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# ON THE POINTWISE CONVERGENCE OF THE INTEGRAL KERNELS IN THE FEYNMAN-TROTTER FORMULA

FABIO NICOLA AND S. IVAN TRAPASSO

ABSTRACT. We study path integrals in the Trotter-type form for the Schrödinger equation, where the Hamiltonian is the Weyl quantization of a real-valued quadratic form perturbed by a potential  $V$  in a class encompassing that - considered by Albeverio and Itô in celebrated papers - of Fourier transforms of complex measures. Essentially,  $V$  is bounded and has the regularity of a function whose Fourier transform is in  $L^1$ . Whereas the strong convergence in  $L^2$  in the Trotter formula, as well as several related issues at the operator norm level are well understood, the original Feynman's idea concerned the subtler and widely open problem of the pointwise convergence of the corresponding probability amplitudes, that are the integral kernels of the approximation operators. We prove that, for the above class of potentials, such a convergence at the level of the integral kernels in fact occurs, uniformly on compact subsets and for every fixed time, except for certain exceptional time values for which the kernels are in general just distributions. Actually, theorems are stated for potentials in several function spaces arising in Harmonic Analysis, with corresponding convergence results. Proofs rely on Banach algebras techniques for pseudo-differential operators acting on such function spaces.

## 1. INTRODUCTION AND MAIN RESULTS

The path integral formulation of Quantum Mechanics is by far one of the major achievements in modern theoretical physics. The first intuition on the issue is attributed to Dirac: in his celebrated 1933 paper [11] he provided several clues indicating that the Lagrangian formulation of classical mechanics should have a quantum counterpart. While it is debatable whether the entire story was already known to him at the time of writing, his program has been finalised by Feynman [16], who explicitly recognized Dirac's remarks as the main source of inspiration for his landmark contribution to a new formulation of non-relativistic quantum mechanics beyond the Schrödinger and Heisenberg pictures.

**1.1. The sequential approach to path integrals.** We could argue that the path integral formulation comes from a profound understanding of the double-slit experiment - in fact, this is precisely the way Feynman introduces the problem in the

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book [17]. While this is an intriguing perspective, we briefly outline this approach from a different starting point. Recall that the state of a non-relativistic particle in the Euclidean space  $\mathbb{R}^d$  at time  $t$  is represented by the wave function  $\psi(t, x)$ ,  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ , such that  $\psi(t, \cdot) \in L^2(\mathbb{R}^d)$ . The dynamics under the real-valued potential  $V$  is regulated by the Cauchy problem for the Schrödinger equation<sup>1</sup>:

$$(1) \quad \begin{cases} i\partial_t \psi = (H_0 + V(x))\psi \\ \psi(0, x) = \varphi(x), \end{cases}$$

where  $H_0 = -\Delta/2$  is the free Hamiltonian of non-relativistic quantum mechanics. Provided that suitable conditions on the potential are satisfied, the dynamics in (1) can be equivalently recast by means of the unitary *propagator*  $U(t) = e^{-itH}$ ,  $t \in \mathbb{R}$ :

$$\psi(t, x) = U(t) \varphi(x).$$

At least on a formal level, one can thus represent  $U(t)$  as an integral operator:

$$\psi(t, x) = \int_{\mathbb{R}^d} u_t(x, x_0) \varphi(x_0) dx_0,$$

where the kernel  $u_t(x, x_0)$  intuitively yields the transition amplitude from the position  $x_0$  at time 0 to the position  $x$  at time  $t$ . The path integral formulation exactly concerns the determination of this kernel: according to Feynman's prescription, one should take into account the many possible *interfering alternative paths* from  $x_0$  to  $x$  that the particle could follow. Each path would contribute to the total probability amplitude with a phase factor proportional to the *action functional* corresponding to the path:

$$S[\gamma] = \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau,$$

where  $L$  is the Lagrangian of the corresponding classical system. In short, a merely formal representation of the kernel is

$$(2) \quad u_t(x, x_0) = \int e^{iS[\gamma]} \mathcal{D}\gamma,$$

namely a sort of integral over the infinite-dimensional space of paths satisfying the aforementioned boundary conditions. In order to shed some light on the heuristics underpinning this formula, let us briefly outline the so-called *sequential approach* to path integrals introduced by Nelson [45], which seems the closest to Feynman's original formulation. First, recall that the free propagator  $e^{-itH_0}$  can be properly identified with a Fourier multiplier and the following integral expression holds:

$$e^{-itH_0} \varphi(x) = \frac{1}{(2\pi it)^{d/2}} \int_{\mathbb{R}^d} \exp\left(i \frac{|x - x_0|^2}{2t}\right) \varphi(x_0) dx_0, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

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<sup>1</sup>We set  $m = 1$  for the mass of the particle and  $\hbar = 1$  for the Planck constant.

Next, under suitable assumptions for the potential  $V$ , the Trotter product formula holds for the propagator generated by  $H = H_0 + V$ :

$$e^{-it(H_0+V)} = \lim_{n \rightarrow \infty} \left( e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V} \right)^n,$$

where the limit is intended in the strong topology of operators in  $L^2(\mathbb{R}^d)$ . Combining these two results gives the following representation of the complete propagator  $e^{-itH}$  as limit of integral operators (cf. [49, Thm. X.66]):

$$(3) \quad e^{-it(H_0+V)}\phi(x) = \lim_{n \rightarrow \infty} \left( 2\pi i \frac{t}{n} \right)^{-\frac{nd}{2}} \int_{\mathbb{R}^{nd}} e^{iS_n(t; x_0, \dots, x_{n-1}, x)} \varphi(x_0) dx_0 \dots dx_{n-1},$$

where

$$S_n(t; x_0, \dots, x_{n-1}, x) = \sum_{k=1}^n \frac{t}{n} \left[ \frac{1}{2} \left( \frac{|x_k - x_{k-1}|}{t/n} \right)^2 - V(x_k) \right], \quad x_n = x.$$

In order to grasp the meaning of the phase  $S_n(t; x_0, \dots, x_n)$ , consider the following argument: given the points  $x_0, \dots, x_{n-1}, x \in \mathbb{R}^d$ , let  $\bar{\gamma}$  be the polygonal path through the vertices  $x_k = \bar{\gamma}(kt/n)$ ,  $k = 0, \dots, n$ ,  $x_n = x$ , parametrized as

$$\bar{\gamma}(\tau) = x_k + \frac{x_{k+1} - x_k}{t/n} \left( \tau - k \frac{t}{n} \right), \quad \tau \in \left[ k \frac{t}{n}, (k+1) \frac{t}{n} \right], \quad k = 0, \dots, n-1.$$

Hence  $\bar{\gamma}$  prescribes a classical motion with constant velocity along each segment. The action for this path is thus given by

$$S[\bar{\gamma}] = \sum_{k=1}^n \frac{1}{2} \frac{t}{n} \left( \frac{|x_k - x_{k-1}|}{t/n} \right)^2 - \int_0^t V(\bar{\gamma}(\tau)) d\tau.$$

According to Feynman's interpretation, (3) can be thought as an integral over all polygonal paths and  $S_n(x_0, \dots, x_n, t)$  is a Riemann-like approximation of the action functional evaluated on them. The limit  $n \rightarrow \infty$  is now intuitively clear: the set of polygonal paths becomes the set of all paths and we recover (2). In fact, it should be noted that the custom in Physics community is to employ the suggestive formula (2) as a placeholder for (3) and the related arguments - see for instance [27, 36].

