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STAR CONFIGURATION POINTS AND GENERIC PLANE CURVES

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Abstract. Let $\ell_1, \ldots \ell_l$ be l lines in $\mathbb{P}^2$ such that no three lines meet in a point. Let $\mathbb{X}(l)$ be the set of points $\{\ell_i \cap \ell_j \mid 1 \leq i < j \leq l\} \subseteq \mathbb{P}^2$. We call $\mathbb{X}(l)$ a star configuration. We describe all pairs (d, l) such that the generic degree d curve in $\mathbb{P}^2$ contains a $\mathbb{X}(l)$.

1. Introduction

The problem of studying subvarieties of algebraic varieties is a crucial one in algebraic geometry, e.g., the case of divisors. The study of subvarieties of hypersurfaces in $\mathbb{P}^n$ has a particularly rich history. For example, one can look for the existence of m dimensional linear spaces on generic hypersurfaces of degree d leading to the theory of Fano varieties and to the well known formula relating n, m and d (e.g., see [10, Theorem 12.8]).

Because linear spaces are complete intersections, it is natural to look for the existence of complete intersection subvarieties on a generic hypersurface. The case of codimension two complete intersections was first studied by by Severi [15], and later generalized and extended to higher codimensions by Noether, Lefschetz [11] and Groethendieck [9]. Recently, in [3], secant vavieties were used to give a complete solution for the existence of complete intersections of codimension r on generic hypersurfaces in $\mathbb{P}^n$ when $2r \leq n+2$. Fewer results are known if the codimension of the complete intersection is large, i.e., the codimension is close to the dimension of the ambient space. In [16] the case of complete intersection curves is studied and completely solved. The case of complete intersection points on generic surfaces in $\mathbb{P}^3$ is considered in [4] where some asymptotic results are presented. The case of complete intersection points in $\mathbb{P}^2$ is a special case of [3], and it is completely solved.

Taking our inspiration from [3, 4], we examine the problem of determining when special configurations of points lie on a generic degree d plane curve in $\mathbb{P}^2$. We shall focus on **star configurations** of points. Consider l lines in the plane $\ell_1, \ldots, \ell_l \subset \mathbb{P}^2$ such that $\ell_i \cap \ell_j \cap \ell_k = \emptyset$. The set of points $\mathbb{X}(l)$ consisting of the $\binom{l}{2}$ pairwise intersections of the lines ℓ_i is called a star configuration. In this paper, we address the following question

Question 1.1. *For what pairs (d, l) does the generic degree d plane curve contain a star configuration $\mathbb{X}(l) \subseteq \mathbb{P}^2$?*

The name star configuration was suggested by A.V. Geramita since $\mathbb{X}(5)$ is the ten points of intersections when drawing a star using five lines (see Figure 6.2.4 in [6]). The configurations $\mathbb{X}(l)$ appeared in the work of Geramita, Migliore, and Sabourin [8] as the

2000 *Mathematics Subject Classification.* 14M05, 14H50.
Key words and phrases. star configurations, generic plane curves.

support of a set of double points whose Hilbert function exhibited an extremal behaviour. More recently, Cooper, Harbourne, and Teitler [6] computed the Hilbert function of any homogeneous set of fatpoints supported on $\mathbb{X}(l)$. Bocci and Harbourne [1] used star configurations (and their generalizations) to compare the symbolic and regular powers of an ideal. Further properties of star configurations continue to be uncovered; e.g., ongoing work of Geramita, Harbourne, Migliore [7].

We can answer Question 1.1 because we can exploit the rich algebraic structure of star configurations. In particular, we will require the fact that one can easily write down a list of generators for $I_{\mathbb{X}(l)}$, the defining ideal of $\mathbb{X}(l)$, as well as the fact that the Hilbert function of $\mathbb{X}(l)$ is the same as the Hilbert function of $\binom{l}{2}$ generic points in $\mathbb{P}^2$. Using these properties, among others, we give the following solution to Question 1.1:

Theorem. 6.3 *Let $l \geq 2$. Then the generic degree d plane curve contains a star configuration $\mathbb{X}(l)$ if and only if*

(i) $l = 2$ *and* $d \geq 1$, *or*
(ii) $l = 3$ *and* $d \geq 2$, *or*
(iii) $l = 4$ *and* $d \geq 3$, *or*
(iv) $l = 5$ *and* $d \geq 5$.

