Resonant Axisymmetric Modes

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Abstract. Axisymmetric modes in shaped tokamak plasmas are normally associated with vertical displacement events. However, not enough attention has been given to the fact that these modes can be resonant in two different ways. Firstly, for a plasma bounded by a divertor separatrix, a generic n=0 ideal-MHD perturbation, ξ, is singular at the divertor X-point(s), where $\mathbf{B_{eq}} \cdot \nabla\xi = \mathbf{0}$, with $\mathbf{B_{eq}}$ the equilibrium magnetic field. As a consequence, n=0 perturbations can give rise to current sheets localized along the divertor separatrix. Secondly, a feedback-stabilized n=0 mode tends to acquire an Alfvénic oscillation frequency. As a result, a resonant interaction with energetic particle orbits can lead to a new type of fast ion instability.

1. Introduction

Present-day tokamak experiments adopt plasma shaping, and magnetic divertors, in order to optimize fusion performance, and also to reduce the adverse effects of plasma-wall interactions. However, cross-section elongation requires careful design of the plasma facing components for passive feedback stabilization, and active feedback control by currents flowing in external coils, in order to avoid vertical displacement events (VDE). Such events may endanger the machine integrity and its safe operation [1–4]. Hence, plasma vertical stability has been the subject of several analytic and numerical theoretical investigations, of which Refs. [5–10] are a sample.

Typically, in these investigations, the plasma is modelled by ideal-MHD, with boundary conditions representing the presence of a nearby resistive wall and external feedback currents. The basic idea is that eddy currents flowing along the wall, and/or in *ad hoc* metallic conductors facing the plasma, can lead to passive stabilization of the ideal-MHD vertical mode. Indeed, passive stabilization is a must, because if somehow vertical displacements were allowed to grow on the Alfvénic time scale, magnetic fluxes generated by external currents placed outside the wall of the vacuum chamber for active feedback control would not have the time to penetrate the wall and counter the fast-growing ideal-MHD instability.

The main objective of the present article is to present an analytic treatment of vertical displacement perturbations on the basis of the reduced ideal-MHD model, with preliminary considerations concerning the impact of the X-point resonance. What has been overlooked in previous theoretical works is that, due to the singular nature of axisymmetric modes at divertor X-points, perturbed currents flowing along the magnetic separatrix may be induced by these perturbations.

Stable vertical displacement modes tend to oscillate with a frequency of the order of the relevant Alfvén frequency. Hence, these modes can be driven unstable by their resonant interaction with fast ions. Therefore, a second objective of this article is a discussion on the nature of this mode-particle resonance and a preliminary analytic assessment and estimate of the resulting instability growth time.

2. A heuristic model: three parallel current wires

Key elements of the vertical displacement dynamics can be understood on the basis of the following heuristic model. Let us consider three parallel current wires. At equilibrium, the *plasma wire* carrying the current I_P is located at $x = 0$, while two equal external current wires, each carrying a current I_{Ext}, are located at $x = \pm l$. Note that we take the x-axis as the *vertical* direction. The currents flow along the z-direction, which mimics the toroidal direction of a tokamak configuration. All three currents are taken to be positive. The external wires are fixed in space, while the plasma wire is free to move along the x-axis. Vacuum is assumed to surround the three wires. Therefore, at equilibrium, no current flows through the two magnetic X-points located at $x = \pm l_X$, with $l_X < l$. This configuration mimics the equilibrium of a *straight* tokamak bounded by a double-null magnetic divertor separatrix.

The equation of motion for the plasma wire is (in c.g.s. units)

$$\mu\, \ddot{x} = (4\, I_P\, I_{Ext}/c^2)\, x/(l^2 - x^2), \tag{1}$$

where μ is the linear mass density, c is the speed of light, and an over-dot signifies time derivative. We neglect self and mutual induction currents. Thus, I_P and I_{Ext} remain constant as the plasma wire is displaced. For small $x \ll l$, the solution of Eq. (1) is $x = x_0\, \mathrm{e}^{\gamma_H\, t}$, where x_0 is an initial displacement, and $\gamma_H = (1/l)(4 I_P I_{Ext}/\mu c^2)^{1/2}$.

If instead of a plasma wire, we consider a vertically elongated plasma column with a uniform current density extending, on the Oxy cross-section, up to an elliptical magnetic surface with minor semi-axis a and major semi-axis b, then, according to the analysis of Ref. [11], a relationship is established between the currents I_P and I_{Ext} and the distance l; namely, $I_{Ext}/I_P = [(b - a)/(b + a)][l^2/(a^2 + b^2)]$. Using this expression, γ_H is found to depend only on the plasma current I_P and on the parameters a and b, but not on l, nor on I_{Ext}. Also, for a cylindrical plasma column with elliptical cross-section, the linear mass density μ should be replaced by $\mu \to \pi\, a\, b\, \varrho_m$, where ϱ_m is the volume mass density. After straightforward algebra, taking the small ellipticity limit, $e_0 \ll 1$, where

$$e_0 = \frac{b^2 - a^2}{b^2 + a^2}, \tag{2}$$

we can rewrite γ_H in terms of more familiar plasma parameters:

$$\gamma_H = {e_0}^{1/2} {\tau_A}^{-1}, \tag{3}$$

where ${\tau_A}^{-1} = {B_P}'/(4\pi\, \varrho_m)^{1/2}$ is the inverse Alfvén time, and ${B_P}'$ is the radial derivative of the poloidal magnetic field, averaged over a flux surface, at the magnetic axis. Note that $0 \le e_0 \le 1$; in the limit of circular flux surfaces, $e_0 = 0$ and the growth rate γ_H vanishes.