We could not hope to frame the vast literature concerning the problem of putting the formula (2) on firm mathematical ground; the interested reader could benefit from the monographs [4, 21, 44] as points of departure. We only remark that there is in general some relationship between the regularity assumptions on the potential and the strength of the convergence of the time-slicing approximation. While the operator theoretic strategy outlined above also allows to treat wild potentials, the convergence in finer operator topologies (for instance, at the level of integral kernels as in Feynman's original formulation) have been an open problem for a long time. Nevertheless, there is a variety of schemes to deal with path integrals and

pointwise convergence of integral kernels can be achieved by means of other sophisticated techniques, at least for smooth potentials - see the works of Fujiwara, Ichinose, Kumano-go and coauthors [19, 20, 30, 31, 32, 33, 37, 38, 39, 40, 41, 42, 43].

**1.2. Main results.** The present contribution aims at investigating the convergence at the level of integral kernels for the time-slicing approximation of path integrals under low regularity assumptions for the involved potentials. We consider the Schrödinger equation

$$(4) \quad i\partial_t\psi = (H_0 + V(x))\psi,$$

where now  $H_0 = a^w$  is the Weyl quantization of a real quadratic form  $a(x, \xi)$  on  $\mathbb{R}^{2d}$  and  $V \in L^\infty(\mathbb{R}^d)$  is complex-valued (so that a linear magnetic potential or a quadratic electric potential are allowed and included in  $H_0$ ). It is well known that the propagator  $U_0(t) = e^{-itH_0}$  for the unperturbed problem ( $V = 0$ ) is a metaplectic operator [18]. By a slight abuse of language (essentially, up to a sign factor), we can suggestively write  $U_0(t) = \mu(\mathcal{A}_t)$ , where  $t \mapsto \mathcal{A}_t \in \text{Sp}(d, \mathbb{R})$  is the one-parameter subgroup of symplectic matrices associated with the solution of the classical equations of motion with Hamiltonian  $H_0$  in phase space and  $\mu$  is the so-called metaplectic representation - see Section 2.3 for the rigorous construction of  $U_0(t)$ . We express the block structure of  $\mathcal{A}_t$ , namely

$$\mathcal{A}_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix},$$

since our results are global in time unless certain exceptional values, namely for any  $t \in \mathbb{R}$  such that  $\det B_t \neq 0$  (equivalently, for any  $t \in \mathbb{R}$  such that  $\mathcal{A}_t$  is a free symplectic matrix - cf. Section 2.3). Consequently, we also introduce the quadratic form

$$(5) \quad \Phi_t(x, y) = \frac{1}{2}xD_tB_t^{-1}x - yB_t^{-1}x + \frac{1}{2}yB_t^{-1}A_t y, \quad x, y \in \mathbb{R}^d.$$

Recall that (cf. [28])  $H_0$  is a self-adjoint operator on its domain

$$D(H_0) = \{\psi \in L^2(\mathbb{R}^d) : H_0\psi \in L^2(\mathbb{R}^d)\}.$$

Hence,  $V$  being bounded, the Trotter product formula holds:

$$(6) \quad e^{-it(H_0+V)} = \lim_{n \rightarrow \infty} E_n(t), \quad E_n(t) = \left( e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V} \right)^n,$$

where the convergence is again in the strong operator topology in  $L^2(\mathbb{R}^d)$  (see e.g. [12, Cor. 2.7]). We denote by  $e_{n,t}(x, y)$  the distribution kernel of  $E_n(t)$  and by  $u_t(x, y)$  that of  $e^{-it(H_0+V)}$ .

In order to state our first result, we need to introduce two spaces of a marked harmonic analysis flavour, defined in terms of the decay of the Fourier transform.

Let  $M_s^\infty(\mathbb{R}^d)$ ,  $s \in \mathbb{R}$ , denote the subspace of temperate distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that, for some non-zero Schwartz function  $g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\|f\|_{M_s^\infty} = \sup_{x, \xi \in \mathbb{R}^d} |\mathcal{F}[f \cdot g(\cdot - x)](\xi)| (1 + |\xi|)^s < \infty,$$

where  $\mathcal{F}$  is the Fourier transform. In addition, consider the space  $\mathcal{FL}_s^1(\mathbb{R}^d)$  of functions with weighted integrable Fourier transforms, namely:

$$f \in \mathcal{FL}_s^1(\mathbb{R}^d) \iff \|f\|_{\mathcal{FL}_s^1} = \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)| (1 + |\xi|)^s d\xi < \infty.$$

While the space  $\mathcal{FL}_s^1$  is rather standard,  $M_s^\infty$  is a special member of a family of Banach spaces, the so-called *modulation spaces*, stemming from the branch of harmonic analysis currently known as time-frequency analysis (cf. Section 2.2 for the details). Modulation spaces proved to be a fruitful environment for the study of PDEs, in particular the Schrödinger equation (see for instance [6, 7, 8, 54] and the references therein), and related problems such as path integrals [46, 47, 48]. We have a convergence result for potentials in this space.

**Theorem 1.1.** *Consider  $H_0$  as specified above and  $V \in M_s^\infty(\mathbb{R}^d)$ , with  $s > 2d$ . For any  $t \in \mathbb{R}$  such that  $\det B_t \neq 0$ :*

- (1) *the distributions  $e^{-2\pi i \Phi_t} e_{n,t}$ ,  $n \geq 1$ , and  $e^{-2\pi i \Phi_t} u_t$  belong to a bounded subset of  $M_s^\infty(\mathbb{R}^{2d})$ ;*
- (2)  *$e_{n,t} \rightarrow u_t$  in  $(\mathcal{FL}_r^1)_{\text{loc}}(\mathbb{R}^{2d})$  for any  $0 < r < s - 2d$ , hence uniformly on compact subsets.*

We notice that for  $s > 2d$  we have  $M_s^\infty(\mathbb{R}^{2d}) \subset (\mathcal{FL}_r^1)_{\text{loc}}(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ , so that the kernels  $e_{n,t}$  and  $u_t$  in the statement are in fact bounded and continuous functions, provided  $\det B_t \neq 0$ .

Also,  $\cap_{s>0} M_s^\infty(\mathbb{R}^d) = C_b^\infty(\mathbb{R}^d)$  is the space of bounded smooth functions with bounded derivatives of any order, which gives the following result.

**Corollary 1.2.** *Let  $H_0$  be as specified above and  $V \in C_b^\infty(\mathbb{R}^d)$ . For any  $t \in \mathbb{R}$  such that  $\det B_t \neq 0$ :*

- (1) *the distributions  $e^{-2\pi i \Phi_t} e_{n,t}$ ,  $n \geq 1$ , and  $e^{-2\pi i \Phi_t} u_t$  belong to a bounded subset of  $C_b^\infty(\mathbb{R}^{2d})$ ;*
- (2)  *$e_{n,t} \rightarrow u_t$  in  $C^\infty(\mathbb{R}^{2d})$ , hence uniformly on compact subsets together with any derivatives.*

The same conclusion of Corollary 1.2 is actually known to hold true for short times, as a consequence of Fujiwara's result [20], but the above result is global in time. The occurrence of a set of exceptional times is to be expected: in these cases, the integral kernel of the propagator degenerates into a distribution. A basic example

of this behaviour is given by the harmonic oscillator, that is<sup>2</sup>  $H_0 = -\frac{1}{4\pi}\Delta + \pi|x|^2$ ,  $V(x) = 0$ , at  $t = k\pi$ ,  $k \in \mathbb{Z}$ . Notice that such exceptional values are exactly those for which the upper-right block of the associated Hamiltonian flow

$$\mathcal{A}_t = \begin{pmatrix} (\cos t)I & (\sin t)I \\ -(\sin t)I & (\cos t)I \end{pmatrix}$$

is non-invertible.