Our proof is broken down into a number of cases. It will follow directly from the generators of $I_{\mathbb{X}(l)}$ that Question 1.1 can have no solution for $d < l - 1$. Using a simple dimension counting argument, Theorem 3.1 shows that for $l \geq 6$, there is no solution to Question 1.1. The cases $l = 2$ and $l = 3$ are trivial cases, so the bulk of the paper will be devoted to the cases that $l = 4$ and $l = 5$.

To prove these cases, we rephrase Question 1.1 into a purely ideal theoretic question (see Lemma 4.3). Precisely, we construct a new ideal I from the generators of $I_{\mathbb{X}(l)}$ and the linear forms defining the lines $\ell_1, \ldots, \ell_l$. We then show that Question 1.1 is equivalent to determining whether $I_d = (\mathbb{C}[x, y, z])_d$. We then answer this new algebraic reformulation.

The proofs for the cases $(d, l) = (3, 4)$ and $(4, 5)$ are, we believe, especially interesting. To prove that Question 1.1 is true for $(3, 4)$, we exploit the natural group structure of the plane cubic curve to find a $\mathbb{X}(4)$ on the curve. To show the non-existence of a solution for $(4, 5)$, we require the classical theory of **Lüroth quartics**. Lüroth quartics are the plane quartics that pass through a $\mathbb{X}(5)$. Lüroth quartics have the property of forming a hypersurface in the space of plane quartics. The existence of this hypersurface is the obstruction for the existence of a solution when $(d, l) = (4, 5)$.

Our paper is structured as follows. In Section 2, we describe the needed algebraic properties of star configurations. In Section 3, we give some asymptotic results. In Section 4, in preparation for the last two sections, we rephrase Question 1.1 into an equivalent algebraic problem. Sections 5 and 6 deal with the cases $l = 4$ and $l = 5$, respectively.

Acknowledgements This paper began when the first author visited the second at Lakehead University. Theorem 6.3 was inspired by computer experiments using CoCoA [5]. The first author was partially supported by the Giovani Ricercatori grant 2008 of the Politecnico di Torino. Both authors acknowledge the financial support of NSERC.

2. Star Configurations

Throughout this paper, we set $S = \mathbb{C}[x, y, z]$ and we denote its dth homogeneous piece with S_d. Moreover, we fix standard monomial bases in each S_d in such a way that $\mathbb{P}S_d$ will be identified with $\mathbb{P}^{N_d}$ where $N_d = \binom{d+2}{2} - 1$. We recall the relevant definitions concerning star configurations of points in $\mathbb{P}^2$.

Let $l \geq 2$ be an integer. The scheme $\mathbb{X}(l) \subset \mathbb{P}^2$ is said to be a **star configuration** if it consists of $\binom{l}{2}$ distinct points which are the pairwise intersections of l distinct lines, say $\ell_1, \ldots, \ell_l$, where no three lines pass through the same point. We will also call $\mathbb{X}(l)$ a **star configuration set of points.**

Note that when $l = 2$, then $\mathbb{X}(2)$ consists of a single point. When $l = 3$, then $\mathbb{X}(3)$ is any set of three points in $\mathbb{P}^2$, provided the three points do not lie on the same line. It is clear that any degree $d \geq 1$ plane curve contains a point. Furthermore, when $d \geq 2$, the generic degree d plane curve will contain three points not lying on a line. These remarks take care of the trivial cases of Theorem 6.3, which we summarize as a lemma:

Lemma 2.1. *The generic degree d plane curve contains a star configuration $\mathbb{X}(l)$ with $l = 2$, respectively $l = 3$, if and only if $d \geq 1$, respectively $d \geq 2$.*

Given a $\mathbb{X}(l)$, for each $i = 1, \ldots, l$, we let L_i denote a linear form in S_1 defining the line ℓ_i. The defining ideal of $I_{\mathbb{X}(l)}$ is then given by

$$I_{\mathbb{X}(l)} = \bigcap_{i \neq j} (L_i, L_j).$$

We will sometimes write a point of $\mathbb{X}(l)$ as $p_{i,j}$, where $p_{i,j}$ is the point defined by the ideal (L_i, L_j), i.e., $p_{i,j}$ is the point of intersection of the lines ℓ_i and ℓ_j. The following lemma describes the generators of $I_{\mathbb{X}(l)}$; we will exploit this fact throughout the paper.

