σ_c should satisfy $-\frac{1}{2} \leq \sigma_c < \frac{1}{2}$ to apply this Sobolev embedding, that is, the case $\sigma_c = 0$ (namely $p = 3$) is included for this embedding.

First, we recall here the local wellposedness result of (1.1) established in Theorem 1.1 of [11].

Proposition 1.1. *Let $p > 1$ and $\psi_0 \in H^1$. Then, there exist $T^* > 0$ and a solution $\psi(x,t)$ to (1.1) on $[0,T^*)$ satisfying for $T < T^*$,*

$$\psi \in C^0_{[0,T]} H^1_x \cap C^0_{\mathbb{R}} H^{\frac{3}{4}}_{(0,T)},$$
$$\partial_x \psi \in C^0_{\mathbb{R}_x \setminus \{0\}} H^{\frac{1}{4}}_{(0,T)}.$$

Here, the derivatives $\partial_x \psi(0^\pm, t) := \lim_{x\to\pm 0} \partial_x \psi(x,t)$, exist in the sense of $H^{\frac{1}{4}}_{(0,T)}$ and ψ satisfies

$$\partial_x \psi(0^+,t) - \partial_x \psi(0^-,t) = -|\psi(0,t)|^{p-1}\psi(0,t)$$

as an equality of $H^{\frac{1}{4}}_{(0,T)}$ functions (not pointwisely in t).

Among all solutions satisfying the above regularity conditions, it is unique. Moreover, the data-to-solution map $\psi_0 \mapsto \psi$, as a map $H^1_x \to C^0_{[0,T]} H^1_x$, is continuous, and if $T^* < +\infty$, then $\lim_{t\uparrow T^*} \|\partial_x \psi(t)\|_{L^2_{\mathbb{R}}} = +\infty$.

Hereafter, the solution to (1.1) satisfying the above regularity condition will be referred to as H^1_x solution to (1.1).

The local virial identity has been also proved in [11]. For any smooth weight function $a(x)$ satisfying $a(0) = \partial_x a(0) = \partial_x^{(3)} a(0) = 0$, the solution ψ to (1.1) satisfies

$$\partial_t^2 \int a(x)|\psi|^2\,dx = 4\int \partial_x^{(2)} a|\partial_x\psi|^2 - 2\partial_x^{(2)} a(0)|\psi(0)|^{p+1} - \int \partial_x^{(4)} a|\psi|^2. \tag{1.2}$$

Proposition 1.2 ([11, Prop 1.3] sharp Gagliardo-Nirenberg inequality)**.** *For any $\psi \in H^1$,*

$$|\psi(0)|^2 \le \|\psi\|_{L^2}\|\partial_x\psi\|_{L^2}\,. \tag{1.3}$$

Equality is achieved if and only if there exist $\theta \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$ such that $\psi(x) = \alpha e^{i\theta}\varphi_0(\beta x)$, where $\varphi_0 = 2^{\frac{1}{p-1}} e^{-|x|}$ is the ground state solution to (1.1) (see [11]*).*

Theorem 1.3 ([11, Prop 1.4] L^2 supercritical global existence/blow-up dichotomy)**.** *Suppose that $\psi(t)$ is an H^1_x solution of (1.1) for $p > 3$ satisfying*

$$M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi_0) < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0). \tag{1.4}$$

Let

$$\eta(t) = \frac{\|\psi\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\psi(t)\|_{L^2}}{\|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\varphi_0\|_{L^2}}$$

Then

(1) *If* $\eta(0) < 1$, *then the solution* $\psi(t)$ *is global in both time directions and* $\eta(t) < 1$ *for all* $t \in \mathbb{R}$.
(2) *If* $\eta(0) > 1$, *then the solution* $\psi(t)$ *blows-up in the negative time direction at some* $T_- < 0$, *blows-up in the positive time direction at some* $T_+ > 0$, *and* $\eta(t) > 1$ *for all* $t \in (T_-, T_+)$.

Remark that if $E(\psi_0) < 0$, then the condition (1.4) is satisfied, and in that case $\eta(t) > 1$ is forced by (1.3), so the condition (2) applies giving the blow-up.

Main result of this paper is the following.

Theorem 1.4. *(asymptotic completeness) Let* $p > 3$. *Let* $\psi_0 \in H^1$ *and let* $\psi(t)$ *be a* H^1_x *solution of (1.1) satisfying*

$$M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi_0) < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$$

and

$$\|\psi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \psi_0\|_{L^2} < \|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \varphi_0\|_{L^2}.$$

Then, there exist $\psi^+, \psi^- \in H^1$ *such that*

$$\lim_{t \to \pm\infty} \|e^{-it\partial_x^2} \psi(t) - \psi^{\pm}\|_{H^1_x} = 0.$$

We only consider the focusing nonlinearity, but the scattering for the defocusing case is similarly proved.

This paper is organized as follows: Below in Section 2, we will discuss the local theory, scattering criterion and long-time perturbation theory. Section 2 includes some preliminary and important results which reflect the smoothing properties of the equation (1.1). We will give in Section 3 the profile decomposition in H^1 in a form well-adapted to our equation. In Section 4, the asymptotic completeness in H^1 will be established using the results in Sections 2 and 3. We sometimes denote all through the paper by $C_{\theta,\dots}$ a constant which depends on θ and so on.

2. Local theory, scattering criterion, and long-time perturbation theory

Write the equation (1.1) in the Duhamel form:

$$\begin{aligned} \psi(x,t) &= e^{it\partial_x^2}\psi_0 + i\int_0^t e^{i(t-s)\partial_x^2}\delta(x)|\psi(x,s)|^{p-1}\psi(x,s)ds \\ &= e^{it\partial_x^2}\psi_0 + i\int_0^t \frac{e^{\frac{ix^2}{4(t-s)}}}{\sqrt{4\pi i(t-s)}}|\psi(0,s)|^{p-1}\psi(0,s)ds. \end{aligned} \tag{2.1}$$

We remark that the equation (1.1) is completely solved once the one-variable complex function $\psi(0,\cdot)$ is known: indeed, specializing (2.1) to the value $x = 0$, one obtains a closed, nonlinear, integral, a Volterra-Abel type equation for $\psi(0,\cdot)$;

$$\psi(0,t) = [e^{it\partial_x^2}\psi_0](0) + i\int_0^t \frac{1}{\sqrt{4\pi i(t-s)}}|\psi(0,s)|^{p-1}\psi(0,s)ds. \tag{2.2}$$

Now, for any $\sigma \in \mathbb{R}$, we define for $f \in \dot{H}^\sigma$, $t, s \in \mathbb{R}$ with $t \geq s$,

$$[\mathcal{L}_s f](x,t) := \int_s^t \frac{e^{\frac{ix^2}{4(t-\tau)}}}{\sqrt{4\pi i(t-\tau)}} f(\tau)d\tau.$$

Similarly, we define, for $t \in \mathbb{R}$,

$$[\Lambda f](x,t) := \int_t^\infty \frac{e^{\frac{ix^2}{4(t-\tau)}}}{\sqrt{4\pi i(t-\tau)}} f(\tau)d\tau.$$

The following smoothing properties of $\mathcal{L}_s$ and Λ will play important roles in what follows.

Proposition 2.1. *Let $\sigma \in \mathbb{R}$.*

(1) $\|[e^{i(t-s)\partial_x^2}f](0)\|_{\dot{H}_t^{\frac{2\sigma+1}{4}}} \lesssim \|f\|_{\dot{H}^\sigma}$, *for any $f \in \dot{H}^\sigma$ and $t, s \in \mathbb{R}$.*

(2) *Assume $-\frac{1}{2} < \frac{2\sigma-1}{4} < \frac{1}{2}$. Let $f \in \dot{H}^{\frac{2\sigma-1}{4}}$ and $s \in \mathbb{R}$.*
   (2a) $\|[\mathcal{L}_s f](0,\cdot)\|_{\dot{H}_t^{\frac{2\sigma+1}{4}}} \lesssim \|\chi_{[s,+\infty)}f\|_{\dot{H}^{\frac{2\sigma-1}{4}}} \lesssim \|f\|_{\dot{H}^{\frac{2\sigma-1}{4}}}$
   (2b) $\|[\Lambda f](0,\cdot)\|_{\dot{H}_t^{\frac{2\sigma+1}{4}}} \lesssim \|f\|_{\dot{H}^{\frac{2\sigma-1}{4}}}$

(3) *Assume $-\frac{1}{2} < \frac{2\sigma-1}{4} < \frac{1}{2}$. Let $f \in \dot{H}^{\frac{2\sigma-1}{4}}$ and $s \in \mathbb{R}$.*
   (3a) $\|\mathcal{L}_s f\|_{L^\infty_{\mathbb{R}_t}\dot{H}^\sigma_x} \lesssim \|f\|_{\dot{H}^{\frac{2\sigma-1}{4}}}$.
   (3b) $\|\Lambda f\|_{L^\infty_{\mathbb{R}_t}\dot{H}^\sigma_x} \lesssim \|f\|_{\dot{H}^{\frac{2\sigma-1}{4}}}$.

For the proof of Proposition 2.1, we need some preparations.

Lemma 2.2. *For any $-\frac{1}{2} < \mu < \frac{1}{2}$, and any $t > 0$, we have*

$$\|\chi_{[0,t]}(s)f(s)\|_{\dot{H}^\mu_s} \lesssim \|f\|_{\dot{H}^\mu_s} \tag{2.3}$$

with implicit constant independent of t.

Proof. First, we claim that it suffices to show

$$\|\chi_{[0,+\infty)}f\|_{\dot{H}^\mu_s} \lesssim \|f\|_{\dot{H}^\mu_s} \tag{2.4}$$

Indeed, suppose that we have proved (2.4). Since $\chi_{[0,t]} = \chi_{[0,+\infty)}\chi_{(-\infty,t]}$, to prove (2.3) we note

$$\begin{aligned}
\|\chi_{[0,t]}f\|_{\dot H^\mu_s} &= \|\chi_{[0,+\infty)}\chi_{(-\infty,t]}f\|_{\dot H^\mu_s} \\
&\lesssim \|\chi_{(-\infty,t]}f\|_{\dot H^\mu_s} && \text{by (2.4)} \\
&= \|\chi_{[0,+\infty)}\tilde f\|_{\dot H^\mu_s}
\end{aligned}$$

where $\tilde f(s) = f(-s+t)$. In the last step, we have used that

$$[\chi_{(-\infty,t]}(s)f(s)]\widehat{\ }(\tau) = e^{-it\tau}[\chi_{[0,\infty)}(s)f(-s+t)]\widehat{\ }(-\tau)$$

We continue and apply (2.4) to obtain

$$\|\chi_{[0,+\infty)}\tilde f\|_{\dot H^\mu_s} \lesssim \|\tilde f\|_{\dot H^\mu_s} = \|f\|_{\dot H^\mu_s}$$

where, in the last step, we used that $\widehat{\tilde f}(\tau) = e^{-it\tau}\hat f(-\tau)$. This completes the proof of (2.3) assuming (2.4).

