The Role of PSL(2, 7) in M-theory: M2-Branes, Englert Equation and the Septuples

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The role of PSL(2, 7) in M-theory: M2-branes, Englert equation and the septuples

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Abstract

Reconsidering the M2-brane solutions of $d = 11$ supergravity with a transverse Englert flux introduced by one of us in 2016, we present a new purely group theoretical algorithm to solve Englert equation based on a specific embedding of the PSL(2, 7) group into Weyl$[\mathfrak{e}_7]$. The aforementioned embedding is singled out by the identification of PSL(2, 7) with the automorphism group of the Fano plane. Relying on the revealed intrinsic PSL(2, 7) symmetry of Englert equation and on the new algorithm we present an exhaustive classification of Englert fluxes. The residual supersymmetries of the corresponding M2-brane solutions associated with the first of the 8 classes into which we have partitioned Englert fluxes are exhaustively analyzed and we show that all residual $d = 3$ supersymmetries with $\mathcal{N} \in \{1, 2, 3, 4, 5, 6\}$ are available. Our constructions correspond to a particular case in the category of M2-brane solutions with transverse self-dual fluxes.

Contents

1 Introduction

The scenario underlying the *gauge/gravity correspondence* [1–11] is multi-faceted and involves many different geometrical aspects. In particular there are two main paradigms:

a) The case of M2-branes solutions of $d = 11$ supergravity, where the eight-dimensional space $\mathcal{M}_8$ transverse to the brane world volume is taken to be the metric cone over a five dimensional compact Einstein manifold $\mathcal{M}_7$ characterized by the metric:

$$ds^2_{(8)} = dr^2 + r^2\, ds^2_{\mathcal{M}_7} \quad ; \quad r \in \mathbb{R}_+ . \tag{1.1}$$

b) The case of D3-brane solutions of type IIB supergravity, where the six-dimensional space $\mathcal{M}_6$ transverse to the brane world volume is taken to be the metric cone over a five-dimensional compact Einstein manifold $\mathcal{M}_5$ characterized by the metric:

$$ds^2_{(6)} = dr^2 + r^2\, ds^2_{\mathcal{M}_5} \quad ; \quad r \in \mathbb{R}_+ . \tag{1.2}$$

In the case of the M2-branes, variants of the above solution included the introduction of a self-dual 4-form flux in the transverse 8-dimensional space and were extensively studied in the literature (see for instance [12–15]). The properties of these solutions, such as supersymmetry, strongly depend on the topology of the transverse space as well as on the structure of the internal flux and only specific examples where analyzed.

A new class of M2-brane solutions with self-dual transverse flux was recently introduced in [16]. Inspired by previous results in $d = 7$ [17, 18], the 11-dimensional manifold at the base of the M2–branes was chosen with the following topology:

$$\mathcal{M}_{11} = \mathrm{Mink}_{1,2} \times \mathbb{R}_+ \times \mathrm{T}^7, \tag{1.3}$$

where $\mathrm{Mink}_{1,2}$ is Minkowski space in $1 + 2$ dimensions and represents the world-volume of the $M2$-brane, while T^7 is a flat compact seven-torus. $\mathbb{R}_+ \times \mathrm{T}^7$ is the eight-dimensional space transverse to the brane. It was shown that one can obtain exact solutions of $d = 11$ supergravity where the metric is of the form:

$$ds^2_{11} = H(y)^{-\frac{2}{3}} \left(d\xi^\mu \otimes d\xi^\nu\, \eta_{\mu\nu}\right) - H(y)^{\frac{1}{3}} \left(dy^I \otimes dy^J\, \delta_{IJ}\right), \tag{1.4}$$

the function $H(y)$ over the transverse eight-dimensional space being defined by an inhomogeneous Laplace equation whose source is provided by the norm of an Englert flux. By this we mean a solution of the following linear equation for a three-form $\mathbf{Y}^{[3]}$ living on the T^7 torus:[1]

$$\star_{\mathrm{T}^7}\, \mathrm{d}\mathbf{Y}^{[3]} = -\frac{\mu}{4}\,\mathbf{Y}^{[3]}, \tag{1.5}$$

which is the natural generalization of Beltrami equation for a 1-form on a T^3-torus:

$$\star_{\mathrm{T}^3}\, \mathrm{d}\mathbf{Y}^{[1]} = -\nu\,\mathbf{Y}^{[1]}. \tag{1.6}$$

[1]The relation between equation (1.5) and the self-duality condition on the 4-for field-strength in the Euclidean 8-dimensional transverse space is illustrated in Appendix B. We shall refer to Eq. (1.5), somewhat improperly, as the *Englert equation*, since it describes the internal flux in the original Englert solution [19], though on a space with a different topology.

Just as in [17, 18, 20], the torus T^3 was chosen to be:

$$T^3 \simeq \frac{\mathbb{R}^3}{\Lambda_{cubic}}, \tag{1.7}$$

where Λ_{cubic} is the cubic lattice, which endowed Beltrami equation with the discrete symmetry provided by the point group of such a lattice, namely the octahedral group O_{24}, in the same way in [16] the torus T^7 was chosen to be:

$$T^7 \simeq \frac{\mathbb{R}^7}{\Lambda^{root}}, \tag{1.8}$$

where Λ^{root} is a root lattice of a suitable Lie algebra, prescribed to admit a point group isomorphic to the simple group PSL(2, 7) of order 168. The main motivation for such an a priori choice performed in [16] was the embedding $\mathrm{PSL}(2,7) \hookrightarrow \mathrm{G}_{2(-14)} \subset \mathrm{SO}(7)$ which appeared to be promising in view of the possible existence of Killing spinors for the corresponding M2-brane solution. In any case just as it happens that Beltrami equation is covariant with respect to the O_{24} group, the adopted point group endows Englert equation with a PSL(2, 7)-symmetry.

In this paper we adopt a substantially new approach to the problem of constructing solutions of this kind. It is based on a deeper understanding of the significance of the group PSL(2, 7), entering in a different role as the *automorphism group of the Fano plane* and, as such, as a subgroup of the Weyl group of $E_{7(7)}$ [21]. This allows for the construction of novel M2-brane solutions with non constant fluxes. The main point of the present work is the realization that the PSL(2, 7)-symmetry of the Englert equation is much less a matter of choice than it appeared to be in the approach of [16]. Indeed, as we explain in section 4, which is a full fledged revisitation of the theory of the PSL(2, 7) group, this latter, in its role as automorphism group of the Fano plane, provides a systematic group-theoretical construction of the solutions to the Englert equation on a flat space. These can be written in terms of elementary solutions, each defined by seven (a *septuple* of) triples of integers $\{n_1, n_2, n_3\}$, $n_i \in \{1, 2, 3, 4, 5, 6, 7\}$, $n_1 < n_2 < n_3$, corresponding to the vertices of a Fano plane (which define a so-called *Steiner triple system*), combined with a suitably defined *complementary* septuple. The elementary solutions are characterized by the property that the non-vanishing internal components $\mathbf{Y}^{[3]}$ of the 3-form field $\mathbf{A}^{[3]}$ are only defined by the triplets of integers in the two septuples. The automorphism group PSL(2, 7) of the Fano plane used for this construction is chosen to be the point group of the torus-lattice as well as the underlying symmetry group of the final solutions. Its action on the $d = 11$ fields (and in particular on the internal components of the 3-form) can be inferred as follows. The 35 components Y_{ijk}, $i, j, k = 1, \ldots, 7$, of $\mathbf{Y}^{[3]}$ on the seven-torus are in one-to-one correspondence with weights of the 35-dimensional representation of $\mathrm{SO}(7) \subset \mathrm{SL}(7, \mathbb{R})$ according to:

$$dx^i \wedge dx^j \wedge dx^k \Leftrightarrow \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array} \Leftrightarrow \mathbf{w_{35}} \in \Lambda^{weight}_{\mathfrak{a}_6}, \tag{1.9}$$

where x^i are the torus coordinates, $\Lambda^{weight}_{\mathfrak{a}_6}$ denotes the weight lattice of the $\mathfrak{a}_6$ Lie algebra and $\mathbf{w_{35}}$ a weight of the 35-dimensional representation. The automorphism group PSL(2, 7) of the chosen Fano plane, being a subgroup of the Weyl group of $\mathfrak{a}_6$, acts in terms of permutations on the seven values of the internal indices, with respect to which the 35 weights $\{\mathbf{w_{35}}\}$ split into an orbit of length 7 plus another of length 28. This embedding of PSL(2, 7) into SL(7, $\mathbb{R}$) is different from the crystallographic embedding considered in [16]. Such observation provides an intrinsic group theoretical algorithm to construct solutions of Englert equation.

In [16] some solutions of the Englert equation were constructed using the obvious uplifting to 7-dimensions of the technique utilized in [17, 18, 20] to construct solutions to Beltrami equation, namely the Fourier expansion of the field $\mathbf{Y}^{[3]}$ and the restriction of the considered momenta to orbits of the PSL(2, 7) in the weight lattice of the $\mathfrak{a}_7$ Lie algebra. Such constructions were particularly cumbersome since they produced rather large parameter spaces that had to be organized a posteriori into irreducible representations of PSL(2, 7) and of its subgroups. Furthermore there was no clear cut strategy for an exhaustive classification.

In this paper, utilizing this new viewpoint and in particular the different inequivalent embedding mentioned above, we have been able to classify all solutions according to 424 generating schemes grouped into 8 classes, each class labeled by an invariant signature. This classification is displayed in Table 4. We have also provided an exhaustive analysis of the residual supersymmetries for the solutions of the first class in which both the original septuple and the complementary one are of Steiner type (they both have signature (0, 21, 0) and define two distinct Fano planes). The result of this analysis is summarized in Table (8.16). It shows that M2-branes with all possible number of supercharges can be obtained from our construction. The analysis of the remaining seven classes of solutions is postponed to a future publication. Similarly, as we discuss in the conclusive section 9, we postpone to a future publication of the possible interpretation of our M2-solutions in various classical contexts of the gauge/gravity correspondence or of the Kaluza-Klein expansion.

Although the approach followed in the present paper and the results are substantially different from those of [16], for the sake of completeness we shall recall some general properties of the group PSL(2, 7) which are illustrated in the same reference.

The paper is organized as follows:

- In section 2 we review the structure of the Ansatz of M2-branes with Englert fluxes.
- In section 3 we study the normal form of Englert three-forms and we introduce the role played by the group PSL(2, 7).
- In sections 4,5 we revisit the entire theory of the group PSL(2, 7) and of its crystallographic irreducible representations. In particular we illustrate the difference between the crystallographic irreducible representation of dimension 7 utilized in [16] and a new crystallographic irreducible representation of dimension 6 which is the key weapon for our algorithm to construct solutions of Englert equation.
- In section 6 we present the intrinsic group theoretical algorithm to solve Englert equation and we arrive at the classification of table 4.
- In section 7 we review the criterion, found in [16], for the preservation of $\mathcal{N} = 2, \ldots, 6$ residual supersymmetries in $d = 3$.
- In section 8 we derive the classification of residual supersymmetries for the solutions of type (0, 21, 0).
- In section 9 we draw our conclusions and we illustrate the perspectives for the interpretation of our M2-brane solutions.

- In appendix B we consider the more general case of M2-brane solutions of $d = 11$ supergravity with a transverse internal flux and we show that Englert fluxes are a particular subclass in this class.

The reader who is only interested in the construction and study of the new solutions and their supersymmetry, can skip the more mathematical sections 4, 5.

2 M2-branes with Englert fluxes

In this section we shortly review the structure of M2-brane solutions of $d = 11$ supergravity with Englert fluxes that were introduced in [16] and constitute the object of study, from a new viewpoint, of the present paper.

In order to describe the general form of these solutions with Englert fluxes we need to consider the effective low energy lagrangian of M-theory, namely $d = 11$ supergravity for which we utilize the geometric rheonomic formulation of [22, 23][2]. Appendix A provides a dictionary between the normalization used in the first paper on $d = 11$ supergravity [25] and those of [22, 23].

2.1 Summary of $d = 11$ supergravity in the rheonomy framework

The complete set of curvatures defining the relevant Free Differential Algebra is given below ([22, 23]):

$$\begin{aligned}
\mathfrak{T}^a &= \mathcal{D}V^a - \mathrm{i}\tfrac{1}{2}\overline{\psi} \wedge \Gamma^a \psi \\
\mathfrak{R}^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \\
\rho &= \mathcal{D}\psi \equiv d\psi - \tfrac{1}{4}\,\omega^{ab} \wedge \Gamma_{ab}\psi \\
\mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} - \tfrac{1}{2}\overline{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \\
\mathbf{F}^{[7]} &= d\mathbf{A}^{[6]} - 15\,\mathbf{F}^{[4]} \wedge \mathbf{A}^{[3]} - \tfrac{15}{2}\,V^a \wedge V^b \wedge \overline{\psi} \wedge \Gamma_{ab}\psi \wedge \mathbf{A}^{[3]} \\
&\quad -\mathrm{i}\,\tfrac{1}{2}\overline{\psi} \wedge \Gamma_{a_1\dots a_5}\psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5}
\end{aligned} \tag{2.1}$$

There is a unique rheonomic parametrization of the curvatures (2.1) which solves the Bianchi identities and it is the following one:

$$\begin{aligned}
\mathfrak{T}^a &= 0 \\
\mathbf{F}^{[4]} &= F_{a_1\dots a_4}\,V^{a_1} \wedge \cdots \wedge V^{a_4} \\
\mathbf{F}^{[7]} &= \tfrac{1}{84}F^{a_1\dots a_4}\,V^{b_1} \wedge \cdots \wedge V^{b_7}\,\epsilon_{a_1\dots a_4 b_1\dots b_7} \\
\rho &= \rho_{a_1a_2}\,V^{a_1} \wedge V^{a_2} - \mathrm{i}\tfrac{1}{3}\left(\Gamma^{a_1a_2a_3}\psi \wedge V^{a_4} + \tfrac{1}{8}\Gamma^{a_1\dots a_4 m}\,\psi \wedge V^m\right)F^{a_1\dots a_4} \\
\mathfrak{R}^{ab} &= R^{ab}{}_{cd}\,V^c \wedge V^d + \mathrm{i}\,\rho_{mn}\left(\tfrac{1}{2}\Gamma^{abmn} - \tfrac{2}{9}\Gamma^{mn[a}\,\delta^{b]c} + 2\,\Gamma^{ab[m}\,\delta^{n]c}\right)\psi \wedge V^c \\
&\quad + \overline{\psi} \wedge \Gamma^{mn}\,\psi\,F^{mnab} + \tfrac{1}{24}\overline{\psi} \wedge \Gamma^{abc_1\dots c_4}\,\psi\,F^{c_1\dots c_4}
\end{aligned} \tag{2.2}$$

[2]For a recent review in modernized notations see [24], Volume II, Chapter 6.

The expressions (2.2) satisfy the Bianchi.s provided the space–time components of the curvatures satisfy the following constraints

$$
\begin{aligned}
0 &= \mathcal{D}_m F^{mc_1c_2c_3} + \tfrac{1}{96}\,\epsilon^{c_1c_2c_3a_1a_8}\,F_{a_1\ldots a_4}\,F_{a_5\ldots a_8} && (2.3)\\
0 &= \Gamma^{abc}\,\rho_{bc} && (2.4)\\
R^{am}{}_{cm} &= 6\,F^{ac_1c_2c_3}\,F^{bc_1c_2c_3} - \tfrac{1}{2}\,\delta^a_b\,F^{c_1\ldots c_4}\,F^{c_1\ldots c_4} && (2.5)
\end{aligned}
$$

which are the space–time field equations.

2.2 M2-brane solutions with $\mathbb{R}_+ \times \mathrm{T}^7$ in the transverse dimensions

Among all the possible solutions to the field equations (2.3-2.5) we are interested in those that describe $M2$-branes of the form described below.

According to the general rules of brane-chemistry (see for instance [24], page 288 and following ones), we introduce the following $d = 11$ metric:

$$
ds^2_{11} = H(y)^{-\frac{4\widetilde{d}}{9\Delta}}\left(d\xi^\mu \otimes d\xi^\nu\,\eta_{\mu\nu}\right) - H(y)^{\frac{4d}{9\Delta}}\left(dy^I \otimes dy^J\,\delta_{IJ}\right) \tag{2.6}
$$

where:

$$
\xi^\mu \quad ; \quad \mu = \underline{0}, \underline{1}, \underline{2} \tag{2.7}
$$

are the coordinates on $\mathrm{Mink}_{1,2}$, while:

$$
y^I \quad ; \quad I = 1, 2, \ldots, 8 \tag{2.8}
$$

are the coordinates of the 8-dimensional transverse space. Since in $d = 11$ there is no dilaton we have

$$
\Delta = 2\frac{\widetilde{d}\,d}{9} = 2\,\frac{6\times 3}{9} = 4 \quad ; \quad d = 3\,; \quad \widetilde{d} = 6 \tag{2.9}
$$

and the appropriate $M2$ Ansatz for the metric becomes (1.4):

$$
ds^2_{11} = H(y)^{-\frac{2}{3}}\left(d\xi^\mu \otimes d\xi^\nu\,\eta_{\mu\nu}\right) - H(y)^{\frac{1}{3}}\left(dy^I \otimes dy^J\,\delta_{IJ}\right) \tag{2.10}
$$

Because of the chosen topology of the transverse space, see Eq. (1.3), it is convenient to set:

$$
y^8 = U \in \mathbb{R}_+ \quad ; \quad y^i = x^i \in \mathrm{T}^7 \quad (i = 1, \ldots, 7) \tag{2.11}
$$

The next point is to choose an appropriate Ansatz for the three-form $\mathbf{A}^{[3]}$. We set:

$$
\mathbf{A}^{[3]} = \frac{2}{H(y)}\,\Omega^{[3]} + e^{-\mu U}\,\mathbf{Y}^{[3]} \tag{2.12}
$$

where:

$$\Omega^{[3]} = \tfrac{1}{6}\epsilon_{\mu\nu\rho}\, d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho \tag{2.13}$$

$$\mathbf{Y}^{[3]} = Y_{ijk}(x)\, dx^i \wedge dx^j \wedge dx^k \tag{2.14}$$

The essential point in the above formula is that the antisymmetric tri-tensor $Y_{ijk}(x)$ depends only on the coordinates x of the seven-torus T^7. The geometry of T^7 is defined by a lattice Λ whose point group is the $\mathrm{PSL}(2,7)$ group to be introduced in the next Sections.

As shown in [16], with the Ansatz (2.12), the non-vanishing components of the 4-form $\mathbf{F}^{[4]}$ are the following ones:

$$F_{\underline{abc}I} = \frac{1}{12} H(y)^{-\frac{7}{6}}\, \partial_I H(y) \tag{2.15}$$

$$F_{8ijk} = -\frac{\mu}{4} e^{-\mu U}\, H(y)^{-\frac{2}{3}}\, Y_{ijk} \tag{2.16}$$

$$F_{ijk\ell} = H(y)^{-\frac{2}{3}}\, e^{-\mu U} \partial_i Y_{jk\ell} \tag{2.17}$$

Then we can easily verify that the Maxwell field equation (2.3) is satisfied provided the following two differential constraints hold:

$$\Box_{\mathbb{R}_+\times \mathrm{T}^7} H(y) = \frac{\mu}{4}\, e^{-2\,\mu\, U} \epsilon^{ijk\ell mnr}\, \partial_i Y_{jk\ell}\, Y_{mnr} \tag{2.18}$$

$$\frac{1}{4!}\epsilon^{pqrijk\ell}\, \partial_i Y_{jk\ell} = -\frac{\mu}{4} Y_{pqr} \tag{2.19}$$

The two equations admit the following index-free rewriting:

$$\Box_{\mathbb{R}_+\times \mathrm{T}^7} H(y) = -\frac{3\,\mu^2}{2}\, e^{-2\,\mu\, U}\ \parallel \mathbf{Y} \parallel^2 \equiv\ J(y) \tag{2.20}$$

$$\star_{\mathrm{T}^7}\, \mathrm{d}\mathbf{Y}^{[3]} = -\frac{\mu}{4}\, \mathbf{Y}^{[3]} \tag{2.21}$$

As we see Eq. (2.21) is the generalization to a 7-dimensional torus of Beltrami equation on the three-dimensional one. It is just Englert equation and in the present work we pursue a new systematic group theoretical approach to the construction of its solutions and the study of their supersymmetries.