Lemma 2.2. *Let $l \geq 2$, and let $\mathbb{X}(l)$ denote the star configuration constructed from the lines $\ell_1, \ldots, \ell_l$. If L_i is a linear form defining ℓ_i for $i = 1, \ldots, l$, then*

$$I_{\mathbb{X}(l)} = (\hat{L}_1, \ldots, \hat{L}_l) \text{ where } \hat{L}_i = \prod_{j \neq i} L_j.$$

For a proof of this fact, see [2, Claim in Proposition 3.4]. From this description of the generators, we see that $I_{\mathbb{X}(l)}$ is generated in degree $l - 1$. Thus, for $d < l - 1$, there are no plane curves of degree d that contain a $\mathbb{X}(l)$ since $(I_{\mathbb{X}(l)})_d = (0)$. Hence, as a corollary, we get some partial information about Question 1.1.

Corollary 2.3. *If $d < l - 1$, then the generic degree d curve does not contain a $\mathbb{X}(l)$.*

It is useful to recall that star configuration points have the same Hilbert function as generic points. More precisely, by [8, Lemma 7.8], we have

Lemma 2.4. *Let $HF(\mathbb{X}(l), t) = \dim_{\mathbb{C}}(S/I_{\mathbb{X}(l)})_t$ denote the Hilbert function of $S/I_{\mathbb{X}(l)}$. Then*

$$HF(\mathbb{X}(l), t) = \min\left\{ \binom{t+2}{2}, \binom{l}{2} \right\} \text{ for all } t \geq 0.$$

3. An asymptotic result

In [3] it is shown that the generic degree d plane curve contains a 0-dimensional complete intersection scheme of type (a, b) whenever $a, b \le d$ (actually the result holds for *any* degree d plane curve). Thus, arbitrarily large complete intersections can be found on generic plane curves of degree high enough. But the same does not hold for star configurations as shown by Theorem 3.1.

Using the previous description of $I_{\mathbb{X}(l)}$ we introduce a quasi-projective variety parametrizing star configurations. Namely, we consider

$$\mathcal{D}_l \subset \underbrace{\check{\mathbb{P}}^2 \times \cdots \times \check{\mathbb{P}}^2}_{l}$$

such that $(\ell_1, \ldots, \ell_l) \in \mathcal{D}_l$ if and only if no three of the lines ℓ_i are passing through the same point; here, $\check{\mathbb{P}}^2$ denotes the dual projective space.

Theorem 3.1. *If $l > 5$ is an integer, then the generic degree d plane curve does not contain any star configuration $\mathbb{X}(l)$.*

Proof. It is enough to consider the case $d \ge l-1$ (see Corollary 2.3). Let $\mathbb{P}S_d$ be the space parametrizing degree d planes curves and define the following incidence correspondence

$$\Sigma_{d,l} = \{(\mathcal{C}, \mathbb{X}(l)) : \mathcal{C} \supset \mathbb{X}(l)\} \subset \mathbb{P}S_d \times \mathcal{D}_l.$$

We also consider the natural projection maps

$$\psi_{d,l} : \Sigma_{d,l} \longrightarrow \mathcal{D}_l \text{ and } \phi_{d,l} : \Sigma_{d,l} \longrightarrow \mathbb{P}S_d.$$

Clearly we have that $\phi_{d,l}$ is dominant if and only if Question 1.1 has an affirmative answer.

Using a standard fiber dimension argument, we see that

$$\dim \Sigma_{d,l} \le \dim \mathcal{D}_l + \dim \psi_{d,l}^{-1}(\mathbb{X}(l))$$

for a generic star configuration $\mathbb{X}(l)$, where $\dim \mathcal{D}_l = 2l$ and by Lemma 2.4

$$\dim \psi_{d,l}^{-1}(\mathbb{X}(l)) = \dim_{\mathbb{C}} \left(I_{\mathbb{X}(l)}\right)_d - 1 = \binom{d+2}{2} - \binom{l}{2} - 1.$$

Hence the map $\phi_{d,l}$ is dominant only if $\dim \Sigma_{d,l} - \dim \mathbb{P}S_d \ge 0$ and this is equivalent to

$$2l - \binom{l}{2} = \frac{l(5-l)}{2} \ge 0.$$

Thus the result is proved. □

By Lemma 2.1 and the above result, we only need to treat the cases $l = 4$ and 5. We postpone these cases to first rephrase Question 1.1 into an equivalent algebraic question.