We now state our main result, which is subtler than Theorem 1.1 and applies to potentials in a lower regularity space known as the Sjöstrand class  $M^{\infty,1}(\mathbb{R}^d)$ : we say that  $f \in \mathcal{S}'(\mathbb{R}^d)$  belongs to  $M^{\infty,1}(\mathbb{R}^d)$  if

$$\|f\|_{M^{\infty,1}} = \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |\mathcal{F}[f \cdot g(\cdot - x)](\xi)| d\xi < \infty,$$

for some non-zero  $g \in \mathcal{S}(\mathbb{R}^d)$ . As a rule of thumb, a function in  $M^{\infty,1}(\mathbb{R}^d)$  is bounded on  $\mathbb{R}^d$  and locally enjoys the mild regularity of the Fourier transform of an  $L^1$  function; in fact

$$(M^{\infty,1})_{\text{loc}}(\mathbb{R}^d) = (\mathcal{FL}^1)_{\text{loc}}(\mathbb{R}^d).$$

Furthermore, we have the following chain of strict inclusions for  $s > d$ :

$$C_b^\infty(\mathbb{R}^d) \subset M_s^\infty(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d) \subset (\mathcal{FL}^1)_{\text{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

Intuitively: we have a scale of low-regularity spaces, the functions in  $M_s^\infty(\mathbb{R}^d)$  becoming less regular as  $s \searrow d$ , until the (fractional) differentiability is completely lost in the “maximal” space  $M^{\infty,1}(\mathbb{R}^d)$ .

It seems worth to highlight that results on the convergence of path integrals are already known for special elements of the Sjöstrand class: for instance, a class of potentials widely investigated by means of different approaches in the papers of Albeverio and coauthors [1, 2, 3] and Itô [35] (see also [34]) is  $\mathcal{FM}(\mathbb{R}^d)$ , namely the space of Fourier transforms of (finite) complex measures on  $\mathbb{R}^d$ . In fact, we have  $\mathcal{FM}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ , cf. Proposition 3.4 below, and the above inclusion is strict; for instance,  $f(x) = \cos|x|$ ,  $x \in \mathbb{R}^d$ , clearly belongs to  $C_b^\infty(\mathbb{R}^d)$ , but it is easy to realize that  $f \notin \mathcal{FM}(\mathbb{R}^d)$  as soon as  $d > 1$ , by the known formula for the fundamental solution of the wave equation [13].

The following result encompasses these potentials and ultimately yields the desired pointwise convergence *at the level of integral kernels* for a wide class of non-smooth potentials.

**Theorem 1.3.** *Let  $H_0$  be as specified above and  $V \in M^{\infty,1}(\mathbb{R}^d)$ . For any  $t \in \mathbb{R}$  such that  $\det B_t \neq 0$ :*

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<sup>2</sup>See Section 2 for the choice of the normalization of the Weyl quantization and the classical flow.

- (1) the distributions  $e^{-2\pi i\Phi_t} e_{n,t}$ ,  $n \geq 1$ , and  $e^{-2\pi i\Phi_t} u_t$  belong to a bounded subset of  $M^{\infty,1}(\mathbb{R}^{2d})$ ;
- (2)  $e_{n,t} \rightarrow u_t$  in  $(\mathcal{FL}^1)_{\text{loc}}(\mathbb{R}^{2d})$ , hence uniformly on compact subsets.

Let us conclude this introduction with a few words on the techniques employed for the proofs. The main idea is to rephrase the problem in terms of pseudodifferential calculus and then to exploit the very rich structure enjoyed by the modulation spaces  $M_s^\infty(\mathbb{R}^{2d})$  (with  $s > 2d$ ) and  $M^{\infty,1}(\mathbb{R}^{2d})$ : in particular, they are Banach algebras for both pointwise multiplication and twisted product of symbols for the Weyl quantization - see the subsequent Section 2.3 for the details.

There is a certain number of questions which seem worthy of further consideration. For example, Theorem 1.1 and Corollary 1.2 should hopefully extend to Hamiltonians  $H_0$  given by the Weyl quantization of a smooth real-valued function with derivatives of order  $\geq 2$  bounded, using techniques from [46]. Also, the potential  $V$  could be replaced by a genuine pseudodifferential operator in suitable classes. We preferred to avoid further technicalities here, since the arguments below are already somewhat involved. Finally we observe that the techniques introduced in the present paper could hopefully be useful to study similar convergence problems of the integral kernels for other approximation formulas arising in semigroup theory; cf. [12].

The paper is organized as follows. Sections 2 and 3 are both devoted to preliminary results and technical lemmas on function spaces and operators involved. In Section 4 we prove Theorem 1.1 and Corollary 1.2. Theorem 1.3 is proved in Section 5.

## 2. PRELIMINARY RESULTS

**2.1. Notation.** We define  $x^2 = x \cdot x$ , for  $x \in \mathbb{R}^d$ , and  $xy = x \cdot y$  is the scalar product on  $\mathbb{R}^d$ . The Schwartz class is denoted by  $\mathcal{S}(\mathbb{R}^d)$ , the space of temperate distributions by  $\mathcal{S}'(\mathbb{R}^d)$ . The brackets  $\langle f, g \rangle$  denote the extension to  $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$  of the inner product  $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx$  on  $L^2(\mathbb{R}^d)$ .

The conjugate exponent  $p'$  of  $p \in [1, \infty]$  is defined by  $1/p + 1/p' = 1$ . The symbol  $\lesssim$  means that the underlying inequality holds up to a positive constant factor  $C > 0$ . For any  $x \in \mathbb{R}^d$  and  $s \in \mathbb{R}$  we set  $\langle x \rangle^s := (1 + |x|^2)^{s/2}$ . We choose the following normalization for the Fourier transform:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}^d.$$

We define the translation and modulation operators: for any  $x, \xi \in \mathbb{R}^d$  and  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$(T_x f)(y) := f(y - x), \quad (M_\xi f)(y) := e^{2\pi i \xi y} f(y).$$

These operators can be extended by duality on temperate distributions. The composition  $\pi(x, \xi) = M_\xi T_x$  constitutes a so-called time-frequency shift.

Denote by  $J$  the canonical symplectic matrix in  $\mathbb{R}^{2d}$ :

$$J = \begin{pmatrix} 0_d & I_d \\ -I_d & 0_d \end{pmatrix} \in \text{Sp}(d, \mathbb{R}),$$

where the symplectic group  $\text{Sp}(d, \mathbb{R})$  is defined as:

$$\text{Sp}(d, \mathbb{R}) = \{M \in \text{GL}(2d, \mathbb{R}) : M^\top J M = J\}$$

and the associated Lie algebra is

$$\mathfrak{sp}(d, \mathbb{R}) := \{M \in \mathbb{R}^{2d \times 2d} : MJ + JM^\top = 0\}.$$

**2.2. Modulation spaces.** The short-time Fourier transform (STFT) of a temperate distribution  $f \in \mathcal{S}'(\mathbb{R}^d)$  with respect to the window function  $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$  is defined by:

$$(7) \quad V_g f(x, \xi) := \langle f, \pi(x, \xi)g \rangle = \mathcal{F}(f \cdot T_x g)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \xi} f(y) \overline{g(y-x)} dy.$$

This is a key instrument for time-frequency analysis; the monograph [23] contains a comprehensive treatment of its mathematical properties, especially those mentioned below. For the sake of conciseness, we only mention that the STFT is deeply connected with other well-known phase-space transforms, in particular the Wigner transform

$$W(f, g)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \xi} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)} dy.$$

For this and other aspects of the connection with phase space analysis, we recommend [10].

Given a non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$ ,  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , the *modulation space*  $M_s^{p,q}(\mathbb{R}^d)$  consists of all temperate distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that  $V_g f \in L_s^{p,q}(\mathbb{R}^{2d})$  (mixed weighted Lebesgue space), that is:

$$\|f\|_{M_s^{p,q}} = \|V_g f\|_{L_s^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p dx \right)^{q/p} \langle \xi \rangle^{qs} d\xi \right)^{1/q} < \infty,$$

with trivial adjustments if  $p$  or  $q$  is  $\infty$ . If  $p = q$ , we write  $M^p$  instead of  $M^{p,p}$ . For the unweighted case, corresponding to  $s = 0$ , we omit the dependence on  $s$ :  $M_0^{p,q} \equiv M^{p,q}$ .