As shown in [16], Einstein equations are also satisfied once Eq.s (2.20-2.21) are satisfied.

As mentioned earlier and shown in detail in Appendix B, these solutions fall in the general class of M2-branes with self-dual transverse flux.

3 Normal form and the role of PSL(2, 7)

Our approach to a systematic study of the solutions to the Englert equation is to construct elementary solutions in which the only non-vanishing components Y_{ijk} (the internal part of the 3-form $\mathbf{A}^{[3]}$) are defined by the *normal form* of the representation **35** with respect to the action of SO(7). The normal form is defined by the subspace of the representation space $V_{\mathbf{35}} = \{Y_{ijk}\}$ of least dimension, in which a generic vector in $V_{\mathbf{35}}$ can be rotated by means of an SO(7) transformation. This subspace has 14 parameters since a generic vector in $V_{\mathbf{35}}$ has a trivial little group in SO(7), so that the number of parameters of a generic element modulo the action of SO(7) is just $14 = 35 - 21$. The normal form of Y_{ijk} can be chosen in various ways, some of which have a special geometric interpretation. It is important to stress that in our solution $Y_{ijk}(x)$ are not constant and thus in general one cannot recover the most general tensor $Y_{ijk}(x)$ from its restriction to the normal form through an SO(7) transformation. Nevertheless the normal form will define elementary tensors satisfying the Englert equation which are the building blocks for our systematic study of its solutions.

As mentioned in the Introduction, the components of Y_{ijk} can be put into one-to-one correspondence with a the weights of the **35** representation of $\mathrm{SL}(7,\mathbb{R})$ group acting linearly on x^i. Formally these weights can be thought of as part of the 63 positive roots of an $\mathfrak{e}_{7(7)}$ Lie algebra. The latter has a special role in $d = 11$ supergravity since it generates the global symmetry group $\mathrm{E}_{7(7)}$ of the $d = 4$ supergravity obtained from the eleven-dimensional one through toroidal reduction [26]. However, it must be emphasized at this point that our solutions in general do not admit an effective $d = 4$ description and that they are covariant only with respect to the $\mathrm{SL}(7,\mathbb{R})$ subgroup of $\mathrm{E}_{7(7)}$. The action of $\mathrm{SL}(7,\mathbb{R})$ will change the metric on the torus into a different constant one. The action of the SO(7) subgroup of $\mathrm{SL}(7,\mathbb{R})$, leaves the metric δ_{ij} on T^7 invariant but transforms the lattice Λ defining it. The latter is left invariant only by its point group which is a subgroup of SO(7) and which will be chosen to be PSL(2, 7).

Let us denote by α_{ijk} the $\mathfrak{e}_{7(7)}$ positive roots corresponding to Y_{ijk}. The action of SO(7) on Y_{ijk} can be fixed by choosing seven non-vanishing components to correspond to a maximal subset of mutually orthogonal roots $\alpha^{(I)}$, $I = 1, \ldots, 7$ among the the 35 that we named α_{ijk}.[3] The normal form is then obtained by complementing this set of components with an other set of seven parameters, so that the total number of independent components amounts to 14.

Adopting this viewpoint the normal form is defined by two septuples of parameters, the first of which is, defined, as we have said by the roots $\alpha^{(I)}$.

Let us recall the main properties of this particular set of seven roots. They define, together with their negative $-\alpha^{(I)}$, an $\mathfrak{sl}(2)^7$ subalgebra of $\mathfrak{e}_7$. Moreover it can be shown that the seven triplets (i, j, k) of indices defining the $\alpha^{(I)}$ among the α_{ijk} form a so-called Steiner triple system and are in one-to-one correspondence with the vertices of a Fano plane (see Figure 6.1 for a particular choice of this septuple). It is at this level that the group PSL(2, 7) enters the game. As we show in the next section 4 entirely devoted to an in depth discussion of PSL(2, 7), of his subgroups and of its representations, this simple group has a crystallographic action on the $\mathfrak{e}_7$ root lattice and actually maps the $\mathfrak{e}_7$ root system $\Delta_{\mathfrak{e}_7}$ into itself, so that it happens to be a subgroup of the Weyl$[\mathfrak{e}_7]$ group.

We anticipate that we can have two distinct conjugacy classes of embeddings:

$$\mathrm{PSL}(2,7) \hookrightarrow \mathrm{Weyl}[\mathfrak{e}_7] \tag{3.1}$$

one based on the 7-dimensional irreducible representation of PSL(2, 7), the other on its 6-dimensional one. With respect to the former embedding there are no orbits of length 7 in the $\mathfrak{e}_7$ root lattice and in particular in the root system $\Delta_{\mathfrak{e}_7}$. With respect to the latter embedding, as discussed below, there are instead orbits of length 7 and the unique such one that is contained in the subset of 35 positive roots α_{ijk} precisely consists

[3]With an abuse of notation we use the same letters to label $\alpha^{(I)}$ and the eight transverse directions to the M2-brane. The different interpretation of these letters will be clear from the context.

of the septuple $\alpha^{(I)}$ of mutually orthogonal roots that define the embedding of the $\mathfrak{sl}(2)^7$ subalgebra into the $\mathfrak{e}_{7(7)}$. This embedding of PSL(2, 7) indeed acts as the automorphism group of the Fano plane associated with this septuple since, as a subgroup of Weyl$[\mathfrak{a}_6] = S_7$, its effect is of permuting the $\alpha^{(I)}$s.

Hence, as firstly shown in [21], there are 135 inequivalent choices of the septuple of commuting roots which is the ratio between the order of Weyl$[\mathfrak{e}_7]$ and the order of the product of $(\mathbb{Z}_2)^7$ (that reverses the sign of each $\alpha^{(I)}$), times the order of PSL(2, 7) that permutes the $\alpha^{(I)}$s in the septuple:

$$135 = \frac{|\text{Weyl}[\mathfrak{e}_7]|}{2^7 \times |\text{PSL}(2,7)|} = \frac{2903040}{2^7 \times 168} . \tag{3.2}$$

Let us refer to the two conjugacy classes of PSL(2, 7) subgroups within Weyl$[\mathfrak{e}_7]$ as:

$$\text{PSL}(2,7)_7 \subset \text{Weyl}[\mathfrak{e}_7] \quad ; \quad \text{PSL}(2,7)_{1+6} \subset \text{Weyl}[\mathfrak{e}_7] . \tag{3.3}$$

The reason for this naming, thoroughly explained in section 4, is that the embedding into the Weyl group occurs via the crystallographic embedding into the point group $\text{SO}(7,\mathbb{Z})_{\mathfrak{e}_7}$ of the root lattice $\Lambda^{\mathbf{r}}_{\mathfrak{e}_7}$:

$$\text{PSL}(2,7)_7 \hookrightarrow \text{SO}(7,\mathbb{Z})_{\mathfrak{e}_7} \quad ; \quad \text{PSL}(2,7)_{1+6} \hookrightarrow \text{SO}(7,\mathbb{Z})_{\mathfrak{e}_7} . \tag{3.4}$$

By $\text{SO}(7,\mathbb{R})_{\mathfrak{e}_7}$ we denote the standard SO(7) Lie group presented in the basis where the invariant metric $\eta = \mathfrak{C}_{\mathfrak{e}_7}$ is the Cartan matrix of the $\mathfrak{e}_7$ Lie algebra:

$$L \in \text{SO}(7,\mathbb{R})_{\mathfrak{e}_7} \Leftrightarrow L^T \mathfrak{C}_{\mathfrak{e}_7} L = \mathfrak{C}_{\mathfrak{e}_7} . \tag{3.5}$$

The point group of the root lattice $\text{SO}(7,\mathbb{Z})_{\mathfrak{e}_7} \subset \text{SO}(7,\mathbb{R})_{\mathfrak{e}_7}$ is the discrete subgroup made by those 7×7 matrices L that satisfy (3.5) and have integer valued entries. The two embeddings (3.4) are distinguished by the fact that the character of the 7 dimensional representation realized by the embedding is the irreducible character $\boldsymbol{\chi}^{irr}_{7}$ of the 7 dimensional representation of PSL(2, 7) in the first case, while it is the sum of the irreducible characters $\boldsymbol{\chi}^{irr}_{6} \oplus \boldsymbol{\chi}^{irr}_{1}$ in the second:

$$\chi\,[\text{PSL}(2,7)_7] = \boldsymbol{\chi}^{irr}_{7} \quad ; \quad \chi\,[\text{PSL}(2,7)_{1+6}] = \boldsymbol{\chi}^{irr}_{6} \oplus \boldsymbol{\chi}^{irr}_{1} . \tag{3.6}$$

Choosing the embedding $\text{PSL}(2,7)_{1+6}$ we obtain that the root lattice of the $\mathfrak{a}_6$ subalgebra of $\mathfrak{e}_7$ is left invariant by the action of $\text{PSL}(2,7)_{1+6}$. This obviously extends to the weight lattice of the same algebra. It follows that the set of positive roots of $\mathfrak{e}_7$ splits into subsets corresponding to irreducible representations of $\mathfrak{a}_6 \sim \mathfrak{sl}(7,\mathbb{R})$. In particular a group of **35** positive roots corresponds to the weights of the 35 irreducible representation of $\mathfrak{sl}(7,\mathbb{R})$, the three times antisymmetric, which means the tensor Y_{ijk}. This is the rigorous definition of the roots α_{ijk} mentioned above.

The **35** dimensional set is invariant under the action of $\text{PSL}(2,7)_{1+6}$ and splits in two orbits:

$$\mathbf{35} \overset{\text{PSL}(2,7)_{1+6}}{\Longrightarrow} \mathbf{7}_A \oplus \mathbf{28}. \tag{3.7}$$

The orbit $\mathbf{7}_A$, group theoretically defined in a unique way, provides, as mentioned above, the set of 7 mutually commuting roots $\alpha^{(I)}$ and, in the correspondence between $\mathfrak{a}_7$ weights and the triples of indices $\{ijk\}$ (see table 3) a first septuple of *Steiner triples*. Such system of triples can be characterized by the property that any two of the seven triplets of indices $\{ijk\}$ must have only one index in common.

We can easily count the possible number of the septuples $\mathbf{7}_A$ measuring the number of conjugate copies of the group $\mathrm{PSL}(2,7)_{1+6}$ inside $\mathrm{Weyl}[\mathfrak{a}_6] \subset \mathrm{Weyl}[\mathfrak{e}_7]$:

$$\#\text{ of septuples } \mathbf{7}_A \;=\; \frac{|\mathrm{Weyl}[\mathfrak{a}_6]|}{2^7 \times |\mathrm{PSL}(2,7)_{1+6}|} \;=\; \frac{7!}{2^7 \times 168} \;=\; 30. \tag{3.8}$$

The second step in order to obtain the 14 parameters of the normal form is to adjoin to septuple $\mathbf{7}_A$ of Steiner triples a second septuple $\mathbf{7}_B$ which is complementary to the first.

The concept of complementarity is briefly described in the lines below.

Let us denote a set of seven triples of indices $\{ijk\}$, $1 \le i < j < k \le 7$, by $\vec{\sigma}$:

$$\vec{\sigma} = \{\vec{\sigma}_I\}_{I=1,\dots,7} \;,\quad \vec{\sigma}_I = (\sigma_I^1, \sigma_I^2, \sigma_I^3)\;,\quad 1 \le \sigma_I^1 < \sigma_I^2 < \sigma_I^3 \le 7\,. \tag{3.9}$$

If P is a permutation of the seven values of the index I labeling the triplets in $\vec{\sigma}$, we shall denote the permuted set of triplets by $\vec{\sigma} \cdot P$:

$$\vec{\sigma} \cdot P = \{\vec{\sigma}_{P(I)}\}_{I=1,\dots,7}\,. \tag{3.10}$$

Two septuples $\vec{\sigma}$ and $\vec{\gamma}$ are *complementary* or *mutually non-local* if there exist two permutations $P,\ P' \in S_7$ such that:

$$\forall I = 1,\dots,7 \;\;:\quad \epsilon^{i_I\, \sigma^1_{P(I)} \sigma^2_{P(I)} \sigma^3_{P(I)} \gamma^1_{P'(I)} \gamma^2_{P'(I)} \gamma^3_{P'(I)}} \neq 0\,, \tag{3.11}$$

where $I \to i_I$ is a mapping of the set $\{1,2,3,4,5,6,7\}$ into itself which need not be onto. For $I = 1,\dots,7$, the numbers i_I are uniquely defined by the condition (3.11).

The selection of a septuple $\mathbf{7}_B$ complementary to the septuple $\mathbf{7}_A$ can be derived automatically in a group theoretical way considering the maximal subgroup of order 21 of $\mathrm{PSL}(2,7)_{1+6}$, denoted G_{21} (see section 4.7.1). Under the action of G_{21} we have:

$$\mathbf{35} \overset{\mathrm{PSL}(2,7)_{1+6}}{\Longrightarrow} \mathbf{7}_A \oplus \mathbf{28} \overset{\mathrm{G}_{21}}{\Longrightarrow} \mathbf{7}_A \oplus \mathbf{7}_B \oplus \mathbf{21} \tag{3.12}$$

and the septuple $\mathbf{7}_B$ is automatically complementary to septuple $\mathbf{7}_B$. How many are the possible choices of $\mathbf{7}_B$ for fixed $\mathbf{7}_A$? There is an easy answer: they are as many as the different subgroups $\mathrm{G}_{21} \subset \mathrm{PSL}(2,7)_{1+6}$ in the unique conjugacy class, namely:

$$\#\text{ of septuples } \mathbf{7}_B \;=\; \frac{|\mathrm{PSL}(2,7)_{1+6}|}{|\mathrm{G}_{21}|} \;=\; \frac{168}{21} \;=\; 8. \tag{3.13}$$

With the above preliminary arguments and anticipations we have illustrated the crucial role played by the group $\mathrm{PSL}(2,7)$ in deriving a normal 14-parameter form of the solution to Englert equation. In particular in Section 6.1 we shall illustrate how to construct a solution from a couple of complementary septuples $\vec{\sigma}$, $\vec{\gamma}$, see Eq. (6.12).

In the next long section we present the theory of $\mathrm{PSL}(2,7)$ in a systematic way, providing a great deal of relevant constructive details about representations, subgroups, crystallographic action on root lattices and orbits that, up to our knowledge, are not available in the mathematical literature. After such a preparation we will return to the explicit construction of the normal form of the solution to Englert equation in section 6.

The reader who is only interested in the construction of the solutions to the Englert equation and the study of their supersymmetry can skip the next two mathematical Sections and move directly to Section 6.

4 Theory of the simple group PSL(2, 7)

Since the finite simple group PSL(2, 7) plays a fundamental role in the derivation of the normal form of solutions to the Englert equation (1.5) we devote the present section and its subsections to the structural theory of this remarkable group. One of its most relevant properties, which turns out to be quite momentous for M–theory and was not duely observed in the mathematical literature, is that it is crystallographic in 7-dimensions. It is also crystallographic in 6 dimensions. In both cases the crystallographic representation corresponds to the irreducible representation of the same dimension predicted by general group theory; furthermore the lattice that is left invariant by the action of the PSL(2, 7) group is, respectively, the root lattice $\Lambda^r_{\mathfrak{a}_7}$ and the root lattice $\Lambda^r_{\mathfrak{a}_6}$, having denoted by $\mathfrak{a}_\ell$ the simple complex Lie algebra whose maximal split real form is $\mathrm{sl}(\ell+1,\mathbb{R})$. Because of duality it follows that also the corresponding weight lattices $\Lambda^w_{\mathfrak{a}_7}$ and $\Lambda^w_{\mathfrak{a}_6}$ are equally preserved by the action of PSL(2, 7) that is provided by integer valued matrices both in the root and in the weight basis. Since the symmetric Cartan matrices $\mathfrak{C}_{\mathfrak{a}_7}$ and $\mathfrak{C}_{\mathfrak{a}_6}$ are left invariant by PSL(2, 7) it follows that this latter has a natural irreducible embedding both in SO(7) and in SO(6). Last but not least, since the root lattice $\Lambda^r_{\mathfrak{a}_7}$ is a sublattice of the $\mathfrak{e}_7$ root lattice it follows that PSL(2, 7) is crystallographic with respect also to this latter and is actually a subgroup of the Weyl group Weyl[$\mathfrak{e}_7$]. It is just this property what provides the link of PSL(2, 7) with exceptional field theory and with the solutions of Englert equation.

4.1 Definition of the group PSL(2, 7)

The finite group:

$$\mathrm{PSL}(2,7) \;\equiv\; \mathrm{PSL}(2,\mathbb{Z}_7) \tag{4.1}$$

is the second smallest simple group after the alternating group A_5 which has 60 elements and coincides with the symmetry group of the regular icosahedron or dodecahedron. PSL(2, 7) has 168 elements: they can be identified with all the possible 2×2 matrices with determinant one whose entries belong to the finite field $\mathbb{Z}_7$, counting them up to an overall sign. In projective geometry, PSL(2, 7) is classified as a *Hurwitz group* since it is the automorphism group of a Hurwitz Riemann surface, namely a surface of genus g with the maximal number $84\,(g-1)$ of conformal automorphisms[4]. The Hurwitz surface pertaining to the Hurwitz group PSL(2, 7) is the Klein quartic, namely the locus $\mathcal{K}_4$ in $\mathbb{P}_2(\mathbb{C})$ cut out by the following quartic polynomial constraint on the homogeneous coordinates $\{x, y, z\}$:

$$x^3\, y + y^3\, z \;+ z^3\, x \;=\; 0 \tag{4.2}$$

Indeed $\mathcal{K}_4$ is a genus $g = 3$ compact Riemann surface and it can be realized as the quotient of the hyperbolic Poincaré plane $\mathbb{H}_2$ by a certain group Γ that acts freely on $\mathbb{H}_2$ by isometries.

[4]Hurwitz's automorphisms theorem proved in 1893 states that the order $|\mathcal{G}|$ of the group $\mathcal{G}$ of orientation-preserving conformal automorphisms, of a compact Riemann surface of genus $g > 1$ admits the following upper bound $|\mathcal{G}| \leq 84(g-1)$

The PSL(2, 7) group, which is also isomorphic to $GL(3, \mathbb{Z}_2)$, has received a lot of attention in Mathematics and it has important applications in algebra, geometry, and number theory: for instance, besides being associated with the Klein quartic, PSL(2, 7) is the automorphism group of the Fano plane.

The reason why we consider PSL(2, 7) in this section is associated with another property of this finite simple group which was proved almost twenty years ago in [27], namely:

$$\mathrm{PSL}(2,7) \subset \mathrm{G}_{2(-14)} \tag{4.3}$$

This means that PSL(2, 7) is a finite subgroup of the compact form of the exceptional Lie group G_2 and the 7-dimensional fundamental representation of the latter is irreducible upon restriction to PSL(2, 7).

As we already mentioned the group PSL(2, 7) is crystallographic in $d = 7$, and in $d = 6$.

4.2 Structure of PSL(2, 7)

For the reasons outlined above we consider the simple group (4.1) and its crystallographic action in $d = 7$. The Hurwitz simple group PSL(2, 7) is abstractly presented as follows[5]:

$$\mathrm{PSL}(2,7) = \left(R, S, T \parallel R^2 = S^3 = T^7 = RST = (TSR)^4 = \mathbf{e}\right) \tag{4.4}$$

and it has order 168:

$$|\ \mathrm{PSL}(2,7)\ | = 168 \tag{4.5}$$

For practical convenience we distinguish the abstract description of the group, from its concrete realization in terms of matrices, by rewriting Eq. (4.4) in terms of abstract generators denoted by the corresponding greek letters:

$$\mathrm{PSL}(2,7) = \left(\rho, \sigma, \tau \parallel \rho^2 = \sigma^3 = \tau^7 = \rho.\sigma.\tau = (\tau.\sigma.\rho)^4 = \epsilon\right) \tag{4.6}$$

In this way we can give an exhaustive enumeration of all the group elements as words in the three symbols ρ,σ,τ.