Ideal-MHD unstable vertical displacements in elongated tokamak plasmas have been observed experimentally (see, eg., Ref. [12]). For values of e_0 that are typical of present-day tokamak experiments, γ_H is indeed a very fast growth rate. For instance, if we take a Hydrogen plasma with $e_0 = 0.2$, $a = 1\,\mathrm{m}$, ${B_p}' = 1\,\mathrm{T/m}$, and number density $n = 10^{20}\,\mathrm{m}^{-3}$, we find $\gamma_H^{-1} \simeq 1\,\mu\mathrm{s}$.

The ideal vertical instability is suppressed in the presence of a perfectly conducting wall, provided this wall is not too far away from the plasma [5]. When the plasma current is displaced from its equilibrium position, image currents are induced at the wall. The sign of these currents is such that the corresponding forces oppose the motion of the plasma wire. From a heuristic point of view, this effect can be mimicked by two currents of opposite sign, $\pm\delta I$, carried by the two external wires, and added to the external currents I_{Ext}. The currents $\pm\delta I$ can be thought of as driven by an induced e.m.f. proportional to $\dot{x}$. In the perfectly conducting limit, $\delta\dot{I} = DI_{Ext}\dot{x}/l$, where D is a dimensionless proportionality constant. In the actual tokamak problem, the parameter D depends on the wall geometry [5] and on other passive stabilization components eventually placed inside the vacuum chamber. Including the effect of the feedback currents $\pm\delta I$, the equation of motion for the plasma wire becomes $\mu\, c^2\, \ddot{x} \approx 4\, I_P\, I_{Ext}\,(1-D)\, x/l^{\,2}$, where we have taken $x \ll l$. Stability is obtained for $D > 1$. When this inequality is satisfied, an initial displacement of the plasma wire from its equilibrium position gives rise to oscillatory motion with characteristic frequency

$$\omega_H = \sqrt{D-1}\,\gamma_H \tag{4}$$

The stability criterion $D > 1$ is satisfied in well-designed tokamaks with elongated cross-sections (see, e.g., Refs. [5, 9]). The effect of triangularity is outside the scope of the present article.

3. Model equations and plasma equilibrium

The magnetic field is represented as $\mathbf{B} = \mathbf{e}_z \times \nabla\psi + B_z\,\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector along the z-direction, and B_z is nearly constant. We assume that all physical quantities are independent of the z coordinate. The plasma flow is represented as $\mathbf{v} = \mathbf{e_z} \times \nabla\varphi$.

The magnetic flux function, ψ, and the stream function, φ, obey the well-known, reduced ideal-MHD model equations [13]. In dimensionless form:

$$\frac{\partial\psi}{\partial t} + [\varphi, \psi] = 0, \tag{5}$$

$$\frac{\partial}{\partial t}\,\nabla\cdot(\varrho\,\nabla\varphi) + [\varphi, U] = [\psi, J]\,, \tag{6}$$

where the brackets are defined as $[\chi, \eta] = \mathbf{e}_z\cdot\nabla\chi\times\nabla\eta$, $J = \nabla^{\,2}\psi$ is the normalized current density, and $U = \nabla^{\,2}\varphi$ is the normalized flow vorticity. Space and time are normalized as $\hat{r} = r/r_0$, where $r_0 = a\,b/[\left(a^{\,2} + b^{\,2}\right)/2]^{1/2}$ is a convenient equilibrium scale length, and $\hat{t} = t/\tau_A$, where τ_A is the relevant Alfvén time as defined below Eq. (3). The dimensionless fields are normalized as $\hat{\psi} = \psi/(B_p^{\,'} r_0^2)$, $\hat{\varphi} = (\tau_A/r_0{}^2)\varphi$; the plasma density is $\hat{\varrho} = \varrho_m/\varrho_{m0}$, with ϱ_{m0} the density on the magnetic axis, and the current density is $\hat{J} = (4\pi/cB_p^{\,'})J_z$. In order to simplify the notation, over-hats are actually dropped in Eqs. (5) and (6), and in the following.

At equilibrium, fields are stationary, and, by assumption, equilibrium plasma flows are absent. In order to solve Eqs. (5,6) analytically, we adopt the relatively simple equilibrium discussed in Ref. [11]. The equilibrium current density, J_{eq}, is assumed to be uniform up to an elliptical boundary with minor semi-axis a and major semi-axis b, and to drop to zero beyond that boundary. External current are assumed to be placed symmetrically along the x-axis (x is the vertical direction), at $x = \pm l$. Clearly, the elliptical boundary must be a magnetic flux-surface, which lies necessarily within the region bounded by the magnetic separatrix. In elliptical coordinates (μ, θ), where $x = A\cosh(\mu)\cos(\theta)$ and $y = A\sinh(\mu)\sin(\theta)$, with $A = \sqrt{b^2 - a^2}$, the elliptical boundary corresponds to $\mu = \mu_b$. Also, $a = A\sinh\mu_b$ and $b = A\cosh\mu_b$. Thus, J_{eq} depends only on the μ coordinate and, consistently with the normalization adopted for the current density, $J_{eq}(\mu) = 2H(\mu_b - \mu)$, where $H(x)$ is the Heaviside unit step function. It can also be stated that the equilibrium current density is a function of the equilibrium flux, ψ_{eq}, and so $[\psi_{eq}, J_{eq}] = 0$, which implies $\nabla^{\,2}\psi_{eq} = J_{eq}(\psi_{eq})$.