4. Restatement of Question 1.1

We derive some technical results, moving our Question 1.1 back and forth between questions in algebra and questions in geometry. First, we notice the following trivial fact:

Lemma 4.1. *Let $\{F=0\}$ be an equation of the degree d curve $\mathcal{C}\subset\mathbb{P}^2$. Then $\mathcal{C}$ contains a star configuration $\mathbb{X}(l)$ only if*

$$F=\sum_{i=1}^{l} M_i\hat{L}_i$$

where the forms M_i have degree $d-l+1$ and the forms $\hat{L}_i$ are defined as $\hat{L}_i=\prod_{j\neq i}L_i$ for some linear forms $L_1,\ldots,L_l$.

Hence, it is natural to perform the following geometric construction. We define a map of affine varieties

$$\Phi_{d,l}:\underbrace{S_1\times\cdots\times S_1}_{l}\times\underbrace{S_{d-l+1}\times\cdots\times S_{d-l+1}}_{l}\longrightarrow S_d$$

such that

$$\Phi_{d,l}\left(L_1,\ldots,L_l,M_1,\ldots,M_l\right)=\sum_{i=1}^{l}M_i\hat{L}_i$$

We then rephrase our problem in terms of the map $\Phi_{d,l}$:

Lemma 4.2. *Let d,l be nonnegative integers with $d\geq l-1$. Then the following are equivalent:*

- *(i) Question 1.1 has an affirmative answer for d and l;*
- *(ii) the map $\Phi_{d,l}$ is a dominant map.*

Proof. Lemma 4.1 proves that (i) implies (ii). To prove the other direction, it is enough to show that for a generic form F, the fiber $\Phi_{d,l}^{-1}(F)$ contains a set of l linear forms defining a star configuration. More precisely, define $\Delta\subset S_1\times\cdots\times S_1\times S_{d-l+1}\times\cdots\times S_{d-l+1}$ as follows:

$$\Delta=\left\{(L_1,\ldots,L_l,M_1,\ldots,M_l)\;\middle|\;\begin{array}{l}\text{there exists } a\neq b\neq c \text{ such that}\\ L_a,L_b,L_c \text{ are linearly dependent}\end{array}\right\}.$$

Then we want to show that $\Phi_{d,l}^{-1}(F)\not\subset\Delta$.

We proceed by contradiction, assuming that the generic fiber of $\Phi_{d,l}$ is contained in Δ. Then Δ would be a component of the domain of $\Phi_{d,l}$. Thus a contradiction as the latter is an irreducible variety being the product of irreducible varieties. □

Using the map $\Phi_{d,l}$ we can now translate Question 1.1 into an ideal theoretic question.

Lemma 4.3. *Let d,l be non-negative integers. Consider generic forms $L_1,\ldots,L_l\in S_1$ and $M_1,\ldots,M_l\in S_{d-l+1}$. Set*

$$\hat{L}_i=\prod_{j\neq i}L_j \text{ and } \hat{L}_{i,j}=\prod_{h\neq i,h\neq j}L_h,\ \text{for } i\neq j.$$

Also set

$$\begin{aligned} Q_1 &= M_2\hat{L}_{1,2} + \cdots + M_l\hat{L}_{1,l} = \sum_{i\neq 1} M_i\hat{L}_{1,i}, \\ &\vdots \\ Q_l &= M_1\hat{L}_{l,1} + \cdots + M_{l-1}\hat{L}_{l,l-1} = \sum_{i\neq l} M_i\hat{L}_{l,i}. \end{aligned}$$

With this notation, form the ideal

$$I = (\hat{L}_1, \ldots, \hat{L}_l, Q_1, \ldots, Q_l) \subset S.$$

Then the following are equivalent:

(*i*) *Question 1.1 has an affirmative answer for d and l;*
(*ii*) $I_d = S_d$.

Proof. Using Lemma 4.2 we just have to show that $\Phi_{d,l}$ is a dominant map. In order to do this we will determine the tangent space to the image of $\Phi_{d,l}$ in a generic point $q = \Phi_{d,l}(p)$, where $p = (L_1, \ldots, L_l, M_1, \ldots, M_l)$ and we denote with T_q this affine tangent space.