The elements of this simple group are organized in six conjugacy classes according to the scheme displayed below:

Conjugacy class	C_1	C_2	C_3	C_4	C_5	C_6
representative of the class	**e**	R	S	TSR	T	SR
order of the elements in the class	1	2	3	4	7	7
number of elements in the class	1	21	56	42	24	24

(4.7)

As one sees from the above table (4.7) the group contains elements of order 2, 3, 4 and 7 and there are two inequivalent conjugacy classes of elements of the highest order. According to the general theory of finite groups, there are 6 different irreducible representations of dimensions 1, 6, 7, 8, 3, 3, respectively. The character table of the group PSL(2, 7) can be found in the mathematical literature. It reads as follows:

[5] In the rest of this section we follow closely the results obtained by the present author in a recent paper [16]

Representation	C_1	C_2	C_3	C_4	C_5	C_6
D_1 [PSL(2, 7)]	1	1	1	1	1	1
D_6 [PSL(2, 7)]	6	2	0	0	−1	−1
D_7 [PSL(2, 7)]	7	−1	1	−1	0	0
D_8 [PSL(2, 7)]	8	0	−1	0	1	1
DA_3 [PSL(2, 7)]	3	−1	0	1	$\frac{1}{2}\left(-1+\mathrm{i}\sqrt{7}\right)$	$\frac{1}{2}\left(-1-\mathrm{i}\sqrt{7}\right)$
DB_3 [PSL(2, 7)]	3	−1	0	1	$\frac{1}{2}\left(-1-\mathrm{i}\sqrt{7}\right)$	$\frac{1}{2}\left(-1+\mathrm{i}\sqrt{7}\right)$

(4.8)

Soon we will retrieve it by constructing explicitly all the irreducible representations

4.3 The 7-dimensional irreducible representation

The two representations most relevant for our purposes are the 7 and the 6-dimensional ones. We begin with the former.

The following three statements are true:

1. The 7-dimensional irreducible representation is crystallographic since all elements $\gamma \in \mathrm{PSL}(2,7)$ are represented by integer valued matrices $D_7(\gamma)$ in a basis of vectors that span a lattice, namely the root lattice $\Lambda^{\mathbf{r}}_{\mathfrak{a}_7}$ of the $\mathfrak{a}_7$ simple Lie algebra.

2. The 7-dimensional irreducible representation provides an immersion $\mathrm{PSL}(2,7) \hookrightarrow \mathrm{SO}(7)$ since its elements preserve the symmetric Cartan matrix of A_7:

$$\begin{aligned} \forall\gamma \in \mathrm{PSL}(2,7) \quad : \quad D_7^T(\gamma)\, \mathfrak{C}_{\mathfrak{a}_7}\, D_7(\gamma) &= \mathfrak{C}_{\mathfrak{a}_7} \\ \mathfrak{C}_{i,j} &= \alpha_i \cdot \alpha_j \qquad (i, j = 1\ldots,7) \end{aligned} \tag{4.9}$$

   defined in terms of the simple roots α_i whose standard construction in terms of the unit vectors ϵ_i of $\mathbb{R}^8$ is recalled below:

$$\begin{array}{lllllllllll} \alpha_1 & = & \epsilon_1-\epsilon_2 & ; & \alpha_2 & = & \epsilon_2-\epsilon_3 & = & ; & \alpha_3 & = \epsilon_3-\epsilon_4 \\ \alpha_4 & = & \epsilon_4-\epsilon_5 & ; & \alpha_5 & = & \epsilon_5-\epsilon_6 & = & ; & \alpha_6 & = \epsilon_6-\epsilon_7 \\ \alpha_7 & = & \epsilon_7-\epsilon_8 \end{array} \tag{4.10}$$

3. Actually the 7-dimensional representation defines an embedding $\mathrm{PSL}(2,7) \hookrightarrow \mathrm{G}_2 \subset \mathrm{SO}(7)$ since there exists a three-index antisymmetric tensor ϕ_{ijk} satisfying the relations of octonionic structure constants that is preserved by all the matrices $D_7(\gamma)$:

$$\forall\gamma \in \mathrm{PSL}(2,7) \quad : \quad D_7(\gamma)_{ii'}\, D_7(\gamma)_{jj'}\;\; D_7(\gamma)_{kk'}\, \phi_{i'j'k'} = \phi_{ijk} \tag{4.11}$$

Let us prove the above statements. It suffices to write the explicit form of the generators R, S and T in the crystallographic basis of the considered root lattice:

$$\mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_7} \quad \Leftrightarrow \quad \mathbf{v} = n_i \, \alpha_i \quad n_i \in \mathbb{Z} \tag{4.12}$$

Explicitly if we set:

$$R_7 = \mathcal{R} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_7 = \mathcal{S} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_7 = \mathcal{T} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.13}$$

we find that the defining relations of PSL(2, 7) are satisfied:

$$\mathcal{R}^2 = \mathcal{S}^3 = \mathcal{T}^7 = \mathcal{R}\mathcal{S}\mathcal{T} = (\mathcal{T}\mathcal{S}\mathcal{R})^4 = \mathbf{1}_{7\times 7} \tag{4.14}$$

and furthermore we have:

$$\mathcal{R}^T \, \mathfrak{C}_{\mathfrak{a}_7} \, \mathcal{R} = \mathcal{S}^T \, \mathfrak{C}_{\mathfrak{a}_7} \, \mathcal{S} = \mathcal{T}^T \, \mathfrak{C}_{\mathfrak{a}_7} \, \mathcal{T} = \mathfrak{C}_{\mathfrak{a}_7} \tag{4.15}$$

where the explicit form of the $\mathfrak{a}_7$ Cartan matrix is recalled below:

$$\mathfrak{C}_{\mathfrak{a}_7} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.16}$$

This proves statements 1) and 2).

In order to prove statement 3) we proceed as follows. In $\mathbb{R}^7$ we consider the antisymmetric three-index tensor ϕ_{ABC} that is required to satisfy the algebraic relations of the octonionic structure constants, namely[6]:

$$\phi_{ABM}\,\phi_{CDM} = \frac{1}{18}\delta^{AB}_{CD} + \frac{2}{3}\Phi_{ABCD} \tag{4.17}$$

$$\phi_{ABC} = -\frac{1}{6}\epsilon_{ABCPQRS}\,\Phi_{ABCD} \tag{4.18}$$

The subgroup of SO(7) which leaves ϕ_{ABC} invariant is, by definition, the compact section $G_{(2,-14)}$ of the complex G_2 Lie group. We mention here two different realizations of the G_2-tensor, ϕ_{ABC} and φ_{ABC}, that we utilize in the sequel in relation with two different irreducible representations of PSL(2, 7):

$$\begin{array}{rcl|rcl} \phi_{1,2,7} & = & \frac{1}{6} & \varphi_{1,2,6} & = & \frac{1}{6} \\ \phi_{1,3,5} & = & \frac{1}{6} & \varphi_{1,3,4} & = & -\frac{1}{6} \\ \phi_{1,4,6} & = & \frac{1}{6} & \varphi_{1,5,7} & = & -\frac{1}{6} \\ \phi_{2,3,6} & = & \frac{1}{6} & \varphi_{2,3,7} & = & \frac{1}{6} \\ \phi_{2,4,5} & = & -\frac{1}{6} & \varphi_{2,4,5} & = & \frac{1}{6} \\ \phi_{3,4,7} & = & \frac{1}{6} & \varphi_{3,5,6} & = & -\frac{1}{6} \\ \phi_{5,6,7} & = & -\frac{1}{6} & \varphi_{4,6,7} & = & -\frac{1}{6} \end{array} \quad ; \quad \text{all other components vanish} \tag{4.19}$$

A particular matrix that transforms the standard orthonormal basis of $\mathbb{R}^7$ into the basis of simple roots α_i is

[6] In this equation the indices of the G_2-invariant tensor are denoted with capital letter of the Latin alphabet, as it was the case in the quoted literature on weak G_2-structures. In the following we will use lower case latin letters, the upper Latin letters being reserved for $d = 8$

the following one:

$$
\mathfrak{M} \;=\; \begin{pmatrix}
\sqrt{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} \\
0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix} \tag{4.20}
$$

since:

$$
\mathfrak{M}^T \, \mathfrak{M} \;=\; \mathfrak{C}_{\mathfrak{a}_7} \tag{4.21}
$$

Defining the transformed tensor:

$$
\widehat{\varphi}_{ijk} \;\equiv\; \left(\mathfrak{M}^{-1}\right)_i^{\;I} \left(\mathfrak{M}^{-1}\right)_j^{\;J} \left(\mathfrak{M}^{-1}\right)_k^{\;K} \varphi_{IJK} \tag{4.22}
$$

we can explicitly verify that:

$$
\begin{aligned}
\widehat{\varphi}_{ijk} &= (\mathcal{R})_i^{\;p} \, (\mathcal{R})_j^{\;q} \, (\mathcal{R})_k^{\;r} \, \widehat{\varphi}_{pqr} \\
\widehat{\varphi}_{ijk} &= (\mathcal{S})_i^{\;p} \, (\mathcal{S})_j^{\;q} \, (\mathcal{S})_k^{\;r} \, \widehat{\varphi}_{pqr} \\
\widehat{\varphi}_{ijk} &= (\mathcal{T})_i^{\;p} \, (\mathcal{T})_j^{\;q} \, (\mathcal{T})_k^{\;r} \, \widehat{\varphi}_{pqr}
\end{aligned} \tag{4.23}
$$

Hence, being preserved by the three-generators $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$, the antisymmetric tensor φ_{ijk} is preserved by the entire discrete group PSL(2, 7) which, henceforth, is a subgroup of $\mathrm{G}_{(2,-14)} \subset \mathrm{SO}(7)$, as it was shown by intrinsic group theoretical arguments in [27]. The other representations of the group PSL(2, 7) were explicitly constructed about ten years ago by Pierre Ramond and his younger collaborators in [28]. They are completely specified by giving the matrix form of the three generators ρ, σ, τ satisfying the defining relations 4.6. For the 6-dimensional representation we will instead use the crystallographic basis provided by the $\mathfrak{a}_6$ root lattice.

4.4 The 6-dimensional representation

Introducing the following short-hand notation:

$$
\begin{aligned}
c_n &= \cos\left[\frac{2\pi}{7}\, n\right] \\
s_n &= \sin\left[\frac{2\pi}{7}\, n\right]
\end{aligned} \tag{4.24}
$$

in [28] the generators of the group PSL(2, 7) in the 6-dimensional irreducible representation were explicitly written as it is displayed below:

$$D[\rho]_6 \quad = \quad \left(\begin{array}{c|c|c|c|c|c} \frac{c_3-1}{\sqrt{2}} & \frac{c_2-1}{\sqrt{2}} & \frac{c_1-1}{\sqrt{2}} & c_3-c_1 & c_1-c_2 & c_2-c_3 \\ \hline \frac{c_2-1}{\sqrt{2}} & \frac{c_1-1}{\sqrt{2}} & \frac{c_3-1}{\sqrt{2}} & c_2-c_3 & c_3-c_1 & c_1-c_2 \\ \hline \frac{c_1-1}{\sqrt{2}} & \frac{c_3-1}{\sqrt{2}} & \frac{c_2-1}{\sqrt{2}} & c_1-c_2 & c_2-c_3 & c_3-c_1 \\ \hline c_3-c_1 & c_2-c_3 & c_1-c_2 & \frac{c_1-1}{\sqrt{2}} & \frac{c_2-1}{\sqrt{2}} & \frac{c_3-1}{\sqrt{2}} \\ \hline c_1-c_2 & c_3-c_1 & c_2-c_3 & \frac{c_2-1}{\sqrt{2}} & \frac{c_3-1}{\sqrt{2}} & \frac{c_1-1}{\sqrt{2}} \\ \hline c_2-c_3 & c_1-c_2 & c_3-c_1 & \frac{c_3-1}{\sqrt{2}} & \frac{c_1-1}{\sqrt{2}} & \frac{c_2-1}{\sqrt{2}} \end{array}\right)$$

$$D[\sigma]_6 \; = \; \left(\begin{array}{c|c|c|c|c|c} \frac{(c_3-1)\rho^2}{\sqrt{2}} & \frac{(c_2-1)\rho^4}{\sqrt{2}} & \frac{(c_1-1)\rho}{\sqrt{2}} & (c_3-c_1)\rho^3 & (c_1-c_2)\rho^5 & (c_2-c_3)\rho^6 \\ \hline \frac{(c_2-1)\rho^2}{\sqrt{2}} & \frac{(c_1-1)\rho^4}{\sqrt{2}} & \frac{(c_3-1)\rho}{\sqrt{2}} & (c_2-c_3)\rho^3 & (c_3-c_1)\rho^5 & (c_1-c_2)\rho^6 \\ \hline \frac{(c_1-1)\rho^2}{\sqrt{2}} & \frac{(c_3-1)\rho^4}{\sqrt{2}} & \frac{(c_2-1)\rho}{\sqrt{2}} & (c_1-c_2)\rho^3 & (c_2-c_3)\rho^5 & (c_3-c_1)\rho^6 \\ \hline (c_3-c_1)\rho^2 & (c_2-c_3)\rho^4 & (c_1-c_2)\rho & \frac{(c_1-1)\rho^3}{\sqrt{2}} & \frac{(c_2-1)\rho^5}{\sqrt{2}} & \frac{(c_3-1)\rho^6}{\sqrt{2}} \\ \hline (c_1-c_2)\rho^2 & (c_3-c_1)\rho^4 & (c_2-c_3)\rho & \frac{(c_2-1)\rho^3}{\sqrt{2}} & \frac{(c_3-1)\rho^5}{\sqrt{2}} & \frac{(c_1-1)\rho^6}{\sqrt{2}} \\ \hline (c_2-c_3)\rho^2 & (c_1-c_2)\rho^4 & (c_3-c_1)\rho & \frac{(c_3-1)\rho^3}{\sqrt{2}} & \frac{(c_1-1)\rho^5}{\sqrt{2}} & \frac{(c_2-1)\rho^6}{\sqrt{2}} \end{array}\right)$$

$$D[\tau]_6 \quad = \quad (D[\rho]_6 \cdot D[\sigma]_6)^{-1} \tag{4.25}$$

and where shown to satisfy the required relations (4.6).

We rather introduce the crystallographic basis in a completely analogous way to the case of the 7-dimensional irreducible representation.

The following two statements are true:

1. The 6-dimensional irreducible representation is crystallographic since all elements $\gamma \in \mathrm{PSL}(2,7)$ are represented by integer valued matrices $D_6(\gamma)$ in a basis of vectors that span a lattice, namely the root lattice $\Lambda^{\mathbf{r}}_{\mathfrak{a}_6}$ of the $\mathfrak{a}_6$ simple Lie algebra.

2. The 6-dimensional irreducible representation provides an immersion $\mathrm{PSL}(2,7) \hookrightarrow \mathrm{SO}(6)$ since its elements preserve the symmetric Cartan matrix of $\mathfrak{a}_6$:

$$\begin{array}{rcrcl} \forall\gamma \in \mathrm{PSL}(2,7) & : & D_6^T(\gamma)\,\mathfrak{C}_{\mathfrak{a}_6}\,D_6(\gamma) & = & \mathfrak{C}_{\mathfrak{a}_6} \\ & & \mathfrak{C}_{i,j} & = & \alpha_i \cdot \alpha_j \qquad (i,j = 1\ldots,6) \end{array} \tag{4.26}$$

defined in terms of the simple roots α_i whose standard construction in terms of the unit vectors ϵ_i of

$\mathbb{R}^7$ is recalled below:

$$\begin{array}{rclcrclclcrcl} \alpha_1 & = & \epsilon_1 - \epsilon_2 & ; & \alpha_2 & = & \epsilon_2 - \epsilon_3 & = & ; & \alpha_3 & = & \epsilon_3 - \epsilon_4 \\ \alpha_4 & = & \epsilon_4 - \epsilon_5 & ; & \alpha_5 & = & \epsilon_5 - \epsilon_6 & = & ; & \alpha_6 & = & \epsilon_6 - \epsilon_7 \end{array} \tag{4.27}$$

Let us prove the above statements. It suffices to write the explicit form of the generators ρ, σ and τ in the crystallographic basis of the considered root lattice:

$$\mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} \quad \Leftrightarrow \quad \mathbf{v} = n_i\,\alpha_i \quad n_i \in \mathbb{Z} \tag{4.28}$$

Explicitly if we set:

$$R_6 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \; ; \; S_6 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.29}$$

we find that the defining relations of PSL(2, 7) are satisfied:

$$R_6^2 = S_6^3 = T_6^7 = (T_6\,S_6\,R_6)^4 = \mathbf{1}_{6\times 6} \tag{4.30}$$

and furthermore we have:

$$R_6^T\,\mathfrak{C}_{\mathfrak{a}_6}\,R_6 = S_6^T\,\mathfrak{C}_{\mathfrak{a}_6}\,S_6 = T_6^T\,\mathfrak{C}_{\mathfrak{a}_6}\,T_6 = \mathfrak{C}_{\mathfrak{a}_6} \tag{4.31}$$

where the explicit form of the $\mathfrak{a}_6$ Cartan matrix is recalled below:

$$\mathfrak{C}_{\mathfrak{a}_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.32}$$

4.5 The 8-dimensional representation

Utilizing the same notations as before in [28] the matrix form of the generators pertaining to the irreducible 8-dimensional representation was given as follows:

$$D[\sigma]_8 = \begin{pmatrix} c_1 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & s_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D[\rho]_8 = \begin{pmatrix} 2-2c_1 & 0 & 2c_1+2c_2-4c_3 & 2-2c_2 & 0 & 2-2c_3 & 0 & 2\sqrt{3}c_1-2\sqrt{3}c_2 \\ 0 & -2c_1+4c_2-2 & 0 & 0 & 2c_2-4c_3+2 & 0 & 4c_1-2c_3-2 & 0 \\ 2c_1+2c_2-4c_3 & 0 & -c_1+2c_2-c_3 & -4c_1+2c_2+2c_3 & 0 & 2c_1-4c_2+2c_3 & 0 & \sqrt{3}c_1-\sqrt{3}c_3 \\ 2-2c_2 & 0 & -4c_1+2c_2+2c_3 & 2-2c_3 & 0 & 2-2c_1 & 0 & 2\sqrt{3}c_2-2\sqrt{3}c_3 \\ 0 & 2c_2-4c_3+2 & 0 & 0 & 4c_1-2c_3-2 & 0 & 2c_1-4c_2+2 & 0 \\ 2-2c_3 & 0 & 2c_1-4c_2+2c_3 & 2-2c_1 & 0 & 2-2c_2 & 0 & 2\sqrt{3}c_3-2\sqrt{3}c_1 \\ 0 & 4c_1-2c_3-2 & 0 & 0 & 2c_1-4c_2+2 & 0 & -2c_2+4c_3-2 & 0 \\ 2\sqrt{3}c_1-2\sqrt{3}c_2 & 0 & \sqrt{3}c_1-\sqrt{3}c_3 & 2\sqrt{3}c_2-2\sqrt{3}c_3 & 0 & 2\sqrt{3}c_3-2\sqrt{3}c_1 & 0 & c_1-2c_2+c_3 \end{pmatrix}$$

$$D[\tau]_8 = (D[\rho]_8 \cdot D[\sigma]_8)^{-1} \tag{4.33}$$

It remains to be seen whether there exists a crystallographic basis also for this irreducible representation. We have not explored the matter but we conjecture that if such a basis exists it is that of the simple roots of $\mathfrak{a}_8$ leading to an embedding into the $\mathfrak{e}_8$ Weyl group.

4.6 The 3-dimensional complex representations

Before passing to other items of PSL(2, 7) theory we mention the last two irreducible representations of this simple group. They are very important in the context of the resolution of $\mathbb{C}^3/\Gamma$ singularities and its relation with the AdS/CFT correspondence (see [29] and [30]). Indeed the two three dimensional irreducible representations are complex and they are conjugate to each other. They define an embedding:

$$\mathrm{PSL}(2,7) \hookrightarrow \mathrm{SU}(3) \tag{4.34}$$

so that the resolution of $\mathbb{C}^3/\mathrm{PSL}(2,7)$ is crepant and defines a Ricci flat Kähler manifold of Calabi Yau type (non-compact).

To define these two representations it suffices to give the form of the generators for one of them. The generators of the conjugate representation are the complex conjugates of the same matrices.