To prove (2.4), we note $\hat\chi_{[0,+\infty)}(\tau) = \mathrm{pv}\,\frac{1}{i\tau} + \pi\delta(\tau)$ and thus

$$[\chi_{[0,+\infty)}f]\widehat{\ }(\tau) = \pi(H\hat f + \hat f)$$

where H denotes the Hilbert transform. Hence

$$\begin{aligned}
\|\chi_{[0,+\infty)}f\|_{\dot H^\mu} &= \||\tau|^\mu[\chi_{[0,+\infty)}f]\widehat{\ }(\tau)\|_{L^2_\tau} \\
&\lesssim \||\tau|^\mu(H\hat f)(\tau)\|_{L^2_\tau} + \||\tau|^\mu\hat f(\tau)\|_{L^2_\tau}
\end{aligned}$$

Since $-\frac12 < \mu < \frac12$, we can apply Corollary of Theorem 2 on page 205 in [18], combined with (6.4) on p. 218 of [18] (for $p = 2$, $n = 1$, $a = 2\mu$) to estimate the above as

$$\|\chi_{[0,+\infty)}f\|_{\dot H^\mu} \lesssim \||\tau|^\mu\hat f\|_{L^2_\tau} = \|f\|_{\dot H^\mu}.$$

□

Proof. (of Proposition 2.1) (1) was already proved in Lemma 1 of [3], but for the sake of completeness we give a proof. We use here the notation ˆ, which means the Fourier transform in space, and $\mathcal F$ is in time. It suffices to show the case $s = 0$. Since the free Schrödinger group is unitary in $\dot H^\sigma_x$ for any $\sigma \in \mathbb R$, We may write

$$[e^{it\partial_x^2}f](0) = \int_{\mathbb R_\xi} e^{-i\xi^2 t}\hat f(\xi)d\xi.$$

By a change of variables this equals

$$\int_0^{+\infty} e^{-ikt}\frac{\hat f(-\sqrt k) + \hat f(\sqrt k)}{2\sqrt k}dk.$$

Thus the Fourier transform in time gives

$$\mathcal F[(e^{it\partial_x^2}f)(0)](\omega) = 2\pi\frac{\hat f(-\sqrt\omega) + \hat f(\sqrt\omega)}{2\sqrt\omega}\chi_{[0,+\infty)}(\omega).$$

Therefore

$$\begin{aligned}\|[e^{it\partial_x^2}f](0)\|^2_{\dot H^\eta} &= \pi^2\int_{\mathbb{R}_\omega}|\omega|^{2\eta-1}|\hat f(-\sqrt{\omega})+\hat f(\sqrt{\omega})|^2\chi_{[0,+\infty)}(\omega)d\omega\\ &\le 2\pi^2\int_{\mathbb{R}_k}|k|^{4\eta-1}|\hat f(k)|^2dk\\ &= C\|f\|_{\dot H^{\frac{4\eta-1}{2}}},\end{aligned}$$

where, again we changed the variables $\pm\sqrt{\omega}=k$ in the second inequality. For (2a), we may write

$$\begin{aligned}[\mathcal{L}_sf](0,t) &= \int_s^t\frac{f(\tau)}{\sqrt{4\pi i(t-\tau)}}d\tau\\ &= \frac{1}{\sqrt{4\pi i}}\int_{-\infty}^{+\infty}(t-\tau)_+^{-\frac12}\chi_{[s,\infty)}(\tau)f(\tau)d\tau=\frac{1}{\sqrt{4\pi i}}(t_+^{-\frac12}*\chi_{[s,+\infty)}f)(t),\end{aligned}$$

where

$$t_+^{-\frac12}:=\begin{cases}t^{-\frac12}, & t>0\\ 0, & t\le 0,\end{cases}\qquad \widehat{t_+^{-\frac12}}(\xi)=(i\xi)^{-\frac12}\Gamma\left(\frac12\right).$$

We operate the Fourier transform and obtain

$$\widehat{[\mathcal{L}_sf](0,\cdot)}(\xi)=\frac{(i\xi)^{-\frac12}}{\sqrt{4i}}\widehat{\chi_{[s,\infty)}f}(\xi).$$

It thus follows that by Lemma 2.2, for $-\frac12<\frac{2\sigma-1}{4}<\frac12$,

$$\|[\mathcal{L}_sf](0,\cdot)\|^2_{\dot H^{\frac{2\sigma+1}{4}}}\le C\|\chi_{[s,+\infty)}f\|^2_{\dot H^{\frac{2\sigma-1}{4}}}\le C\|f\|^2_{\dot H^{\frac{2\sigma-1}{4}}}.$$

The proof of (2b) is similar, since

$$[\Lambda f](0,t)=\frac{-i}{\sqrt{4\pi i}}((-t)_+^{-\frac12}*f)(t).$$

For (3a), it suffices to prove that for any $g\in\dot H_x^{-\sigma}(\mathbb{R})$ with $\|g\|_{\dot H_x^{-\sigma}}=1$,

$$\langle\mathcal{L}_sf,g\rangle\le\|f\|_{\dot H_t^{\frac{2\sigma-1}{4}}}.$$

The left hand side can be estimated as follows.

$$\begin{aligned}\langle\mathcal{L}_sf,g\rangle &= \frac{1}{\sqrt{4\pi i}}\int_{-\infty}^{+\infty}\chi_{[s,t]}(\tau)f(\tau)[e^{i(t-\tau)\partial_x^2}\bar g](0)d\tau\\ &\le C\|\chi_{[s,t]}f\|_{\dot H^{\frac{2\sigma-1}{4}}}\|[e^{i(t-\cdot)\partial_x^2}\bar g](0)\|_{\dot H^{-\frac{2\sigma-1}{4}}}\\ &\le C\|f\|_{\dot H^{\frac{2\sigma-1}{4}}}\|g\|_{\dot H_x^{-\sigma}}\end{aligned}$$

where we have used (1) with the unitary property of free Schrödinger group in $\dot H_x^s$ for any $s\in\mathbb{R}$, and Lemma 2.2 in the last inequality. Since (3b) can be similarly proved, we omit the proof, but we remark that for any $\sigma \in \mathbb{R}$, (that is, without the restriction $-\frac{1}{2} < \frac{2\sigma-1}{4} < \frac{1}{2}$),

$$\|\Lambda f\|_{\dot{H}^\sigma_x} \lesssim \|\chi_{[t,+\infty)} f\|_{\dot{H}^{\frac{2\sigma-1}{4}}}. \tag{2.5}$$

holds. □

From now on, we prepare some basic facts in order to prove the asymptotic completeness. For the sake of simplicity we will study the following Propositions 2.3-2.5 only in the case $t>0$, but we can consider the negative time $t<0$ similarly.

Proposition 2.3 (small data global well-posedness)**.** *Let $p \geq 3$. There exists $\delta_{sd} > 0$ such that if $\psi_0 \in \dot{H}^{\sigma_c}$ and $\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} \leq \delta_{sd}$, then $\psi \in \dot{H}^{\sigma_c}$ solving* (1.1) *is global in $\dot{H}^{\sigma_c}$ and*

$$\|\psi(0,t)\|_{L^q_{t>0}} \leq 2\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}}$$
$$\|\psi(x,t)\|_{C^0_{[0,\infty)}\dot{H}^{\sigma_c}_x} \leq 2\|\psi_0\|_{\dot{H}^{\sigma_c}}.$$

(Note that by Proposition 2.1 (1) and Sobolev embedding, the smallness assumption $\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} \leq \delta_{\mathrm{sd}}$ is satisfied if $\|\psi_0\|_{\dot{H}^{\sigma_c}} \leq C\delta_{\mathrm{sd}}$.)

Proof. Define a map: for a $\psi_0 \in \dot{H}^{\sigma_c}$ given,

$$\mathcal{T}_{\psi_0}\psi(t) := [e^{it\partial_x^2}\psi_0](0) + i[\mathcal{L}_0(|\psi|^{p-1}\psi)](t).$$

By Proposition 2.1 and Sobolev embedding, we have

$$\begin{aligned}
\|\mathcal{T}_{\psi_0}\psi\|_{L^q_{t>0}} &\leq \|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} + \|\mathcal{L}_0(|\psi|^{p-1}\psi)(0,\cdot)\|_{L^q_{t>0}} \\
&\leq \|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} + C\|[\mathcal{L}_0(|\psi|^{p-1}\psi)](0,\cdot)\|_{\dot{H}_t^{\frac{2\sigma_c+1}{4}}} \\
&\leq \|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} + C\|\chi_{[0,\infty)}|\psi|^p\|_{\dot{H}_t^{\frac{2\sigma_c-1}{4}}} \\
&\leq \|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} + C\|\psi(0,\cdot)\|^p_{L^q_{t>0}}.
\end{aligned}$$

Let

$$B := \{\phi \in L^q_{t>0} :\ \|\phi\|_{L^q_{t>0}} \leq 2\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}}\}.$$

If $\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} \leq \delta_{\mathrm{sd}}$ then $\mathcal{T}_{\psi_0}\psi \in B$ for any $\psi \in B$, taking δ_{sd} sufficiently small.

The difference $\|\mathcal{T}_{\psi_0}\psi - \mathcal{T}_{\psi_0}\tilde{\psi}\|_{L^q_t}$ is similarly estimated by

$$\|[\mathcal{T}_{\psi_0}(|\psi|^{p-1}\psi - |\tilde{\psi}|^{p-1}\tilde{\psi})](\cdot)\|_{L^q_{t>0}} \leq C(\|\psi\|^{p-1}_{L^q_{t>0}} + \|\tilde{\psi}\|^{p-1}_{L^q_{t>0}})\|\psi-\tilde{\psi}\|_{L^q_{t>0}}$$

for $\psi, \tilde{\psi} \in B$. Again taking δ_{sd} sufficiently small, we conclude that $\mathcal{T}_{\psi_0}$ is a contraction on B. There thus exists a unique solution $\tilde{\psi} \in B$ such that $\mathcal{T}_{\psi_0}\tilde{\psi} = \tilde{\psi}$.

For the last inequality in the proposition, we use Eq. (2.1) for the unique solution $\tilde{\psi}$ obtained above in B. Inserting $\tilde{\psi}$ as the value of $\psi(0,t)$ at time t in the RHS of (2.1), The values of $\psi(x,t)$ for any x can be expressed as

$$\psi(x,t) = e^{it\partial_x^2}\psi_0 + i\int_0^t \frac{e^{\frac{ix^2}{4(t-s)}}}{\sqrt{4\pi i(t-s)}}|\psi(0,s)|^{p-1}\psi(0,s)ds,$$

with $\psi(0,\cdot)\in B$. Then, Sobolev embedding and Proposition 2.1 implies

$$\begin{aligned}
\|\psi\|_{\dot H_x^{\sigma_c}} &\le \|e^{it\partial_x^2}\psi_0\|_{\dot H_x^{\sigma_c}} + \|\mathcal{L}_0(|\psi|^p\psi)(\cdot,t)\|_{\dot H_x^{\sigma_c}} \\
&\le \|e^{it\partial_x^2}\psi_0\|_{\dot H_x^{\sigma_c}} + C\|\chi_{[0,t]}|\psi|^{p-1}\psi\|_{\dot H^{\frac{2\sigma_c-1}{4}}} \\
&\le \|\psi_0\|_{\dot H_x^{\sigma_c}} + C\|\chi_{[0,t]}|\psi|^{p-1}\psi\|_{L^q_{\mathbb{R}}} \\
&\le \|\psi_0\|_{\dot H_x^{\sigma_c}} + \|\psi(0,\cdot)\|^p_{L^q_{t>0}}. \qquad (2.6)
\end{aligned}$$

Since $\psi(0,\cdot)\in B$ with $\|[e^{it\partial_x^2}\psi_0](0,t)\|_{L^q_{t>0}} \le \delta_{\rm sd}$, by Sobolev embedding and Proposition 2.1(1),

$$\|\psi(0,\cdot)\|^p_{L^q_{t>0}} \le 2^p\delta_{\rm sd}^{p-1}\|[e^{it\partial_x^2}\psi_0](0)\|_{L^q_{t>0}} \le 2^p\delta_{\rm sd}^{p-1}\|e^{it\partial_x^2}\psi_0(0)\|_{\dot H_t^{\frac{2\sigma_c+1}{4}}} \le 2^p\delta_{\rm sd}^{p-1}\|\psi_0\|_{\dot H_x^{\sigma_c}}.$$

Taking $\delta_{\rm sd}$ sufficiently small, the RHS of (2.6) is bounded by $2\|\psi_0\|_{\dot H_x^{\sigma_c}}$. Note that the time continuity property follows from the fundamental solution, and this concludes

$$\|\psi(x,t)\|_{C^0_{[0,\infty)}\dot H_x^{\sigma_c}} \le 2\|\psi_0\|_{\dot H_x^{\sigma_c}}.$$