It can be proved that  $M_s^{p,q}(\mathbb{R}^d)$  is a Banach space whose definition does not depend on the choice of the window  $g$ . Just to get acquainted with this family, it is worth to mention that many common function spaces can be equivalently designed as modulation spaces: for instance,

- (i)  $M^2(\mathbb{R}^d)$  coincides with the Hilbert space  $L^2(\mathbb{R}^d)$ ;

- (ii)  $M_s^2(\mathbb{R}^d)$  coincides with the usual  $L^2$ -based Sobolev space  $H^s(\mathbb{R}^d)$ ;
- (iii) the following continuous embeddings with Lebesgue spaces hold:

$$M_r^{p,q}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M_s^{p,q}(\mathbb{R}^d), \quad r > d/q' \text{ and } s < -d/q.$$

In particular,

$$M^{p,1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,\infty}(\mathbb{R}^d).$$

For these and other properties we address the reader to [14, 15, 23].

For a fixed window  $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ , the STFT operator  $V_g$  is clearly bounded from  $M_s^{p,q}(\mathbb{R}^d)$  to  $L_s^{p,q}(\mathbb{R}^{2d})$ . The *adjoint operator* of  $V_g$ , defined by

$$V_g^* F = \int_{\mathbb{R}^{2d}} F(x, \xi) \pi(x, \xi) g \, dx d\xi,$$

continuously maps the Banach space  $L_s^{p,q}(\mathbb{R}^{2d})$  into  $M_s^{p,q}(\mathbb{R}^d)$ , the integral above to be intended in a weak sense.

The *inversion formula for the STFT* can be conveniently expressed as follows: for any  $f \in M_s^{p,q}(\mathbb{R}^d)$ ,

$$(8) \quad f = \frac{1}{\|g\|_{L^2}^2} V_g^* V_g f,$$

again in a weak sense.

The Sjöstrand's class, originally defined in [52], coincides with the choice  $p = \infty$ ,  $q = 1$ ,  $s = 0$ . We have that  $M^{\infty,1}(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and it is a Banach algebra under pointwise product. In fact, precise conditions are known on  $p$ ,  $q$  and  $s$  in order for  $M_s^{p,q}$  to be a Banach algebra with respect to pointwise multiplication:

**Lemma 2.1** ([50, Thm. 3.5]). *Let  $1 \leq p, q \leq \infty$  and  $s \in \mathbb{R}$ . The modulation space  $M_s^{p,q}(\mathbb{R}^d)$  is a Banach algebra for pointwise multiplication if and only if either  $s = 0$  and  $q = 1$  or  $s > d/q'$ .*

Therefore, the Sjöstrand's class  $M^{\infty,1}(\mathbb{R}^d)$  and the modulation spaces  $M_s^\infty(\mathbb{R}^d)$  with  $s > d$  are Banach algebras for pointwise multiplication. It is worth to point out that the condition required in Lemma 2.1 are in fact equivalent to assume  $M_s^{p,q} \hookrightarrow L^\infty$  - cf. [50, Cor. 2.2].

**Remark 2.2.** *We clarify once for all that the preceding results concern the conditions under which the embedding  $M_s^{p,q} \cdot M_s^{p,q} \hookrightarrow M_s^{p,q}$  is continuous, hence there exists a constant  $C > 0$  such that*

$$\|fg\|_{M_s^{p,q}} \leq C \|f\|_{M_s^{p,q}} \|g\|_{M_s^{p,q}}, \quad \forall f, g \in M_s^{p,q}.$$

*Thus, the algebra property holds up to a constant. It is a well known general fact that one can provide an equivalent norm for which the previous estimate holds with  $C = 1$  and the unit element of the algebra has norm equal to 1 (cf. [51, Thm. 10.2]).*

From now on, we assume to work with such equivalent norm whenever concerned with a Banach algebra.

An important subspace of both  $M^{\infty,1}(\mathbb{R}^d)$  and  $M_s^\infty(\mathbb{R}^d)$  is the space

$$C_b^\infty(\mathbb{R}^{2d}) := \{f \in C^\infty(\mathbb{R}^{2d}) : |\partial^\alpha f| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{2d}\} = \bigcap_{s \geq 0} M_s^\infty(\mathbb{R}^d);$$

see e.g. [26, Lem. 6.1] for this characterization.

We briefly mention that the image of modulation spaces under Fourier transform yields another important family of function spaces for the purposes of real harmonic analysis, which are a very special type of *Wiener amalgam spaces*: for any  $1 \leq p, q \leq \infty$ , we set

$$W^{p,q}(\mathbb{R}^d) := \mathcal{F}M^{p,q}(\mathbb{R}^d).$$

One can prove that  $W^{p,q}(\mathbb{R}^d)$  is a Banach space under the same norm of  $M^{p,q}(\mathbb{R}^d)$  but with flipped order of integration with respect to the time and frequency variables:

$$\|f\|_{W^{p,q}} := \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p(\xi) d\xi \right)^{q/p} dx \right)^{1/q},$$

for  $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ , as usual.

**2.3. Weyl operators.** The usual definition of the Weyl transform of the symbol  $\sigma : \mathbb{R}^{2d} \rightarrow \mathbb{C}$  is

$$\sigma^w f(x) := \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

The meaning of this formal integral operator heavily relies on the function space to which the symbol  $\sigma$  belongs. Instead, we adopt the following definition via duality for symbols  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ :

$$(9) \quad \sigma^w : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d), \quad \langle \sigma^w f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

In particular,  $M^{\infty,1}(\mathbb{R}^{2d})$  and  $M_s^\infty(\mathbb{R}^{2d})$  are suitable symbol classes. It is worth to mention that the classical symbol classes investigated within the long tradition of pseudodifferential calculus are usually defined by means of decay/smoothness conditions (see for instance the general Hörmander classes  $S_{\rho,\delta}^m(\mathbb{R}^{2d})$  - [29]), while the fruitful interplay with time-frequency analysis allows to cover very rough symbols too - cf. [24].

**Remark 2.3.** Notice that the multiplication by  $V(x)$  is a special example of Weyl operator with symbol

$$\sigma_V(x, \xi) = V(x) = (V \otimes 1)(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

It is not difficult to prove that the correspondence  $V \mapsto \sigma_V$  is continuous from  $M_s^\infty(\mathbb{R}^d)$  (resp.  $M^{\infty,1}(\mathbb{R}^d)$ ) to  $M_s^\infty(\mathbb{R}^{2d})$  (resp.  $M^{\infty,1}(\mathbb{R}^{2d})$ ). In the rest of the paper

this identification shall be implicitly assumed; by a slight abuse of notation, we will write  $V$  also for  $\sigma_V^w$  for the sake of legibility.

The composition of Weyl transforms provides a bilinear form on symbols, the so-called *twisted product*:

$$\sigma^w \circ \rho^w = (\sigma \# \rho)^w.$$

Although explicit formulas for the twisted product of symbols can be derived (cf. [55]), we will not need them hereafter. Anyway, this is a fundamental notion in order to establish an algebra structure on symbol spaces. It is a distinctive property of  $M^{\infty,1}(\mathbb{R}^{2d})$ , as well as  $M_s^\infty(\mathbb{R}^{2d})$  with  $s > 2d$ , to enjoy a double Banach algebra structure:

- a commutative one with respect to the pointwise multiplication as detailed above;
- a non-commutative one with respect to the twisted product of symbols ([26, 52]).