According to the analysis of Ref. [11], it can be shown that the solution ψ_{eq} of the equilibrium problem in the limit where the parameter $\varepsilon = [(a^2 + b^2)/l^2]e_0$ is small reduces to Gajewski's solution [14], represented as follows. Inside the elliptical boundary, where $\mu < \mu_b$ and $\psi = \psi_{eq}^-$, the solution of $\nabla^2\psi_{eq}^- = 2$ that is well behaved on the magnetic axis and that reduces to a constant on the elliptical boundary is best written in terms of Cartesian components:

$$\psi_{eq}^-(x,y) = \frac{1}{2}\left(\frac{x^2}{b^2} + \frac{y^2}{a^2}\right) \tag{7}$$

Outside the elliptical boundary, where $\mu > \mu_b$ and $\psi_{eq} = \psi_{eq}^+$, $\nabla^2\psi_{eq}^+ = 0$. We require that $\psi_{eq}(\mu,\theta)$ and $\partial\psi_{eq}/\partial n = \mathbf{n}\cdot\nabla\psi_{eq}$ be continuous across the elliptical boundary, as we assume that no equilibrium current sheet is present on the boundary. Here, the unit vector $\mathbf{n}$ is the normal to the boundary. A suitable analytic solution can be obtained:

$$\psi_{eq}^+(\mu,\theta) = \frac{1}{2} + \alpha^2\left\{\mu - \mu_b - \frac{e_0}{2}\sinh\left[2(\mu-\mu_b)\right]\cos(2\theta)\right\} \tag{8}$$

with $\alpha^2 = ab/r_0^2$ and e_0 the ellipticity parameter defined in Eq. (2). The magnetic flux surfaces $\psi_{eq}(\mu,\theta) = \text{const}$ exhibit a magnetic separatrix for $\psi_{eq}(\mu,\theta) = \psi_X = \mu_b\,\alpha^2$, with X-points located at $\mu = \mu_X = 2\mu_b$ and $\theta = \theta_X = (0,\pi)$. Gajewski's solution approximates well the complete solution given in Ref. [11] up to values of $|x|$ and $|y|$ much smaller than l, which, for $\varepsilon \ll 1$, include the separatrix region.

4. Normal mode analysis of vertical displacements: the base scenario

The simplest assumption for the equilibrium plasma density is that its profile is the same as that of the equilibrium current density, i.e., $\varrho_{eq} = H(\mu_b - \mu)$, with $H(x)$ the unit step function. We refer to this as the "base scenario". This scenario has two important implications: (i) We can show that a rigid-shift vertical displacement is indeed the analytic solution of the linearized reduced ideal-MHD model. For this displacement, the stream function in Cartesian coordinate can be represented as $\varphi = -\gamma\xi y$, where the constant ξ is the actual distance by which the magnetic axis, and indeed the whole plasma within the elliptical boundary $\mu = \mu_b$, move along the vertical x-direction. Furthermore, we can show analytically, on the basis of the reduced ideal-MHD model, that the relevant growth rate, γ, as well as the feedback-stabilized oscillation frequency, are in very good agreement with those obtained heuristically in Sec. 2; (ii) Since the plasma density vanishes at the X-points, the X-point singularity does not play a role for the base scenario. This is the reason why the results for this scenario agree so well with the results obtained by the heuristic three-wire model of Sec. 2. There are, however, some interesting results obtained for the base scenario that cannot be captured by the heuristic model. The more realistic scenario where the plasma density extends to the X-points will be considered in Sec. 6.

Let $\psi(\mu,\theta,t) = \psi_{eq}(\mu,\theta) + \tilde{\psi}(\mu,\theta)\,\mathrm{e}^{\gamma t}$ and $\varphi(\mu,\theta,t) = \tilde{\phi}(\mu,\theta)\,\mathrm{e}^{\gamma t}$, where the over-tilde denotes small perturbations. To first order in perturbed quantities, Eqs. (5) and (6) yield

$$\gamma\,\tilde{\psi} + [\tilde{\varphi},\psi_{eq}] = 0, \tag{9}$$

$$\gamma\,\nabla\cdot(\varrho_{eq}\nabla\tilde{\varphi}) = \left[\tilde{\psi}, J_{eq}\right] + \left[\psi_{eq}, \tilde{J}\right]. \tag{10}$$

In the region inside the equilibrium elliptical boundary, $\mu < \mu_b$, the stream function corresponding to a rigid vertical shift, in elliptical coordinates, is

$$\tilde{\varphi}(\mu,\theta) = -\gamma\,\xi\,A\sinh\mu\sin\theta. \tag{11}$$

Clearly, the vorticity field $\nabla\tilde{\varphi} = 0$ for the rigid-shift stream function, and so, since the plasma density is assumed to be constant inside the elliptical region, also the l.h.s. of Eq. (10) vanishes. Furthermore, $[\tilde{\psi}, J_{eq}] = 0$ within that region. Therefore, a consistent solution must also satisfy $[\psi_{eq}, \tilde{J}] = 0$, or more simply $\tilde{J} = 0$, inside the elliptical boundary.