. □

Proposition 2.4 (scattering criterion). *Let $p\ge 3$. Suppose that $\psi_0\in H^1$ and $\psi\in H_x^1$ solving* (1.1) *is forward global with*

$$\|\psi(0,\cdot)\|_{L^q_{t>0}} < \infty$$

and with a uniform H_x^1 bound

$$\sup_{t\ge 0}\|\psi(\cdot,t)\|_{H_x^1} \le B.$$

Then $\psi(t)$ scatters in H_x^1 as $t\nearrow +\infty$. This means that there exists $\psi^+\in H_x^1$ such that

$$\lim_{t\nearrow+\infty}\|\psi(t) - e^{it\partial_x^2}\psi^+\|_{H_x^1} = 0.$$

Proof. Using the equation (2.1), we may write

$$\psi(t) - e^{it\partial_x^2}\psi^+ = -i\int_t^{+\infty} e^{i(t-s)\partial_x^2}\delta(x)|\psi(s)|^{p-1}\psi(s)ds, \qquad (2.7)$$

where

$$\psi^+ := \psi_0 + i\int_0^{+\infty} e^{-is\partial_x^2}\delta(x)|\psi(s)|^{p-1}\psi(s)ds.$$

Therefore,

$$\begin{aligned}\|\psi(t)-e^{it\partial_x^2}\psi^+\|_{H_x^1} &= \|\int_t^{+\infty} e^{i(t-s)\partial_x^2}\delta(x)|\psi(s)|^{p-1}\psi(s)ds\|_{H_x^1}\\ &= \|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{H_x^1}.\end{aligned}$$

Thus we shall estimate $\|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{L_x^2}$ and $\|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{\dot H_x^1}$. First, $\|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{L_x^2}$ is estimated by (3b) of Proposition 2.1 and the Sobolev embedding as follows. For any $t>0$,

$$\begin{aligned}\|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{L_x^2} &\le \|\chi_{[t+\infty)}|\psi|^{p-1}\psi\|_{\dot H^{-\frac14}}\\ &\le C\|\chi_{[t,+\infty)}|\psi|^{p-1}\psi\|_{L_{\mathbb{R}}^{\tilde q}}\\ &\le C\|\psi\|^p_{L^q_{(t,+\infty)}}.\end{aligned}\tag{2.8}$$

Second, by the Sobolev embedding and fractional chain rule [8], for any $t>0$,

$$\begin{aligned}\|\Lambda(|\psi|^{p-1}\psi)(\cdot,t)\|_{\dot H_x^1} &\le C\|\chi_{[t,+\infty)}|\psi|^{p-1}\psi\|_{\dot H_t^{\frac14}}\\ &\le C\|\chi_{[t,+\infty)}|\psi|^{p-1}\|_{L_{\mathbb{R}_t}^{r_1}}\||\nabla|^{\frac14}\chi_{[t,+\infty)}\psi\|_{L_{\mathbb{R}_t}^{r_2}}\end{aligned}\tag{2.9}$$

with $\frac12=\frac{1}{r_1}+\frac{1}{r_2}$, $1<r_1,r_2<+\infty$. Taking $q<r_1<+\infty$ and $2<r_2<4$, by interpolation,

$$\begin{aligned}\|\chi_{[t,+\infty)}|\psi|^{p-1}\|_{L_{\mathbb{R}_t}^{r_1}} &\le C\|\psi\|^{\frac{q}{r_1}}_{L^q_{(t,+\infty)}}\sup_{s\ge t}|\psi(0,s)|^{(1-\frac{q}{r_1})}\\ &\le C\|\psi\|^{\frac{q}{r_1}}_{L^q_{(t,+\infty)}}\sup_{s\ge t}\|\psi(s)\|^{(1-\frac{q}{r_1})}_{L^\infty_{\mathbb{R}_x}}\\ &\le C\|\psi\|^{\frac{q}{r_1}}_{L^q_{(t,+\infty)}}\sup_{s\ge t}\|\psi(s)\|^{(1-\frac{q}{r_1})}_{H_x^1}\le C_B\|\psi\|^{\frac{q}{r_1}}_{L^q_{(t,+\infty)}}\end{aligned}$$

where we have used the Sobolev embedding $H^1(\mathbb{R}_x)\subset L^\infty(\mathbb{R}_x)$. Again by interpolation

$$\begin{aligned}\||\nabla|^{\frac14}\chi_{[t,+\infty)}\psi\|_{L_{\mathbb{R}_t}^{r_2}} &\le \|\chi_{[t,+\infty)}\psi\|^{\frac{2}{r_2}}_{\dot H_t^{\frac14}}\||\nabla|^{\frac14}\chi_{[t,+\infty)}\psi\|^{(1-\frac{2}{r_2})}_{L^\infty_{\mathbb{R}_t}}\\ &\le C\|\chi_{[t,+\infty)}\psi\|^{\frac{2}{r_2}}_{\dot H^{\frac14}}\left(\|\chi_{[t,+\infty)}\psi\|_{\dot H^{\frac14}}+\|\chi_{[t,+\infty)}\psi\|_{\dot H^{\frac34}}\right)^{(1-\frac{2}{r_2})}\end{aligned}$$

where we have used the Sobolev embedding $H^1(\mathbb{R}_t)\subset L^\infty(\mathbb{R}_t)$ in the second inequality. We go back to the equation (2.7), evaluating at $x=0$, to estimate

$$\begin{aligned}\|\chi_{[t,+\infty)}\psi\|_{\dot H^{\frac14}} &\le \|\chi_{[t,+\infty)}[e^{it\partial_x^2}\psi^+](0)\|_{\dot H^{\frac14}}+\|\chi_{[t,+\infty)}\Lambda(|\psi|^{p-1}\psi)(0,\cdot))\|_{\dot H^{\frac14}}\\ &\le \|\psi^+\|_{L_x^2}+\|\chi_{[t,+\infty)}|\psi|^{p-1}\psi\|_{\dot H^{-\frac14}}\\ &\le \|\psi^+\|_{L_x^2}+\|\psi\|^p_{L^q_{t>0}},\end{aligned}$$

and

$$\begin{aligned}\|\chi_{[t,+\infty)}\psi\|_{\dot H^{\frac34}} &\le \|\chi_{[t,+\infty)}[e^{it\partial_x^2}\psi^+](0)\|_{\dot H^{\frac34}} + \|\chi_{[t,+\infty)}\Lambda(|\psi|^{p-1}\psi)(0,\cdot))\|_{\dot H^{\frac34}}\\ &\le \|\psi^+\|_{H^1_x} + \|\chi_{[t,+\infty)}|\psi|^{p-1}\psi\|_{\dot H^{\frac14}}.\end{aligned}$$

Note that we used Lemma 2.2, and Proposition 2.1 (2b). Plugging these results into (2.9), we see that for $t>0$ sufficiently large, $\|\chi_{[t,+\infty)}|\psi|^{p-1}\psi\|_{\dot H^{\frac14}}$ is small. This completes the proof combining with (2.8). □

Proposition 2.5 (long-time perturbation theory)**.** *Let $p\ge 3$. For each $A\gg 1$, there exists $\epsilon_0=\epsilon_0(A)\ll 1$ and $c=c(A)\gg 1$ such that the following holds. Let $\psi\in H^1_x$ for all t solving*

$$i\partial_t\psi+\partial_x^2\psi+\delta|\psi|^{p-1}\psi=0.$$

Let $\tilde\psi\in H^1_x$ for all t and suppose that there exists $e\in L^{\tilde q}_{t>0}$ such that

$$i\partial_t\tilde\psi+\partial_x^2\tilde\psi+\delta(|\tilde\psi|^{p-1}\tilde\psi-e)=0.$$

If

$$\|\tilde\psi(0,\cdot)\|_{L^q_{t>0}}\le A\,,\quad \|e(0,\cdot)\|_{L^{\tilde q}_{t>0}}\le\epsilon_0$$

and

$$\|[e^{i(t-t_0)\partial_x^2}(\psi(t_0)-\tilde\psi(t_0))](0)\|_{L^q_{t_0\le t<\infty}}\le\epsilon_0$$

for some $t_0\ge 0$, then

$$\|\psi(0,\cdot)\|_{L^q_{t>0}}\le c=c(A)<\infty.$$

Proof. Put $w=\psi-\tilde\psi$. Then w satisfies

$$i\partial_t w+\partial_x^2 w+W=0, \tag{2.10}$$

where

$$W=\delta(|\tilde\psi+w|^{p-1}(\tilde\psi+w)-|\tilde\psi|^{p-1}\tilde\psi+e).$$

Since $\|\tilde\psi(0,\cdot)\|_{L^q_{[t_0,+\infty)}}\le A$, there exists a $N=N(A)$ so that the interval $[t_0,+\infty)$ may be divided into the sum of $N(A)$ intervals. Namely, $[t_0,+\infty)=\cup_{j=1}^{N(A)}I_j$ with $I_j=[t_j,t_{j+1}]$ $(j=0,1,2,..)$ so that $\|\tilde\psi(0,\cdot)\|_{L^q_{I_j}}\le\eta$ (η is small to be determined later). Let $t\in I_j$. Write the equation (2.10) in the integral form.

$$w(t)=e^{i(t-t_j)\partial_x^2}w(t_j)+i\int_{t_j}^t e^{i(t-s)\partial_x^2}W(s)ds. \tag{2.11}$$

We estimate the time L^q norm of w evaluated at $x=0$.

$$\|w(0,\cdot)\|_{L^q_{I_j}}\le\|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{I_j}}+\left\|\int_{t_j}^t e^{i(t-s)\partial_x^2}W(s)ds|_{x=0}\right\|_{L^q_{I_j}}.$$

The last term can be written as, taking into account for the delta potential in W,

$$\left\| \int_{t_j}^{t} e^{i(t-s)\partial_x^2} W(s)ds|_{x=0} \right\|_{L^q_{I_j}} = \|[\mathcal{L}_{t_j}(|\tilde{\psi}+w|^{p-1}(\tilde{\psi}+w)(0,\cdot)-|\tilde{\psi}|^{p-1}\tilde{\psi}(0,\cdot)+e(\cdot))](0,\cdot)\|_{L^q_{I_j}}$$

and then we estimate as follows.

$$\begin{aligned}
&\|[\mathcal{L}_{t_j}(|\tilde{\psi}+w|^{p-1}(\tilde{\psi}+w)-|\tilde{\psi}|^{p-1}\tilde{\psi}+e)](0,\cdot)\|_{L^q_{I_j}} \\
\leq\ & C\|\tilde{\psi}+w|^{p-1}(\tilde{\psi}+w)-|\tilde{\psi}|^{p-1}\tilde{\psi}\|_{L^{\tilde{q}}_{I_j}} + \|e\|_{L^{\tilde{q}}_{I_j}} \\
\leq\ & C(\|\tilde{\psi}^{p-1}w(0,\cdot)\|_{L^{\tilde{q}}_{I_j}} + \|w^p(0,\cdot)\|_{L^{\tilde{q}}_{I_j}}) + \|e\|_{L^{\tilde{q}}_{I_j}},
\end{aligned}$$

where, in the first inequality, we have used, by density of $C_0^\infty(I_j) \subset L^{\tilde{q}}(I_j)$, Sobolev embedding, and Proposition 2.1 (2a).