The elements of the tangent space T_q are obtained as

$$\frac{d}{dt}\bigg|_{t=0} \Phi_{d,l}\left(L_1 + tL'_1, \ldots, L_l + tL'_l, M_1 + tM'_1, \ldots, M_l + tM'_l\right)$$

when we vary the forms $L'_i \in S_1$ and $M'_i \in S_{d-l+1}$. By a direct computation we see that the elements of T_q have the form

$$\begin{gathered} M'_1\hat{L}_1 + \cdots + M'_l\hat{L}_l + L'_1(M_2\hat{L}_{1,2} + \cdots + M_l\hat{L}_{1,l}) + \cdots + \\ +L'_j(M_1\hat{L}_{1,j} + \cdots + M_l\hat{L}_{j,l}) + \cdots + L'_l(M_1\hat{L}_{1,2} + \cdots + M_{l-1}\hat{L}_{1,l-1}), \end{gathered}$$

where $\hat{L}_i = \prod_{j\neq i} L_i$ and $\hat{L}_{i,j} = \prod_{h\neq i, h\neq j} L_h$, for $i \neq j$.

Since the $L'_i \in S_1$ and $M'_i \in S_{d-l+1}$ can be chosen freely, we obtain that $I_d = T_q$. $\square$

Remark 4.4. Lemma 4.3 is an effective tool to give a positive answer to each special issue of Question 1.1. Given d and l we construct the ideal I as described by choosing forms L_i and M_i. Then we compute $\dim_{\mathbb{C}} I_d$ using a computer algebra system, e.g., CoCoA. If $\dim_{\mathbb{C}} I_d = \dim_{\mathbb{C}} S_d$, by upper semi-continuity of the dimension, we have proved that our question has an affirmative answer for these given d and l. In fact, the dimension can decrease only on a proper closed subset. On the contrary, if $\dim_{\mathbb{C}} I_d < \dim_{\mathbb{C}} S_d$ we do not have any proof. Question 1.1 can both have a negative (for any choice of forms the inequality holds) or an affirmative answer (our choice of forms is too special and another choice will give an equality). But, if we choose our forms generic enough, we do have a strong indication that the answer to Question 1.1 should be negative.

Example 4.5. We use CoCoA [5] to consider the example $(d, l) = (4, 4)$.

```
--we define the ring we want to use
Use S::=QQ[x,y,z];
--we choose our forms
L1:=x;
```

```
L2:=y;
L3:=z;
L4:=x+y+z;
M1:=x+y-z;
M2:=-x+2y+2z;
M3:=2x-y-z;
M4:=x+y+2z;
--we build the forms Qi
Q1:=M2*L3*L4+M3*L2*L4+M4*L2*L3;
Q2:=M1*L3*L4+M3*L1*L4+M4*L1*L3;
Q3:=M1*L3*L4+M2*L1*L4+M4*L1*L2;
Q4:=M1*L2*L3+M2*L1*L3+M3*L1*L2;
--we define the ideal I
I:=Ideal(L2*L3*L4,L1*L3*L4,L1*L2*L4,L1*L2*L3,Q1,Q2,Q3,Q4);
--we compute the Hilbert function of S/I in degree 4
Hilbert(S/I,4);
```

Evaluating the code above we get

```
Hilbert(S/I,4);
0
-------------------------------
```

and this means that $\dim_{\mathbb{C}} S_4 - \dim_{\mathbb{C}} I_4 = 0$. Hence we showed that the generic plane quartic contains a $\mathbb{X}(4)$.

5. The $l = 4$ case

In this section we consider Question 1.1 when $l = 4$. We begin with a special instance of Question 1.1, that is, when $d = 3$, since it has a nice geometric proof which takes advantage of the group structure on the curve.

Lemma 5.1. *Let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth cubic curve. Then there exists a star configuration $\mathbb{X}(4) \subset \mathcal{C}$.*

Proof. We will use the group law on $\mathcal{C}$ and hence we fix a point $p_0 \in \mathcal{C}$ serving as identity. Then we choose a point $p_2 \in \mathcal{C}$ such that $2p_2 = p_0$ and a generic point $p_1 \in \mathcal{C}$. Consider the line joining p_1 and p_2 and let p_3 be the third intersection point with $\mathcal{C}$. Joining p_3 and p_0 and taking the third intersection we get the point $p_1 + p_2$. Now join $p_1 + p_2$ and p_2 and let p_4 be the third intersection point. Then joining p_4 and p_0 we get $p_1 + 2p_2 = p_1$. Hence, the points $p_0, p_1, p_2, p_1 + p_2, p_3, p_4 \in \mathcal{C}$ are a star configuration. □

We now consider the general situation:

Theorem 5.2. *Let $\mathcal{C} \subset \mathbb{P}^2$ be a generic degree $d \geq 3$ plane curve. Then there exists a star configuration $\mathbb{X}(4) \subset \mathcal{C}$.*

Proof. The case $d = 3$ is treated in Lemma 5.1, while for $d > 3$ we will use Lemma 4.3 and the notation introduced therein. For $d = 4$, we produced an explicit example in Example 4.5 where $I_4 = S_4$, and this is enough to conclude by semi-continuity.