We wish to underline that the latter algebra structure has been deeply investigated from a time-frequency analysis perspective. Indeed, it is subtly related to a characterizing property satisfied by pseudodifferential operators with symbols in those spaces, namely *almost diagonalization* with respect to time-frequency shifts: we have  $\sigma \in M_s^\infty(\mathbb{R}^{2d})$  if and only if

$$|\langle \sigma^w \pi(z)\varphi, \pi(w)\varphi \rangle| \leq C \langle w - z \rangle^{-s}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}, \quad z, w \in \mathbb{R}^d.$$

Similarly,  $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$  if and only if there exists  $H \in L^1(\mathbb{R}^{2d})$  such that

$$|\langle \sigma^w \pi(z)\varphi, \pi(w)\varphi \rangle| \leq H(w - z), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}, \quad z, w \in \mathbb{R}^d.$$

We address the reader to [5, 6, 9, 25, 26] for further discussions on these aspects.

**Remark 2.4.** *To unambiguously fix the notation: whenever concerned with a product of elements  $a_1, \dots, a_N$  in a Banach algebra  $(A, \star)$ , we write*

$$\prod_{k=1}^N a_k := a_1 \star a_2 \star \dots \star a_N.$$

*This relation is meant to hold even when  $(A, \star)$  is a non-commutative algebra, provided that the symbol on the LHS exactly designates the ordered product on the RHS.*

**2.4. Metaplectic operators.** Given a symplectic matrix  $\mathcal{A} \in \text{Sp}(d, \mathbb{R})$ , we say that the unitary operator  $\mu(\mathcal{A}) \in \mathcal{U}(L^2(\mathbb{R}^d))$  is a *metaplectic operator* associated with  $\mathcal{A}$  if it does satisfy the following intertwining relation:

$$\pi(\mathcal{A}z) = \mu(\mathcal{A})\pi(z)\mu(\mathcal{A})^{-1}, \quad \forall z \in \mathbb{R}^{2d}.$$

Strictly speaking, the previous formula defines a whole set of unitary operators up to a constant phase factor:  $\{c_{\mathcal{A}}\mu(\mathcal{A}) : c_{\mathcal{A}} \in \mathbb{C}, |c_{\mathcal{A}}| = 1\}$ . The phase factor can be adjusted to either  $c_{\mathcal{A}} = 1$  or  $c_{\mathcal{A}} = -1$ , namely:

$$\mu(AB) = \pm\mu(A)\mu(B), \quad \forall A, B \in \mathrm{Sp}(d, \mathbb{R}).$$

That is,  $\mu$  provides a double-valued unitary representation of  $\mathrm{Sp}(d, \mathbb{R})$  or, better, a representation of the double covering  $\mathrm{Mp}(d, \mathbb{R})$  of  $\mathrm{Sp}(d, \mathbb{R})$ ; we will denote by  $\rho^{\mathrm{Mp}} : \mathrm{Mp}(d, \mathbb{R}) \rightarrow \mathrm{Sp}(d, \mathbb{R})$  the projection. We refer to [10, 18] for a comprehensive discussion of these aspects.

We confine ourselves to recall that the metaplectic operator corresponding to special symplectic matrices can be explicitly written as a quadratic Fourier transform. We say that  $\mathcal{A} \in \mathrm{Sp}(d, \mathbb{R})$ , with

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is a *free symplectic matrix* whenever  $\det B \neq 0$ . We have the following integral formula<sup>3</sup> for  $\mu(\mathcal{A})$ .

**Theorem 2.5** ([10, Sec. 7.2.2]). *Let  $\mathcal{A} \in \mathrm{Sp}(d, \mathbb{R})$  be a free symplectic matrix. Then,*

$$(10) \quad \mu(\mathcal{A})f(x) = c|\det B|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_{\mathcal{A}}(x, \xi)} f(y) dy, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where  $c \in \mathbb{C}$  is a suitable complex factor of modulus 1 and  $\Phi_{\mathcal{A}}$  is the quadratic form given by

$$\Phi_{\mathcal{A}}(x, y) = \frac{1}{2}xDB^{-1}x - yB^{-1}x + \frac{1}{2}yB^{-1}Ay.$$

Incidentally, notice that  $\mu(J) = c\mathcal{F}^{-1}$ .

It is important to recall that a truly distinctive property of the Weyl quantization is its *symplectic covariance* [10, Thm. 215], namely: for any  $\mathcal{A} \in \mathrm{Sp}(d, \mathbb{R})$  and  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ , the following relation holds:

$$(11) \quad (\sigma \circ \mathcal{A})^{\mathrm{w}} = \mu(\mathcal{A})^{-1} \sigma^{\mathrm{w}} \mu(\mathcal{A}).$$

Let now  $a$  be a real-valued, time-independent, quadratic homogeneous polynomial on  $\mathbb{R}^{2d}$ , namely:

$$a(x, \xi) = \frac{1}{2}xAx + \xi Bx + \frac{1}{2}\xi C\xi,$$

---

<sup>3</sup>We underline that the following quadratic Fourier transform, up to a sign in  $\Phi_{\mathcal{A}}$ , is actually the point of departure for the construction of the metaplectic representation in [10].

where  $A, C \in \mathbb{R}^{d \times d}$  are symmetric matrices and  $B \in \mathbb{R}^{d \times d}$ . The phase-space flow determined by the Hamilton equations<sup>4</sup>

$$2\pi\dot{z} = J\nabla_z a(z) = \mathcal{A}, \quad \mathcal{A} = \begin{pmatrix} B & C \\ -A & -B^\top \end{pmatrix} \in \mathfrak{sp}(d, \mathbb{R}),$$

defines a mapping  $\mathbb{R} \ni t \mapsto \mathcal{A}_t = e^{(t/2\pi)\mathcal{A}} \in \mathrm{Sp}(d, \mathbb{R})$ . It follows from the general theory of covering manifolds that this path can be lifted in a unique way to a mapping  $\mathbb{R} \ni t \mapsto M_0(t) \in \mathrm{Mp}(d, \mathbb{R})$ ,  $M_0(0) = I$ ; hence  $\rho^{\mathrm{Mp}}(M_0(t)) = \mathcal{A}_t$ . Then, with  $H_0 = a^w$ , the Schrödinger equation

$$\begin{cases} i\partial_t \psi = H_0 \psi \\ \psi(0, x) = \varphi(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \end{cases}$$

is solved by

$$\psi(t, x) = e^{-itH_0} \varphi(x) = \mu(M_0(t)) \varphi(x),$$

see [10, Sec. 15.1.3]. By a slight abuse of language we will write  $\mu(\mathcal{A}_t)$  in place of  $\mu(M_0(t))$ . We recommend [8, 10, 18] for further details on the matter.

**2.5. Operators and kernels.** Consider the space  $\mathcal{L}(X, Y)$  of all continuous linear mappings between two Hausdorff topological vector spaces  $X$  and  $Y$ . It can be endowed with different topologies [53], in which cases we write:

- (1)  $\mathcal{L}_b(X, Y)$ , if equipped with the topology of *bounded convergence*, that is uniform convergence on bounded subsets of  $X$ ;
- (2)  $\mathcal{L}_c(X, Y)$ , if equipped with the topology of *compact convergence*, that is uniform convergence on compact subsets of  $X$ ;
- (3)  $\mathcal{L}_s(X, Y)$ , if equipped with the topology of *pointwise convergence*, that is uniform convergence on finite subsets of  $X$ .