The first term of RHS is estimated by Hölder inequality as follows.

$$\|\tilde{\psi}^{p-1}w(0,\cdot)\|_{L^{\tilde{q}}_{I_j}} \leq \|\tilde{\psi}(0,\cdot)\|^{p-1}_{L^q_{I_j}}\|w(0,\cdot)\|_{L^q_{I_j}}.$$

Thus, we have

$$\begin{aligned}
\|w(0,\cdot)\|_{L^q_{I_j}} \leq\ & \|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{I_j}} + C\eta^{p-1}\|w(0,\cdot)\|_{L^q_{I_j}} \\
& + C\|w(0,\cdot)\|^p_{L^q_{I_j}} + C\epsilon_0.
\end{aligned}$$

We then obtain

$$\|w(0,\cdot)\|_{L^q_{I_j}} \leq 2\|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{I_j}} + 2C\epsilon_0, \tag{2.12}$$

provided

$$\eta < \left(\frac{1}{2C}\right)^{\frac{1}{p-1}}$$

and

$$\|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{I_j}} + C\epsilon_0 \leq \left(\frac{1}{2C}\right)^{\frac{1}{p-1}}. \tag{2.13}$$

Now take $t = t_{j+1}$ in (2.11), apply $e^{i(t-t_{j+1})\partial_x^2}$ to both hands,

$$e^{i(t-t_{j+1})\partial_x^2}w(t_{j+1}) = e^{i(t-t_j)\partial_x^2}w(t_j) + i\int_{t_j}^{t_{j+1}} e^{i(t-s)\partial_x^2}W(s)ds,$$

and we take $L^q(\mathbb{R}_t)$ norm of this equation after evaluating at $x = 0$,

$$\begin{aligned}
\|[e^{i(t-t_{j+1})\partial_x^2}w(t_{j+1})](0)\|_{L^q_{\mathbb{R}_t}} \leq\ & \|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{\mathbb{R}_t}} + C\eta^{p-1}\|w(0,\cdot)\|_{L^q_{I_j}} \\
& + C\|w(0,\cdot)\|^p_{L^q_{I_j}} + C\epsilon_0.
\end{aligned}$$

Thus, by (2.12),

$$\|[e^{i(t-t_{j+1})\partial_x^2}w(t_{j+1})](0)\|_{L^q_{\mathbb{R}_t}} \leq 2\|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{\mathbb{R}_t}} + 2C\epsilon_0.$$

Iterating this inequalty starting from $j=0$, we have

$$\|[e^{i(t-t_j)\partial_x^2}w(t_j)](0)\|_{L^q_{\mathbb{R}_t}} \leq 2^{j+2}C\epsilon_0.$$

To satisfy (2.13) for all I_j with $0 \leq j \leq N-1$, we require $\epsilon_0 = \epsilon_0(N)$ to be sufficiently small such that $2^{N+2}C\epsilon_0 < \left(\frac{1}{2C}\right)^{\frac{1}{p-1}}$ (i.e. ϵ_0 needs to be taken in terms of A), and we obtain

$$\|\psi(0,t)\|_{L^q_{t>0}} \leq c = c(A).$$

□

3. Profile decomposition

Proposition 3.1 (profile decomposition)**.** *Let $p \geq 3$. Suppose that $\{\psi_n\}$ is a uniformly bounded sequence in H^1_x. Then for each M, there exists a subsequence of $\{\psi_n\}$, also denoted $\{\psi_n\}$ and*

(1) *for each $1 \leq j \leq M$, there exists a (fixed in n) profile $\phi^j \in H^1$*
(2) *for each $1 \leq j \leq M$, there exists a sequence (in n) of time shifts t_n^j*
(3) *there exists a sequence (in n) of remainders $w_n^M(x)$ in H^1 such that*

$$\psi_n = \sum_{j=1}^{M} e^{-it_n^j\partial_x^2}\phi^j + w_n^M$$

The time sequences have a pairwise divergence property: for $1 \leq i \neq j \leq M$, we have

$$\lim_{n\to\infty} |t_n^i - t_n^j| = +\infty.$$

The remainder sequence $\{w_n^M\}_n$ has the following asymptotic smallness property

$$\lim_{M\to\infty}\left[\lim_{n\to\infty}\|[e^{it\partial_x^2}w_n^M](0)\|_{L^q_{\mathbb{R}_t}}\right] = 0.$$

For fixed M and any $0 \leq \sigma_c \leq 1$, we have the asymptotic $\dot{H}^{\sigma_c}$ decoupling

$$\|\psi_n\|^2_{\dot{H}^{\sigma_c}} = \sum_{j=1}^{M}\|\phi^j\|^2_{\dot{H}^{\sigma_c}} + \|w_n^M\|^2_{\dot{H}^{\sigma_c}} + o_n(1), \tag{3.1}$$

also we have

$$|\psi_n(0)|^{p+1} = \sum_{j=1}^{M}|[e^{-it_n^j\partial_x^2}\phi^j](0)|^{p+1} + |w_n^M(0)|^{p+1} + o_n(1). \tag{3.2}$$

Proof. For $R > 0$, let $\chi_R(\xi)$ be a smooth cutoff to $R^{-1} < |\xi| < R$. Let $A = \limsup_{n\to\infty} \|\psi_n\|_{H^1_x}$ and $B_1 = \lim_{n\to\infty} \|[e^{it\partial_x^2}\psi_n](0)\|_{L^q_{\mathbb{R}_t}}$. If $B_1 = 0$, the proof is done. Let $B_1 > 0$. Since for $0 \le \sigma_c \le 1$,

$$\int_{|\xi|<R^{-1}} |\hat{\psi}_n(\xi)|^2 |\xi|^{2\sigma_c}\, d\xi \le R^{-2\sigma_c} \|\psi_n\|_{L^2}^2 \le A^2 R^{-2\sigma_c}$$

$$\int_{|\xi|>R} |\hat{\psi}_n(\xi)|^2 |\xi|^{2\sigma_c}\, d\xi \le R^{2(\sigma_c-1)} \|\psi_n\|_{\dot{H}^1}^2 \le A^2 R^{2(\sigma_c-1)}.$$

We may take a R_1 large enough so that $AR_1^{-\sigma_c} \le B_1/2$ and $AR_1^{\sigma_c-1} \le B_1/2$, specifically $R_1 = \langle 2AB_1^{-1}\rangle^{\max\{\frac{1}{\sigma_c},\frac{1}{1-\sigma_c}\}}$ so that

$$\lim_{n\to\infty} \|[e^{it\partial_x^2}(\delta - \check{\chi}_{R_1}) * \psi_n](0)\|_{L^q_{\mathbb{R}_t}} \le \frac{1}{2}B_1.$$

It thus follows, using Proposition 2.1(1),

$$\begin{aligned}\left(\frac{1}{2}B_1\right)^q &\le \lim_{n\to\infty} \|[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0)\|_{L^q_{\mathbb{R}_t}}^q \\ &\le \lim_{n\to\infty} \|[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0)\|_{L^2_{\mathbb{R}_t}}^2 \|[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0)\|_{L^\infty_{\mathbb{R}_t}}^{q-2}.\end{aligned}$$

For the factor $\|[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0)\|_{L^2_{t>0}}^2$, we use again the smoothing estimate of Proposition 2.1(1) to bound by

$$\|\check{\chi}_{R_1} * \psi_n\|_{\dot{H}_x^{-1/2}}^2 \le R_1 \|\check{\chi}_{R_1} * \psi_n\|_{L^2_x}^2 \le R_1 A^2.$$

Thus, we see $\lim_{n\to\infty} \|[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0)\|_{L^\infty_{\mathbb{R}_t}} > (R_1A^2)^{-\frac{1}{q-2}}(B_1/2)^{\frac{q}{q-2}}$, and we take a sequence $\{t_n^1\}_n$ such that

$$[\check{\chi}_{R_1} * e^{it\partial_x^2}\psi_n](0, t_n^1) = \int \check{\chi}_{R_1}(-y)(e^{it_n^1\partial_x^2}\psi_n)(y)\, dy,$$

and

$$\frac{1}{2}(R_1A^2)^{-\frac{1}{q-2}} \left(\frac{B_1}{2}\right)^{\frac{q}{q-2}} \le \left| \int \check{\chi}_{R_1}(-y) e^{it_n^1\partial_x^2}\psi_n(y)\, dy \right|. \tag{3.3}$$

Consider the sequence $\{e^{it_n^1\partial_x^2}\psi_n\}_n$, which is uniformly bounded in H^1_x, and pass to subsequence such that $e^{it_n^1\partial_x^2}\psi_n$ converges weakly in H^1_x to some $\phi^1 \in H^1$. By Cauchy-Schwarz inequality, using that $\|\check{\chi}_{R_1}\|_{\dot{H}^{-\sigma_c}} \lesssim R_1^{\frac{1}{2}-\sigma_c}$ and (3.3),

$$\|\phi^1\|_{\dot{H}^{\sigma_c}} \ge (R_1^{\frac{1}{2}-\sigma_c})^{-1}(R_1A^2)^{-\frac{1}{q-2}} \left(\frac{B_1}{2}\right)^{\frac{q}{q-2}} \frac{1}{2}.$$

Then for any $0 \le \sigma_c \le 1$

$$\lim_{n\to\infty} \|\psi_n - e^{-it_n^1\partial_x^2}\phi^1\|_{\dot{H}^{\sigma_c}}^2 = \|\psi_n\|_{\dot{H}^{\sigma_c}}^2 - \|\phi^1\|_{\dot{H}^{\sigma_c}}^2.$$

If $|t_n^1| \to +\infty$, since $\|[e^{-it\partial_x^2}\phi^1](0)\|_{L^q_{\mathbb{R}_t}} \leq \|\phi^1\|_{\dot H_x^{\sigma_c}}$, possibly taking a subsequence, we have $|[e^{-it_n^1\partial_x^2}\phi^1](0)|^q \to 0$ as $n \to +\infty$. On the other hand, since ψ_n is uniformly bounded in H_x^1, there is a weak limit $\tilde\psi \in H_x^1$ and $\psi_n(0) \to \tilde\psi(0)$ as $n \to \infty$ by Proposition 4.1 of [11]. Then, we have

$$\begin{aligned}
&\lim_{n\to\infty} |[\psi_n - e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1} \\
= \;& \lim_{n\to\infty} \{(\psi_n(0) - [e^{-it_n^1\partial_x^2}\phi^1](0))\overline{(\psi_n(0) - [e^{-it_n^1\partial_x^2}\phi^1](0))}\}^{\frac{p+1}{2}} \\
= \;& |\tilde\psi(0)|^{p+1} = \lim_{n\to\infty}(|\psi_n(0)|^{p+1} - |[e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1}),
\end{aligned}$$

i.e.

$$\lim_{n\to\infty}[|\psi_n(0)|^{p+1} - |[e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1} - |w_n^1(0)|^{p+1}] = 0. \tag{3.4}$$

If $t_n^1 \to t^*$ for some finite t^*, by the time continuity of free Schrödinger group, $\lim_{n\to\infty}\psi_n(0) = \tilde\psi(0) = [e^{-it^*\partial_x^2}\phi^1](0)$. Thus we may write

$$\begin{aligned}
\lim_{n\to\infty} |[\psi_n - e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1} &= \lim_{n\to\infty}(|\psi_n(0)|^2 - |[e^{-it_n^1\partial_x^2}\phi^1](0)|^2)^{\frac{p+1}{2}} \\
&= 0 = \lim_{n\to\infty}(|\psi_n(0)|^{p+1} - |[e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1}),
\end{aligned}$$

which again gives (3.4).