To deal with the general case $d \geq 5$, we use the structure of the coordinate ring of a star configuration. Given four generic linear forms L_1, L_2, L_3, L_4 we consider the star configuration they define, say $\mathbb{X} = \{p_{i,j} : 1 \leq i < j \leq 4\}$. The coordinate ring of $\mathbb{X}$ is

$$A = \frac{S}{\left(\hat{L}_1, \ldots, \hat{L}_4\right)}.$$

When $d \geq 2$, $\dim_{\mathbb{C}} A_d = 6$. To prove that $I_d = S_d$, we want to find linear forms $N_1, \ldots, N_6$ such that the six forms N_iQ_i are linearly independent in A_4. To check whether elements in A are linearly independent it is enough to consider their evaluations at the points $p_{i,j}$. Consider the following evaluation matrix

(5.1)

	Q_1	Q_2	Q_3	Q_4
$p_{1,2}$	$M_2L_3L_4$	$M_1L_3L_4$	0	0
$p_{1,3}$	$M_3L_2L_4$	0	$M_1L_2L_4$	0
$p_{1,4}$	$M_4L_2L_3$	0	0	$M_1L_2L_3$
$p_{2,3}$	0	$M_3L_1L_4$	$M_2L_1L_4$	0
$p_{2,4}$	0	$M_4L_1L_3$	0	$M_2L_1L_3$
$p_{3,4}$	0	0	$M_4L_1L_2$	$M_3L_1L_2$

obtained by evaluating each Q_i at the points of the configuration where we denote, by abuse of notation, $M_hL_iL_j(p_{m,n})$ with $M_hL_iL_j$. For example, since $Q_2 = M_1L_3L_4 + M_3L_1L_4 + M_4L_1L_3$, $Q_2(p_{2,3}) = M_3L_1L_4(p_{2,3})$ because L_3 vanishes at $p_{2,3}$, while $Q_2(p_{1,4}) = 0$ since L_4 and L_1 vanish at $p_{1,4}$. Now choose the forms M_i with $\deg M_i = d-3 \geq 2$ so that

$$M_2(p_{2,3}) = M_3(p_{3,4}) = M_4(P_{2,4}) = 0$$

and no other vanishing occurs at the points of the star configuration. Notice that this is possible as $\dim_{\mathbb{C}} A_d = 6$ for $d \geq 2$. The matrix (5.1) can be represented as

(5.2)

	Q_1	Q_2	Q_3	Q_4
$p_{1,2}$	$*$	$*$	0	0
$p_{1,3}$	$*$	0	$*$	0
$p_{1,4}$	$*$	0	0	$*$
$p_{2,3}$	0	$*$	0	0
$p_{2,4}$	0	0	0	$*$
$p_{3,4}$	0	0	$*$	0

where $*$ denotes a non-zero scalar. As this matrix has rank four, then the forms Q_i are linearly independent in A.

To represent N_iQ_i in A it is enough to multiply the j-th element of the i-th column of (5.1) by the evaluation of N_i at $p_{j,i}$. Hence, if we choose the forms N_i such that they do not vanish at any point of $\mathbb{X}$, the evaluation matrix of the N_iQ_i has the same non-zero pattern as (5.2). Thus the forms $N_1Q_1, \ldots, N_4Q_4$ are linearly independent in A.

To complete the proof we need two more forms, and we choose L_1Q_3 and L_1Q_2 whose evaluation matrix at the points $p_{i,j}$ is

(5.3)

	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	$p_{2,3}$	$p_{2,4}$	$p_{3,4}$
L_1Q_3	0	0	0	0	0	$*$
L_1Q_2	0	0	0	$*$	0	0

These rows are linearly independent with the columns of (5.2), completing the proof. □

6. The $l = 5$ case

It remains to consider the case that $l = 5$. The case $l = 5$ and $d = 4$ was classically studied. A quartic containing a star configuration of ten points, that is, a $\mathbb{X}(5)$. For more details, see [12, 13], and for a modern treatment we refer to [14]. In particular, we require the following property of Lüroth quartics:

Theorem (Theorem 11.4 of [14]). *Lüroth quartics form a hypersurface of degree* 54 *in the space of plane quartics.*