Setting:

$$\psi \equiv e^{\frac{2i\pi}{7}} \tag{4.35}$$

we have the following form for the representation **3**:

$$\begin{aligned}
D[\rho]_3 &= \begin{pmatrix} \frac{i(\psi^2-\psi^5)}{\sqrt{7}} & \frac{i(\psi-\psi^6)}{\sqrt{7}} & \frac{i(\psi^4-\psi^3)}{\sqrt{7}} \\ \frac{i(\psi-\psi^6)}{\sqrt{7}} & \frac{i(\psi^4-\psi^3)}{\sqrt{7}} & \frac{i(\psi^2-\psi^5)}{\sqrt{7}} \\ \frac{i(\psi^4-\psi^3)}{\sqrt{7}} & \frac{i(\psi^2-\psi^5)}{\sqrt{7}} & \frac{i(\psi-\psi^6)}{\sqrt{7}} \end{pmatrix} \\
D[\sigma]_3 &= \begin{pmatrix} \frac{i(\psi^3-\psi^6)}{\sqrt{7}} & \frac{i(\psi^3-\psi)}{\sqrt{7}} & \frac{i(\psi-1)}{\sqrt{7}} \\ \frac{i(\psi^2-1)}{\sqrt{7}} & \frac{i(\psi^6-\psi^5)}{\sqrt{7}} & \frac{i(\psi^6-\psi^2)}{\sqrt{7}} \\ \frac{i(\psi^5-\psi^4)}{\sqrt{7}} & \frac{i(\psi^4-1)}{\sqrt{7}} & \frac{i(\psi^5-\psi^3)}{\sqrt{7}} \end{pmatrix} \\
D[\tau]_3 &= \begin{pmatrix} -ie^{\frac{3i\pi}{14}} & 0 & 0 \\ 0 & -ie^{-\frac{i\pi}{14}} & 0 \\ 0 & 0 & -e^{-\frac{i\pi}{7}} \end{pmatrix}
\end{aligned} \tag{4.36}$$

4.7 The proper subgroups of PSL(2, 7)

The crystallographic nature of the group in $d = 7$ has already been stressed. We introduce the $\mathfrak{a}_7$ weight lattice which, by definition, is just the dual of the root lattice. According with

$$\boldsymbol{\pi} \in \Lambda^{\mathbf{w}}_{\mathfrak{a}_7} \Leftrightarrow \boldsymbol{\pi} = n_i \lambda^i \quad : \quad n^i \in \mathbb{Z} \tag{4.37}$$

the root lattice is spanned by the simple weights that are implicitly defined by the relations:

$$\lambda^i \cdot \alpha_j \;=\; \delta^i_j \quad \Rightarrow \quad \lambda^i \;=\; \left(\mathfrak{C}^{-1}_{\mathfrak{a}_7}\right)^{ij} \alpha_j \tag{4.38}$$

Since the group PSL(2, 7) is crystallographic on the root lattice, by necessity it is crystallographic also on the weight lattice. Given the generators of the group PSL(2, 7) in the basis of simple roots we obtain the same in the basis of simple weights through the following transformation:

$$\mathcal{R}_w \;=\; \mathfrak{C}_{\mathfrak{a}_7}\,\mathcal{R}\,\mathfrak{C}^{-1}_{\mathfrak{a}_7} \quad ; \quad \mathcal{S}_w \;=\; \mathfrak{C}_{\mathfrak{a}_7}\,\mathcal{S}\,\mathfrak{C}^{-1}_{\mathfrak{a}_7} \quad ; \quad \mathcal{T}_w \;=\; \mathfrak{C}_{\mathfrak{a}_7}\,\mathcal{T}\,\mathfrak{C}^{-1}_{\mathfrak{a}_7} \tag{4.39}$$

Explicitly we find:

$$\mathcal{R}_w \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \mathcal{S}_w \;=\; \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{T}_w \;=\; \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{4.40}$$

Given the weight basis, which is useful in several constructions, let us continue our survey of the remarkable simple group PSL(2, 7) by a discussion of its subgroups, none of which, obviously, is normal.

PSL(2, 7) contains maximal subgroups only of index 8 and 7, namely of order 21 and 24. The order 21 subgroup G_{21} is the unique non-abelian group of that order and abstractly it has the structure of the semidirect product $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$. Up to conjugation there is only one subgroup G_{21} as we have explicitly verified with the computer. On the other hand, up to conjugation, there are two different groups of order 24 that are both isomorphic to the octahedral group $O_{24} \sim S_4$.

4.7.1 The maximal subgroup G_{21}

The group G_{21} has two generators $\mathcal{X}$ and $\mathcal{Y}$ that satisfy the following relations:

$$\mathcal{X}^3 = \mathcal{Y}^7 = \mathbf{1} \quad ; \quad \mathcal{X}\mathcal{Y} = \mathcal{Y}^2\mathcal{X} \tag{4.41}$$

The organization of the 21 group elements into conjugacy classes is displayed below:

ConjugacyClass	C_1	C_2	C_3	C_4	C_5
representative of the class	e	$\mathcal{Y}$	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
order of the elements in the class	1	7	7	3	3
number of elements in the class	1	3	3	7	7

(4.42)

As we see there are five conjugacy classes which implies that there should be five irreducible representations the square of whose dimensions should sum up to the group order 21. The solution of this problem is:

$$21 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 \tag{4.43}$$

and the corresponding character table is mentioned below:

0	e	$\mathcal{Y}$	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
$D_1\,[G_{21}]$	1	1	1	1	1
$DX_1\,[G_{21}]$	1	1	1	$-(-1)^{1/3}$	$(-1)^{2/3}$
$DY_1\,[G_{21}]$	1	1	1	$(-1)^{2/3}$	$-(-1)^{1/3}$
$DA_3\,[G_{21}]$	3	$\frac{1}{2}i\left(i+\sqrt{7}\right)$	$-\frac{1}{2}i\left(-i+\sqrt{7}\right)$	0	0
$DB_3\,[G_{21}]$	3	$-\frac{1}{2}i\left(-i+\sqrt{7}\right)$	$\frac{1}{2}i\left(i+\sqrt{7}\right)$	0	0

(4.44)

In the weight-basis the two generators of the G_{21} subgroup of PSL(2, 7) can be chosen to be the following matrices and this fixes our representative of the unique conjugacy class:

$$\mathcal{X} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \quad \mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.45}$$

The embedding of G_{21} into PSL(2, 7) can be unambiguously fixed by writing the two generators of the former as words in the generators of the latter. We have:

$$\mathcal{Y} = \rho \cdot \tau \cdot \tau \cdot \tau \cdot \sigma \cdot \rho \quad ; \quad \mathcal{X} = \sigma \cdot \rho \cdot \sigma \cdot \rho \cdot \tau \cdot \tau \tag{4.46}$$

Eq.(4.46) allows to restrict any given representation of PSL(2, 7) to its maximal subgroup G_{21}.

4.7.2 The maximal subgroups O_{24A} and O_{24B}

The octahedral group O_{24} has two generators $\mathfrak{s}$ and $\mathfrak{t}$ that satisfy the following relations:

$$\mathfrak{s}^2 = \mathfrak{t}^3 = (\mathfrak{s}\mathfrak{t})^4 = \mathbf{1} \tag{4.47}$$

The 24 elements are organized in five conjugacy classes according to the scheme displayed below:

Conjugacy Class	C_1	C_2	C_3	C_4	C_5
representative of the class	e	$\mathfrak{t}$	$\mathfrak{s}\mathfrak{t}\mathfrak{s}\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{s}\mathfrak{t}$
order of the elements in the class	1	3	2	2	4
number of elements in the class	1	8	3	6	6

(4.48)

The character table where we also mention a standard representative of each conjugacy class is the following one:

0	e	$\mathfrak{t}$	$\mathfrak{s}\mathfrak{t}\mathfrak{s}\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{s}\mathfrak{t}$
$D_1\,[O_{24}]$	1	1	1	1	1
$D_2\,[O_{24}]$	1	1	1	−1	−1
$D_3\,[O_{24}]$	2	−1	2	0	0
$D_4\,[O_{24}]$	3	0	−1	−1	1
$D_5\,[O_{24}]$	3	0	−1	1	−1

(4.49)

By computer calculations we have verified that there are just two disjoint conjugacy classes of O_{24} maximal subgroups in PSL(2, 7) that we have named A and B, respectively. We have chosen two standard representatives, one for each conjugacy class, that we have named O_{24A} and O_{24B} respectively. To fix these subgroups it suffices to mention the explicit form of their generators in the weight basis.

For the group O_{24A}, we chose:

$$\mathfrak{t}_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \quad \mathfrak{s}_A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \tag{4.50}$$

For the group O_{24B}, we chose:

$$\mathfrak{t}_B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \mathfrak{s}_B = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.51}$$

Just as in the case of the subgroup G_{21} we can uniquely fix the embedding of the two octahedral subgroups into PSL(2, 7) in any given of its representations by writing the two generators of the subgroup as words in the generators of the bigger group. Explicitly we have:

$$\begin{aligned} \mathfrak{t}_A &= \rho \cdot \sigma \cdot \rho \cdot \tau \cdot \tau \cdot \sigma \cdot \rho \cdot \tau \quad ; \quad \mathfrak{s}_A = \tau \cdot \tau \cdot \sigma \cdot \rho \cdot \tau \cdot \sigma \cdot \sigma \\ \mathfrak{t}_B &= \rho \cdot \tau \cdot \sigma \cdot \rho \cdot \tau \cdot \tau \cdot \sigma \cdot \rho \cdot \tau \quad ; \quad \mathfrak{s}_B = \sigma \cdot \rho \cdot \tau \cdot \sigma \cdot \rho \cdot \tau \end{aligned} \tag{4.52}$$

4.7.3 The tetrahedral subgroup $T_{12} \subset O_{24}$

Every octahedral group O_{24} has, up to O_{24}-conjugation, a unique tetrahedral subgroup T_{12} whose order is 12. The abstract description of the tetrahedral group is provided by the following presentation in terms of two generators:

$$T_{12} = \left(s, t \,\middle|\, s^2 = t^3 = (st)^3 = 1\right) \tag{4.53}$$

The 12 elements are organized into four conjugacy classes as displayed below:

Classes	C_1	C_2	C_3	C_4
standard representative	1	s	t	t^2s
order of the elements in the conjugacy class	1	2	3	3
number of elements in the conjugacy class	1	3	4	4

(4.54)

We do not display the character table since we will not use it. The two tetrahedral subgroups $\mathrm{T}_{12\mathrm{A}} \subset \mathrm{O}_{24\mathrm{A}}$ and $\mathrm{T}_{12\mathrm{B}} \subset \mathrm{O}_{24\mathrm{B}}$ are not conjugate under the big group PSL(2, 7). Hence we have two conjugacy classes of tetrahedral subgroups of PSL(2, 7).

4.7.4 The dihedral subgroup $\mathrm{Dih}_3 \subset \mathrm{O}_{24}$

Every octahedral group O_{24} has a dihedral subgroup Dih_3 whose order is 6. The abstract description of the dihedral group Dih_3 is provided by the following presentation in terms of two generators:

$$\mathrm{Dih}_3 = \left(A, B \,\middle|\, A^3 = B^2 = (BA)^2 = 1\right) \tag{4.55}$$

The 6 elements are organized into three conjugacy classes as displayed below:

ConjugacyClasses	C_1	C_2	C_3
standard representative of the class	1	A	B
order of the elements in the class	1	3	2
number of elements in the class	1	2	3

(4.56)

We do not display the character table since we will not use it. Differently from the case of the tetrahedral subgroups the two dihedral subgroups $\mathrm{Dih}_{3\mathrm{A}} \subset \mathrm{O}_{24\mathrm{A}}$ and $\mathrm{Dih}_{3\mathrm{B}} \subset \mathrm{O}_{24\mathrm{B}}$ turn out to be conjugate under the big group PSL(2, 7). Actually there is just one PSL(2, 7)-conjugacy class of dihedral subgroups Dih_3.

4.8 Enumeration of the possible subgroups and orbits in the $\mathfrak{a}_7$ and $\mathfrak{a}_6$ weight lattices

In $d = 3$ the orbits of the octahedral group acting on the cubic lattice are the vertices of regular geometrical figures. Since PSL(2, 7) has a crystallographic action on the mentioned 7-dimensional and 6-dimensional weight lattices, its orbits $\mathcal{O}$ in $\Lambda^{\mathbf{w}}_{\mathfrak{a}_7}$ and $\Lambda^{\mathbf{w}}_{\mathfrak{a}_6}$ correspond to the analogue of the regular geometrical figures in $d = 7$ and in $d = 6$. Every orbit is in correspondence with a coset G/H where G is the big group and H one of its possible subgroups. Indeed H is the stability subgroup of an element of the orbit.

Since the maximal subgroups of PSL(2, 7) are of index 7 or 8 we can have subgroups $\mathrm{H} \subset$ PSL(2, 7) that are either G_{21} or O_{24} or subgroups thereof. Furthermore, as we know, the order |H| of any subgroup $\mathrm{H} \subset \mathrm{G}$ must be a divisor of |G|. Hence we conclude that

$$|\mathrm{H}| \in \{1, 2, 3, 4, 6, 7, 8, 12, 21, 24\} \tag{4.57}$$

Correspondingly we might have PSL(2, 7)-orbits $\mathcal{O}$ in the weight lattices $\Lambda^{\mathbf{w}}_{\mathfrak{a}_{7,6}}$, whose length is one of the following 10 numbers:

$$\ell_{\mathcal{O}} \in \{168, 84, 56, 42, 28, 24, 21, 14, 8, 7\} \tag{4.58}$$

Combining the information about the possible group orders (4.57) with the information that the maximal subgroups are of index 8 or 7, we arrive at the following list of possible subgroups H (up to conjugation) of the group PSL(2, 7):

Order 24) Either $\mathrm{H} = \mathrm{O}_{24A}$ or $\mathrm{H} = \mathrm{O}_{24B}$.

Order 21) The only possibility is $\mathrm{H} = \mathrm{G}_{21}$.

Order 12) The only possibilities are $\mathrm{H} = \mathrm{T}_{12A}$ or $\mathrm{H} = \mathrm{T}_{12B}$ where T_{12} is the tetrahedral subgroup of the octahedral group O_{24}.

Order 8) Either $\mathrm{H} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathrm{H} = \mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathrm{H} = \mathrm{Dih}_4$ where Dih_4 denotes the dihedral subgroup of index 3 of the octahedral group O_{24}.

Order 7) The only possibility is $\mathbb{Z}_7$.

Order 6) Either $\mathrm{H} = \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathrm{H} = \mathrm{Dih}_3$, where Dih_3 denotes the dihedral subgroup of index 4 of the octahedral group O_{24}.

Order 4) Either $\mathrm{H} = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathrm{H} = \mathbb{Z}_4$.

Order 3) The only possibility is $\mathrm{H} = \mathbb{Z}_3$

Order 2) The only possibility is $\mathrm{H} = \mathbb{Z}_2$.

Quite curiously and inspiringly the various possibilities are realized in a partially mutually exclusive pattern in 7 and 6 dimensions as recalled in the following two subsections and summarized in table 1.

4.8.1 Synopsis of the PSL(2, 7) orbits in the weight lattice $\Lambda^{\mathbf{w}}_{\mathfrak{a}_7}$

In [16], the author presented the results, obtained by means of computer calculations, on the orbits of the considered simple group acting on the $\mathfrak{a}_7$ weight lattice. They are briefly summarized below:

1. Orbits of length 8 (one parameter**n**; stability subgroup $\mathrm{H}^s = \mathrm{G}_{21}$)
2. Orbits of length 14 (two types A & B) (one parameter **n**; stability subgroup $\mathrm{H}^s = \mathrm{T}_{12A,B}$)
3. Orbits of length 28 (one parameter **n** ; stability subgroup $\mathrm{H}^s = \mathrm{Dih}_3$)
4. Orbits of length 42 (one parameter **n**; stability subgroup $\mathrm{H}^s = \mathbb{Z}_4$))
5. Orbits of length 56 (three parameters **n,m,p**; stability subgroup $\mathrm{H}^s = \mathbb{Z}_3$)
6. Orbits of length 84 (three parameters **n,m,p**; stability subgroup $\mathrm{H}^s = \mathbb{Z}_2$)
7. Generic orbits of length 168 (seven parameters ; stability subgroup $\mathrm{H}^s = \mathbf{1}$)

Table 1: Summary of the PSL(2, 7) orbits of vectors existing in the $\mathfrak{a}_7$ and $\mathfrak{a}_6$ weight lattices. All possible lengths enumerated in Eq. (4.58) are realized, except for $\ell = 24$, yet not at the same time in $d = 7$ and $d = 6$. Most of the lower length orbits corresponding to the largest stability subgroups are realized in either one of the two crystallographic irreducible representations, $d = 7$ or $d = 6$

Orbit length	Subgroup	d=7	d=6
7	O_{24A}	No	Yes
7	O_{24B}	No	Yes
8	G_{21}	Yes	No
14	T_{12A}	Yes	No
14	T_{12B}	Yes	No
21	Dih_4	No	Yes
24	$\mathbb{Z}_7$	No	No
28	Dih_3	Yes	Yes
42	$\mathbb{Z}_4$	Yes	No
56	$\mathbb{Z}_3$	Yes	No
84	$\mathbb{Z}_2$	Yes	Yes
168	Id	Yes	Yes

Also in this case the above list is in some sense the 6-dimensional analogue of Platonic solids. It is only in some sense, since it is a complete classification for the group PSL(2, 7), yet we are not aware of a classification of the other crystallographic subgroups of SO(6), if any.

4.8.2 Synopsis of the PSL(2, 7) orbits in the weight lattice $\Lambda^{w}_{\mathfrak{a}_6}$

Complementing the work done in [16], we obtained, also by means of computer calculations, the orbits of $PSL(2, 7)$ acting through its irreducible 6-dimensional representation on the $\mathfrak{a}_6$ weight lattice. They are briefly summarized below:

1. Orbits of length 7 (one parameter **n**; stability subgroup $H^s = O_{24A}$)
2. Orbits of length 7 (one parameter **n**; stability subgroup $H^s = O_{24B}$)
3. Orbits of length 28 (one parameter **n** ; stability subgroup $H^s = \mathrm{Dih}_3$)
4. Orbits of length 21 (two parameters **m, n**; stability subgroup $H^s = \mathrm{Dih}_3$)
5. Orbits of length 84 (four parameters **n,m,p,q**; stability subgroup $H^s = \mathbb{Z}_2$)
6. Generic orbits of length 168 (six parameters ; stability subgroup $H^s = \mathbf{1}$)

Also in this case the above list is in some sense the 6-dimensional analogue of Platonic solids. It is only in some sense, since it is a complete classification for the group PSL(2, 7), yet we are not aware of a classification of the other crystallographic subgroups of SO(6), if any.

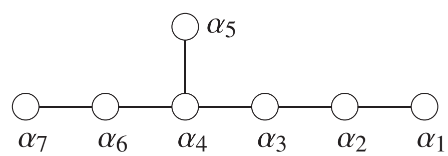
5 Embedding of the group PSL(2, 7) into $\mathfrak{e}_{7(7)}$

Above we considered the simple group PSL(2, 7) showing that it acts crystallographically on $\Lambda^{\mathbf{r}}_{\mathfrak{a}_{7,6}}$ and, consequently, also on the dual weight lattices $\Lambda^{\mathbf{w}}_{\mathfrak{a}_{7,6}}$. In view of our goals pursued within the context of exceptional field theory we show next how the action of PSL(2, 7), can be extended to the root and weight lattices of the exceptional Lie algebra $\mathfrak{e}_7$.

5.1 Embedding of PSL(2, 7) into Weyl[$\mathfrak{e}_7$]

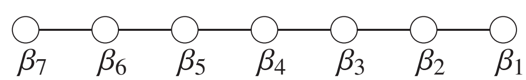
Let us consider the Dynkin diagrams of the three Lie algebras $\mathfrak{e}_7$, $\mathfrak{a}_7$ and $\mathfrak{a}_6$.