Repeat the process, keeping the same A but switching to B_2 obtaining R_2 in terms of B_2. Basically this amounts to replacing ψ_n by $\psi_n - e^{-it_n^1\partial_x^2}\phi^1$ and rewriting the above to obtain t_n^2 and ϕ^2 where

$$\phi^2 = \text{weak}\lim[e^{it_n^2\partial_x^2}(\psi_n - e^{-t_n^1\partial_x^2}\phi^1)] \quad \text{in } H_x^1.$$

As a result,

$$\begin{aligned}
\lim_{n\to\infty}\|\psi_n - e^{-it_n^1\partial_x^2}\phi^1 - e^{-it_n^2\partial_x^2}\phi^2\|^2_{\dot H^{\sigma_c}} &= \lim_{n\to\infty}\|\psi_n - e^{-it_n^1\partial_x^2}\phi^1\|_{\dot H^{\sigma_c}} - \|\phi^2\|^2_{\dot H^{\sigma_c}} \\
&= \lim_{n\to\infty}\|\psi_n\|^2_{\dot H^{\sigma_c}} - \|\phi^1\|^2_{\dot H^{\sigma_c}} - \|\phi^2\|^2_{\dot H^{\sigma_c}},
\end{aligned}$$

and same for

$$\begin{aligned}
&\lim_{n\to\infty}|[\psi_n - e^{-it_n^1\partial_x^2}\phi^1 - e^{-it_n^2\partial_x^2}\phi^2](0)|^{p+1} \\
&= \lim_{n\to\infty}(|\psi_n(0)|^{p+1} - |[e^{-it_n^1\partial_x^2}\phi^1](0)|^{p+1} - |[e^{-it_n^2\partial_x^2}\phi^2](0)|^{p+1}).
\end{aligned}$$

If $t_n^2 - t_n^1$ converged to something finite (say t^*), then ϕ^2 would be the weak limit of $e^{it^*\partial_x^2}[e^{it_n^1\partial_x^2}\psi_n - \phi^1]$, which is zero, contradicting the lower bound. Hence $|t_n^1 - t_n^2| \to \infty$ and thus

$$\langle e^{-it_n^1\partial_x^2}\phi^1, e^{-it_n^2\partial_x^2}\phi^2\rangle_{\dot H^{\sigma_c}} \to 0.$$

Again repeat this process, we have

$$\|\phi^1\|^2_{\dot H^{\sigma_c}} + \|\phi^1\|^2_{\dot H^{\sigma_c}} + \cdots + \|\phi^M\|^2_{\dot H^{\sigma_c}} + \lim_{n\to+\infty}\|w_n^M\|^2_{\dot H^{\sigma_c}} = \lim_{n\to+\infty}\|\psi_n\|^2_{\dot H^{\sigma_c}}.$$

Let $B_{M+1} := \lim_{n\to+\infty} \|[e^{it\partial_x^2} w_n^M](0)\|_{L^q_{\mathbb{R}_t}}$ and we wish to show that $B_{M+1} \to 0$. Note that from the above equality and the lower bound for $\|\phi^M\|_{\dot H^{\sigma_c}}$, we obtain

$$\sum_{M=1}^{\infty} R_M^{-\theta} B_M^{\frac{q}{q-2}} \le 2A^{\frac{2(q-1)}{q-2}}, \quad \theta = \frac{1}{q-2} + \frac{1}{2} - \sigma_c = \frac{1}{2(p-2)} + \frac{1}{2} - \sigma_c > 0,$$

whose LHS diverges if B_M does not converge to 0. □

Lemma 3.2. *With w_n^M as defined in Proposition 3.1 (in particular, $w_n^0 = \psi_n$), let*

$$B_M = \lim_{n\to\infty} \|[e^{it\partial_x^2} w_n^{M-1}](0)\|_{L^q_{\mathbb{R}_t}}.$$

Then

$$\lim_{n\to\infty} \|[e^{i(t-t_n^M)\partial_x^2} \phi^M](0)\|_{L^q_{\mathbb{R}_t}} \le 2B_M.$$

Proof. We will write the argument for $M = 1$ (the general case is analogous). As in the proof of Proposition 3.1, let

$$A = \lim_{n\to\infty} \|\psi_n\|_{H^1_x}$$

and

$$R_1 = \langle 2AB_1^{-1}\rangle^{\max(\frac{1}{\sigma_c}, \frac{1}{1-\sigma_c})}$$

and $\chi_{R_1}(\xi)$ be a cutoff to $R_1^{-1} \le |\xi| \le R_1$. As in the beginning of the proof of Proposition 3.1,

$$\begin{aligned}
\|(\delta - \check\chi_{R_1}) * e^{i(t-t_n^1)\partial_x^2}\phi^1(0)\|^2_{L^q_{\mathbb{R}_t}} &\lesssim \|[(\delta - \check\chi_{R_1}) * e^{it\partial_x^2}\phi^1](0)\|^2_{\dot H_t^{\frac{2\sigma_c+1}{4}}} \\
&\lesssim \|(\delta - \check\chi_{R_1}) * \phi^1\|^2_{\dot H_x^{\sigma_c}} \lesssim {R_1}^{-2\sigma_c}\|\phi^1\|^2_{L^2} + {R_1}^{-2(1-\sigma_c)}\|\phi^1\|^2_{\dot H^1} \\
&\le A^2({R_1}^{-2\sigma_c} + {R_1}^{-2(1-\sigma_c)}) \le \frac{1}{4}B_1^2
\end{aligned}$$

This, and the similar estimates at the beginning of the proof of Proposition 3.1, show that it suffices to prove

$$\lim_{n\to\infty} \|\check\chi_{R_1} * e^{i(t-t_n^1)\partial_x^2}\phi^1(0)\|^2_{L^q_{\mathbb{R}_t}} \le \frac{1}{4}B_1^2, \tag{3.5}$$

and this can be seen as follows. By the translation invariance of $L^q_{\mathbb{R}_t}$ norm,

$$\|\check\chi_{R_1} * e^{i(t-t_n^1)\partial_x^2}\phi^1(0)\|_{L^q_{\mathbb{R}_t}} = \|\check\chi_{R_1} * e^{it\partial_x^2}\phi^1(0)\|_{L^q_{\mathbb{R}_t}}$$

and by Sobolev embedding and Proposition 2.1, we have,

$$\begin{aligned}
\|\check\chi_{R_1} * e^{it\partial_x^2}\phi^1(0)\|_{L^q_{\mathbb{R}_t}} &\lesssim \|\check\chi_{R_1} * e^{it\partial_x^2}\phi^1(0)\|_{\dot H_t^{\frac{2\sigma_c+1}{4}}} \\
&\lesssim \|\check\chi_{R_1} * \phi^1\|_{\dot H_x^{\sigma_c}} \\
&\lesssim \left(A^2 R_1^{-2(1-\sigma_c)}\right)^{\frac{1}{2}} \le B_1/2.
\end{aligned}$$

□

4. Minimal non scattering solution

In this section we will prove that there exists a *minimal non scattering solution*. For this purpose we prepare the following lemma which gives additional estimates under the situation (1) of Theorem 1.3. We recall that φ_0 is the ground state to (1.1). It is known that $\varphi_0(x) = 2^{\frac{1}{p-1}} e^{-|x|}$ (see (1.9) of [11]).

Lemma 4.1. *Let $p > 3$ and $\psi_0 \in H^1_x$. Assume (1.4) and $\eta(0) < 1$. If ψ is a H^1_x solution to (1.1), then for all $t \in \mathbb{R}$,*

$$\frac{(p-1)}{2(p+1)} \|\partial_x \psi(t)\|^2_{L^2} \leq E(\psi(t)) \leq \frac{1}{2} \|\partial_x \psi(t)\|^2_{L^2}. \tag{4.1}$$

Furthermore, if we take $\delta > 0$ such that $M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi_0) \leq (1-\delta) M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$, then there exists $c_\delta > 0$ such that for all $t \in \mathbb{R}$,

$$4\|\partial_x \psi\|^2_{L^2} - 2|\psi(0,t)|^{p+1} \geq c_\delta \|\partial_x \psi_0\|^2_{L^2}. \tag{4.2}$$

Proof. The upper bound of the energy in (4.1) follows by the definition of Energy E and the focusing nonlinearity. Use the sharp Gagliardo-Nirenberg inequality and $\eta(t) < 1$ for the lower bound, i.e.,

$$\begin{aligned} E(\psi) &\geq \frac{1}{2} \|\partial_x \psi\|^2_{L^2} \Big(1 - \frac{1}{p+1} \|\psi\|^{\frac{p+1}{2}}_{L^2} \|\partial_x \psi\|^{\frac{p-3}{2}}_{L^2}\Big) \\ &> \frac{1}{2} \|\partial_x \psi\|^2_{L^2} \Big(1 - \frac{1}{p+1} \|\varphi_0\|^{\frac{p+1}{2}}_{L^2} \|\partial_x \varphi_0\|^{\frac{p-3}{2}}_{L^2}\Big) \\ &= \frac{p-1}{2(p+1)} \|\partial_x \psi\|^2_{L^2}, \end{aligned}$$

where we have used the fact $\|\partial_x \varphi_0\|_{L^2} = \|\varphi_0\|_{L^2} = 2^{\frac{1}{p-1}}$ in the last equality (see [11]). Next, we show (4.2). We may take $\delta_1 = \delta_1(\delta) > 0$ such that

$$\|\psi_0\|^{\frac{1-\sigma_c}{\sigma_c}}_{L^2} \|\partial_x \psi(t)\|_{L^2} \leq (1-\delta_1) \|\varphi_0\|^{\frac{1-\sigma_c}{\sigma_c}}_{L^2} \|\partial_x \varphi_0\|_{L^2}, \tag{4.3}$$

for all $t \in \mathbb{R}$. Let

$$h(t) := \frac{1}{\|\varphi_0\|^{\frac{2(1-\sigma_c)}{\sigma_c}}_{L^2} \|\partial_x \varphi_0\|^2_{L^2}} \big(4\|\psi_0\|^{\frac{2(1-\sigma_c)}{\sigma_c}}_{L^2} \|\partial_x \psi(t)\|^2_{L^2} - 2\|\psi_0\|^{\frac{2(1-\sigma_c)}{\sigma_c}}_{L^2} |\psi(0,t)|^{p+1}\big).$$

By Gagliardo-Nirenberg inequality,

$$h(t) \geq g\left(\frac{\|\psi_0\|^{\frac{(1-\sigma_c)}{\sigma_c}}_{L^2} \|\partial_x \psi(t)\|_{L^2}}{\|\varphi_0\|^{\frac{(1-\sigma_c)}{\sigma_c}}_{L^2} \|\partial_x \varphi_0\|_{L^2}} \right),$$

where $g(y) := 4(y^2 - y^{\frac{p+1}{2}})$. The inequality (4.3) implies the variable y of $g(y)$ is in the interval $0 \leq y \leq 1 - \delta_1$ and then we see that there exists a constant $c = c_{\delta_1} > 0$ such that $g(y) \geq c y^2$ if $0 \leq y \leq 1 - \delta_1$. $\square$

Lemma 4.2. *(Existence of wave operator) Let $p>3$. Suppose $\psi^+\in H^1_x$ and*

$$\frac{1}{2}\|\psi^+\|_{L^2}^{\frac{2(1-\sigma_c)}{\sigma_c}}\|\partial_x\psi^+\|_{L^2}^2 < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}}E(\varphi_0). \tag{4.4}$$

There exists $\psi_0\in H^1_x$ such that ψ solving (1.1) with initial data ψ_0 is global in H^1_x, with

$$M(\psi)=\|\psi^+\|_{L^2}^2,\quad E(\psi)=\frac{1}{2}\|\partial_x\psi^+\|_{L^2}^2,$$
$$\|\partial_x\psi(t)\|_{L^2}\|\psi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} < \|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\varphi_0\|_{L^2}$$

and

$$\lim_{t\nearrow+\infty}\|\psi(t)-e^{it\partial_x^2}\psi^+\|_{H^1_x}=0.$$

Moreover, if $\|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{t>0}}\le\delta_{sd}$, then

$$\|\psi_0\|_{\dot H^{\sigma_c}}\le 2\|\psi^+\|_{\dot H^{\sigma_c}},\quad \|\psi(0,\cdot)\|_{L^q_{t>0}}\le 2\|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{t>0}}.$$

The statement above is for the case $t>0$, but the case $t<0$ can be similarly proved.

