The Pentagonal Numbers and their Link to an Integer Sequence which contains the Primes of Form 6n-1

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A binary operation is a calculus that combines two elements to obtain another elements. It seems quite simple for numbers, because we usually imagine it as a simple sum or product. However, also in the case of numbers, a binary operation can be extremely fascinating if we consider it in a generalized form. Previously, several examples have been proposed of generalized sums for different sets of numbers (Fibonacci, Mersenne, Fermat, q-numbers, repunits and others). These sets can form groupoid which possess different binary operators. Here we consider the pentagonal numbers (OEIS A000326). We will see the binary operation for them. This generalized sum involves another integer sequence, OEIS A016969, and this sequence contains OEIS A007528, that is the primes of the 6n-1.

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In mathematics, a binary operation is a calculation that combines two elements to obtain another element. In particular, this operation acts on a set in a manner that its two domains and its codomain are the same set. Examples of binary operations include the familiar arithmetic operations of addition and multiplication. Let us note that binary operations are the keystone of most algebraic structures: semigroups, monoids, groups, rings, fields, and vector spaces.

The binary operations that we have proposed in some previous calculations for different sets of numbers (Fibonacci, Mersenne, Fermat, q-numbers, repunits and others), are generalizations of the sum, therefore named also as "generalized sums". The approach was inspired by the generalized sums used for entropy [1,2]. The analyses of sequences of integers and q-numbers have been collected in [3].

Let us repeat here just one of these generalized sums, that concerning the Mersenne numbers [4]. These numbers are given by: $M_n = 2^n - 1$. The generalized sum is:

$$M_m \oplus M_n = M_{m+n} = M_m + M_n + M_m M_n$$

In particular:

$$M_n \oplus M_1 = M_{n+1} = M_n + M_1 + M_n M_1$$

Being $M_1 = 1$, we have:

$$M_{n+1} = 2 M_n + 1$$

Let us consider here the pentagonal numbers. A pentagonal number is a figurate number that extends the concept of triangular [5] and square numbers to the pentagon.

A pentagonal number is given by the formula:

$$p_n = \frac{3 n^2 - n}{2}$$

The first few pentagonal numbers are: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, 425, 477, 532, 590, 651, 715, 782, 852, 925, 1001, 1080, 1162, 1247, 1335, 1426, 1520, 1617, 1717, 1820, 1926, 2035, 2147, 2262, 2380, 2501, 2625, 2752, 2882, 3015, 3151, 3290, 3432, 3577, 3725, 3876, 4030, 4187... (sequence A000326 in the OEIS, The On-Line Encyclopedia of Integer Sequences).

Let us determine the generalized sum for the pentagonal numbers.

$$\frac{2}{3}\left(\frac{3}{2} n^2 - \frac{n}{2} - p_n\right) = 0 \quad \text{then:} \quad n^2 - \frac{n}{3} - \frac{2}{3} p_n = 0$$

$$n = \frac{1}{2}\left(\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{8}{3} p_n}\right) = \frac{1}{6} \pm \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3} p_n}$$

We can immediately note that we have to consider just the positive sign in the above

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result.

Let us consider $p_n = 1$, $1 = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3}1} = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{25}{9}} = \frac{1}{6} + \frac{5}{6}$.

The pentagonal number 1 is linked to prime 5.

In the case of $p_2 = 5$, $2 = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3}5} = \frac{1}{6} + \frac{11}{6}$,

then the pentagonal number 5 is linked to prime 11.

Let us do the same for $p_3 = 12$; in this case $3 = \frac{1}{6} + \frac{17}{6}$. The link is to 17.

For $p_4 = 22$, we find the integer 23. For $p_5 = 35$, we have 29, and so on.

It seems that there are some prime numbers linked to the pentagonal numbers. However, before discussing the sequence, let us write the generalized sum.