$\mathfrak{e}_7$



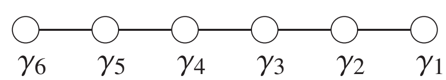
(5.1)

$\mathfrak{a}_7$



(5.2)

$\mathfrak{a}_6$



(5.3)

The Lie algebras $\mathfrak{a}_7$ has the same rank as $\mathfrak{e}_7$ and the former is regularly embedded into the latter, having the same Cartan subalgebra. Indeed given any set of simple roots α_i fulfilling the relations imposed by the Dynkin diagram (5.1), we immediately construct a set of simple roots β_j fulfilling the relations imposed by the Dinkin diagram (5.2) by setting:

$$\begin{aligned}
\beta_1 &= \alpha_2 + 2\,\alpha_3 + 3\,\alpha_4 + 2\,\alpha_5 + 2\,\alpha_6 + 2\,\alpha_7 \\
\beta_2 &= \alpha_1 \\
\beta_3 &= \alpha_2 \\
\beta_4 &= \alpha_3 \\
\beta_5 &= \alpha_4 \\
\beta_6 &= \alpha_6 \\
\beta_7 &= \alpha_7
\end{aligned} \tag{5.4}$$

As one notices the $\mathfrak{a}_7$ simple roots are integer valued linear combinations of the $\mathfrak{e}_7$ simple roots, hence they all belong to the $\mathfrak{e}_7$ root lattice $\Lambda^{\mathbf{r}}_{\mathfrak{e}_7}$. It follows that $\Lambda^{\mathbf{r}}_{\mathfrak{a}_7}$ is a sublattice of the former:

$$\Lambda^{\mathbf{r}}_{\mathfrak{a}_7} \subset \Lambda^{\mathbf{r}}_{\mathfrak{e}_7} \tag{5.5}$$

From Eq. (5.4) we immediately read off the matrix that performs the change of basis of PSL(2, 7) group elements from the basis β_i of $\mathfrak{a}_7$ simple roots to the basis α_i of $\mathfrak{e}_7$ simple roots. It is the following one:

$$\mathbf{\Pi} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

Setting:

$$\mathbf{R}^{\mathbf{r}}_{\mathfrak{e}_7} = \mathbf{\Pi}\,\mathcal{R}\,\mathbf{\Pi}^{-1} \quad ; \quad \mathbf{S}^{\mathbf{r}}_{\mathfrak{e}_7} = \mathbf{\Pi}\,\mathcal{S}\,\mathbf{\Pi}^{-1} \quad ; \quad \mathbf{T}^{\mathbf{r}}_{\mathfrak{e}_7} = \mathbf{\Pi}\,\mathcal{T}\,\mathbf{\Pi}^{-1} \tag{5.7}$$

where $\mathcal{R}, \mathcal{S}, \mathcal{T}$ are the generators of the irreducible representation of PSL(2, 7) in the $\mathfrak{a}_7$ root basis, we obtain the generators of the same representation in the $\mathfrak{e}_7$ root basis. The explicit form of these 7×7 matrices is given below:

$$\mathbf{T}^{\mathbf{r}}_{\mathfrak{e}_7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad ; \quad \mathbf{S}^{\mathbf{r}}_{\mathfrak{e}_7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 & 1 & 0 & -3 \\ 0 & -1 & 0 & 1 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & -1 & 0 & 0 & 2 & 0 & -2 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{5.8}$$

$$\mathbf{R}^{\mathbf{r}}_{\mathfrak{e}_7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & 1 & -1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.9}$$

We can now easily verify that PSL(2, 7) is crystallographic with respect to the $\mathfrak{e}_7$-root lattice. It suffices to check that the above generators satisfy:

$$\left(\mathbf{T}^{\mathbf{r}}_{\mathfrak{e}_7}\right)^T \mathfrak{C}_{\mathfrak{e}_7} \mathbf{T}^{\mathbf{r}}_{\mathfrak{e}_7} = \left(\mathbf{S}^{\mathbf{r}}_{\mathfrak{e}_7}\right)^T \mathfrak{C}_{\mathfrak{e}_7} \mathbf{S}^{\mathbf{r}}_{\mathfrak{e}_7} = \left(\mathbf{R}^{\mathbf{r}}_{\mathfrak{e}_7}\right)^T \mathfrak{C}_{\mathfrak{e}_7} \mathbf{R}^{\mathbf{r}}_{\mathfrak{e}_7} = \mathfrak{C}_{\mathfrak{e}_7} \tag{5.10}$$

where:

$$\mathfrak{C}_{\mathfrak{e}_7} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{5.11}$$

is the Cartan matrix of $\mathfrak{e}_7$.

This construction guarantees that via its 7-dimensional irreducible representation the group PSL(2, 7) is embedded into the Weyl group of $\mathfrak{e}_7$. So that we can write:

$$\mathrm{PSL}(2,7) \overset{\text{Irrep }7}{\hookrightarrow} \mathrm{Weyl}\left[\mathfrak{a}_7\right] \subset \mathrm{Weyl}\left[\mathfrak{e}_7\right] \tag{5.12}$$

5.2 The second embedding of PSL(2, 7) into Weyl[$\mathfrak{e}_7$]

There is another embedding of the PSL(2, 7) group into Weyl [$\mathfrak{e}_7$] which is governed by the crystallographic 6-dimensional representation and which turns out to be the relevant one to construct solutions of Englert equations utilizing exceptional field theory:

$$\mathrm{PSL}(2,7) \overset{\text{Irrep 6}}{\hookrightarrow} \mathrm{Weyl}\,[\mathfrak{a}_6] \subset \mathrm{Weyl}\,[\mathfrak{e}_7] \tag{5.13}$$

To understand this second embedding let us compare the Dynkin diagram of $\mathfrak{e}_7$, in Eq. (5.1) with that of $\mathfrak{a}_6$ in Eq. (5.3). It is clear that the Lie algebra $\mathfrak{a}_6$ is also regularly embedded in $\mathfrak{e}_7$ since it suffices to identify the simple roots of the former with a subset of the simple roots of the latter:

$$\gamma_{1,2,3,4} = \alpha_{1,2,3,4} \quad ; \quad \gamma_5 = \alpha_6 \quad ; \quad \gamma_6 = \alpha_7 \tag{5.14}$$

It follows that the root lattice $\Lambda^{\mathbf{r}}_{\mathfrak{a}_6}$ of $\mathfrak{a}_6$ is a sublattice of $\Lambda^{\mathbf{r}}_{\mathfrak{e}_7}$. Indeed we have:

$$\mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} \subset \Lambda^{\mathbf{r}}_{\mathfrak{e}_6} \Leftrightarrow \mathbf{v} = v_i\,\alpha^i \quad \text{with} \;\; v_5 = 0 \;\; \text{and} \;\; v_{1,2,3,4,6,7} \in \mathbb{Z} \tag{5.15}$$

What we need is an orthogonal decomposition of the root lattice of $\mathfrak{e}_7$ into the root lattice $\mathfrak{a}_6$ plus its one-dimensional complement:

$$\Lambda^{\mathbf{r}}_{\mathfrak{e}_7} \supset \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} \oplus \Lambda^{\mathbf{r}}_{\mathbf{1}} \tag{5.16}$$

Orthogonality is obviously meant with respect to the Cartan matrix $\mathfrak{C}_{\mathfrak{e}_7}$. Imposing the condition that a vector $\mathbf{w} \in \Lambda^{\mathbf{r}}_{\mathbf{1}}$ should have vanishing scalar product with any vector $\mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6}$:

$$0 = (\mathbf{v}, \mathbf{w}) \equiv v_i\, w_j\, \mathfrak{C}^{ij}_{\mathfrak{e}_7} \tag{5.17}$$

we immediately find the solution. The sublattice $\Lambda^{\mathbf{r}}_{\mathbf{1}}$ is spanned by all vectors of the form

$$w^i = \{3m, 6m, 9m, 12m, 7m, 8m, 4m\} \quad ; \quad m \in \mathbb{Z} \tag{5.18}$$

It is convenient to use a permutation and rename the simple root α_5 as the last one α_7, so that the first six roots span the $\mathfrak{a}_6$ root lattice. This is done by the matrix:

$$P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.19}$$

In the permuted basis of simple roots the $\mathfrak{e}_7$ Cartan matrix becomes:

$$\widehat{\mathfrak{C}}_{\mathfrak{e}_7} = (P^{-1})^T \mathfrak{C}_{\mathfrak{e}_7} P^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{5.20}$$

In this basis the orthogonal decomposition (5.16) of the $\mathfrak{e}_7$ root lattice is represented as follows:

$$\begin{aligned} \mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} &\Leftrightarrow \mathbf{v} = \{v_1, v_2, v_3, v_4, v_5, v_6, 0\} \quad v_i \in \mathbb{Z} \\ \mathbf{w} \in \Lambda^{\mathbf{r}}_{\mathbf{1}} &\Leftrightarrow \{3m, 6m, 9m, 12m, 8m, 4m, 7m\} \quad m \in \mathbb{Z} \end{aligned} \tag{5.21}$$

Using this basis we can introduce the embedding of the 6-dimensional crystallographic representation of the PSL(2, 7) into the point group of the $\mathfrak{e}_7$ root lattice. We write the following form of the three generators of the considered group:

$$\mathcal{G} \equiv \{\rho, \sigma, \tau\} = \left\{\mathbf{R}^{\mathbf{r}}_{6+1}, \mathbf{S}^{\mathbf{r}}_{6+1}, \mathbf{T}^{\mathbf{r}}_{6+1}\right\} \tag{5.22}$$

where

$$\mathbf{R}^{\mathbf{r}}_{6+1} = \left(\begin{array}{cccccc|c} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbf{S}^{\mathbf{r}}_{6+1} = \left(\begin{array}{cccccc|c} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & -1 & 2 \\ -1 & 0 & 1 & -1 & 1 & -1 & 2 \\ -1 & 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbf{T}^{\mathbf{r}}_{6+1} = \left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{5.23}$$

which have the following properties:

1) The defining relations of PSL(2, 7) displayed in Eq. (4.6) are satisfied.

2) The generators preserve the Cartan matrix of $\mathfrak{e}_7$:

$$\mathcal{G}_i^T \, \widehat{\mathfrak{C}}_{\mathfrak{e}_7} \, \mathcal{G}_i = \widehat{\mathfrak{C}}_{\mathfrak{e}_7} \quad \text{for} \quad i = \rho, \sigma, \tau \tag{5.24}$$

3) The generators preserve the splitting (5.21) of the root lattice, namely they map any vector belonging to the sublattice $\Lambda^{\mathbf{r}}_{\mathfrak{a}_6}$ into a vector belonging to the same sublattice and leave invariant any vector belonging to $\Lambda^{\mathbf{r}}_1$

$$\begin{array}{rcl} \mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} & \Rightarrow & \mathcal{G}_i \, \mathbf{v} \in \Lambda^{\mathbf{r}}_{\mathfrak{a}_6} \quad \text{for} \quad i = \rho, \sigma, \tau \\ \mathbf{w} \in \Lambda^{\mathbf{r}}_1 & \Rightarrow & \mathcal{G}_i \, \mathbf{w} = \mathbf{w} \quad \text{for} \quad i = \rho, \sigma, \tau \end{array} \tag{5.25}$$

4) The first 6×6 blocks of the 7-dimensional matrices $\mathcal{G}_i$ are, respectively, the matrices R_6, S_6, T_6 displayed in Eq.s (4.29) and generating the irreducible 6-dimensional crystallographic representation of PSL(2, 7) that maps the $\mathfrak{a}_6$ root lattice into itself.

5.2.1 Change of basis

Once the embedding of the 6-dimensional representation of PSL(2, 7) has been done in one basis it can be transformed to any other basis. We are interested in the weight basis of the $\mathfrak{a}_7$-lattice; hence we introduce the following product of transformation matrices:

$$\mathfrak{M} = P \cdot \mathbf{\Pi} \cdot \mathfrak{C}^{-1}_{\mathfrak{a}_7} \tag{5.26}$$

where the first factor brings back to the standard labeling of $\mathfrak{e}_7$ roots, as in Eq. (5.1), the second converts to the $\mathfrak{a}_7$ root lattice and the last converts from the root to the $\mathfrak{a}_7$ weight lattice. Setting:

$$\begin{aligned} \mathbf{R}^{\mathbf{w}}_{6+1} &= \mathfrak{M}^{-1} \, \mathbf{R}^{\mathbf{r}}_{6+1} \, \mathfrak{M} \\ \mathbf{S}^{\mathbf{w}}_{6+1} &= \mathfrak{M}^{-1} \, \mathbf{S}^{\mathbf{r}}_{6+1} \, \mathfrak{M} \\ \mathbf{T}^{\mathbf{w}}_{6+1} &= \mathfrak{M}^{-1} \, \mathbf{T}^{\mathbf{r}}_{6+1} \, \mathfrak{M} \end{aligned} \tag{5.27}$$

we obtain:

$$\mathbf{R}^{\mathbf{r}}_{6+1} = \left(\begin{array}{c|cccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$\mathbf{S}^{\mathbf{r}}_{6+1} = \left(\begin{array}{c|cccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \end{array}\right)$$

$$\mathbf{T}^{\mathbf{w}}_{6+1} = \left(\begin{array}{c|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \end{array}\right) \tag{5.28}$$

Naming, respectively, $\mathbf{R}^{\mathbf{w}}_{6}$, $\mathbf{S}^{\mathbf{w}}_{6}$ and $\mathbf{T}^{\mathbf{w}}_{6}$ the lower 6×6 blocks of the above three matrices (they are separated by lines in Eq.s (5.28) we obtain the three generators of the 6-dimensional representation of PSL(2, 7) which is crystallographic with respect to the weight lattice $\Lambda^{w}_{\mathfrak{a}_6}$. In the $\mathfrak{a}_7$ weight basis the invariant sublattice is spanned by the vectors of the following form:

$$\mathfrak{w} = \mathfrak{M}^{-1} \{3m, 6m, 9m, 12m, 8m, 4m, 7m\} = \{4m, 0, 0, 0, 0, 0, 0\} \tag{5.29}$$

and the group PSL(2, 7) generated by the 7×7 matrices (5.28) leave the orthogonal complement (5.29) invariant.

6 Constructing the elementary solution

We come next to the construction of solutions to Englert equation utilizing, as building blocks, the minimal solutions whose structure is governed by Eq. (3.7) that we presently retrieve.To this effect, having constructed the explicit form of the two isomorphic groups $\mathrm{PSL}(2,7)_7$ and $\mathrm{PSL}(2,7)_{1+6}$ we let them act on the complete $\mathfrak{e}_7$ root system Δ_{126} containing 126 roots and we observe how this latter splits into orbits.

$\mathbf{PSL(2,7)_7}$-case We consider first the case where PSL(2, 7) is embedded into Weyl[$\mathfrak{e}_7$] through its seven-dimensional irreducible representation. Under the action of this group we find the following four orbits:

$$\Delta_{\mathbf{126}} = O_{14A} \oplus O_{14B} \oplus O_{42} \oplus O_{56} \tag{6.1}$$

whose explicit content is displayed below:

$$O_{14A} = \{11,33,34,40,41,47,57\}_{\mathrm{neg}} \bigcup \{11,33,34,40,41,47,57\}_{\mathrm{pos}} \tag{6.2}$$

$$O_{14B} = \{19,21,30,42,43,52,54\}_{\mathrm{neg}} \bigcup \{19,21,30,42,43,52,54\}_{\mathrm{pos}} \tag{6.3}$$

$$\begin{aligned} O_{42} = & \{5,16,24,25,26,29,31,35,36,37,38,39,44,45,46,48,49,50,51,53,59\}_{\mathrm{neg}} \\ & \bigcup \\ & \{5,16,24,25,26,29,31,35,36,37,38,39,44,45,46,48,49,50,51,53,59\}_{\mathrm{pos}} \end{aligned} \tag{6.4}$$

$$\begin{aligned} O_{56} = & \{1,2,3,4,6,7,8,9,10,12,13,14,15,17,18,20,22,23,27,28,32,55,56,58,60,61,62,63\}_{\mathrm{neg}} \\ & \bigcup \\ & \{1,2,3,4,6,7,8,9,10,12,13,14,15,17,18,20,22,23,27,28,32,55,56,58,60,61,62,63\}_{\mathrm{pos}} \end{aligned} \tag{6.5}$$

In the above equations we have utilized the following notation: the numbers from 1 to 63 refer to the positive roots as listed in table 2. The suffix pos/neg indicates whether the roots in the brackets are the positive ones or their negatives enumerated in the same order.

The key point is that no subset of purely positive roots is left invariant by the group $\mathrm{PSL}(2,7)_7$. This shows that this embedding is inconvenient in order to utilize the group $\mathrm{PSL}(2,7)_7$ as a classifier for fields Y_{ijk}. Indeed in the compactification of M-theory on a T^7 torus the massless fields are in correspondence with the positive roots.

$\mathbf{PSL(2,7)_{1+6}}$-case If we embed PSL(2, 7) into Weyl[$\mathfrak{e}_7$] through its six-dimensional irreducible representation, the scenario of orbits changes considerably. Under the action of $\mathrm{PSL}(2,7)_{1+6}$ the set of 126 $\mathfrak{e}_7$ roots splits into the following orbits:

$$\Delta_{\mathbf{126}} = O^+_{7A} \oplus O^-_{7A} \oplus O^+_{7C} \oplus O^-_{7C} \oplus O^+_{28} \oplus O^-_{28} \oplus O_{42} \tag{6.6}$$

1	{1, 0, 0, 0, 0, 0, 0}	33	{1, 1, 1, 1, 1, 1, 0}
2	{0, 1, 0, 0, 0, 0, 0}	34	{0, 1, 1, 1, 1, 1, 1}
3	{0, 0, 1, 0, 0, 0, 0}	35	{0, 1, 1, 2, 1, 1, 0}
4	{0, 0, 0, 1, 0, 0, 0}	36	{0, 0, 1, 2, 1, 1, 1}
5	{0, 0, 0, 0, 1, 0, 0}	37	{1, 1, 1, 1, 1, 1, 1}
6	{0, 0, 0, 0, 0, 1, 0}	38	{1, 1, 1, 2, 1, 1, 0}
7	{0, 0, 0, 0, 0, 0, 1}	39	{0, 1, 1, 2, 1, 1, 1}
8	{1, 1, 0, 0, 0, 0, 0}	40	{0, 1, 2, 2, 1, 1, 0}
9	{0, 1, 1, 0, 0, 0, 0}	41	{0, 0, 1, 2, 1, 2, 1}
10	{0, 0, 1, 1, 0, 0, 0}	42	{1, 1, 1, 2, 1, 1, 1}
11	{0, 0, 0, 1, 1, 0, 0}	43	{1, 1, 2, 2, 1, 1, 0}
12	{0, 0, 0, 1, 0, 1, 0}	44	{0, 1, 1, 2, 1, 2, 1}
13	{0, 0, 0, 0, 0, 1, 1}	45	{0, 1, 2, 2, 1, 1, 1}
14	{1, 1, 1, 0, 0, 0, 0}	46	{1, 1, 1, 2, 1, 2, 1}
15	{0, 1, 1, 1, 0, 0, 0}	47	{1, 1, 2, 2, 1, 1, 1}
16	{0, 0, 1, 1, 1, 0, 0}	48	{1, 2, 2, 2, 1, 1, 0}
17	{0, 0, 1, 1, 0, 1, 0}	49	{0, 1, 2, 2, 1, 2, 1}
18	{0, 0, 0, 1, 0, 1, 1}	50	{1, 1, 2, 2, 1, 2, 1}
19	{0, 0, 0, 1, 1, 1, 0}	51	{1, 2, 2, 2, 1, 1, 1}
20	{1, 1, 1, 1, 0, 0, 0}	52	{0, 1, 2, 3, 1, 2, 1}
21	{0, 1, 1, 1, 1, 0, 0}	53	{1, 1, 2, 3, 1, 2, 1}
22	{0, 1, 1, 1, 0, 1, 0}	54	{1, 2, 2, 2, 1, 2, 1}
23	{0, 0, 1, 1, 0, 1, 1}	55	{0, 1, 2, 3, 2, 2, 1}
24	{0, 0, 1, 1, 1, 1, 0}	56	{1, 1, 2, 3, 2, 2, 1}
25	{0, 0, 0, 1, 1, 1, 1}	57	{1, 2, 2, 3, 1, 2, 1}
26	{1, 1, 1, 1, 1, 0, 0}	58	{1, 2, 2, 3, 2, 2, 1}
27	{1, 1, 1, 1, 0, 1, 0}	59	{1, 2, 3, 3, 1, 2, 1}
28	{0, 1, 1, 1, 0, 1, 1}	60	{1, 2, 3, 3, 2, 2, 1}
29	{0, 1, 1, 1, 1, 1, 0}	61	{1, 2, 3, 4, 2, 2, 1}
30	{0, 0, 1, 1, 1, 1, 1}	62	{1, 2, 3, 4, 2, 3, 1}
31	{0, 0, 1, 2, 1, 1, 0}	63	{1, 2, 3, 4, 2, 3, 2}
32	{1, 1, 1, 1, 0, 1, 1}		

Table 2: Enumeration of the 63 positive roots of $\mathfrak{e}_7$ displayed in the simple root basis.

where:

$$O^{\pm}_{7A} = \{5, 31, 42, 44, 45, 48, 50\}_{\frac{\text{pos}}{\text{neg}}} \tag{6.7}$$

$$O^{\pm}_{7C} = \{55, 56, 58, 60, 61, 62, 63\}_{\frac{\text{pos}}{\text{neg}}} \tag{6.8}$$

$$O^{\pm}_{28} = \{11, 16, 19, 21, 24, 25, 26, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 46, 47, 49, 51, 52, 53, 54, 57, 59\}_{\frac{\text{pos}}{\text{neg}}} \tag{6.9}$$

$$O_{42} = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 22, 23, 27, 28, 32\}_{\text{pos}} \bigcup \{1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 22, 23, 27, 28, 32\}_{\text{neg}} \tag{6.10}$$

where the notation for the roots is the same as that utilized before.