Proof. It suffices to solve the integral equation:

$$\psi(t)=e^{it\partial_x^2}\psi^+ - i\Lambda(|\psi(0)|^{p-1}\psi(0))(t)$$

for $t\ge T$ with T large. Since

$$\|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{t>0}}\lesssim\|[e^{it\partial_x^2}\psi^+](0)\|_{\dot H_t^{\frac{2\sigma_c+1}{4}}}\le\|\psi^+\|_{\dot H_x^{\sigma_c}},$$

there exists a large $T>0$ such that $\|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{[T,\infty)}}\le\delta_{\mathrm{sd}}$. Thus we may solve as in the proof of Proposition 2.3.

$$\begin{aligned}\|\psi(0,\cdot)\|_{L^q_{[T,+\infty)}} &\le \|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{[T,\infty)}}+C\|\Lambda(|\psi(0)|^{p-1}\psi(0))(\cdot)\|_{L^q_{[T,+\infty)}}\\ &\le \|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{[T,\infty)}}+C\|\psi(0,\cdot)\|^p_{L^q_{[T,+\infty)}}.\end{aligned}$$

If T is sufficiently large, we have $\|\psi(0,\cdot)\|_{L^q_{[T,+\infty)}}<2\|[e^{it\partial_x^2}\psi^+](0)\|_{L^q_{[T,+\infty)}}$. Using this, similarly as in the proof of Proposition 2.4, we obtain if $t\ge T$,

$$\|\psi(t)-e^{it\partial_x^2}\psi^+\|_{L^2_x}\le C\|\Lambda(|\psi(0)|^{p-1}\psi(0))\|_{L^2_x}\le\|\psi(0,\cdot)\|^p_{L^q_{[T,+\infty)}}\le C\delta^p_{\mathrm{sd}},$$

$$\|\psi(t)-e^{it\partial_x^2}\psi^+\|_{\dot H^1_x}\le C\|\chi_{[T,+\infty)}|\psi|^{p-1}\psi\|_{\dot H_t^{1/4}},$$

which are small if T is sufficiently large. Thus, $\psi(t)-e^{it\partial_x^2}\psi^+\to 0$ in H^1_x as $t\to+\infty$. Note that $\|\partial_x e^{it\partial_x^2}\psi^+\|_{L^2_x}=\|\partial_x\psi^+\|_{L^2}$. On the other hand, since $[e^{it\partial_x^2}\psi^+](0)$ is uniformly bounded in $L^q_{t>0}$, there exists a sequence $\{t_n\}_n\to+\infty$ such that $[e^{it_n\partial_x^2}\psi^+](0)\to 0$ as $n\to+\infty$. Together with all these facts, we have

$$E(\psi(t))=\lim_{n\to+\infty}\left\{\frac{1}{2}\|\partial_x e^{it_n\partial_x^2}\psi^+\|_{L^2_x}-\frac{1}{p+1}|e^{it_n\partial_x^2}\psi^+(0)|^{p+1}\right\}=\frac{1}{2}\|\partial_x\psi^+\|_{L^2_x}.$$

Similarly, $M(\psi(t)) = \|\psi^+\|_{L_x^2}^2$. It now follows from (4.4) that

$$M(\psi(t))^{\frac{1-\sigma_c}{\sigma_c}} E(\psi(t)) < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0),$$

and

$$\begin{aligned}
\lim_{t\to+\infty} \|\partial_x \psi(t)\|_{L_x^2}^2 \|\psi(t)\|_{L_x^2}^{\frac{2(1-\sigma_c)}{\sigma_c}} &= \lim_{t\to+\infty} \|\partial_x e^{it\partial_x^2}\psi^+\|_{L_x^2}^2 \|e^{it\partial_x^2}\psi^+\|_{L_x^2}^{\frac{2(1-\sigma_c)}{\sigma_c}} \\
&= \|\partial_x \psi^+\|_{L_x^2}^2 \|\psi^+\|_{L_x^2}^{\frac{2(1-\sigma_c)}{\sigma_c}} \\
&< 2M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0) = \frac{p-3}{p+1} \|\partial_x \varphi_0\|_{L_x^2}^2 \|\varphi_0\|_{L_x^2}^{\frac{2(1-\sigma_c)}{\sigma_c}}
\end{aligned}$$

We can take a large T such that $\|\partial_x \psi(T)\|_{L_x^2} \|\psi(T)\|_{L_x^2}^{\frac{1-\sigma_c}{\sigma_c}} < \|\partial_x \varphi_0\|_{L_x^2} \|\varphi_0\|_{L_x^2}^{\frac{1-\sigma_c}{\sigma_c}}$. Then, applying Theorem 1.3 we evolve $\psi(t)$ from T back to the time 0. $\square$

We are now in position to enter in the main subject of this section. If the initial data ψ_0 to (1.1) satisfies $M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi_0) \le \frac{p-1}{2(p+1)} \delta_{sd}$ and $\eta(0) < 1$, we have

$$\|\psi_0\|_{\dot{H}_x^{\sigma_c}(\mathbb{R})}^{2/\sigma_c} \le \|\psi_0\|_{L_x^2}^{\frac{2(1-\sigma_c)}{\sigma_c}} \|\partial_x \psi_0\|_{L^2}^2 \le M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi_0) \le \delta_{sd},$$

and the scattering holds by the small data scattering, Proposition 2.3. Now let A be the infimum of $M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi)$, taken over all evolution of ψ which does not scatter. In what follows $\mathrm{NLS}(t)\psi$ denotes the solution to (1.1) with initial data ψ. By the above argument, $0 < \frac{p-1}{2(p+1)} \delta_{sd} \le A$, and moreover due to Proposition 2.4, A satisfies

(1) For any ψ such that $M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi) < A$, it holds $\|[\mathrm{NLS}(t)\psi](0,\cdot)\|_{L^q_{\mathbb{R}_t}} < \infty$,
(2) For any $A' > A$, there exists a non scattering $\mathrm{NLS}(t)\psi$ for which

$$A \le M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi) \le A'.$$

If $A \ge M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$, Theorem 1.4 is true. We therefore proceed with the proof by assuming $A < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$.

The first task is to apply the profile decomposition to show that there exists ψ such that $M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi) = A$ and $\mathrm{NLS}(t)\psi$ does not scatter. We will call such a solution a *minimal non scattering solution*. Take a sequence of initial data $\psi_{0,n}$, with $1 > \eta_n(0) := \|\psi_{0,n}\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \psi_{0,n}\|_{L^2} / \|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \varphi_0\|_{L^2}$, each evolving to non scattering solutions, for which $M(\psi_{0,n}) = 1$, $E(\psi_{0,n}) \ge A$ and $E(\psi_{0,n}) \to A$. Apply the profile decomposition to $\psi_{0,n}$ which is uniformly bounded in H^1 to obtain, extracting a subsequence,

$$\psi_{0,n} = \sum_{j=1}^{M} e^{-it_n^j \partial_x^2} \phi^j + w_n^M, \tag{4.5}$$

$$E(\psi_{0,n}) = \sum_{j=1}^{M} E(e^{-it_n^j \partial_x^2} \phi^j) + E(w_n^M) + o_n(1), \tag{4.6}$$

where M will be taken large later. Remark that each term in (4.6) is non negative by the same reason for (4.1), using the decompositions (3.1) and (3.2) in $\eta_n(0) < 1$. Taking the limit $n \to \infty$ in both hand sides,

$$\lim_{n\to\infty} \sum_{j=1}^{M} E(e^{-it_n^j \partial_x^2} \phi^j) \leq A \tag{4.7}$$

for all j. Also, by $\sigma_c = 0$ in (3.1), we have

$$\sum_{j=1}^{M} M(\phi^j) + \lim_{n\to\infty} M(w_n^M) = \lim_{n\to\infty} M(\psi_{0,n}) = 1. \tag{4.8}$$

Here we consider two cases.

Case 1 There are at least two indexes j such that ϕ^j is not zero.

Case 2 Only one profile is non zero, i.e. without loss of generality $\phi^1 \neq 0$, and $\phi^j = 0$ for all $j \geq 2$.

We begin with Case 1. By (4.8), we necessarily have $0 \leq M(\phi^j) < 1$ for each j which, by (4.7), implies that for n sufficiently large

$$M(e^{-it_n^j \partial_x^2} \phi^j)^{\frac{1-\sigma_c}{\sigma_c}} E(e^{-it_n^j \partial_x^2} \phi^j) \leq A_j, \tag{4.9}$$

with each $A_j < A$. For a given j, there are two possibilities. Case a) $|t_n^j| \to \infty$ as $n \to \infty$ and Case b) there is a finite limit t_* such that $t_n^j \to t_*$ as $n \to \infty$. Both cases allow us to ensure the existence of a new profile $\tilde{\phi}^j \in H^1$ associated to ϕ^j such that

$$\|\mathrm{NLS}(-t_n^j)\tilde{\phi}^j - e^{-it_n^j \partial_x^2} \phi^j\|_{H^1} \to 0, \quad n \to \infty;$$

indeed, if Case a) occurs, by the uniform L^q integrability in time of $[e^{-it\partial_x^2}\phi^j](0)$ (cf. the same argument in Proposition 3.1), passing to a subsequence of t_n^j,

$$|[e^{-it_n^j \partial_x^2} \phi^j](0)| \to 0, \quad n \to \infty$$

and thus

$$\frac{1}{2}\|\phi^j\|_{L^2}^{\frac{2(1-\sigma_c)}{\sigma_c}} \|\partial_x \phi^j\|_{L^2}^2 < A.$$

Since $A < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$, ϕ^j satisfies the assumption of Lemma 4.2. Namely, there exists $\tilde{\phi}^j \in H^1$ such that

$$\|\mathrm{NLS}(-t_n^j)\tilde{\phi}^j - e^{-it_n^j \partial_x^2} \phi^j\|_{H^1} \to 0, \quad n \to \infty$$

with

$$M(\tilde{\phi}^j) = \|\phi^j\|_{L^2}^2, \quad E(\tilde{\phi}^j) = \frac{1}{2}\|\partial_x \phi^j\|_{L^2}^2,$$

$$\|\partial_x \mathrm{NLS}(t)\tilde{\phi}^j\|_{L^2}\|\tilde{\phi}^j\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} < \|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\varphi_0\|_{L^2},$$

and thus

$$M(\tilde{\phi}^j)^{\frac{1-\sigma_c}{\sigma_c}}E(\tilde{\phi}^j) < A.$$

Therefore by the definition of threshold A, we have

$$\|\mathrm{NLS}(t)\tilde{\phi}^j(0)\|_{L^q_{\mathbb{R}_t}} < +\infty. \tag{4.10}$$

If the Case b), by the time continuity in H^1_x norm of the linear flow, we know

$$e^{-it_n^j\partial_x^2}\phi^j \to e^{-it_*\partial_x^2}\phi^j \text{ in } H^1_x.$$

Thus it suffices to put $\tilde{\phi}^j := \mathrm{NLS}(t_*)[e^{-it_*\partial_x^2}\phi^j]$. Then this $\tilde{\phi}^j$ again satisfies (4.10). To see this, note first that by the H^1 continuity of the flow, sending $n \to \infty$ in (4.9) gives

$$M(e^{-it_*\partial_x^2}\phi^j)^{\frac{1-\sigma_c}{\sigma_c}}E(e^{-it_*\partial_x^2}\phi^j) \le A_j < A$$

By (3.1) applied for $\sigma_c = 0$ and $\sigma_c = 1$, and the assumption that $\eta_n(0) < 1$ for every n, we obtain that

$$\frac{\|\phi^j\|_{L^2_x}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\phi^j\|_{L^2_x}}{\|\varphi_0\|_{L^2_x}^{\frac{1-\sigma_c}{\sigma_c}}\|\partial_x\varphi_0\|_{L^2_x}} < 1.$$

By the defining property of the threshold A, we have that the NLS flow with initial data $e^{-it_*\partial_x^2}\phi^j$ scatters, i.e.