The perturbed magnetic flux, $\tilde{\psi}^-$, can be obtained from the flux-freezing Eq. (9), which, in Cartesian coordinates, yields $\tilde{\psi}^- = -\xi_0 x/b$, where $\xi_0 = \xi/b$, and in elliptical coordinates,

$$\tilde{\psi}^-(\mu,\theta) = -\xi_0 \frac{\cosh\mu}{\cosh\mu_b}\cos\theta \tag{12}$$

The vorticity Eq. (10) is, thereby, fulfilled as a consequence of the fact that $\tilde{U} = \nabla^2\tilde{\varphi}$, $\tilde{J} = \nabla^2\tilde{\psi}$, $\nabla\varrho_{eq}$ and ∇J_{eq} all vanish inside the elliptical boundary. Note that in elliptical coordinates, $\nabla^2\chi = h^{-2}(\partial^2\chi/\partial\mu^2 + \partial^2\chi/\partial\theta^2)$, where $h = 1/|\nabla\mu| = 1/|\nabla\theta|$ is a scale factor. We still have to make sure that the three terms in Eq. (10) balance also on the elliptical boundary, where each term is proportional to a delta function, $\delta(\mu-\mu_b)$. This is shown next. As a byproduct, the growth rate γ is determined by the balance condition.

In the vacuum region beyond the elliptical boundary, $\mu > \mu_b$, the equation for perturbed flux function $\tilde{\psi}^+$ is $[\psi_{eq}, \tilde{J}^+] = 0$, which reduces to $\tilde{J}^+ = \nabla^2\tilde{\psi}^+ = 0$. The solution that satisfies the continuity condition at the elliptical boundary and the regularity condition at infinity is

$$\tilde{\psi}^+(\mu,\theta) = -\xi_0\, \mathrm{e}^{-(\mu-\mu_b)}\cos\theta \tag{13}$$

The derivative of the perturbed flux function with respect to the μ coordinate exhibits a discontinuity at the elliptical boundary, $\mu = \mu_b$, which gives rise to a perturbed current sheet along the "toroidal" z-direction:

$$\tilde{J}(\mu,\theta) = \tilde{j}_b(\theta)\delta(\mu-\mu_b) = \frac{1}{h^2}\left(\frac{\partial\tilde{\psi}^+}{\partial\mu} - \frac{\partial\tilde{\psi}^-}{\partial\mu}\right)\Bigg|_{\mu_b}\delta(\mu-\mu_b), \tag{14}$$

where $\delta(x)$ is the Dirac delta function. A simple calculation gives

$$\tilde{j}_b(\theta) = \frac{2(a+b)}{b(a^2+b^2)}\frac{\xi_0\cos\theta}{1-e_0\cos 2\theta}. \tag{15}$$

In addition, the inertial term, as well as the term $[\tilde{\psi}, J_{eq}]$ in Eq. (10), are proportional to $\delta(\mu-\mu_b)\sin\theta$, the latter since $dJ_{eq}/d\mu = -2\delta(\mu-\mu_b)$. Multiplying both sides of Eq. (10) by the scale factor h^2, and integrating across the elliptical boundary over an infinitesimal interval in μ, allows us to pick up the delta-function contributions and to determine the mode growth rate γ :

$$\lim_{\delta\mu\to 0}\int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\,\gamma\,\nabla\cdot(\varrho\,\nabla\tilde{\varphi})\,d\mu = \lim_{\delta\mu\to 0}\left\{\int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\left[\tilde{\psi}, J_{eq}\right]d\mu + \int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\left[\psi_{eq}, \tilde{J}\right]d\mu\right\} \tag{16}$$

We can evaluate the three terms above separately as follows:

$$\lim_{\delta\mu\to 0}\int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\,\gamma\,\nabla\cdot(\varrho\,\nabla\tilde{\varphi})\,d\mu = -\gamma\left.\frac{\partial\tilde{\varphi}}{\partial\mu}\right|_{\mu_b^-} = \gamma^2\, b^2\,\xi_0\,\sin\theta \tag{17}$$

$$\lim_{\delta\mu\to 0}\int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\left[\tilde{\psi}, J_{eq}\right]d\mu = J_{eq}\left.\frac{\partial\tilde{\psi}}{\partial\theta}\right|_{\mu_b} = 2\,\xi_0\,\sin\theta \tag{18}$$

$$\lim_{\delta\mu\to 0}\int_{\mu_b-\delta\mu}^{\mu_b+\delta\mu} h^2\left[\psi_{eq}, \tilde{J}\right]d\mu = \frac{d}{d\theta}\left[\tilde{j}(\theta)\left.\frac{\partial\psi_{eq}}{\partial\mu}\right|_{\mu_b}\right] = -\frac{a+b}{ab^2}\,\xi_0\,\sin\theta \tag{19}$$

Reintroducing, for clarity, physical dimensions for the growth rate γ and for the plasma minor and major semi-axis a and b, we obtain $\gamma^2 = ({r_0}^4/a^2b^2)(1-a/b)\tau_A^{-2}$, which can also be written in terms of the ellipticity parameter e_0 as

$$\gamma^2 = (1-e_0)(1+e_0-\sqrt{1-e_0^2}\,)\tau_A^{-2}. \tag{20}$$

The result in Eq. (20) is valid for arbitrary values of e_0 within the interval $0 \le e_0 \le 1$. In the limit of small e_0, Eq. (20) reduces to $\gamma^2 \approx e_0\tau_A^{-2}$, which agrees very well with the growth rate (3) obtained heuristically in Sec. 2. An alternative form of the growth rate can be expressed in terms of the elongation parameter, $\kappa = b/a \ge 1$, i.e., $\gamma^2 = 4\kappa(\kappa-1)/(1+\kappa^2)^2\tau_A^{-2}$.