The group theoretical and physical interpretation of the above splitting is clear. The orbit of 42 roots is made by the roots of $\mathfrak{a}_6$, that is to say of the Lie algebra of the subgroup $\mathrm{SL}(7,\mathbb{R}) \subset \mathrm{E}_{7(7)}$ parameterizing through the coset $\frac{\mathrm{SL}(7,\mathbb{R})}{\mathrm{SO}(7)}$ the metrics on the T^7-torus. The orbit O^+_{7C} is characterized, as it can be seen from table 2, by the fact that all its elements have $n_5 = 2$, namely their grading with respect to the root α_5 (see Eq. (5.1)) is 2. Projecting these vectors onto the fundamental weights of $\mathfrak{a}_6$ they turn out to be the weights of the fundamental defining representation of $\mathrm{SL}(7,\mathbb{R})$. On the other hand the roots in the two orbits O^+_{7A} and O^+_{7A} are characterized by the fact that their grading with respect to α_5 is 1 (see table 2). Projecting these 35 roots on the fundamental weights of $\mathfrak{a}_6$ we find the weights of the **35**-dimensional representation enumerated in table 3 and there put into one–to–one correspondence with the components of a three–time antisymmetric tensor, namely with a triple of different integer numbers in the range $1, 2, 3, 4, 5, 6, 7$. This antisymmetric tensor is Y_{ijk}, namely the 3-form defined over the 7-torus that is supposed to satisfy Englert equation. Summarizing we have:

$$\mathbf{35} = O^+_{7A} \oplus O^+_{28} \tag{6.11}$$

which is equation (3.7).

As we already stressed this is the starting point in the construction of minimal solutions

6.1 The Minimal Solutions

Let us now illustrate step by step how to construct a set of solutions starting from a normal form of Y_{ijk} in which seven components correspond to a Steiner triple system. The solutions will fit orbits with respect to the PSL(2, 7) invariance group of this septuple. To this end we introduce the relevant notation.

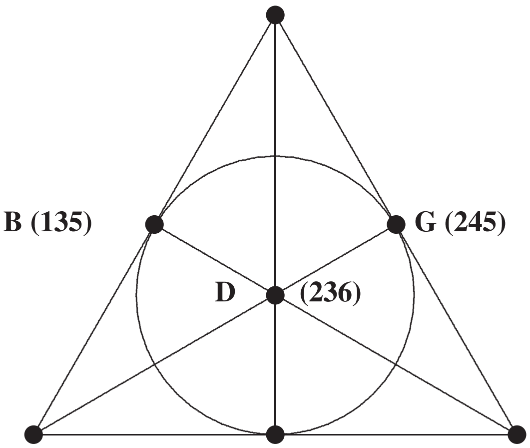
A septuple is conveniently characterized in terms of a distinctive *signature* (n_0, n_1, n_2) which is defined as follows: n_ℓ, $\ell = 0, 1, 2$, is the number of couples of triplets which have ℓ indices in common. The Steiner triples have signature $(0, 21, 0)$.

The *automorphism group* of a septuple is the subgroup of the permutation group S_7 acting on the internal indices i, j, k, which leaves the set of seven triplets invariant, though changing the order. PSL(2, 7) is the

Enumeration	Triple	Corresponding weight	Enumeration	Triple	Corresponding weight
1	$\{1,2,3\}$	$\{0,0,1,0,0,0\}$	19	$\{2,3,7\}$	$\{-1,0,1,0,0,-1\}$
2	$\{1,2,4\}$	$\{0,1,-1,1,0,0\}$	20	$\{2,4,5\}$	$\{-1,1,-1,0,1,0\}$
3	$\{1,2,5\}$	$\{0,1,0,-1,1,0\}$	21	$\{2,4,6\}$	$\{-1,1,-1,1,-1,1\}$
4	$\{1,2,6\}$	$\{0,1,0,0,-1,1\}$	22	$\{2,4,7\}$	$\{-1,1,-1,1,0,-1\}$
5	$\{1,2,7\}$	$\{0,1,0,0,0,-1\}$	23	$\{2,5,6\}$	$\{-1,1,0,-1,0,1\}$
6	$\{1,3,4\}$	$\{1,-1,0,1,0,0\}$	24	$\{2,5,7\}$	$\{-1,1,0,-1,1,-1\}$
7	$\{1,3,5\}$	$\{1,-1,1,-1,1,0\}$	25	$\{2,6,7\}$	$\{-1,1,0,0,-1,0\}$
8	$\{1,3,6\}$	$\{1,-1,1,0,-1,1\}$	26	$\{3,4,5\}$	$\{0,-1,0,0,1,0\}$
9	$\{1,3,7\}$	$\{1,-1,1,0,0,-1\}$	27	$\{3,4,6\}$	$\{0,-1,0,1,-1,1\}$
10	$\{1,4,5\}$	$\{1,0,-1,0,1,0\}$	28	$\{3,4,7\}$	$\{0,-1,0,1,0,-1\}$
11	$\{1,4,6\}$	$\{1,0,-1,1,-1,1\}$	29	$\{3,5,6\}$	$\{0,-1,1,-1,0,1\}$
12	$\{1,4,7\}$	$\{1,0,-1,1,0,-1\}$	30	$\{3,5,7\}$	$\{0,-1,1,-1,1,-1\}$
13	$\{1,5,6\}$	$\{1,0,0,-1,0,1\}$	31	$\{3,6,7\}$	$\{0,-1,1,0,-1,0\}$
14	$\{1,5,7\}$	$\{1,0,0,-1,1,-1\}$	32	$\{4,5,6\}$	$\{0,0,-1,0,0,1\}$
15	$\{1,6,7\}$	$\{1,0,0,0,-1,0\}$	33	$\{4,5,7\}$	$\{0,0,-1,0,1,-1\}$
16	$\{2,3,4\}$	$\{-1,0,0,1,0,0\}$	34	$\{4,6,7\}$	$\{0,0,-1,1,-1,0\}$
17	$\{2,3,5\}$	$\{-1,0,1,-1,1,0\}$	35	$\{5,6,7\}$	$\{0,0,0,-1,0,0\}$
18	$\{2,3,6\}$	$\{-1,0,1,0,-1,1\}$			

Table 3: In this table we enumerate the 35 weights of the irreducible representation of $\mathfrak{a}_6$ corresponding to an antisymmetric tensor Y_{ijk} in $d = 7$. We associate each weight vector to the corresponding triple $\{i, j, k\}$ of indices.

automorphism group of the Fano plane and thus of the Steiner system defining the multiplication table of the octonions $\Longrightarrow \mathrm{PSL}(2,7) \subset \mathrm{G}_2 \subset \mathrm{SO}(7)$. Below we represent the Fano plane with identification of its vertices with the seven triplets



The construction of the solutions to the Englert equation proceeds as follows.

1. We consider the case in which one of the two septuples, say $\vec{\sigma}$, defines the embedding of an $\mathfrak{sl}(2)^7$ group inside $E_{7(7)}$ through the set of positive roots $\alpha_{ijk} = \alpha_{\vec{\sigma}_I}$. As pointed out earlier, this system of triples is of Steiner type and defines the group PSL(2, 7) with respect to which we construct the orbits of the solutions.

2. Then we choose the second set of 7 parameters of the minimal solution by choosing a septuple $\vec{\gamma}$ which is complementary to $\vec{\sigma}$. We shall classify in the sequel the independent choices of such septuples;

3. Given the couple of complementary septuples $\vec{\sigma}$ and $\vec{\gamma}$, we construct a solution $Y^{(\gamma\sigma)}$ through the formula:

$$
\begin{aligned}
Y^{(\gamma\sigma)}_{\vec{\sigma}_{P(I)}}(x^{i_I}) &= \Big(f_I\ \cos(\mu x^{i_I}) + g_I\ \sin(\mu x^{i_I})\Big)\ , \\
Y^{(\gamma\sigma)}_{\vec{\gamma}_{P'(I)}}(x^{i_I}) &= \varepsilon_I\ \Big(f_I \sin(\mu x^{i_I}) - g_I \cos(\mu x^{i_I})\Big)\ ,\quad I = 1,\ldots,7,
\end{aligned}
\tag{6.12}
$$

   with $\varepsilon_I = \epsilon^{i_I\,\sigma^1_{P(I)}\sigma^2_{P(I)}\sigma^3_{P(I)}\gamma^1_{P'(I)}\gamma^2_{P'(I)}\gamma^3_{P'(I)}}$.

4. Being the Englert equation linear, a linear combination of solutions $Y^{(\sigma\gamma)}$ corresponding to different choices of complementary $\vec{\sigma}$ and $\vec{\gamma}$, is still a solution.

Given an elementary solution $Y^{(\gamma\sigma)}$ of the form (6.12), we note that it sources the warp factor by a term which does not depend on the internal coordinates x^i since:

$$
\frac{1}{6}(Y^{(\gamma\sigma)}\cdot Y^{(\gamma\sigma)}) = Y^{(\gamma\sigma)}_{\vec{\sigma}_{P(I)}} Y^{(\gamma\sigma)}_{\vec{\sigma}_{P(I)}} + Y^{(\gamma\sigma)}_{\vec{\gamma}_{P'(I)}} Y^{(\gamma\sigma)}_{\vec{\gamma}_{P'(I)}} = \sum_{I=1}^{7}(f_I^2 + g_I^2),
\tag{6.13}
$$

and therefore:

$$
H = 1 - \frac{9}{4} e^{-2\mu U} \sum_{I=1}^{7}(f_I^2 + g_I^2).
\tag{6.14}
$$

Notice that for this kind of solution $H = H(U)$, i.e. H does not depend on the torus coordinates x^i. This is reminiscent of what happens for the 2-brane solution of seven-dimensional minimal supergravity, studied in [18], when the internal 1-form flux, which satisfies the Arnold-Beltrami equation on a 3-torus, corresponds to the so-called *ABC solution*. Combining elementary solutions, the warp factor acquires a non-trivial dependence on x^i.

Classifying the second septuple. The first septuple can be identified with the orbit $\mathbf{7}_A$ in the decomposition (3.7) of the 35 roots α_{ijk} with respect to the action of the corresponding $\mathrm{PSL}(2,7)_{1+6}$ automorphism group. The automorphism group of the second septuple will intersect $\mathrm{PSL}(2,7)_{1+6}$ in a subgroup H of the latter. We classify the second septuple by the possible choices of H in $\mathrm{PSL}(2,7)_{1+6}$. The condition on this subgroup is that the decomposition of the **28** orbit in (3.7) with respect to it should contain an order-7 orbit $\mathbf{7}_B$, which is mutually non-local with respect to $\mathbf{7}_A$. The septuple $\mathbf{7}_B$ may also result from a combination of smaller H-orbits. We considered the possible simple subgroups H classified in section 4.8 and found the following results:

- $\mathrm{H} = \mathrm{O}_{24\mathrm{A}}$ and $\mathrm{O}_{24\mathrm{B}}$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{12} + \mathbf{12} + \mathbf{4}\,. \tag{6.15}$$

  This case is not relevant to our analysis since the above decomposition contains no order-7 orbit $\mathbf{7}_B$;

- $\mathrm{H} = \mathrm{T}_{12\mathrm{A}}$ and $\mathrm{T}_{12\mathrm{B}}$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{12} + \mathbf{6} + \mathbf{6} + \mathbf{4}\,. \tag{6.16}$$

  Also this decomposition contains no orbit $\mathbf{7}_B$;

- $\mathrm{H} = \mathrm{Dih}_3$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{3} + \mathbf{3} + \mathbf{3} + \mathbf{6} + \mathbf{6} + \mathbf{6} + \mathbf{1}\,. \tag{6.17}$$

  In this case we checked that no-one of the combinations of orbits on the right hand side, with seven elements realized either as $\mathbf{3} + \mathbf{3} + \mathbf{1}$ or as $\mathbf{6} + \mathbf{1}$, is mutually non-local with respect to $\mathbf{7}_A$;

- $\mathrm{H} = \mathrm{G}_{21}$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{7} + \mathbf{21}\,. \tag{6.18}$$

  The order-7 orbit in the decomposition is mutually non-local with respect to $\mathbf{7}_A$ and thus is a viable septuple $\mathbf{7}_B$ for constructing a minimal solution. Moreover this $\mathbf{7}_B$ is of Steiner type;

- $\mathrm{H} = \mathbb{Z}_7$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{7} + \mathbf{7} + \mathbf{7} + \mathbf{7}\,. \tag{6.19}$$

  All the four order-7 orbits in the decomposition are mutually non-local with respect to $\mathbf{7}_A$. One is of Steiner type and coincides with the one in (6.18), being $\mathbb{Z}_7$ a subgroup of G_{21}. The other three are not of Steiner type and have signature $(7,7,7)$. Therefore all these four orbits are viable choices for $\mathbf{7}_B$;

Table 4: Multiplicities of the elementary solutions based on pair of septuples $\mathbf{7}_A \oplus \mathbf{7}_B$ with fixed signature for the second septuple. In the table we mention the invariance group of the pair of septuples

Aut	sign.	mult.	n. coord.s
G_{21}	$(0,21,0)$	8	7
$\mathbb{Z}_7$	$(7,7,7)$	24	7
$\mathbb{Z}_3$	$(6,9,6)$	56	7
$\mathbb{Z}_3$	$(3,15,3)$	56	7
$\mathbb{Z}_3$	$(0,15,6)$	112	4
$\mathbb{Z}_3$	$(0,18,3)$	56	4
$\mathbb{Z}_3$	$(3,12,6)$	112	4
Total number		**424**	

- $\mathrm{H} = \mathbb{Z}_3$. The **28** decomposes as:

$$\mathbf{28} \to \mathbf{1} + 9 \times \mathbf{3}\,. \tag{6.20}$$

  Also this decomposition contains septuples, realized as $\mathbf{1} + \mathbf{3} + \mathbf{3}$, which are mutually non-local with respect to $\mathbf{7}_A$. Some are of Steiner type and coincide with the one in (6.18), being $\mathbb{Z}_3$ a subgroup of G_{21}, for isomorphic choices of G_{21} inside $\mathrm{PSL}(2,7)$. The decomposition also features non-Steiner septuples with signatures $(3,15,3)$, $(6,9,6)$, $(0,15,6)$, $(0,18,3)$, $(3,12,6)$. The last three classes of $\mathbf{7}_B$ are distinguished from the first two in that the coordinates complementary to $\mathbf{7}_A$ and $\mathbf{7}_B$ are not 7 but 4. This means that the corresponding minimal solution would only depend on 4 coordinates and the mapping $I \longrightarrow i_I$ is not onto.

Summarizing, we found viable septuples $\mathbf{7}_B$ only when the group H is either G_{21} or one of its subgroups. Then we counted the possible septuples $\mathbf{7}_B$ for various isomorphic choices of H in $\mathrm{PSL}(2,7)$ and found the multiplicities displayed in table 4. The total number of independent minimal solutions is then 424. Only

$$144 = 8 + 56 + 56 + 8$$

of these solutions depending on all the 7 coordinates. Among these latter, only 8 consists of two Steiner systems. Notice that in general the solutions do not preserve any supersymmetry. However, for particular choices of the parameters the solutions can admit $\mathcal{N} = 1,2,3,4,5,6$ supersymmetries. This is quite different from the original Englert solution, which does not preserve any supersymmetry. In the next sections we recall the criterion for preservation of supersymmetries in the context of these M2-brane solutions that was derived in [16] and we apply it systematically to the solutions of type $(0,21,0)$ obtaining just only from this sector a rich spectrum of possibilities encompassing all available values of $\mathcal{N}$. The analysis of the remaining solutions is postponed to a future publications.

1st sept.	1st sept.	1st sept.	1st sept.	1st sept.	1st sept.	1st sept.	1st sept.
1 2 7	1 2 7	1 2 7	1 2 7	1 2 7	1 2 7	1 2 7	1 2 7
1 3 5	1 3 5	1 3 5	1 3 5	1 3 5	1 3 5	1 3 5	1 3 5
1 4 6	1 4 6	1 4 6	1 4 6	1 4 6	1 4 6	1 4 6	1 4 6
2 3 6	2 3 6	2 3 6	2 3 6	2 3 6	2 3 6	2 3 6	2 3 6
2 4 5	2 4 5	2 4 5	2 4 5	2 4 5	2 4 5	2 4 5	2 4 5
3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7	3 4 7
5 6 7	5 6 7	5 6 7	5 6 7	5 6 7	5 6 7	5 6 7	5 6 7
2nd sept.	2nd sept.	2nd sept.	2nd sept.	2nd sept.	2nd sept.	2nd sept.	2nd sept.
1 2 5	1 2 4	1 2 6	1 2 4	1 2 6	1 2 3	1 2 5	1 2 3
1 3 4	1 3 7	1 3 4	1 3 6	1 3 7	1 4 5	1 3 6	1 4 7
1 6 7	1 5 6	1 5 7	1 5 7	1 4 5	1 6 7	1 4 7	1 5 6
2 3 7	2 3 5	2 3 5	2 3 7	2 3 4	2 4 7	2 3 4	2 4 6
2 4 6	2 6 7	2 4 7	2 5 6	2 5 7	2 5 6	2 6 7	2 5 7
3 5 6	3 4 6	3 6 7	3 4 5	3 5 6	3 4 6	3 5 7	3 4 5
4 5 7	4 5 7	4 5 6	4 6 7	4 6 7	3 5 7	4 5 6	3 6 7

Table 5: The eight pairs of mutually non local Steiner septuples produced by the orbits of the 8 different conjugate copies of subgroups ${G_{21}}^I \subset \mathrm{PSL}(2,7)_{1+6}$ $(I = 1, \ldots, 8)$.

7 The Killing spinor equation of M2-branes with Englert fluxes

As announced we review here the discussion of the Killing spinor equation presented in [16].

In order to analyze the structure of the Killing spinor equation in the background of the M2-branes with Englert fluxes, we need a basis of gamma matrices that is well-adapted to the splitting of the 11-dimensional manifold mentioned in Eq. (1.3).

Such a well adapted basis is provided by the following nested hierarchy.

7.1 Gamma matrices

At the bottom of the hierarchy we have the Pauli matrices.

Pauli matrices. We use the following conventions:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{7.1}$$

Gamma matrices on the $d = 3$ world-volume. Next we construct the set of 2×2 gamma matrices in $d = 3$ in the following way

$$\left\{\gamma_{\underline{a}}, \gamma_{\underline{b}}\right\} = 2\eta_{\underline{ab}} \quad ; \quad \gamma = \{\sigma_2, \mathrm{i}\,\sigma_1, \mathrm{i}\,\sigma_3\} \ , \quad \underline{a}, \underline{b} = 1, 2, 3 \,. \tag{7.2}$$

Gamma matrices in $d = 7$ In $d = 7$ we choose gamma matrices that are real and antisymmetric and fulfill the following Clifford algebra:

$$\left\{\tau_i, \tau_j\right\} = -2\delta_{ij} \ , \quad i, j = 1, \ldots, 7 \,. \tag{7.3}$$

The explicit basis utilized is that one where we express the τ-matrices in terms of ϕ_{ijk}, namely of the G_2-invariant three-tensor:

$$\begin{aligned} (\tau_i)_{jk} &= \phi_{ijk} \\ (\tau_i)_{j8} &= \delta_{ij} \quad ; \quad (\tau_i)_{8j} = -\delta_{ij} \end{aligned} \tag{7.4}$$

The explicit form of the ϕ_{ijk} tensor is given in Eq. (4.19) and it is the one well-adapted to the immersion of the discrete group which acts crystallographically on T^7 into the compact G_2 Lie group, namely according to the canonical immersion $PSL(2, 7) \longrightarrow G_{2(-14)}$.