$$\|\mathrm{NLS}(t)\tilde{\phi}^j(0)\|_{L^q_{\mathbb{R}_t}} = \|\mathrm{NLS}(t+t_*)e^{-it_*\partial_x^2}\phi^j(0)\|_{L^q_{\mathbb{R}_t}} < \infty.$$

Now replace $e^{-it_n^j\partial_x^2}\phi^j$ by $\mathrm{NLS}(-t_n^j)\tilde{\phi}^j$ in (4.5), and we have

$$\psi_{0,n} = \sum_{j=1}^M \mathrm{NLS}(-t_n^j)\tilde{\phi}^j + \tilde{w}_n^M,$$

with

$$\tilde{w}_n^M = w_n^M + \sum_{j=1}^M (e^{-it_n^j\partial_x^2}\phi^j - \mathrm{NLS}(-t_n^j)\tilde{\phi}^j).$$

Note that by Sobolev embedding and Proposition 2.1 (1),

$$\begin{aligned}
&\|[e^{it\partial_x^2}\tilde{w}_n^M](0)\|_{L^q_{\mathbb{R}_t}} \\
\le\ &\|[e^{it\partial_x^2}w_n^M](0)\|_{L^q_{\mathbb{R}_t}} + \sum_{j=1}^{M}\|[e^{it\partial_x^2}(-\mathrm{NLS}(-t_n^j)\tilde{\phi}^j + e^{-it_n^j\partial_x^2}\phi^j)](0)\|_{L^q_{\mathbb{R}_t}} \\
\le\ &\|[e^{it\partial_x^2}w_n^M](0)\|_{L^q_{\mathbb{R}_t}} + \sum_{j=1}^{M}\|\mathrm{NLS}(-t_n^j)\tilde{\phi}^j - e^{-it_n^j\partial_x^2}\phi^j\|_{\dot{H}_x^{\sigma_c}}, \\
\le\ &\|[e^{it\partial_x^2}w_n^M](0)\|_{L^q_{\mathbb{R}_t}} + \sum_{j=1}^{M}\|\mathrm{NLS}(-t_n^j)\tilde{\phi}^j - e^{-it_n^j\partial_x^2}\phi^j\|_{H_x^1}.
\end{aligned}$$

Thus we obtain,

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|[e^{it\partial_x^2}\tilde{w}_n^M](0)\|_{L^q_{\mathbb{R}_t}}] = 0.$$

From this way of writing we might approximately see

$$\mathrm{NLS}(t)\psi_{n,0} \approx \sum_{j=1}^{M}\mathrm{NLS}(t-t_n^j)\tilde{\phi}^j.$$

However, from (4.10), the RHS is finite in $L^q_{\mathbb{R}_t}$ norm, while the LHS cannot scatter by assumption, and so a contradiction could be deduced. We shall justify this argument by Proposition 2.5.

Let $v^j(t) := \mathrm{NLS}(t)\tilde{\phi}^j$, $\psi_n := \mathrm{NLS}(t)\psi_{0,n}$, and $\tilde{\psi}_n = \sum_{j=1}^{M} v^j(t-t_n^j)$. Then, $\tilde{\psi}_n$ satisfies

$$i\partial_t\tilde{\psi}_n + \partial_x^2\tilde{\psi}_n + \delta(|\tilde{\psi}_n|^{p-1}\tilde{\psi}_n + e_n) = 0.$$

Here,

$$e_n := -|\tilde{\psi}_n|^{p-1}\tilde{\psi}_n + \sum_{j=1}^{M}|v^j(t-t_n^j)|^{p-1}v^j(t-t_n^j).$$

We are going to show that

1 there exists a large constant A independent of M satisfying the following property: for any M there is $n_0 = n_0(M)$ such that if $n > n_0$, $\|\tilde{\psi}_n(0,\cdot)\|_{L^q_{\mathbb{R}_t}} \le A$.
2 For each M and $\varepsilon > 0$ there exists $n_1 = n_1(M,\varepsilon)$ such that for $n > n_1$, $\|e_n\|_{L^{\tilde{q}}_{\mathbb{R}_t}} \le \varepsilon$.

Remark that there exists $M_1 = M_1(\varepsilon)$ such that for each $M > M_1$, there exists $n_2 = n_2(M)$ such that if $n > n_2$, $\|[e^{it\partial_x^2}(\tilde{\psi}_n(0) - \psi_n(0))](0)\|_{L^q_{\mathbb{R}_t}} \le \varepsilon$. Thus, if the above 1 and 2 hold, it follows from Proposition 2.5 that for n and M sufficiently large,

$\|\psi_n\|_{L^q_{\mathbb{R}_t}} < \infty$, which gives a contradiction. Therefore it is enough to prove the above claims 1 and 2. First we prove the claim 1. Take M_0 large enough so that

$$\|[e^{it\partial_x^2} w_n^{M_0}](0)\|_{L^q_{\mathbb{R}_t}} \le \delta_{\rm sd}/2.$$

Then, by Lemma 3.2, for each $j > M_0$, we have $\|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)\|_{L^q_{\mathbb{R}_t}} \le \delta_{\rm sd}$. Thus by Lemma 4.2 we obtain, for each $j > M_0$, and for large n,

$$\|v^j(0,\cdot - t_n^j)\|_{L^q_{\mathbb{R}_t}} \le 2\|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)\|_{L^q_{\mathbb{R}_t}}. \tag{4.11}$$

By Minkowski inequality (since $p > 3$),

$$\begin{aligned}
&\|\tilde{\psi}_n(0,\cdot)\|^q_{L^q_{\mathbb{R}_t}} \\
\le\ & C_q\Big(\Big\|\sum_{j=1}^{M_0} v^j(0,\cdot - t_n^j)\Big\|^q_{L^q_{\mathbb{R}_t}} + \Big\|\sum_{j=M_0+1}^{M} v^j(0,\cdot - t_n^j)\Big\|^q_{L^q_{\mathbb{R}_t}}\Big) \\
\le\ & C_q\Big(\sum_{j=1}^{M_0}\|v^j(0,\cdot - t_n^j)\|^2_{L^q_{\mathbb{R}_t}} + \sum_{j=M_0+1}^{M}\|v^j(0,\cdot - t_n^j)\|^2_{L^q_{\mathbb{R}_t}} \\
& + \sum_{j\neq m, j,m=1}^{M_0}\|v^j(0,\cdot - t_n^j)v^m(0,\cdot - t_n^m)\|^{q/2}_{L^{q/2}_{\mathbb{R}_t}} \\
& + \sum_{j\neq m, j,m=M_0+1}^{M}\|v^j(0,\cdot - t_n^j)v^m(0,\cdot - t_n^m)\|^{q/2}_{L^{q/2}_{\mathbb{R}_t}}\Big) \\
\le\ & C_q\Big(\sum_{j=1}^{M_0}\|v^j(0,\cdot - t_n^j)\|^2_{L^q_{\mathbb{R}_t}} + \sum_{j=M_0+1}^{M}\|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)\|^2_{L^q_{\mathbb{R}_t}} \\
& + \sum_{j\neq m, j,m=1}^{M_0}\|v^j(0,\cdot - t_n^j)v^m(0,\cdot - t_n^m)\|^{q/2}_{L^{q/2}_{\mathbb{R}_t}} \\
& + \sum_{j\neq m, j,m=M_0+1}^{M}\|v^j(0,\cdot - t_n^j)v^m(0,\cdot - t_n^m)\|^{q/2}_{L^{q/2}_{\mathbb{R}_t}}\Big)
\end{aligned}$$

where we have used (4.11). The last terms $\sum_{j\neq m}\|v^j v^m\|_{L_t^{q/2}}$ can be made small if n is large (see the argument below for the claim 2). On the other hand, using (4.5), the same argument for (3.2) allows us to obtain

$$|[e^{it\partial_x^2}\psi_{0,n}](0)|^q = \sum_{j=1}^{M}|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)|^q + |[e^{it\partial_x^2}w_n^M](0)|^q + o_n(1),$$

thus, integrating in time,

$$\begin{aligned}\|[e^{it\partial_x^2}\psi_{0,n}](0)\|_{L^q_{\mathbb{R}_t}} &= \sum_{j=1}^{M_0}\|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)\|_{L^q_{\mathbb{R}_t}} \\ &\quad + \sum_{j=M_0+1}^{M}\|[e^{i(t-t_n^j)\partial_x^2}\phi^j](0)\|_{L^q_{\mathbb{R}_t}} + \|[e^{it\partial_x^2}w_n^M](0)\|_{L^q_{\mathbb{R}_t}} + o_n(1)\end{aligned}$$

which shows that $\sum_{j=M_0+1}^{M}\|e^{i(t-t_n^j)\partial_x^2}\phi^j\|^2_{L^q_{\mathbb{R}_t}}$ is bounded independently of M if $n>n_0$ since $\|[e^{it\partial_x^2}\psi_{0,n}](0)\|_{L^q_{\mathbb{R}_t}} \le \|\psi_{0,n}\|_{\dot H^{\sigma_c}}$. Recall that $\|v^j(0,\cdot - t_n^j)\|_{L^q_{\mathbb{R}_t}} = \|\mathrm{NLS}(t)\tilde\phi^j(0)\|_{L^q_{\mathbb{R}_t}} < \infty$. Therefore $\|\tilde{\psi}_n(0,\cdot)\|^q_{L^q_{\mathbb{R}_t}}$ is bounded independently of M provided $n>n_0$.

We next prove the claim 2. We see that e_n is estimated using Hölder inequality with $\frac{1}{\tilde q} = \frac{p-2}{q} + \frac{2}{q}$ as follows.

$$\begin{aligned}&\|e_n\|_{L^{\tilde q}_{\mathbb{R}_t}} \\ \le\ & C_p \sum_{j=1}^{M}\Big(\|v^j\|^{p-2}_{L^q_{\mathbb{R}_t}} + \Big\|\sum_{j=1}^{M} v^j\Big\|^{p-2}_{L^q_{\mathbb{R}_t}}\Big)\|(v^1+\cdots+v^{j-1}+v^{j+1}+\cdots+v^M)v^j\|_{L^{q/2}_{\mathbb{R}_t}}\end{aligned}$$

where we abbreviated $v^j(0,t-t_n^j)$ as v^j. Here, note that by (4.10), for any $\varepsilon>0$, there exists a large $R>0$ such that

$$\|\mathrm{NLS}(t-t_n^k)\tilde\phi^k(0)\|_{L^q(\{t:|t-t_n^k|>R\})} < \varepsilon.$$

Thus, taking large n such that $|t_n^j - t_n^k| > 2R$ with $j\neq k$ for such a $R>0$, we can estimate $\|v^jv^k\|_{L^{q/2}_{\mathbb{R}_t}}$ as follows:

$$\begin{aligned}\|v^jv^k\|_{L^{q/2}_{\mathbb{R}_t}} &\le \|[\mathrm{NLS}(t-t_n^j)\tilde\phi^j](0)[\mathrm{NLS}(t-t_n^k)\tilde\phi^k](0)\|_{L^{q/2}_{\mathbb{R}_t}} \\ &\le \|\mathrm{NLS}(t-t_n^j)\tilde\phi^j(0)\|_{L^q(\{t:|t-t_n^j|>R\})}\|\mathrm{NLS}(t-t_n^k)\tilde\phi^k(0)\|_{L^q_{\mathbb{R}_t}} \\ &\quad + \|\mathrm{NLS}(t-t_n^j)\tilde\phi^j(0)\|_{L^q_{\mathbb{R}_t}}\|\mathrm{NLS}(t-t_n^k)\tilde\phi^k(0)\|_{L^q(\{t:|t-t_n^k|>R\})} \\ &\le C\varepsilon.\end{aligned}$$

This shows that there exists n_1 such that the $L^{\tilde q}$ norm of e_n is small if $n>n_1(M,\varepsilon)$.