Corollary 6.1. *Question 1.1 has a negative answer for* $(d, l) = (4, 5)$*.*

Proof. We notice that the previous theorem is enough to give a negative answer to our question for $l = 5$ and $d = 4$. In fact, a generic plane quartic is not a Lüroth quartic, and hence, no star configuration $\mathbb{X}(5)$ can be found on it. □

For the remaining values $d > 4$, we give an affirmative answer to Question 1.1:

Theorem 6.2. *The generic degree* $d > 4$ *plane curve contains a star configuration* $\mathbb{X}(5)$*.*

Proof. For the case $d = 5$ we produce an explicit example and then we conclude by semi-continuity. For the general case, we produce a proof similar to Theorem 5.2.

For the $d = 5$ case, let our linear forms be given by

$$L_1 = x,\ L_2 = y, L_3 = z,\ L_4 = x + y + z,\ \text{and } L_5 = 2x - 3y + 5z.$$

To construct the polynomials $Q_1, \ldots, Q_5$, we make use of the following forms:

$$M_1 = x+y-z,\ M_2 = -x+2y+2z,\ M_3 = 2x-y-z,\ M_4 = 3x+y-z,\ \text{and } M_5 = 4x-4y+3z.$$

Form the ideal I as in Lemma 4.3. Using CoCoA [5] to compute I_5, we find that $I_5 = S_5$.

For the $d \geq 6$ case, we use the notation of Lemma 4.3, and fix linear forms $L_1, \ldots, L_5$, and use $p_{i,j}$ to denote the ten points of $\mathbb{X}(5)$ defined by the linear forms L_i. We construct the evaluation table

(6.1)

	Q_1	Q_2	Q_3	Q_4	Q_5
$p_{1,2}$	$M_2L_3L_4L_5$	$M_1L_3L_4L_5$	0	0	0
$p_{1,3}$	$M_3L_2L_4L_5$	0	$M_1L_2L_4L_5$	0	0
$p_{1,4}$	$M_4L_2L_3L_5$	0	0	$M_1L_2L_3L_5$	0
$p_{1,5}$	$M_5L_2L_3L_4$	0	0	0	$M_1L_2L_3L_4$
$p_{2,3}$	0	$M_3L_1L_4L_5$	$M_2L_1L_4L_5$	0	0
$p_{2,4}$	0	$M_4L_1L_3L_5$	0	$M_2L_1L_3L_5$	0
$p_{2,5}$	0	$M_5L_1L_3L_4$	0	0	$M_2L_1L_3L_4$
$p_{3,4}$	0	0	$M_4L_1L_2L_5$	$M_3L_1L_2L_5$	0
$p_{3,5}$	0	0	$M_5L_1L_2L_4$	0	$M_3L_1L_2L_4$
$p_{4,5}$	0	0	0	$M_5L_1L_2L_3$	$M_4L_1L_2L_3$

obtained by evaluating each Q_i at the points of the configuration where we denote, by abuse of notation, $M_hL_iL_jL_r(p_{m,n})$ with $M_hL_iL_jL_r$.

We work in the coordinate ring of the star configuration $A = \frac{S}{(\hat{L}_1,\ldots,\hat{L}_5)}$. Now consider

$$L_5Q_1, L_2Q_1, L_1Q_2, L_3Q_2, L_2Q_3, L_4Q_3, L_3Q_4, L_5Q_4, L_4Q_5, L_1Q_5 \tag{6.2}$$

and we want to show that they are linearly independent in A for a generic choice of the forms M_i, $\deg M_i = d-4$. Again, it is enough to show this for a special choice of forms. We choose the forms M_i in such a way that

$$M_1(p_{1,5}) = M_4(p_{3,4}) = M_5(p_{4,5}) = 0,$$

$$M_2(p_{1,2}) = M_2(p_{2,5}) = M_3(p_{1,3}) = M_3(p_{2,3}) = 0,$$

and the following are non-zero

$$M_1(p_{1,2}), M_1(p_{1,3}), M_1(p_{1,4}),\ \ M_2(p_{2,3}), M_2(p_{2,4}),\ \ M_3(p_{3,4}), M_1(p_{3,5}),$$

$$M_4(p_{1,4}), M_4(p_{2,4}), M_4(p_{4,5}),\ \ M_5(p_{1,5}), M_5(p_{2,5}), M_5(p_{3,5}).$$