Gamma matrices in $d = 8$ Because of our splitting $11 = 3 \oplus 1 \oplus 7$ we need also the gamma matrices in $d = 8$ corresponding to the transverse space to the M2-brane, namely $\mathbb{R}_+ \otimes T^7$. We choose the following Clifford algebra:

$$\{T_I, T_J\} = -2\delta_{IJ} \ , \quad I, J = 1, \ldots, 8 \,, \tag{7.5}$$

and we utilize the following explicit realization:

$$\begin{aligned} T_i &= \sigma_1 \otimes \tau_i \\ T_8 &= \mathrm{i}\sigma_2 \otimes \mathbf{1}_{8\times 8} \\ T_9 &= \sigma_3 \otimes \mathbf{1}_{8\times 8} \end{aligned} \tag{7.6}$$

The last matrix is the $d = 8$ chirality operator which plays an important role in the discussion of the Killing spinor equation.

Gamma matrices in $d = 11$ At the top of the hierarchy we have the $d = 11$ gamma matrices, obeying the following Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \ , \quad a, b = 0, \ldots, 10 \,. \tag{7.7}$$

For them we utilize the following explicit realization:

$$\begin{aligned} \Gamma_{\underline{a}} &= \gamma_{\underline{a}} \otimes T_9 \\ \Gamma_I &= \mathbf{1}_{2\times 2} \otimes T_I \end{aligned} \tag{7.8}$$

With these choices the charge conjugation matrix, takes the following form:

$$
\begin{aligned}
C &= \mathrm{i}\sigma_2 \otimes \mathbf{1}_{2\times 2} \otimes \mathbf{1}_{8\times 8} \\
C\,\Gamma_a C^{-1} &= -\Gamma_a{}^T
\end{aligned}
\tag{7.9}
$$

Equipped with this set of properly chosen gamma matrices we can turn to the investigation of the Killing spinor equation.

7.2 The tensor structure of the Killing spinor equation

The rheonomic solution of the $d = 11$ Bianchi identities (see Eq. (2.2)) allows us to write the Killing spinor equation in the following general form:

$$
\mathcal{D}\xi - \frac{\mathrm{i}}{3}\Gamma^{\mathrm{abc}} V^d F_{\mathrm{abcd}}\xi - \frac{\mathrm{i}}{24}\Gamma_{\mathrm{abcdf}} F^{\mathrm{abcd}} V^f \xi = 0 \tag{7.10}
$$

where

$$
\mathcal{D}\xi \equiv \mathrm{d}\xi - \frac{1}{4}\omega^{\mathrm{ab}}\Gamma_{\mathrm{ab}}\xi \tag{7.11}
$$

is the Lorentz covariant differential in $d = 11$.

Equation (7.10) can be usefully rewritten as follows:

$$
\nabla\xi \equiv \mathrm{d}\xi + \Omega\xi = 0 \tag{7.12}
$$

where Ω is a generalized connection in the 32–dimensional spinor space, defined as follows:

$$
\Omega \equiv \Theta_L + \Theta_1^{[F]} + \Theta_2^{[F]} \tag{7.13}
$$

In the above equation we have introduced the following definitions:

$$
\begin{aligned}
\Theta_L &\equiv -\frac{1}{4}\omega^{\mathrm{ab}}\Gamma_{\mathrm{ab}} \\
\Theta_1^{[F]} &\equiv -\frac{\mathrm{i}}{3}\Gamma^{\mathrm{abc}} V^d F_{\mathrm{abcd}} \\
\Theta_2^{[F]} &\equiv -\frac{\mathrm{i}}{24}\Gamma_{\mathrm{abcdf}} F^{\mathrm{abcd}} V^f
\end{aligned}
\tag{7.14}
$$

Next let us make another splitting of the overall generalized connection:

$$
\Omega = \Omega_H + \Omega_Y \tag{7.15}
$$

where Ω_H depends only on the (inhomogeneous)-harmonic function H and it is obtained from Ω by setting $Y_{ijk} \to 0$. Instead, the other part Ω_Y, is just the difference and it depends linearly on Y_{ijk}

7.2.1 M2-branes without Englert fluxes: tensor structure of Ω_H

As shown in [16], by introducing the following operators:

$$
\begin{aligned}
V \circ \gamma &= V^{\underline{a}} \gamma_{\underline{a}} && (7.16)\\
\mathbb{P}_{\pm} &= \frac{1}{2}\left(\mathbf{1}_{16} \pm T_9\right) && (7.17)\\
\partial H \circ T &= \frac{1}{3} H^{-\frac{7}{6}} \partial_I H T^I && (7.18)\\
V \diamond \partial H \circ T &= -\frac{1}{12} H^{-\frac{7}{6}} V_{[I} \partial_{J]} H T^{IJ} && (7.19)\\
\mathbf{d}\mathrm{H} &= \frac{1}{6} H^{-\frac{7}{6}} \sum_{I=1}^{8} \partial_I H V^I && (7.20)
\end{aligned}
$$

we get that the H-part of the generalized connection has the following tensor structure:

$$
\Omega_H = V \circ \gamma \otimes \partial H \circ T \, \mathbb{P}_- + \mathbf{1_2} \otimes \; V \diamond \partial H \circ T \, \mathbb{P}_- + \mathbf{1}_2 \otimes \mathbf{d}\mathrm{H} \, T_9 \tag{7.21}
$$

From equation (7.21) one readily derives the form of the Killing spinors for pure M2-brane solutions. Writing the 32 component Killing spinor as a tensor product:

$$
\xi = \epsilon \otimes \chi \tag{7.22}
$$

we find that, in the absence of Y-fields, the Killing spinor equation is satisfied provided:

$$
\begin{aligned}
T_9 \chi &= \chi \quad \Rightarrow \quad \mathbb{P}_- \chi = 0 && (7.23)\\
\chi &= H^{-\frac{1}{6}} \chi_0 && (7.24)
\end{aligned}
$$

where H is the (inhomogeneous)-harmonic function appearing in the metric (2.6) and χ_0 is a constant spinor with commuting components. Indeed, in view of our 2-brane interpretation of these backgrounds, we assume that the two–component spinors ϵ are the anticommuting objects.

Using the tensor structure of the $d = 8$ T-matrices we set:

$$
\chi = \kappa \otimes \lambda \tag{7.25}
$$

where κ is a two component spinor:

$$
\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \tag{7.26}
$$

with commuting components, while λ is an eight–component spinor:

$$
\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\} \tag{7.27}
$$

also with commuting components.

In this language the most general 32–component spinor has the form:

$$\xi = \epsilon \otimes \kappa \otimes \lambda \tag{7.28}$$

and the general solution for the Killing spinor at $Y_{ijk} = 0$ is obtained by setting:

$$\kappa_2 = 0 \quad ; \quad \kappa_1 = H^{-\frac{1}{6}} \tag{7.29}$$

This shows that the M2-branes without Englert-fluxes preserve 16 supersymmetries, namely $\frac{1}{2}$ of the total SUSY.

7.2.2 M2-branes with Englert fluxes: tensor structure of Ω_Y

We come next to analyze the structure of the Y-part of the connection Ω_Y.

We begin by introducing two $d = 7$ operators constructed with the Englert field Y_{ijk}, the flat 8-dimensional vielbein $\widehat{V}^I \equiv dy^I$ and the τ-matrices:

$$\mathcal{B} \equiv \tau_{ijk} Y_{ijk} \quad ; \quad \mathcal{T} = \widehat{V}^i \tau_i \tag{7.30}$$

in [16] it was shown that Ω_Y can be written as follows:

$$\begin{aligned} \Omega_Y \;=\; & \mathrm{i}\frac{1}{12}\mu e^{-U\mu} H^{-2/3} \times \\ & \left[V \circ \gamma \begin{pmatrix} 2\mathcal{B} & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{1} \otimes \begin{pmatrix} \widehat{V}^0\mathcal{B} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\mathbf{1} \otimes \begin{pmatrix} 0 & 3\mathcal{B}\mathcal{T} \\ -\mathcal{T}\mathcal{B} & 0 \end{pmatrix} \right] \end{aligned} \tag{7.31}$$

Eq. (7.31) reveals the mechanism behind the preservation of supersymmetry by M2-branes with Englert fluxes. Writing the candidate Killing spinor in the tensor product form (7.28) we see that the connection Ω_Y annihilates it if $\kappa = \begin{pmatrix} H^{-\frac{1}{6}} \\ 0 \end{pmatrix}$ as we already established from consideration of the H-part of the connection and if the 8-component λ is a null-vector of $\mathcal{B}$:

$$\mathcal{B}\,\lambda = 0 \tag{7.32}$$

This is the only possibility to integrate the Killing spinor equation. Indeed the term with $\mathrm{V}\circ\gamma$ which mixes the internal coordinateswith the world volume ones has to vanish since it cannot be compensated in any other way. This implies Eq. (7.32). The magic thing is that the precise values of the coefficients provided by the rheonomic solution of Bianchi identities in $d = 11$, produce the structure in Eq. (7.31). In this way the condition (7.32) suffices to annihilate also the action of the other terms in the connection.

In conclusion M2-branes with Englert fluxes preserve part of the Killing spinors existing in the case of $Y = 0$ if and only if the operator $\mathcal{B}$ has a non trivial Null-Space, namely if the Rank of $\mathcal{B}$ is < 8. Every λ satisfying (7.32) corresponds to a preserved supersymmetry.

In Appendix B the above conditions on the matrix $\mathcal{B}$ for the solution to preserve an amount of supersymmetry are shown to be a special case of the general supersymmetry conditions worked out in the literature on M2-branes with self-dual fluxes.

8 Supersymmetry of the solutions of type $(0, 21, 0)$

In this section we present the results we have obtained for the supersymmetry of solutions of type $(0, 21, 0)$, mentioned in table 4.

According to the previously explained rules for the construction of minimal solutions we have derived each of the eight 14-parameter solutions for the three-form $\mathbf{Y}$ obtained by pairing the standard septuple $\mathbf{7}_A$ with one of the eight different septuples $\mathbf{7}_B^I$ displayed in table 5. Let us name them $\mathbf{Y}_I^{14}$, $I = 1, \ldots, 8$. Since Englert equation is linear, the sum of these solutions is also a solution:

$$\widehat{\mathbf{Y}} = \sum_{I=1}^{8} \mathbf{Y}_I^{14} \tag{8.1}$$

which apparently depends on $8 \times 14 = 112$ parameters. Actually the independent combinations of differentials $dx^i \wedge dx^j \wedge dx^k$ with the trigonometric functions $\cos(\mu x^\ell)$ and $\sin(\mu x^\ell)$ that is produced in this sum are not 112 but rather 56, since each combination appears twice. Renaming δ_α, $\alpha = 1, \ldots, 56$ the coefficients of the independent combinations $\mathbf{B}^\alpha$ (they are listed in table 6), we have obtained a general solution of the following form:

$$\mathbf{Y}^{56}(\boldsymbol{x}|\boldsymbol{\delta}) = \sum_{\alpha=1}^{56} \delta_\alpha \, \mathbf{B}^\alpha \tag{8.2}$$

$\mathbf{B}^1$	$\cos(\mu x^4) dx^1 \wedge dx^2 \wedge dx^3$	$\mathbf{B}^{29}$	$\cos(\mu x^6) dx^1 \wedge dx^3 \wedge dx^7$
$\mathbf{B}^2$	$\sin(\mu x^4) dx^1 \wedge dx^2 \wedge dx^3$	$\mathbf{B}^{30}$	$\sin(\mu x^6) dx^1 \wedge dx^3 \wedge dx^7$
$\mathbf{B}^3$	$\cos(\mu x^3) dx^1 \wedge dx^2 \wedge dx^4$	$\mathbf{B}^{31}$	$\cos(\mu x^7) dx^1 \wedge dx^4 \wedge dx^5$
$\mathbf{B}^4$	$\sin(\mu x^3) dx^1 \wedge dx^2 \wedge dx^4$	$\mathbf{B}^{32}$	$\sin(\mu x^7) dx^1 \wedge dx^4 \wedge dx^5$
$\mathbf{B}^5$	$\cos(\mu x^6) dx^1 \wedge dx^2 \wedge dx^5$	$\mathbf{B}^{33}$	$\cos(\mu x^2) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^6$	$\sin(\mu x^6) dx^1 \wedge dx^2 \wedge dx^5$	$\mathbf{B}^{34}$	$\cos(\mu x^3) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^7$	$\cos(\mu x^5) dx^1 \wedge dx^2 \wedge dx^6$	$\mathbf{B}^{35}$	$\cos(\mu x^5) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^8$	$\sin(\mu x^5) dx^1 \wedge dx^2 \wedge dx^6$	$\mathbf{B}^{36}$	$\cos(\mu x^7) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^9$	$\cos(\mu x^3) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{37}$	$\sin(\mu x^2) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^{10}$	$\cos(\mu x^4) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{38}$	$\sin(\mu x^3) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^{11}$	$\cos(\mu x^5) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{39}$	$\sin(\mu x^5) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^{12}$	$\cos(\mu x^6) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{40}$	$\sin(\mu x^7) dx^1 \wedge dx^4 \wedge dx^6$
$\mathbf{B}^{13}$	$\sin(\mu x^3) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{41}$	$\cos(\mu x^5) dx^1 \wedge dx^4 \wedge dx^7$
$\mathbf{B}^{14}$	$\sin(\mu x^4) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{42}$	$\sin(\mu x^5) dx^1 \wedge dx^4 \wedge dx^7$
$\mathbf{B}^{15}$	$\sin(\mu x^5) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{43}$	$\cos(\mu x^2) dx^1 \wedge dx^5 \wedge dx^6$
$\mathbf{B}^{16}$	$\sin(\mu x^6) dx^1 \wedge dx^2 \wedge dx^7$	$\mathbf{B}^{44}$	$\sin(\mu x^2) dx^1 \wedge dx^5 \wedge dx^6$
$\mathbf{B}^{17}$	$\cos(\mu x^2) dx^1 \wedge dx^3 \wedge dx^4$	$\mathbf{B}^{45}$	$\cos(\mu x^4) dx^1 \wedge dx^5 \wedge dx^7$
$\mathbf{B}^{18}$	$\sin(\mu x^2) dx^1 \wedge dx^3 \wedge dx^4$	$\mathbf{B}^{46}$	$\sin(\mu x^4) dx^1 \wedge dx^5 \wedge dx^7$
$\mathbf{B}^{19}$	$\cos(\mu x^2) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{47}$	$\cos(\mu x^3) dx^1 \wedge dx^6 \wedge dx^7$
$\mathbf{B}^{20}$	$\cos(\mu x^4) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{48}$	$\sin(\mu x^3) dx^1 \wedge dx^6 \wedge dx^7$
$\mathbf{B}^{21}$	$\cos(\mu x^6) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{49}$	$\cos(\mu x^1) dx^2 \wedge dx^3 \wedge dx^4$
$\mathbf{B}^{22}$	$\cos(\mu x^7) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{50}$	$\sin(\mu x^1) dx^2 \wedge dx^3 \wedge dx^4$
$\mathbf{B}^{23}$	$\sin(\mu x^2) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{51}$	$\cos(\mu x^7) dx^2 \wedge dx^3 \wedge dx^5$
$\mathbf{B}^{24}$	$\sin(\mu x^4) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{52}$	$\sin(\mu x^7) dx^2 \wedge dx^3 \wedge dx^5$
$\mathbf{B}^{25}$	$\sin(\mu x^6) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{53}$	$\cos(\mu x^1) dx^2 \wedge dx^3 \wedge dx^6$
$\mathbf{B}^{26}$	$\sin(\mu x^7) dx^1 \wedge dx^3 \wedge dx^5$	$\mathbf{B}^{54}$	$\cos(\mu x^4) dx^2 \wedge dx^3 \wedge dx^6$
$\mathbf{B}^{27}$	$\cos(\mu x^7) dx^1 \wedge dx^3 \wedge dx^6$	$\mathbf{B}^{55}$	$\cos(\mu x^5) dx^2 \wedge dx^3 \wedge dx^6$
$\mathbf{B}^{28}$	$\sin(\mu x^7) dx^1 \wedge dx^3 \wedge dx^6$	$\mathbf{B}^{56}$	$\cos(\mu x^7) dx^2 \wedge dx^3 \wedge dx^6$

Table 6: List of the addends $\mathbf{B}_\alpha$ in the general solution of Englert equation corresponding to septuples of signature $(0, 21, 0)$.

The action of the group PSL(2, 7) on the Englert form $\mathbf{Y}^{56}(\boldsymbol{x}|\boldsymbol{\delta})$ is generated by the action of the group on the seven coordinates x^i which is only by means of permutations. The explicit form of this action which is consistent with the action on the **35** representation considered as weights of the $\mathfrak{a}_6$ Lie algebra, according with the conversion rule of table 3, is that provided by the following identification of the three generators:

$$\begin{array}{lcl} R & \imath & \{x^1 \to x^3,\ x^2 \to x^2,\ x^3 \to x^1,\ x^4 \to x^4,\ x^5 \to x^5,\ x^6 \to x^7,\ x^7 \to x^6\} \\ S & \imath & \{x^1 \to x^7,\ x^2 \to x^1,\ x^3 \to x^4,\ x^4 \to x^5,\ x^5 \to x^3,\ x^6 \to x^6,\ x^7 \to x^2\} \\ T & \imath & \{x^1 \to x^5,\ x^2 \to x^7,\ x^3 \to x^2,\ x^4 \to x^3,\ x^5 \to x^4,\ x^6 \to x^1,\ x^7 \to x^6\} \end{array} \quad (8.3)$$

Let us name $\mathfrak{g}_7 \in \mathrm{PSL}(2,7)$ any element of the group in the 7-dimensional representation generated by the transformations (8.3). Since the basis forms $\mathbf{B}^\alpha$ are permuted among themselves by this action it follows that $\mathfrak{g}_7 \in \mathrm{PSL}(2,7)$ induces a corresponding linear transformation $\mathfrak{g}_{56}$ on the 56 parameters δ_α according with:

$$\mathbf{Y}^{56}\left(\mathfrak{g}_7 \boldsymbol{x}|\boldsymbol{\delta}\right) = \mathbf{Y}^{56}\left(\boldsymbol{x}|\mathfrak{g}_{56}\boldsymbol{\delta}\right) \quad (8.4)$$

In this way we obtain a 56-dimensional representation of the group PSL(2, 7) group of which we can consider the decomposition into irreducible representations. We obtain:

$$\mathbf{56} \stackrel{\mathrm{PSL}(2,7)}{\Longrightarrow} 4\,\mathrm{D}_7 + 2\,\mathrm{D}_8 + 2\,\mathrm{DA}_3 + 2\,\mathrm{DB}_3 \quad (8.5)$$

This clearly means that there are in this sector no Englert fields that are invariant under the full PSL(2, 7) group, since no singlets do appear in the above decomposition. Calculating instead the decomposition of the same representation under the maximal subgroup $\mathrm{G}_{21} \subset \mathrm{PSL}(2,7)$ we obtain the following decomposition:

$$\mathbf{56} \stackrel{\mathrm{G}_{21}}{\Longrightarrow} 4\,\mathrm{D}_1 + 8\,\mathrm{DA}_3 + 8\mathrm{DB}_3 + 2\,\mathrm{DX}_1 + 2\,\mathrm{DY}_1 \quad (8.6)$$

This means that there exists a 4-parameter solution of Englert equation that is invariant with respect to the full group G_{21}. As we are going to see a 2-parameter subspace of this solution preserves also $\mathcal{N} = 1$ supersymmetry.