When feedback stabilization is present, the perturbed flux function involves an additional term due to the external feedback currents. This term is continuous across the elliptical boundary, and can be conveniently written as

$$\tilde{\psi}^{ext} = e_0\xi_1\frac{b}{a+b}\frac{\cosh\mu}{\cosh\mu_b}\cos\theta, \tag{21}$$

where ξ_1 is a constant that depends on the strength of the feedback currents. Therefore, the perturbed flux is modified by feedback into

$$\tilde{\psi}_f^- = \tilde{\psi}^- + \tilde{\psi}^{ext} = -\left(\xi_0 - e_0\frac{b}{a+b}\xi_1\right)\frac{\cosh\mu}{\cosh\mu_b}\cos\theta \tag{22}$$

$$\tilde{\psi}_f^+ = \tilde{\psi}^+ + \tilde{\psi}^{ext} = -\xi_0 e^{-(\mu-\mu_b)}\cos\theta + e_0\xi_1\frac{b}{a+b}\frac{\cosh\mu}{\cosh\mu_b}\cos\theta \tag{23}$$

Similarly, the stream function is modified by feedback into

$$\tilde{\varphi}_f^- = -\gamma\left(\xi_0 - e_0\frac{b}{a+b}\xi_1\right)ab\frac{\sinh\mu}{\sinh\mu_b}\sin\theta. \tag{24}$$

Since $\tilde{\psi}^{ext}$ is continuous across the elliptical boundary, the perturbed current sheet, Eq. (15), is not modified by feedback. However, $\partial\tilde{\psi}/\partial\mu$ and $\partial\tilde{\psi}/\partial\theta$ at $\mu = \mu_b$ do change. Using the expressions (22), (23) and (24) in Eq. (16), the growth rate is modified into

$$\gamma^2 = {\gamma_0}^2\frac{1-(\xi_1/\xi_0)}{1-e_0\left[b\xi_1/(a+b)\xi_0\right]}, \tag{25}$$

where ${\gamma_0}^2$ is the growth rate in the absence of feedback as given in Eq. (20). Therefore, the criterion for feedback stabilization is

$$\xi_1/\xi_0 > 1. \tag{26}$$

We now show that the ratio ξ_1/ξ_0 equals the parameter D introduced in Sec. 2. Following the discussion at the end of Sec. 2, the perturbed flux, associated with the two currents $\pm\delta I$, is $\tilde{\psi}^{ext} = (\delta I/\pi)(A/l)\cosh\mu\cos\theta$, to first order in r/l, with $r = (x^2+y^2)^{1/2}$. As in Sec. 2, we let $\delta I = DI_{ext}\xi/l$. Relating I_{Ext} to the plasma current I_P, see above Eq. (2), using $I_P = \pi abJ_{eq}$ and the proper normalization for the dimensionless magnetic flux, see below Eq. (6), we obtain

$$\tilde{\psi}^{ext} = e_0 D\xi_0\frac{b}{a+b}\frac{\cosh\mu}{\cosh\mu_b}\cos\theta. \tag{27}$$

Comparing this expression with that given by Eq. (21) reveals that $D = \xi_1/\xi_0$, as expected. Therefore, in the limit of small ellipticity, the feedback stabilization criterion (26) is well approximated by the criterion $D > 1$ obtained heuristically in Sec. 2.

5. Interaction of vertical displacements with energetic ions

As shown in Secs. 2 and 4, feedback stabilization converts the n=0 unstable perturbation into a neutrally stable MHD oscillation with frequency ω_H given by Eq. (4), or better still, $\omega_H = (-\gamma_f^2)^{1/2}$, with γ_f in Eq. (25), provided $D = \xi_1/\xi_0 > 1$.

This mode may indeed have been observed in recent JET tokamak plasma experiments designed to study the confinement of fusion products [15]. Finite amplitude $n = 0$ oscillations with frequency $\omega_{obs} \approx 2.0 \times 10^6 s^{-1}$ were reported. The D-^{3}He plasmas were heated by a combination of neutral beam injection (NBI) and ion cyclotron resonance heating (ICRH). The resulting fast ion population reached energies in the MeV range. Using the JET parameters of Ref. [15], namely, plasma minor radius $a \simeq 0.9m$, major radius $R_0 \simeq 3m$, average ion density $\overline{n}_i \sim 3.5 \times 10^{19} m^{-3}$, average ion mass $\overline{m}_i \sim 2.1 m_H$, ellipticity $e_0 \sim 0.25$, and estimating $B_P^{'} \sim 1T/m$, we find $\gamma_H \sim 1.3 \times 10^6 s^{-1}$, and so, $\omega_H \approx \omega_{obs}$ if we take $D \approx 1.6$. This is indeed a reasonable value, if we compare it, e.g., with the valued of D inferred from Ref. [5].

We suggest that JET observations of finite-amplitude $n = 0$ oscillations may be explained by the resonant interaction of these modes with MeV fast ions. Fast ions with trapped "banana" orbits may also contribute, but in general this involves higher fast ion energies, and therefore the trapped ion resonance will be neglected in this preliminary analysis. The relevant resonance involves the transit frequency, ω_t, of circulating fast ions. In the following, we sketch the derivation of the dispersion relation for $n = 0$ perturbations in the presence of energetic particles, which can be conveniently written as $\delta K = -\delta W_{MHD} - \delta W_{hot}(\omega)$, where $\delta K = \int d^3x \rho_m \boldsymbol{\xi}^* \cdot (\partial^2 \boldsymbol{\xi}/\partial t^2)/2 = -\rho_m \omega^2 |\xi|^2 V/2$ is the perturbed plasma kinetic energy and V is the plasma volume. In the latter expression for δK, we have assumed a constant plasma density and the rigid-shift displacement derived in Sec. 4. Furthermore, δW_{MHD} is the usual ideal-MHD potential energy functional, modified by feedback, as discussed, e.g., in Ref. [16], and δW_{hot} is the fast ion potential energy, of which a general expression can be found, e.g., in Ref. [17].