Notice that these conditions can be satisfied because $\deg M_i \geq 2$ and $\dim_{\mathbb{C}} A_2 = 6, \dim_{\mathbb{C}} A_e = 10$ for $e \geq 3$. Then, evaluating the forms in (6.2) at the points $p_{i,j}$, we obtain the matrix

(6.3)

	L_5Q_1	L_2Q_1	L_1Q_2	L_3Q_2	L_2Q_3	L_4Q_3	L_3Q_4	L_5Q_4	L_4Q_5	L_1Q_5
$p_{1,2}$	0	0	0	$*$	0	0	0	0	0	0
$p_{1,3}$	0	0	0	0	$*$	$*$	0	0	0	0
$p_{1,4}$	$*$	$*$	0	0	0	0	$*$	$*$	0	0
$p_{1,5}$	0	$*$	0	0	0	0	0	0	0	0
$p_{2,3}$	0	0	0	0	0	$*$	0	0	0	0
$p_{2,4}$	0	0	$*$	$*$	0	0	$*$	$*$	0	0
$p_{2,5}$	0	0	$*$	$*$	0	0	0	0	0	0
$p_{3,4}$	0	0	0	0	0	0	0	$*$	0	0
$p_{3,5}$	0	0	0	0	$*$	$*$	0	0	$*$	$*$
$p_{4,5}$	0	0	0	0	0	0	0	0	0	$*$

where $*$ denotes a non-zero scalar. One can verify that (6.3) has rank ten and hence the forms in (6.2) are linearly independent in A and the thesis follows. □

We can now prove our main theorem.

Theorem 6.3. *Let $l \geq 2$. Then the generic degree d plane curve contains a star configuration $\mathbb{X}(l)$ if and only if*

(i) $l = 2$ *and* $d \geq 1$*, or*
(ii) $l = 3$ *and* $d \geq 2$*, or*
(iii) $l = 4$ *and* $d \geq 3$*, or*
(iv) $l = 5$ *and* $d \geq 5$*.*

Proof. Combine Lemmas 2.1 and 5.1, Corollaries 2.3 and 6.1, and Theorems 3.1, 5.2, and 6.2. □

References

[1] C. Bocci, B. Harbourne, Comparing Powers and Symbolic Powers of Ideals. (2007) To appear Journal of Algebraic Geometry. `arXiv:0706.3707`
[2] E. Carlini, Codimension one decompositions and Chow varieties. In *Projective varieties with unexpected properties*, 67–79, Walter de Gruyter GmbH & Co. KG, Berlin, 2005.
[3] E. Carlini, L. Chiantini, A.V. Geramita, Complete intersections on general hypersurfaces. Mich. Math. J. **57** (2008), 121-136.
[4] E. Carlini, L. Chiantini, A.V. Geramita, Complete intersection points on general surfaces in $\mathbb{P}^3$. (2008) To appear Mich. Math. J. `arXiv:0811.2233v1`
[5] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it`
[6] S. Cooper, B. Harbourne, Z. Teitler, Using residuation and collinearity to bound Hilbert functions of fat points in the plane. (2009) Preprint. `arXiv:0912.1915v1`
[7] A.V. Geramtia, B. Harbourne, J. Migliore. Personal Communication.
[8] A.V. Geramita, J. Migliore L. Sabourin. The first infinitesimal neighborhood of a linear configuration of points in $\mathbb{P}^2$. J. Algebra **298** (2006), no. 2, 563-611.
[9] A. Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 4. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.
[10] J. Harris. *Algebraic geometry, A first course.* GTM. Springer-Verlag, New York, 1992.
[11] S. Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties. Trans. Amer. Math. Soc. **22** (1921), no. 3, 327–406.
[12] J. Lüroth, Einige Eigenschaften einer gewissen Gattung von Curven vierter Ordnung. Math. Ann. **1** (1869), no. 1, 37–53.
[13] F. Morley, On the Lüroth Quartic Curve. Amer. J. Math. **41** (1919), no. 4, 279–282.
[14] G. Ottaviani, E. Sernesi, On the hypersurface of Luroth quartics. (2009) To appear Mich. Math. J. `arXiv:0903.5149v2`
[15] F. Severi. Una proprieta' delle forme algebriche prive di punti muiltipli. Rend. Accad. Lincei, II **15** (1906) 691–696.
[16] E. Szabó. Complete intersection subvarieties of general hypersurfaces. Pacific J. Math. **175** (1996), no. 1, 271–294.

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