Notice that if  $Y = \mathbb{C}$ ,  $\mathcal{L}_b(X, Y) = X'_b$  (the strong dual of  $X$ ), while  $\mathcal{L}_s(X, Y) = X'_s$  (the weak dual of  $X$ ). We will be mainly concerned with the case  $X = \mathcal{S}(\mathbb{R}^d)$  and  $Y = \mathcal{S}'(\mathbb{R}^d)$ , the latter always endowed with the strong topology unless otherwise specified (i.e.,  $\mathcal{S}'(\mathbb{R}^d) = \mathcal{L}_b(\mathcal{S}(\mathbb{R}^d), \mathbb{C})$ ). The celebrated Schwartz kernel theorem is usually invoked for proving that any reasonably well-behaved operator is indeed an integral transform, though in a distributional sense. In the following we will need this identification but at the topological level [22, 53], that is, a linear map  $A : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$  is continuous if and only if it is generated by a (unique) temperate distribution  $K \in \mathcal{S}'(\mathbb{R}^{2d})$ , namely:

$$(12) \quad \langle Af, g \rangle = \langle K, g \otimes \bar{f} \rangle, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d),$$

---

<sup>4</sup>The factor  $2\pi$  derives from the normalization of the Fourier transform adopted in the paper.

and the correspondence  $K \mapsto A$  above is a topological isomorphism between  $\mathcal{S}'(\mathbb{R}^{2d})$  and the space  $\mathcal{L}_b(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ . As mentioned above,  $\mathcal{S}'(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^{2d})$  are endowed with the strong topology.

**Proposition 2.6.** *Let  $A_n \rightarrow A$  in  $\mathcal{L}_s(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ . Then we have convergence in  $\mathcal{S}'(\mathbb{R}^{2d})$  of the corresponding distribution kernels.*

*Proof.* Since  $\mathcal{S}(\mathbb{R}^d)$  is a Fréchet space and  $A_n$ , being a sequence, defines a filter with countable basis on  $\mathcal{L}_s(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ , from [53, Cor. at pag. 348] we have that  $A_n \rightarrow A$  also in  $\mathcal{L}_c(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ , which is in turn equivalent to convergence in  $\mathcal{L}_b(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$  since  $\mathcal{S}(\mathbb{R}^d)$  is a Montel space - cf. [53, Prop. 34.4 and 34.5]. The desired conclusion then follows from Schwartz' kernel theorem.  $\square$

### 3. TECHNICAL LEMMAS

The following lemma extends [6, Lem. 2.2 and Prop. 5.2].

**Lemma 3.1.** *Let  $X$  denote either  $M_s^\infty(\mathbb{R}^{2d})$ ,  $s \geq 0$ , or  $M^{\infty,1}(\mathbb{R}^{2d})$ .*

(i) *Let  $\sigma \in X$  and  $t \mapsto \mathcal{A}_t \in \text{Sp}(d, \mathbb{R})$  be a continuous mapping defined on the compact interval  $[-T, T] \subset \mathbb{R}$ ,  $T > 0$ . For any  $t \in [-T, T]$ , we have  $\sigma \circ \mathcal{A}_t \in X$ , with*

$$(13) \quad \|\sigma \circ \mathcal{A}_t\|_X \leq C(T) \|\sigma\|_X.$$

(ii) *Let  $\sigma \in X$  and  $A, B, C$  be real  $d \times d$  matrices with  $B$  invertible, and set*

$$\Phi(x, y) = \frac{1}{2}xAx + yBx + \frac{1}{2}yCy.$$

*There exists a unique symbol  $\tilde{\sigma} \in X$  such that, for any  $f \in \mathcal{S}(\mathbb{R}^d)$ :*

$$(14) \quad \sigma^w \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,y)} f(y) dy = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,y)} \tilde{\sigma}(x, y) f(y) dy.$$

*Furthermore, the map  $\sigma \mapsto \tilde{\sigma}$  is bounded on  $X$ .*

*Proof.* The case  $X = M^{\infty,1}(\mathbb{R}^{2d})$  is covered by [6, Lem. 2.2]. We prove here the claim for  $X = M_s^\infty(\mathbb{R}^{2d})$ .

(i) For any non-zero window function  $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$  and  $\mathcal{A} \in \text{Sp}(d, \mathbb{R})$  we have

$$\begin{aligned}
 \|\sigma \circ \mathcal{A}\|_{M_s^\infty} &= \sup_{z, \zeta \in \mathbb{R}^{2d}} |\langle \sigma \circ \mathcal{A}, M_\zeta T_z \Phi \rangle| \langle \zeta \rangle^s \\
 &= \sup_{z, \zeta \in \mathbb{R}^{2d}} \left| \left\langle \sigma, M_{(\mathcal{A}^{-1})^\top \zeta} T_{\mathcal{A}z} (\Phi \circ \mathcal{A}^{-1}) \right\rangle \right| \langle \zeta \rangle^s \\
 &= \sup_{z, \zeta \in \mathbb{R}^{2d}} \left| \left\langle \sigma, M_\zeta T_z (\Phi \circ \mathcal{A}^{-1}) \right\rangle \right| \langle \mathcal{A}^\top \zeta \rangle^s \\
 &\leq \|\mathcal{A}^\top\|^s \|V_{\Phi \circ \mathcal{A}^{-1}} \sigma\|_{M_s^\infty} \\
 &\lesssim \|\mathcal{A}\|^s \|V_{\Phi \circ \mathcal{A}^{-1}} \Phi\|_{L_s^1} \|\sigma\|_{M_s^\infty},
 \end{aligned}$$

where we used the estimate  $\langle \mathcal{A}^\top \zeta \rangle^s \leq \|\mathcal{A}^\top\|^s \langle \zeta \rangle^s$  (here  $\|\mathcal{B}\|$  denotes the operator norm of the matrix  $\mathcal{B}$ ) and the change-of-window Lemma [23, Lem. 11.3.3] ( $\|\cdot\|_{L_s^1}$  denoting the weighted  $L^1$  norm with weight  $\langle \zeta \rangle^s$ ).

We now prove the uniformity with respect to the parameter  $t$ , when  $\mathcal{A} = \mathcal{A}_t$ . The subset  $\{\mathcal{A}_t : t \in [-T, T]\} \subset \text{Sp}(d, \mathbb{R})$  is bounded and thus  $\|\mathcal{A}_t\| \leq C_1(T)$ . Furthermore,  $\{V_{\Phi \circ \mathcal{A}_t^{-1}} \Phi : t \in [-T, T]\}$  is a bounded subset of  $\mathcal{S}(\mathbb{R}^{2d})$  (this follows at once by inspecting the Schwartz seminorms of  $\Phi \circ \mathcal{A}_t^{-1}$ ), hence  $\|V_{\Phi \circ \mathcal{A}_t^{-1}} \Phi\|_{L_s^1} \leq C_2(T)$ .

(ii) The proof is similar to that of the case  $X = M^{\infty,1}(\mathbb{R}^{2d})$  in [6, Prop. 5.2]. In particular,  $\tilde{\sigma}$  is explicitly derived from  $\sigma$  as follows:  $\tilde{\sigma} = \mathcal{U}_2 \mathcal{U} \mathcal{U}_1 \sigma$ , where  $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2$  are the mappings

$$\mathcal{U}_1 \sigma(x, y) = \sigma(x, y + Ax), \quad \mathcal{U}_2 \sigma(x, y) = \sigma(x, B^\top y), \quad \widehat{\mathcal{U}} \sigma(\xi, \eta) = e^{\pi i \xi \eta} \widehat{\sigma}(\xi, \eta).$$

$\mathcal{U}_1$  and  $\mathcal{U}_2$  are isomorphisms of  $M_s^\infty(\mathbb{R}^{2d})$ , as a consequence of the previous item. For what concerns  $\mathcal{U}$ , an inspection of the proof of [23, Cor. 14.5.5] shows that any modulation space  $M_s^{p,q}(\mathbb{R}^{2d})$  is invariant under the action of  $\mathcal{U}$ .  $\square$

We will also make use of the following easy result.