In order to study residual supersymmetry of the considered solutions we have proceeded as follows. Naming $Y^{56}_{ijk}\left(\boldsymbol{x}|\boldsymbol{\delta}\right)$ the components of the form (8.2) we have constructed the corresponding symmetric 8×8 matrix $\mathcal{B}$:

$$\mathcal{B}\left[\boldsymbol{\delta},\boldsymbol{x}\right] = \tau^{ijk} Y^{56}_{ijk}\left(\boldsymbol{x}|\boldsymbol{\delta}\right) \quad (8.7)$$

The condition of $\mathcal{N} = 1$ supersymmetry is provided by requiring that, independently from the point $\boldsymbol{x}$, one should have:

$$\mathcal{B}\left[\boldsymbol{\delta},\boldsymbol{x}\right]_{I,8} = 0 \quad ; \quad I = 1,\dots,8 \quad (8.8)$$

This yields 14 linear conditions on the 56 parameters $\boldsymbol{\delta}$. We can view this as an orthogonal splitting of the 56-dimensional parameter space $\mathcal{M}^{56}$ of the following type:

$$\begin{aligned} \mathcal{M}^{56} &= \mathcal{M}_{N=1} \oplus \mathcal{M}^{\perp}_{N=1} && (8.9)\\ \dim\mathcal{M}_{N=1} &= 42 && (8.10)\\ \dim\mathcal{M}^{\perp}_{N=1} &= 14 && (8.11) \end{aligned}$$

The 42-dimensional subspace $\mathcal{M}_{N=1}$ is the space of $\mathcal{N} = 1$ supersymmetric Englert solutions. We can inquire what is the subgroup $\mathrm{G} \subset \mathrm{PSL}(2,7)$ that preserves the splitting (8.9), namely:

$$\mathrm{G} \quad : \quad \mathcal{M}_{N=1} \longrightarrow \mathcal{M}_{N=1} \quad ; \quad \mathrm{G} \quad : \quad \mathcal{M}^{\perp}_{N=1} \longrightarrow \mathcal{M}^{\perp}_{N=1} \tag{8.12}$$

By explicit calculation we find that $\mathrm{G} \sim \mathrm{G}_{21}$, namely it is one of the eight conjugate copies of G_{21} contained in $PSL(2,7)$. We already know from Eq. (8.6) that with respect to this group there are invariant Englert solutions and indeed we find that the invariant subspace:

$$\mathcal{M}^{inv}_{N=1} \subset \mathcal{M}_{N=1} \tag{8.13}$$

of those Englert fields that preserve $\mathcal{N} = 1$ supersymmetry and are invariant under the full group G_{21} stabilizing the space $\mathcal{M}_{N=1}$ has dimension:

$$\dim \mathcal{M}^{inv}_{N=1} = 2 \tag{8.14}$$

In other words there is a 2-parameter G_{21}-invariant solution of Englert equation that preserves $\mathcal{N} = 1$ supersymmetry.

The scan of various supersymmetries was performed along these same lines defining:

$$\boldsymbol{\delta} \in \mathcal{M}_N \quad \Leftrightarrow \quad \mathcal{B}\left[\boldsymbol{\delta}, \boldsymbol{x}\right]_{I,9-K} = 0 \quad ; \quad I = 1, \ldots, 8 \quad ; \quad K = 1, \ldots, N \tag{8.15}$$

The result of this scan are summarized in the table here below:

SUSY	Stability subgroup of $\mathcal{M}_N$	Order of G	dim of $\mathcal{M}^{inv}_N$	dim of $\mathcal{M}_N$	dim of $\mathcal{M}^{\perp}_N$	Max inv. of N sol	Order of Γ
N	$\mathrm{G} \subset \mathrm{PSL}(2,7)$	$\lvert G \rvert$	n^{inv}_N	n_N	$n^{\perp}_N$	$\Gamma \subset \mathrm{G}$	$\lvert\Gamma\rvert$
1	G_{21}	21	2	42	14	G_{21}	21
2	Dih_3	6	2	30	26	Dih_3	6
3	$\mathbb{Z}_3$	3	8	20	36	$\mathbb{Z}_3$	3
4	T_{12}	12	4	12	44	$\mathbb{Z}_3 \subset \mathrm{T}_{12}$	3
5	$\mathbb{Z}_3$	3	2	6	50	$\mathbb{Z}_3$	3
6	Dih_3	6	2	2	54	$\mathbb{Z}_3 \subset \mathrm{Dih}_3$	3
7	$\mathrm{PSL}(2,7)$	168	0	0	56	$\mathrm{PSL}(2,7)$	168

(8.16)

Let us comment on the notation. The names of the subgroups G are those used in the previous sections and need no explanation. By definition we name $\Gamma \subset \mathrm{G}$ the subgroup with respect to which the space $\mathcal{M}_N$ contains singlets. Except for the cases $N = 4, 6$ the subgroup Γ coincides with the full group G.

Table (8.16) suffices to show that we have a rich collection of solutions to Englert equation solutions leading to exact M2-brane solutions of $d = 11$ supergravity endowed with prescribed $\mathcal{N} = N = 1, 2, 3, 4, 5, 6$ supersymmetries and possessing also a non trivial group Γ of discrete symmetries. The complete analysis of all the 424 solutions classified in previous sections is postponed to a future publication.

9 Conclusions and outlook

In this paper we have achieved an exhaustive classification of all $M2$-brane solutions of $d = 11$ supergravity of the type described in equations (2.6),(2.12), (2.20-2.21). The key item in this classification of M2-branes is the exhaustive classification of solutions to Englert equation on a 7-torus which is precisely what we have obtained utilizing the properties of the discrete group PSL(2, 7). We have also shown that this rich collection of solutions possesses equally rich subclasses with three-dimensional supersymmetries of all types from $\mathcal{N} = 1$ to $\mathcal{N} = 6$. These exact solutions are of a genuinely new type, so far never considered in M-theory.

The open problem is that of the possible interpretation of our new solutions in the following contexts:

1. The conformal gauge/gravity correspondence in the case a suitable change of coordinates revealed an asymptotic factorization of the $d = 11$ space of the following form:

$$\mathcal{M}_{11} \stackrel{asymptotically}{\longrightarrow} \mathrm{AdS}_4 \times \mathrm{SE}_7 \tag{9.1}$$

   SE_7 denoting some Sasaki-Einstein 7-manifold.

2. The domain-wall/quantum field theory correspondence if by means of some other suitable change of coordinates we succeeded in achieving domain wall configurations.

3. Effective four-dimensional gauged supergravity description if suitable conditions on the parameters were revealed for which our solutions admit a well-defined $d = 4$ limit.

Independently of the above listed possibilities a mandatory analysis of the physical content of our new class of M-theory solutions is the systematic derivation of their Kaluza-Klein spectrum. Indeed seven of the eleven dimensions are chosen to be those corresponding to a compact 7-torus and an expansion in the corresponding normal modes is well-defined and natural. We plan to perform such analysis in a forthcoming future publication.

Last but not least let us remark that one key algebraic item of our constructions is the discovery that not only the 7-dimensional irreducible representation of PSL(2, 7) is crystallographic with respect to the $\mathfrak{a}_7$ and $\mathfrak{e}_7$ root lattices, but also the 6-dimensional one is crystallographic with respect to the $\mathfrak{a}_6$ lattice. An appealing conjecture is that also the 8-dimensional irreducible representation might be crystallographic with respect to the $\mathfrak{a}_8$ and $\mathfrak{e}_8$ root lattices. This might lead to interesting consequences for $\mathrm{E}_{(8,8)}/\mathrm{SO}(16)$ sigma model representing supergravity degrees of freedom in three dimensions.

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A Main Formulas of $d = 11$ Supergravity and Conventions

In this Appendix we recall the main formulas of $d = 11$ supergravity [25] and give the dictionary relating the relevant quantities in the formalism of the original paper to those of the rheonomic Free-Differential Algebra formulation [22] that were utilized in [16] as well as in the present paper. The former will be distinguished from the latter by a tilde, when different. The $d = 11$ supergravity bosonic fields consist in the metric $\widehat{g}_{\widehat{\mu\nu}}$ and the 3-form field $\widetilde{A}_{\widehat{\mu\nu\rho}}$ and the bosonic action reads

$$\widehat{e}^{-1}\mathcal{L} = -\frac{1}{4}\,\widetilde{R} - \frac{1}{48}\,\widetilde{F}_{\widehat{\mu\nu\rho\sigma}}\widetilde{F}^{\widehat{\mu\nu\rho\sigma}} + \frac{2}{\widehat{e}\,(12)^4}\,\epsilon^{\widehat{\mu}_1\ldots\widehat{\mu}_{11}}\widetilde{F}_{\widehat{\mu}_1\ldots\widehat{\mu}_4}\widetilde{F}_{\widehat{\mu}_5\ldots\widehat{\mu}_8}\widetilde{A}_{\widehat{\mu}_9\widehat{\mu}_{10}\widehat{\mu}_{11}}\,, \tag{A.1}$$

where $\widehat{e} \equiv \sqrt{|\det(\widehat{g}_{\widehat{\mu\nu}})|}$, $\widehat{\mu}, \widehat{\nu}, \cdots = 0, \ldots, 10$. We use the "mostly minus" notation and $\epsilon_{01\ldots\,10} = \epsilon^{01\ldots\,10} = +1$.

The Einstein equation and the field equation for the 3-form read:

$$\begin{aligned}
\widetilde{R}_{\widehat{\mu\nu}} &= -\frac{1}{3}\,\widetilde{F}_{\widehat{\mu}\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3}\widetilde{F}_{\widehat{\nu}}{}^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3} + \frac{1}{36}\,\widehat{g}_{\widehat{\mu\nu}}\,\widetilde{F}_{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}\widetilde{F}^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}\,,\\
\partial_{\widehat{\mu}}\left(\widehat{e}\,\widetilde{F}^{\widehat{\mu\nu\rho\sigma}}\right) &= -\frac{3}{(12)^3}\epsilon^{\widehat{\nu\rho\sigma}\widehat{\mu}_1\ldots\widehat{\mu}_8}\,\widetilde{F}_{\widehat{\mu}_1\ldots\widehat{\mu}_4}\widetilde{F}_{\widehat{\mu}_5\ldots\widehat{\mu}_8}\,.
\end{aligned} \tag{A.2}$$

Below we give the dictionary between this notation and that of [16], in which the relevant quantities are denoted by untilded symbols:

$$\begin{aligned}
\widetilde{R}_{\widehat{\mu\nu}} &= -2\,R_{\widehat{\mu\nu}}\,,\\
\widetilde{F}_{\widehat{\mu\nu\rho\sigma}} &= 4\,\partial_{[\widehat{\mu}}\widetilde{A}_{\widehat{\nu\rho\sigma}]} = 6\,F_{\widehat{\mu\nu\rho\sigma}} = 6\,\partial_{[\widehat{\mu}}A_{\widehat{\nu\rho\sigma}]}\\
\widetilde{A}_{\widehat{\nu\rho\sigma}} &= \frac{3}{2}\,A_{\widehat{\nu\rho\sigma}}\\
\widetilde{\mathbf{F}}^{[4]} &= d\widetilde{\mathbf{A}}^{[3]} = \frac{1}{4}\,F^{[4]} = \frac{1}{4}\,dA^{[3]}\,,
\end{aligned} \tag{A.3}$$

where we have defined:

$$\begin{aligned}
\widetilde{\mathbf{F}}^{[4]} &\equiv \frac{1}{4!}\,\widetilde{F}_{\widehat{\mu\nu\rho\sigma}}\,dx^{\widehat{\mu}}\wedge\cdots\wedge dx^{\widehat{\sigma}}\,,\\
\widetilde{\mathbf{A}}^{[3]} &\equiv \frac{1}{3!}\,\widetilde{A}_{\widehat{\mu\nu\rho}}\,dx^{\widehat{\mu}}\wedge dx^{\widehat{\nu}}\wedge dx^{\widehat{\rho}}\,,\\
F^{[4]} &\equiv F_{\widehat{\mu\nu\rho\sigma}}\,dx^{\widehat{\mu}}\wedge\cdots\wedge dx^{\widehat{\sigma}}\,,\\
A^{[3]} &\equiv A_{\widehat{\mu\nu\rho}}\,dx^{\widehat{\mu}}\wedge dx^{\widehat{\nu}}\wedge dx^{\widehat{\rho}}\,.
\end{aligned} \tag{A.4}$$

We also introduce the 6-form $\widetilde{\mathbf{A}}^{[6]}$ dual to $\widetilde{\mathbf{A}}^{[3]}$ by Legendre transforming the $d = 11$ action. Its 7-form field strength reads:

$$\widetilde{\mathbf{F}}^{[7]} = d\widetilde{\mathbf{A}}^{[6]} + \widetilde{\mathbf{F}}^{[4]}\wedge\widetilde{\mathbf{A}}^{[3]} = {}^*\widetilde{\mathbf{F}}^{[4]}\,. \tag{A.5}$$

In components:

$$\begin{aligned}
\widetilde{\mathbf{F}}^{[7]} &= \frac{1}{7!}\widetilde{F}_{\widehat{\mu}_1\ldots\widehat{\mu}_7}\,dx^{\widehat{\mu}_1}\wedge\cdots\wedge dx^{\widehat{\mu}_7}\,,\\
\widetilde{F}_{\widehat{\mu}_1\ldots\widehat{\mu}_7} &= 7\,\partial_{[\widehat{\mu}_1}\widetilde{A}_{\widehat{\mu}_2\ldots\widehat{\mu}_7]} + 35\,\widetilde{F}_{[\widehat{\mu}_1\ldots\widehat{\mu}_4}\widetilde{A}_{\widehat{\mu}_5\ldots\widehat{\mu}_7]} = \frac{e}{4!}\epsilon_{\widehat{\mu}_1\ldots\widehat{\mu}_7\widehat{\mu}_8\ldots\widehat{\mu}_{11}}\,\widetilde{F}^{\widehat{\mu}_8\ldots\widehat{\mu}_{11}}\,.
\end{aligned} \tag{A.6}$$

As for the fermionic sector, the gamma matrices $\widetilde{\Gamma}^{\widehat{\mu}}$ in [25] differ by an overall sign from those in the present paper $\Gamma^{\widehat{\mu}}$:

$$\widetilde{\Gamma}^{\widehat{\mu}} = -\Gamma^{\widehat{\mu}}\,, \tag{A.7}$$

while the gravitino field is the same in the two notations. The supersymmetry variation of the latter field therefore reads:

$$\begin{aligned}\delta\Psi_{\widehat{\mu}} &= \mathcal{D}_{\widehat{\mu}}\epsilon + \frac{i}{144}\left(\widetilde{\Gamma}^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}{}_{\widehat{\mu}} - 8\,\widetilde{\Gamma}^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3}\delta^{\widehat{\mu}_4}_{\widehat{\mu}}\right)\Psi\,\widetilde{F}_{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4} = \\ &= \mathcal{D}_{\widehat{\mu}}\epsilon - \frac{i}{24}\left(\Gamma^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}{}_{\widehat{\mu}} - 8\,\Gamma^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3}\delta^{\widehat{\mu}_4}_{\widehat{\mu}}\right)\Psi\,F_{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}\,.\end{aligned} \tag{A.8}$$

B M2-Brane Solutions with Transverse Flux

The $M2$-brane solutions considered in the present work are part of a general class of solutions characterized by the presence of a self-dual 4-form flux along the transverse eight-dimensional space [12–15]. The Ansatz for the $d = 11$ metric is the one given in Eq. (1.4) while the 3-form field has the following general expression:

$$\mathbf{A}^{[3]} = \frac{2}{H(y)}\,\Omega^{[3]} + \mathring{\mathbf{A}}^{[3]}(y)\,, \tag{B.1}$$

where $H(y)$ is a function of the eight transverse coordinates y^I and $\mathring{\mathbf{A}}^{[3]}(y)$ is a 3-form in the transverse space. The 4-form field strength reads:

$$\mathbf{F}^{[4]} = d\mathbf{A}^{[3]} = -\frac{2}{H(y)^2}\,\partial_I H\, dy^I \wedge \Omega^{[3]} + \mathring{\mathbf{F}}^{[4]}(y)\,, \tag{B.2}$$

where

$$\mathring{\mathbf{F}}^{[4]} = d\mathring{\mathbf{A}}^{[3]} = \mathring{F}(y)_{IJKL}\, dy^I \wedge dy^J \wedge dy^K \wedge dy^L\,.$$

We require $\mathring{\mathbf{F}}^{[4]}$ to be self-dual in the transverse Euclidean space: [7]

$$\star_8\, \mathring{\mathbf{F}}^{[4]} = \mathring{\mathbf{F}}^{[4]}\,. \tag{B.3}$$

Plugging the above Ansatz in the field equations we find for $H(y)$

$$\Box_8 H \;=\; -3\,\mathring{F}_{IJKL}\mathring{F}^{IJKL}\,, \tag{B.4}$$

where $\Box_8$ is the d'Alembertian in the flat transverse space: $\Box_8 H \equiv \partial_I\partial_I H$.

Supersymmetry. Substituting the above Ansatz in the Killing spinor equation:

$$\mathcal{D}_{\widehat{\mu}}\xi - \frac{i}{24}\left(\Gamma^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4}{}_{\widehat{\mu}} - 8\,\Gamma^{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3}\delta^{\widehat{\mu}_4}_{\widehat{\mu}}\right)\xi\,F_{\widehat{\mu}_1\widehat{\mu}_2\widehat{\mu}_3\widehat{\mu}_4} = 0\,, \tag{B.5}$$

[7] Had we chosen the $M2$-brane with the opposite charge with respect to $\mathbf{A}^{[3]}$, we should have taken $\mathring{\mathbf{F}}^{[4]}$ to be anti-self-dual.

and writing $\xi = \epsilon \otimes \chi$, as in (7.22), after some algebra, one finds the following conditions:

$$\mathbb{P}_- \chi = 0\,, \quad \mathring{F}_{IJKL}\, T^{IJKL}\, \chi = 0\,, \quad \mathring{F}_{I_1\dots I_4} T^{I_1\dots I_4}\, T_I\, \chi = 0\,, \tag{B.6}$$

where $\mathbb{P}_- \equiv \frac{1}{2}(\mathbf{1}_{16} - T_9)$. One can show, following [12], that the above conditions can be recast in the following equivalent form:

$$\mathbb{P}_- \chi = 0\,, \ \ \mathring{F}_{IJKL}\, T^{JKL}\, \chi = 0\,. \tag{B.7}$$

The self-duality condition of $\mathring{F}$ further simplifies equations (B.6) since

$$\mathring{F}_{IJKL}\, T^{IJKL} = \mathring{F}_{IJKL}\, T^{IJKL}\, \mathbb{P}_+\,. \tag{B.8}$$

Therefore, choosing χ so that $\mathbb{P}_- \chi = 0$, the last of Eq.s (B.6) is automatically satisfied, and the supersymmetry conditions reduce to:

$$\mathbb{P}_- \chi = 0\,, \ \ \mathring{F}_{IJKL}\, T^{IJKL}\, \chi = 0\,. \tag{B.9}$$

The existence of solutions to the above equations depends on the detailed structure of $\mathring{F}_{IJKL}$. As we have shown, the form of the self-dual flux in the class of solutions considered here does allow for solutions with different degrees of supersymmetry. Let us show below how the Englert equation implements the self-duality condition (B.3) for the class of solutions discussed in the present work.

***M*2-branes with Englert fluxes.** These solutions are obtained by choosing the transverse space of the form $\mathbb{R}_+ \times T^7$, splitting $(y^I) = (x^i, U)$ and further specializing the Ansatz (B.1) by choosing the inner components of the 3-form as follows:

$$\mathring{\mathbf{A}}^{[3]} = e^{-\mu U}\, Y_{ijk}(x)\, dx^i \wedge dx^j \wedge dx^k\,, \tag{B.10}$$

where $i, j, k = 1, \dots, 7$. The self-duality condition (B.1) then reduces to the Englert equation in Y_{ijk}. Formally this amounts to a Scherk-Schwarz reduction [31] from the Euclidean eight-dimensional transverse space to the seven-torus and the original self-duality condition reduces to the "self-duality" in odd-dimensions of [32].

As far as the supersymmetry conditions (B.9) are concerned, if we further split $\chi = \kappa \otimes \lambda$, as in Eq. (7.25), where now $\sigma_3 \kappa = \kappa$, condition $\mathbb{P}_- \chi = 0$ is satisfied, being $T_9 = \sigma_3 \otimes \mathbf{1}_8$. The last of equations (B.9) now boils down to:

$$0 = \mathring{F}_{0ijk}\, T^{0ijk}\, \chi \propto \kappa \otimes \mathcal{B}\lambda \ \Leftrightarrow \ \ \mathcal{B}\lambda = 0\,, \tag{B.11}$$

where $\mathcal{B} \equiv \tau^{ijk}\, Y_{ijk}$. We then retrieve the equation (7.32), whose solutions have been studied in the present work.

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