The generalized sum of the pentagonal numbers is obtained starting from:

$$n+m = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3}p_{m+n}} = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3}p_m} + \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9} + \frac{8}{3}p_n}$$

Let us call $\frac{1}{9} + \frac{8}{3}p_n = A_n$. Then:

$$p_m \oplus p_n = p_{m+n} = p_m + p_n + \frac{1}{12} + \frac{1}{4}\left(\sqrt{A_m} + \sqrt{A_n}\right) + \frac{3}{4}\sqrt{A_m A_n}$$

Moreover: $$p_n \oplus p_1 = p_n + p_1 + \frac{1}{12} + \frac{1}{4}\left(\sqrt{A_n} + \sqrt{A_1}\right) + \frac{3}{4}\sqrt{A_n A_1}$$

Being: $p_1 = \frac{3-1}{2} = 1$, $A_1 = \frac{1}{9} + \frac{8}{3} = \frac{25}{9}$, the recurrence is:

$$p_n \oplus p_1 = p_n + 1 + \frac{1}{12} + \frac{1}{4}\left(\sqrt{A_n} + \frac{5}{3}\right) + \frac{5}{4}\sqrt{A_n}$$

Now, let us consider the numbers $B_n = 9\left(\frac{1}{9} + \frac{8}{3}p_n\right) = 9A_n$.

These numbers are: $B_n = 36n^2 - 12n + 1$.

Let us consider the OEIS sequence A016969, where numbers are $c_n = 6n+5$.

We have:

$$B_n = (c_{n-1})^2$$

Therefore, the pentagonal numbers are linked to the sequence A016969:

$$p_n \oplus p_1 = p_n + 1 + \frac{1}{12} + \frac{1}{4}\left(\sqrt{\frac{B_n}{9}} + \frac{5}{3}\right) + \frac{5}{4}\sqrt{\frac{B_n}{9}} = p_n + 1 + \frac{1}{12} + \frac{1}{4}\left(\frac{c_{n-1}}{3} + \frac{5}{3}\right) + \frac{5}{4}\frac{c_{n-1}}{3}$$

The sequence A016969 is made by: 5, 11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95, 101, 107, 113, 119, 125, 131, 137, 143, 149, 155, 161, 167, 173, 179, 185, 191, 197, 203, 209, 215, 221, 227, 233, 239, 245, 251, 257, 263, 269, 275, 281, 287, 293, 299, 305, 311, 317, 323, 329, 335, ...

In this sequence, we con find prime numbers, those of A007528, with the primes of the form $6n-1$: 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179, 191, 197, 227, 233, 239, 251, 257, 263, 269, 281, 293, 311, 317, 347, 353, 359, 383, 389, 401, 419, 431, 443, 449, 461, 467, 479, 491, 503, 509, 521, 557, 563, 569, 587, ...

In fact: $c_n = 6n+5 = 6(n+1)-1$.

"Intruders" are: 35, 65, 77, 95, 119, 125, 143, 155, 161, 185, 203, 209, 215, 221, 227, 245, 275, 287, 299, 305, 323, 329, 335 ...

In [6], Euler considered also the pentagonal numbers:

$$p_n = \frac{3n^2+n}{2}$$

In OEIS, this is the sequence A005449, of the "second pentagonal numbers".

The first few second pentagonal numbers are: 0, 2, 7, 15, 26, 40, 57, 77, 100, 126, 155, 187, 222, 260, 301, 345, 392, 442, 495, 551, 610, 672, 737, 805, 876, 950, 1027, 1107, 1190, 1276, 1365, 1457, 1552, 1650, 1751, 1855, 1962, 2072, 2185, 2301, 2420, 2542, 2667, 2795, 2926, 3060, 3197, 3337, 3480, ...

Let us determine the generalized sum. From $n^2+\frac{n}{3}-\frac{2}{3}p_n=0$, we have:

$$n = -\frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{9}+\frac{8}{3}p_n}$$

The generalized sum of the pentagonal numbers is obtained as:

$$p_m \oplus p_n = p_{m+n} = p_m + p_n + \frac{1}{12} - \frac{1}{4}\left(\sqrt{A_m}+\sqrt{A_n}\right) + \frac{3}{4}\sqrt{A_m A_n}$$

Moreover:

$$p_n \oplus p_1 = p_n + p_1 + \frac{1}{12} - \frac{1}{4}\left(\sqrt{A_n}+\sqrt{A_1}\right) + \frac{3}{4}\sqrt{A_n A_1}$$

and

$$p_n \oplus p_0 = p_n + p_0 + \frac{1}{12} - \frac{1}{4}\left(\sqrt{A_n}+\sqrt{A_0}\right) + \frac{3}{4}\sqrt{A_n A_0} = p_n$$

p_0 is the neutral element of the generalised sum.

Also for the second pentagonal numbers we have again OEIS sequence A016969.

Of OEIS A000326, OEIS A016969, OEIS A007528, OEIS A016969, see please the detailed discussion and references given in the On-Line Encyclopedia of Integer Sequences

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