**Lemma 3.2.** *Let  $A$  be a Banach algebra of complex-valued functions on  $\mathbb{R}^d$  with respect to pointwise multiplication and assume  $u \in A$ . For any real  $t$  and integer  $n \geq 1$  we have*

$$e^{-i \frac{t}{n} u} = 1 + i \frac{t}{n} u_0,$$

where  $u_0 \in A$  and the following estimate holds:

$$\|u_0\| \leq \|u\| e^{t\|u\|}.$$

*Proof.* We have

$$e^{-i \frac{t}{n} u} = \sum_{k=0}^{\infty} \left( -i \frac{t}{n} \right)^k \frac{u^k}{k!} = 1 + i \frac{t}{n} u_0$$

with

$$u_0 = -u \sum_{k=0}^{\infty} \left(-i \frac{t}{n}\right)^k \frac{u^k}{(k+1)!}.$$

We can estimate the norm of  $u_0$  as follows:

$$\begin{aligned} \|u_0\| &\leq \|u\| \left( \sum_{k=0}^{\infty} \frac{|t|^k \|u\|^k}{(k+1)!} \right) \\ &= \frac{1}{|t|} (e^{|t|\|u\|} - 1) \leq \|u\| e^{|t|\|u\|}. \end{aligned}$$

□

Thanks to the subsequent result, we are able to treat Theorem 1.3 as a perturbation of Theorem 1.1.

**Lemma 3.3.** *For any  $\epsilon > 0$  and  $f \in M^{\infty,1}(\mathbb{R}^d)$ , there exist  $f_1 \in C_b^\infty(\mathbb{R}^d)$  and  $f_2 \in M^{\infty,1}(\mathbb{R}^d)$  such that*

$$f = f_1 + f_2, \quad \|f_2\|_{M^{\infty,1}} \leq \epsilon.$$

*Proof.* Fix  $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$  with  $\|g\|_{L^2} = 1$ , and set

$$(15) \quad f_1(y) = V_g^* (V_g f \cdot 1_{A_R})(y) = \int_{A_R} V_g f(x, \xi) e^{2\pi i y \xi} g(y-x) dx d\xi,$$

in the sense of distributions, where  $1_{A_R}$  denotes the characteristic function of the set  $A_R = \{(x, \xi) \in \mathbb{R}^{2d} : |\xi| \leq R\}$ , and  $R > 0$  will be chosen later, depending on  $\epsilon$ .

The integral in (15) actually converges for every  $y$  and defines a bounded function. Indeed, setting

$$S(\xi) = \sup_{x \in \mathbb{R}^d} |V_g f(x, \xi)|$$

we have  $S \in L^1(\mathbb{R}^d)$  by the assumption  $f \in M^{\infty,1}(\mathbb{R}^d)$ , and for any  $y \in \mathbb{R}^d$ ,

$$\begin{aligned} |f_1(y)| &= \left| \int_{A_R} V_g f(x, \xi) e^{2\pi i y \xi} g(y-x) dx d\xi \right| \\ &\leq \int_{A_R} |V_g f(x, \xi)| |g(y-x)| dx d\xi \\ &\leq \left( \int_{\mathbb{R}^d} |g(y-x)| dx \right) \left( \int_{|\xi| \leq R} S(\xi) d\xi \right) \leq \|g\|_{L^1} \|S\|_{L^1}. \end{aligned}$$

Similarly one shows that all the derivatives  $\partial^\alpha f_1$  are bounded, using that  $\xi^\alpha S(\xi)$  is integrable on  $|\xi| \leq R$ . Differentiation under the integral sign is permitted because for  $y$  in a neighbourhood of any fixed  $y_0 \in \mathbb{R}^d$  and every  $N$ ,

$$|V_g f(x, \xi) \partial_y^\alpha [e^{2\pi i y \xi} g(y-x)]| \leq C_N (1 + |\xi|)^{|\alpha|} S(\xi) (1 + |y_0 - x|)^{-N},$$

which is integrable in  $A_R$ . Hence  $f_1 \in C_b^\infty(\mathbb{R}^d)$ .

Now, let

$$f_2 = f - f_1 = V_g^* (V_g f \cdot 1_{A_R^c})$$

where in the second equality we used the inversion formula for the STFT (8). The continuity of  $V_g^* : L^{\infty,1}(\mathbb{R}^{2d}) \rightarrow M^{\infty,1}(\mathbb{R}^d)$  yields

$$\begin{aligned} \|f_2\|_{M^{\infty,1}} &= \|V_g^* (V_g f \cdot 1_{A_R^c})\|_{M^{\infty,1}} \\ &\lesssim \|V_g f \cdot 1_{A_R^c}\|_{L^{\infty,1}} \\ &= \int_{|\xi|>R} S(\xi) d\xi \leq \epsilon \end{aligned}$$

provided that  $R = R_\epsilon$  is large enough.  $\square$

As already claimed in the Introduction, we prove that the Sjöstrand class includes the Fourier transforms of (finite) complex measures.

**Proposition 3.4.** *Let  $\mathcal{M}(\mathbb{R}^d)$  denote the space of complex Radon measures on  $\mathbb{R}^d$ . The image of  $\mathcal{M}(\mathbb{R}^d)$  under the Fourier transform is contained in  $M^{\infty,1}(\mathbb{R}^d)$ , that is:  $\mathcal{FM}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ .*

*Proof.* We regard  $\mathcal{M}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ . Therefore, for any non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$  we can explicitly write the STFT of  $\mu$ :

$$V_g \mu(x, \xi) = \langle \mu, M_\xi T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} \overline{g(y-x)} d\mu(y).$$

In view of the relation between the Wiener amalgam space  $W^{p,q}(\mathbb{R}^d)$  and  $M^{p,q}(\mathbb{R}^d)$ , the claimed result is equivalent to prove that  $\mathcal{M}(\mathbb{R}^d) \subset W^{\infty,1}(\mathbb{R}^d)$ . Indeed,

$$\begin{aligned} \|\mu\|_{W^{\infty,1}} &= \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} |V_g \mu(x, \xi)| dx \\ &\leq \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} |e^{-2\pi i x \xi} \overline{g(y-x)}| d|\mu|(y) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(y-x)| d|\mu|(y) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(y-x)| dx d|\mu|(y) \\ &= \|g\|_{L^1} |\mu|(\mathbb{R}^d). \end{aligned}$$

$\square$

## 4. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

4.1. **Proof of Theorem 1.1.** Recall that  $H_0 = a^w$  is the Weyl quantization of the real quadratic form  $a(x, \xi)$  on  $\mathbb{R}^{2d}$  and we are assuming  $V \in M_s^\infty(\mathbb{R}^d)$ , with  $s > 2d$  (the multiplication by  $V$  coincides with  $\sigma_V^w$ , as discussed in Remark 2.3). The proof will be carried on for  $t > 0$ , since the case  $t < 0$  is similar. Actually, the upper-right block of the matrix  $\mathcal{A}_{-t} = \mathcal{A}_t^{-1}$  is  $-B_t^\top$  (cf. [10, Eq. (2.6)]), hence  $\det B_t \neq 0$  if and only if  $\det B_{-t} \neq 0$ .