Analytic progress is possible assuming that the fast ion density, n_h, is relatively low as compared with the thermal plasma density, n_i, so that $|\delta W_{MHD}| \gg |\delta W_{hot}|$. When this inequality is satisfied, the fast ion contribution can be treated perturbatively, with n_h/n_i as small expansion parameter. To zeroth order in this parameter, the rigid shift displacement obtained in Sec. 4 can indeed be used to evaluate δK and δW_{MHD}. Then, only the zeroth-order rigid-shift displacement is required to evaluate the first order term δW_{hot}. The real part of δW_{hot} will give rise to a small correction to the oscillation frequency. On the other hand, the imaginary part of δW_{hot} will introduce a growth rate, or damping, of the $n = 0$ mode. Therefore, in the following, we will concentrate on the calculation of $\mathcal{I}m(\delta W_{hot})$.

Let us introduce $\delta \hat{K} = \delta K/[(\rho_m |\xi|^2 V)/2] = \omega^2$, $\delta \hat{W}_{MHD} = \delta W_{MHD}/[(\rho_m |\xi|^2 V)/2] = \omega_H^2$, and $\delta \hat{W}_{hot} = \delta W_{hot}/[(\rho_m |\xi|^2 V)/2]$. The dispersion relation takes the form

$$\omega^2 = \omega_H^2 + \delta \hat{W}_{hot}(\omega). \tag{28}$$

We can use for ω_H^2 the value obtained in Eq. (25). Following Ref. [17], we can write $\delta W_{hot}(\omega) = \delta W_1 + \delta W_2(\omega)$, where the fluid-like part δW_1 is a real quantity, while δW_2 is a complex quantity as it contains the contribution of resonant fast particles. For $n = 0$, δW_2 reduces to (cf. Eq.(76) of Ref. [17])

$$\delta W_2 = -\frac{2\pi^2 c}{Zem^2} \sum_\sigma \int dP_\varphi d\mathcal{E} d\mu_\perp \tau_t \omega \frac{\partial F}{\partial \mathcal{E}} \sum_{p=-\infty}^{+\infty} \frac{|\Upsilon_p|^2}{\omega + p\omega_t} \tag{29}$$

The integration variables are invariants of the particle motion: toroidal canonical momentum P_φ, kinetic energy $\mathcal{E}$ and magnetic moment $\mu_\perp$. In general, the equilibrium distribution function

F is a function of these three invariants and of the index $\sigma = \pm 1$, where $\sigma = 1$ corresponds to co-circulating orbits and $\sigma = -1$ to counter-circulating orbits (with respect to the direction of the toroidal magnetic field). The transit time is defined as $\tau_t = 2\pi/|\omega_t|$. The Fourier coefficients Υ_p are associated with the expansion of the perturbed guiding center Lagrangian, $\tilde{\mathcal{L}}^{(1)}$ over the particle orbit periodicity, namely:

$$\Upsilon_p(\mathcal{E}, \mu, P_\varphi) = \left\langle \tilde{\mathcal{L}}^{(1)} \exp\left(ip\omega_t\tau\right)\right\rangle \tag{30}$$

where $\langle\chi\rangle = \oint d\tau\chi/\tau_t$ represents averaging over unperturbed orbits. The perturbed Lagrangian for $n = 0$ ideal-MHD perturbations reduces to $\mathcal{L}^{(1)} \simeq -(mv_\parallel^2 + \mu_\perp B)\boldsymbol{\xi}\cdot\boldsymbol{\kappa}$, with $\boldsymbol{\kappa}$ the magnetic curvature vector and $v_\parallel$ the particle velocity along magnetic field lines [17].

Equation (29) indicates that the relevant mode-particle resonance is $\omega = -p\omega_t$. The $p = \pm 1$ harmonics give the largest contribution to the imaginary part of δW_2. Since $|\Upsilon_1|^2 = |\Upsilon_{-1}|^2$, Eq. (29) can be rewritten as:

$$\delta W_2 = -\frac{4\pi^2 c}{Zem^2}\int dP_\varphi d\mathcal{E} d\mu_\perp \tau_t\omega\frac{\partial F}{\partial\mathcal{E}}\frac{|\Upsilon_1|^2}{\omega - \omega_t} \tag{31}$$

Toroidicity gives a contribution to the curvature vector that is orthogonal to the vertical displacement $\boldsymbol{\xi}$. Therefore, only the "straight tokamak" cylindrical curvature is relevant for the calculation of the scalar product $\boldsymbol{\xi}\cdot\boldsymbol{\kappa}$ appearing in the perturbed Lagrangian, which, after straightforward algebra, can be further approximated as: $\mathcal{L}^{(1)} \approx \epsilon^2\mathcal{E}\left(2-\Lambda\right)\xi\sin\left(\vartheta\right)/rq(r)^2$, where $\Lambda = \mu_\perp B_0/\mathcal{E}$, B_0 is the magnetic field on axis, $\epsilon = r/R_0$, R_0 is tokamak major radius, ξ is the constant amplitude of the vertical displacement, $q(r)$ is the safety factor and (r, ϑ) are standard poloidal coordinates. It follows that

$$\Upsilon^{(1)} = i\frac{\epsilon^2}{q(r)^2}\mathcal{E}(2-\Lambda)\frac{\xi}{r}\left\langle\sin\left(\vartheta\right)\sin\left(\tau\omega_t\right)\right\rangle \tag{32}$$

The thin radial orbit width approximation has been used in Eq. (32). In this limit, the particle orbit does not depart significantly from $r =$ const magnetic surfaces (in the limit of small ellipticity), and the variable P_φ can be replaced by the coordinate r.