Now we consider Case 2. In this case, we have $M(\phi^1)\le 1$ and $\lim_{n\to\infty} E(e^{-it_n^1\partial_x^2}\phi^1) \le A$. As in the Case 1, by the existence of wave operator, there is $\tilde\phi^1 \in H^1_x$ such that

$$\|\mathrm{NLS}(-t_n^1)\tilde\phi^1 - e^{-it_n^1\partial_x^2}\phi^1\|_{H^1} \to 0, \quad n\to+\infty.$$

Put

$$\tilde w_n^M := w_n^M - \mathrm{NLS}(-t_n^1)\tilde\phi^1 + e^{-it_n^1\partial_x^2}\phi^1$$

Then we can write

$$\psi_{0,n} = e^{-it_n^1\partial_x^2}\phi^1 + w_n^M = \mathrm{NLS}(-t_n^1)\tilde\phi^1 + \tilde w_n^M,$$

with

$$\lim_{M\to\infty}\lim_{n\to\infty}\|[e^{it\partial_x^2}\tilde{w}_n^M](0)\|_{L^q_{\mathbb{R}_t}}=0.$$

Let ψ_c be the solution to (1.1) with initial data $\psi_c(0)=\tilde{\phi}^1$. Now we claim that $\|\psi_c(0,\cdot)\|_{L^q_{\mathbb{R}_t}}=+\infty$ (and thus $M(\psi_c)^{\frac{1-\sigma_c}{\sigma_c}}E(\psi_c)=A$). We proceed as in the Case 1. Suppose $A:=\|\psi_c(0,\cdot)\|_{L^q_{\mathbb{R}_t}}<\infty$. By definition, $\|\mathrm{NLS}(t)\tilde{\phi}^1(0)\|_{L^q_{\mathbb{R}_t}}=\|\psi_c(0,\cdot)\|_{L^q_{\mathbb{R}_t}}=A$. For any shift t', we can say $\|\mathrm{NLS}(t-t')\tilde{\phi}^1(0)\|_{L^q_{\mathbb{R}_t}}=\|\mathrm{NLS}(t)\tilde{\phi}^1(0)\|_{L^q_{\mathbb{R}_t}}$, thus we take in particular $t'=t_n^1$ and operate $\mathrm{NLS}(t)$ to $\psi_{0,n}=\mathrm{NLS}(-t_n^1)\tilde{\phi}^1+\tilde{w}_n^M$. We apply the perturbation argument by Proposition 2.5 to

$$\psi_n=\tilde{\psi}_n+\mathrm{NLS}(t)\tilde{w}_n^M,$$

with $\tilde{\psi}_n=\mathrm{NLS}(t-t_n^1)\tilde{\phi}^1$ and $\|\tilde{\psi}_n(0,\cdot)\|_{L^q_{\mathbb{R}_t}}=A<+\infty$. For n and M sufficiently large, we have

$$\|[e^{it\partial_x^2}(\psi_n(0)-\tilde{\psi}_n(0))](0)\|_{L^q_{\mathbb{R}_t}}=\|[e^{it\partial_x}\tilde{w}_n^M](0)\|_{L^q_{\mathbb{R}_t}}\le\epsilon_0,$$

and also the $L_t^{\tilde{q}}$ norm of the corresponding error term is estimated by ϵ_0, where $\epsilon_0=\epsilon_0(A)$ is obtained in Proposition 2.5. Then, by Proposition 2.5, we have $\|\psi_n(0,\cdot)\|_{L^q_{\mathbb{R}_t}}<\infty$, and this is a contradiction to non scattering assumption on ψ_n.

On the other hand, the proof of Lemma 5.6 in [10] allows us to have also,

Lemma 4.3. *Suppose $\{\psi(t,x),t\ge 0\}$ is precompact in H^1_x. Then for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that*

$$\sup_{t\ge 0}\int_{|x|\ge R_\varepsilon}(|\psi(x,t)|^2+|\partial_x\psi(t,x)|^2)dx\le\varepsilon.$$

Using this Lemma and the local viriel identity (1.2), we conclude the following proposition.

Proposition 4.4. *Let $p>3$. Assume $\psi_0\in H^1$ satisfies (1.4) and $\eta(0)<1$. Let $\psi(t,x)$ be the global solution to (1.1) with the initial data ψ_0 satisfying the precompactness: for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that*

$$\int_{|x|\ge R_\varepsilon}(|\psi(x,t)|^2+|\partial_x\psi(x,t)|^2)dx\le\varepsilon,\quad \textit{for all}\quad t\ge 0. \tag{4.12}$$

Then $\psi_0\equiv 0$.

Proof. Take $a(x)$ in the localized virial (1.2), as, for $R>0$ (which will be determined later), and for all $x\in\mathbb{R}$,

$$a(x)=R^2\chi\Big(\frac{|x|}{R}\Big),$$

where $\chi \in C_0^\infty(\mathbb{R}^+)$, $\chi(r) = r^2$ for $r \le 1$, and $\chi(r) = 0$ for $r \ge 2$. Put $z_R(t) := \int_{\mathbb{R}} a(x)|\psi|^2 dx$, then we have

$$z_R'(t) = -2R \operatorname{Im} \int_{\mathbb{R}} \chi'\Big(\frac{|x|}{R}\Big)\partial_x\psi\overline{\psi}dx,$$

and

$$\begin{aligned}
z_R''(t) &= 8\int_{|x|\le R}|\partial_x\psi|^2dx + 4\int_{R<|x|<2R}\chi''\Big(\frac{|x|}{R}\Big)|\partial_x\psi|^2dx \\
&\quad -\frac{1}{R^2}\int_{R<|x|<2R}\chi^{(4)}\Big(\frac{|x|}{R}\Big)|\psi|^2dx - 4|\psi(0)|^{p+1} \\
&\ge 2\{4\int_{|x|\le R}|\partial_x\psi|^2dx - 2|\psi(0)|^{p+1}\} - C_0\int_{R<|x|<2R}(|\partial_x\psi|^2 + \frac{1}{R^2}|\psi|^2)dx \\
&\ge 2\{4\int_{|x|\le R}|\partial_x\psi|^2dx - 2|\psi(0)|^{p+1}\} - C_0\int_{R<|x|}(|\partial_x\psi|^2 + \frac{1}{R^2}|\psi|^2)dx \qquad (4.13)
\end{aligned}$$

with a constant $C_0 = C_0(\|\chi''\|_{L^\infty}, \|\chi^{(4)}\|_{L^\infty})$ uniform in R.

Take $0 < \delta < 1$ such that

$$M(\psi_0)^{\frac{1-\sigma_c}{\sigma_c}}E(\psi_0) \le (1-\delta)M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}}E(\varphi_0),$$

then by (4.2), there exists $c_\delta > 0$ such that for any $t \in \mathbb{R}$

$$4\int_{|x|\le R}|\partial_x\psi|^2dx - 2|\psi(0)|^{p+1} \ge c_\delta\|\partial_x\psi_0\|_{L^2}^2 - 4\int_{|x|>R}|\partial_x\psi|^2dx. \qquad (4.14)$$

Now, we choose $\varepsilon = \frac{c_\delta}{8+C_0}\|\partial_x\psi_0\|_{L^2}^2$ in (4.12), then for sufficiently large $R_1 > \max\{1, R_\varepsilon\}$,

$$\int_{|x|>R_1}\Big(|\partial_x\psi|^2 + \frac{1}{R_1^2}|\psi|^2\Big)dx \le \int_{|x|>R_1}\Big(|\partial_x\psi|^2 + |\psi|^2\Big)dx \le \varepsilon = \frac{c_\delta}{8+C_0}\|\partial_x\psi_0\|_{L^2}^2.$$

Thus, by the choice of $R = R_1$, we have (4.14) $\ge c_\delta\|\partial_x\psi_0\|_{L^2}^2 - 4\varepsilon$ and so

$$z_{R_1}''(t) \ge c_\delta\|\partial_x\psi_0\|_{L^2}^2.$$

Integration in time then implies

$$z_{R_1}'(t) - z_{R_1}'(0) \ge c_\delta t\|\partial_x\psi_0\|_{L^2}^2.$$

On the other hand,

$$|z_{R_1}'(t) - z_{R_1}'(0)| \le CR_1$$

where C depends on p, $\|\psi_0\|_{L^2}$, and $\|\partial_x\psi_0\|_{L^2}$. This is absurd except the case $\psi_0 \equiv 0$. □

Finally we complete our arguments with

Proposition 4.5.

$$K = \{\psi_c(t), t \ge 0\} \subset H_x^1$$

with ψ_c obtained above as the minimal non scattering solution, is precompact in H_x^1.

The proof for this proposition is similar to the proof for the existence of ψ_c, and we omit it. We apply Proposition 4.4 to ψ_c, and we have $\psi_c(0) \equiv 0$, which contradicts the fact that $\|\psi_c(0,\cdot)\|_{L^q_{\mathbb{R}_t}} = +\infty$. This concludes the statement of Theorem 1.4. $\square$

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References

[1] R. Adami, R. Carlone, M. Correggi, L. Tentarelli, *Blow-up for the pointwise NLS in dimension two: absence of critical power*, preprint arXiv:1808.10343 (2018).

[2] R. Adami, G. Dell'Antonio, R. Figari, A. Teta, *The Cauchy Problem for the Schrödinger Equation in Dimension Three with Concentrated Nonlinearity*, Ann. Inst. H. Poincaré (C) An. Nonlin. 20 (2003), no. 3, pp. 477–500.

[3] R. Adami and A. Teta, *A class of nonlinear Schrödinger equations with concentrated nonlinearity*, J. Funct. Anal. 180 (2001), no. 1, pp. 148–175.

[4] V. Banica and N. Visciglia, *Scattering for NLS with a delta potential*, J. Differential Equations. 260 (2016), no.5, pp. 44104439.

[5] V.S. Buslaev, A.I. Komech, E. A. Kopylova, D. Stuart, *On asymptotic stability of solitary waves in Schrödinger equation coupled to nonlinear oscillator*, Commun. in PDE 33 (2008), no. 4, pp. 669–705.

[6] R. Carlone, M. Correggi, L. Tentarelli, *Well-posedness of the two-dimensional nonlinear equation with concentrated nonlinearity*, Ann. IHP (c) An. Nonlin. 36 2019 no. 1, pp. 257–294.

[7] C. Cacciapuoti, D. Finco, D. Noja and A. Teta, *The NLS equation in dimension one with spatially concentrated nonlinearities: the pointlike limit*, Lett. Math. Phys. 104 (2014), no. 12, pp. 1557–1570.

[8] F.M. Christ and M. I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. 100 (1991) pp. 87-109.

[9] T. Duyckaerts, J. Holmer, and S. Roudenko, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*, Math. Res. Lett. 15 (2008), no. 6, pp. 1233–1250.

[10] J. Holmer and S. Roudenko, *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys. 282 (2008), no. 2, pp. 435–467.

[11] J. Holmer, C. Liu, *Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity I: Basic theory*, preprint.

[12] J. Holmer and C. Liu, *Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity II: Supercritical blowup profiles*, preprint.

[13] D. Hennig, G.P Tsironis, M.I. Molina, and H. Gabriel, *A nonlinear quasiperiodic Kronig- Penney model*, Physics Letters A 190 (1994)

[14] , A.I. Komech, E. A. Kopylova, D. Stuart, *On asymptotic stability of solitons in a nonlinear Schrödinger equation*, Commun. in Pure App. An. 11 (2012), no. 3, pp. 1063–1079.

[15] C. E. Kenig, F. Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. 166 (2006), no. 3, pp. 645–675.

[16] M. I. Molina and C. A. Bustamante, *The attractive nonlinear delta-function potential*, arXiv:physics/0102053 [physics.ed-ph] (2001).

[17] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2*, J. Funct. Anal. 169 (1999) 201-225.

[18] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, with the assistance of Timothy S. Murphy*, Princeton University Press, Princeton, New Jersey (1993).