A preliminary analytic assessment of δW_2 can be obtained using the distribution function:

$$F(r, \Lambda, v) = \mathcal{C}H(r_h - r)\delta(\Lambda)\exp\left[-\left(\frac{v - v_0}{\delta v_0}\right)^2\right] \tag{33}$$

where $\mathcal{C}$ is a normalization constant, the unit step function $H(r_h - r)$ represents a uniform radial distribution of energetic particles, $\mathcal{E}_0 = mv_0^2/2$ is the typical energy of fast ions and the parameter δ is assumed to be relatively small. When $\Lambda = 0$, the transit frequency reduces to $\omega_t = \sigma v/[R_0 q(r)]$, and $\langle\sin\left(\vartheta\right)\sin\left(\tau\omega_t\right)\rangle = \pi$. For $q = 1$ and $R_0 = 3m$, JET resonance condition $\omega_t \sim \omega_{obs} \approx 2.0\times 10^6 s^{-1}$ involves fast ions with energies $\mathcal{E}_0 \approx 410 keV$.

Let us write $\mathcal{I}m(\delta\hat{W}_{hot}) = \omega_H^2\lambda_{hot}$ and $\omega = \omega_R + i\gamma$. Assuming $\lambda_{hot} \ll 1$, and substituting in the dispersion relation (28), yields $\omega_R \approx \omega_H$ and the growth rate

$$\gamma = \omega_H\lambda_{hot}/2 \tag{34}$$

The sign of λ_{hot} depends only on the ratio v^*/v_0, where $v^* = R_0 q\omega_H$ is the resonant velocity. Assuming a nearly constant $q \approx 1$, it is possible to show that

$$\lambda_{hot} \sim \mathcal{C}\epsilon^2\left(\frac{r_h}{a}\right)^4\frac{\partial}{\partial v}\exp\left[-\left(\frac{v - v_0}{\delta v_0}\right)^2\right]\Bigg|_{v=v^*} \tag{35}$$

For $\delta < 1$, $\mathcal{C} \sim \delta^{-1} n_h/n_c$. Clearly, λ_{hot} vanishes for $v^*/v_0 = 1$. The controlling factor $\exp\{-[(v^* - v_0)/\delta v_0]^2\}$ is of order unity when $(v^*/v_0) - 1 \sim \delta$. For these values of v^*/v_0, assuming $\delta \sim 0.1$, $n_h/n_c \sim 10^{-2}$ and $r_h \sim 0.3m$, and typical parameters of the JET experiments reported in Ref. [15], we estimate $\lambda_{hot} \sim 10^{-2} - 10^{-3}$, corresponding to $n = 0$ modes destabilized by fast ions, growing on a time scale γ^{-1} in the range $0.1 - 1$ ms. This growth rate is comparable with linear growth rates of toroidal Alfven eigenmodes and of $m = n = 1$ fishbone modes destabilized by fast ions.

6. X-point magnetic resonance

Work is in progress to assess the impact of the X-point resonance on vertical displacements. We consider the scenario where the equilibrium plasma density is uniform, but extends all the way to the magnetic separatrix, $\varrho = H(\psi_X - \psi)$. In this scenario, correct treatment of the X-point resonance becomes essential. Analytic progress is possible if the equilibrium current density and magnetic structure are assumed to be the same as those discussed in Secs. 3 and 4. In particular, the equilibrium current density is zero beyond the elliptical boundary. However, perturbed currents are now allowed to flow along the separatrix.

Also for this scenario, the rigid-shift solution for the perturbed stream function, $\tilde{\varphi}$, discussed in Sec. 4, see Eq. (11), is found to be a valid solution of the perturbed ideal-MHD reduced model; indeed, its validity now extends all the way to the separatrix. As a consequence, the perturbed magnetic flux $\tilde{\psi}(\mu, \theta)$ in the region inside the magnetic separatrix also involves a single harmonic of the elliptical angle θ, and it satisfies the ideal-MHD constraint at the X-points, $\tilde{\psi}(\mu_X, \theta_X) = 0$, where $\mu_X = 2\mu_b$, $\theta_X = 0, \pi$ are the X-points coordinates. However, since the separatrix is not a $\mu =$ const surface, several θ harmonics will be coupled on the separatrix and into the vacuum solution.

Orthogonal magnetic flux coordinates (u, v) can be introduced in the region outside the elliptical boundary, where the equilibrium current density is assumed to vanish, and all the way to infinity. Thus, $u = \alpha^{-2}(\psi_{eq} - \psi_X)$, where the parameter α was defined in Sec. 3, while $\partial v/\partial \theta = \partial u/\partial \mu$. A dispersion relation can be obtained following a procedure at all similar to that in Sec. 4, the main difference being that now the perturbed equation of motion involves terms that are zero everywhere except on the magnetic separatrix, where they are proportional to delta functions, $\delta(u)$. Detailed of the algebra will be presented in a separate publication. Here, we summarize the main results:

(i) A perturbed current sheet forms at the separatrix, $\tilde{J}(u, v) = \tilde{j}_X(v)\delta(u)$, with $\tilde{j}_X(v)$ an even function of v, at least in the vicinity of the X-points. Near the X-point at $v = 0$ ($\theta = 0$), one finds $\tilde{j}_X(v) \sim -e_0^{3/2}|v|^{1/2}$. This non-analytic form of $\tilde{j}_X(v)$ for small v is indicative of the fact that several harmonics in the angle v make up the spatial structures of the perturbed flux and current density near the separatrix.

(ii) Since the derivative of $\tilde{j}_X(v)$ with respect to the v coordinate is negative for small v, also γ^2 is negative. Therefore, it is found that the perturbed currents, that are induced along the magnetic separatrix as a consequence of the ideal-MHD X-point resonance, are capable of suppressing the growth of vertical displacements, at least on the ideal-MHD time scale. Therefore, passive feedback stabilization of the fast-growing ideal-MHD instability would not be required in this case. The stabilized $n = 0$ mode is found to oscillate with a frequency scaling with ellipticity as $\omega \sim e_0 \omega_A$.

(iii) A special solution can also be found, such that no current sheet forms along the separatrix. In this case, $\gamma^2 = 0$. Therefore, this special solution corresponds to a new ideal-MHD equilibrium, where the current-carrying plasma column is shifted vertically.

7. Conclusions

This article contains three main results. First of all, we have shown that a rigid-shift displacement is indeed the analytic solution for the normal mode analysis of axisymmetric $n = 0$ modes. This result has been obtained on the basis of the linearized, reduced ideal-MHD model, starting from a relatively simple "straight tokamak" equilibrium. Our result for the growth rate of ideal-MHD unstable vertical displacements, valid for arbitrary values of the ellipticity parameter e_0, is given in Eq. (20). In the limit of small e_0, the growth rate scales as $\gamma \sim e_0^{1/2}\tau_A^{-1}$, where τ_A is the relevant Alfvén time. With feedback stabilization, the $n = 0$ mode becomes oscillatory in nature, with a frequency $\omega \sim e_0^{1/2}\omega_A$, where $\omega_A = \tau_A^{-1}$. This is what we have called in Sec. 4 the "base scenario" for $n = 0$ vertical modes.

Secondly, in Sec. 5, we have investigated the possibility that the feedback-stabilized $n = 0$ modes interact with fast ions. The relevant resonance condition is $\omega \sim \pm\omega_t$, where ω_t is the transit frequency of circulating fast ion orbits. In JET experiments, where finite amplitude $n = 0$ modes were observed [15], the resonance condition can be satisfied by MeV fast ions. Preliminary analysis (Sec. 5) suggests that the resonant mode-fast particle interaction may indeed drive $n = 0$ modes unstable, with a growth time of order 0.1 - 1 ms.

Finally, in Sec. 6, we have considered the realistic scenario where the equilibrium plasma density extends to the X-points of the magnetic divertor separatrix. In this circumstance, special care is needed to treat the X-point resonance correctly, as $n = 0$ perturbed current sheets may form along the separatrix. It is found that these currents are such to completely stabilize ideal-MHD vertical displacements, replacing in a way the effect of passive feedback stabilization associated with wall-image and/or external currents.

Clearly, the fact that perturbed current sheets are likely to form along the magnetic separatrix suggests that plasma resistivity may have a profound impact on the stability of $n = 0$ vertical displacements. Resistive effects are important in a narrow boundary later extending along the magnetic separatrix. The work presented here provides the "outer", ideal-MHD mode structure for asymptotic matching with the boundary layer solution.

References

[1] Clauser C F, Jardin S C and Ferraro N M 2019 *Nucl. Fusion* **59** 126037
[2] Neyatani Y, Yoshino R, Nakamura Y and Sakurai S 1999 *Nucl. Fusion* **39** 559
[3] Knight P J, Castle G G, Morris A W, Caloutsis A and Gimblett C G 2000 *Nucl. Fusion* **40** 325
[4] Riccardo V and JET EFDA Contributors 2009 *Nucl. Fusion* **49** 055012
[5] Laval G, Pellat R and Soule J L 1974 *Phys. Fluids* **17** 835
[6] Perrone M and Wesson J 1981 *Nucl. Fusion* **21** 871
[7] Lazarus E A, Lister J B and Neilson G H 1990 *Nucl. Fusion* **30** 111
[8] Paccagnella R, Strauss H R and Breslau J 2009 *Nucl. Fusion* **49** 035003
[9] Portone A 2017 *Nucl. Fusion* **57** 126060
[10] Krebs I e al 2020 *Phys. Plasmas* **10** 930
[11] Porcelli F and Yolbarsop A 2019 *Phys. Plasmas* **27** 022505
[12] Robinson D C and Wootton A J 1978 *Nucl. Fusion* **18** 11
[13] Strauss H R 1976 *Phys. Fluids* **19** 134
[14] Gajewski R 1972 *The Physics of Fluids* **15** 70–74
[15] Oliver H J C, Sharapov S E, Breizman B N and Zheng L J 2017 *Physics of Plasmas* **24** 122505
[16] Coelho R and Porcelli F 2003 *Physics of Plasmas* **10** 930–933
[17] Porcelli F, Stankiewicz R, Kerner W and Berk H L 1994 *Physics of Plasmas* **1** 470–480