Having in mind the framework outlined in the introductory Section 1.2, we start from Trotter formula (6). We employ Lemma 3.2 and the notation  $e^{-itH_0} = \mu(\mathcal{A}_t)$  from Section 2.4 in order to write

$$E_n(t) = \left( e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V} \right)^n = \left( \mu(\mathcal{A}_{t/n}) \left( 1 + i\frac{t}{n}V_0 \right) \right)^n$$

for a suitable  $V_0 = V_{0,n,t}$ . According to Remark 2.3, we identify  $1 + i\frac{t}{n}V_0$  with the Weyl operator with symbol  $1 + i\frac{t}{n}\sigma_{V_0}$ , where  $\sigma_{V_0} = V_0 \otimes 1$ . By the assumption  $V \in M_s^\infty(\mathbb{R}^d)$ , Remark 2.3 and Lemma 3.2 we have

$$(16) \quad \|\sigma_{V_0}\|_{M_s^\infty} \leq C(t)$$

for some constant  $C(t) > 0$  independent of  $n$ . By applying (11) repeatedly, the ordered product of operators in  $E_n(t)$  can be expanded as

$$E_n(t) = \left[ \prod_{k=1}^n \left( I + i\frac{t}{n} \left( \sigma_{V_0} \circ \mathcal{A}_{-k\frac{t}{n}} \right)^w \right) \right] \mu(\mathcal{A}_{t/n})^n = a_{n,t}^w \mu(\mathcal{A}_t),$$

where, for any  $t$  and  $n \geq 1$ :

$$\begin{aligned} \|a_{n,t}\|_{M_s^\infty} &= \left\| \prod_{k=1}^n \left( 1 + i\frac{t}{n} \left( \sigma_{V_0} \circ \mathcal{A}_{-k\frac{t}{n}} \right) \right) \right\|_{M_s^\infty} \\ &\leq \prod_{k=1}^n \left( 1 + \frac{t}{n} \|\sigma_{V_0} \circ \mathcal{A}_{-k\frac{t}{n}}\|_{M_s^\infty} \right), \end{aligned}$$

where in the first product symbol we mean the twisted product  $\#$  of symbols - cf. Section 2.3.

By Lemma 3.1 applied with  $T = t$  and (16), we then have

$$(17) \quad \|a_{n,t}\|_{M_s^\infty} \leq \left( 1 + \frac{t}{n} C(t) \right)^n \leq e^{C(t)t},$$

for some new locally bounded constant  $C(t) > 0$  independent of  $n$ .

Since  $\mathcal{A}_t$  is a free symplectic matrix by assumption, by (10) and (14) we explicitly have

$$\begin{aligned} E_n(t) \psi(x) &= a_{n,t}^w \mu(\mathcal{A}_t) \psi(x) \\ &= c(t) |\det B_t|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)} \widetilde{a}_{n,t}(x,y) \psi(y) dy, \end{aligned}$$

where  $\Phi_t$  is given in (5) and  $c(t)$  is a suitable complex factor of modulus 1.

Therefore, we managed to write  $E_n(t)$  as an integral operator with kernel

$$e_{n,t}(x,y) = c(t) |\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} \widetilde{a}_{n,t}(x,y),$$

Now, consider the integral kernel  $u_t$  of the propagator  $U(t) = e^{-it(H_0+V)}$  and define for consistency  $\widetilde{a}_t \in \mathcal{S}'(\mathbb{R}^{2d})$  such that

$$u_t(x,y) = c(t) |\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} \widetilde{a}_t(x,y).$$

Since we know by the usual Trotter formula (6) that for any fixed  $t$

$$\|E_n(t)\psi - U(t)\psi\|_{L^2} \rightarrow 0, \quad \forall \psi \in L^2(\mathbb{R}^d),$$

we have  $E_n(t) \rightarrow U(t)$  in  $\mathcal{L}_s(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ , because  $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ . As a consequence of Proposition 2.6, we get  $e_{n,t} \rightarrow u_t$  in  $\mathcal{S}'(\mathbb{R}^d)$ . This is equivalent to

$$(18) \quad \widetilde{a}_{n,t} \rightarrow \widetilde{a}_t \quad \text{in } \mathcal{S}'(\mathbb{R}^{2d}).$$

Therefore, for any non-zero  $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$  we have pointwise convergence of the corresponding short-time Fourier transforms: for any fixed  $(z, \zeta) \in \mathbb{R}^{4d}$ ,

$$(19) \quad V_\Psi \widetilde{a}_{n,t}(z, \zeta) = \langle \widetilde{a}_{n,t}, M_\zeta T_z \Psi \rangle \rightarrow \langle \widetilde{a}_t, M_\zeta T_z \Psi \rangle = V_\Psi \widetilde{a}_t(z, \zeta).$$

By (17) and Lemma 3.1 we see that the sequence  $\widetilde{a}_{n,t}$ , for any fixed  $t$ , is bounded in  $M_s^\infty(\mathbb{R}^{2d})$ . Hence, there exists a constant  $C = C(t)$  independent of  $n$  such that

$$(20) \quad |V_\Psi \widetilde{a}_{n,t}(z, \zeta)| \leq C \langle \zeta \rangle^{-s}, \quad \forall z, \zeta \in \mathbb{R}^{2d}.$$

Combining this estimate with (19) immediately yields  $\widetilde{a}_t \in M_s^\infty(\mathbb{R}^{2d})$  as well, hence the first claim of Theorem 1.1.

For the remaining part, we argue as follows: choose a non-zero window  $\Psi \in C_c^\infty(\mathbb{R}^{2d})$  and set  $\Theta \in C_c^\infty(\mathbb{R}^{2d})$  with  $\Theta = 1$  on  $\text{supp} \Psi$ ; for any fixed  $z \in \mathbb{R}^{2d}$  and  $0 < r < s - 2d$ , we have

$$\begin{aligned} \|\mathcal{F}[(e_{n,t} - u_t) \overline{T_z \Psi}]\|_{L_r^1} &= |\det B_t|^{-1/2} \|\mathcal{F}[e^{2\pi i \Phi_t} (\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]\|_{L_r^1} \\ &= |\det B_t|^{-1/2} \|\mathcal{F}[(T_z \Theta e^{2\pi i \Phi_t}) (\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]\|_{L_r^1} \\ &= |\det B_t|^{-1/2} \|\mathcal{F}[T_z \Theta e^{2\pi i \Phi_t}] * \mathcal{F}[(\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]\|_{L_r^1} \\ &\lesssim |\det B_t|^{-1/2} \|\mathcal{F}[T_z \Theta e^{2\pi i \Phi_t}]\|_{L_r^1} \|\mathcal{F}[(\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]\|_{L_r^1}, \end{aligned}$$

where the convolution inequality in the last step is an easy consequence of *Peetre's inequality*<sup>5</sup>.

Clearly,  $T_z \Theta e^{2\pi i \Psi_t} \in C_c^\infty(\mathbb{R}^{2d})$ , while

$$\|\mathcal{F}[(\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]\|_{L_r^1} \rightarrow 0$$

by dominated convergence, using (19) and

$$\begin{aligned} |\mathcal{F}[(\widetilde{a}_{n,t} - \widetilde{a}_t) \overline{T_z \Psi}]| \langle \zeta \rangle^r &= |V_\Psi(\widetilde{a}_{n,t} - \widetilde{a}_t)(z, \zeta)| \langle \zeta \rangle^r \\ &\leq C \langle \zeta \rangle^{r-s} \in L^1(\mathbb{R}^{2d}), \end{aligned}$$

because  $s - r > 2d$ , where in the last inequality we used (20) and the fact that  $\widetilde{a}_t \in M_s^\infty(\mathbb{R}^{2d})$ .

This gives the claimed convergence in  $(\mathcal{F}L_r^1)_{\text{loc}}(\mathbb{R}^{2d})$ .

To conclude, we have that

$$\|(e_{n,t} - u_t) \overline{T_z \Psi}\|_{L^\infty} \leq \|V_\Psi(e_{n,t} - u_t)(z, \cdot)\|_{L^1} \rightarrow 0,$$

and in particular this yields uniform convergence on compact subsets: for any compact  $K \subset \mathbb{R}^{2d}$ , choose  $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ ,  $\Psi = 1$  on  $K$ .

**4.2. Proof of Corollary 1.2.** The proof of Corollary 1.2 is then immediate, since  $C_b^\infty(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} M_s^\infty(\mathbb{R}^{2d})$  and  $C^\infty(\mathbb{R}^{2d}) = \bigcap_{r > 0} (\mathcal{F}L_r^1)_{\text{loc}}(\mathbb{R}^{2d})$ ; we leave the easy proof of the latter equality to the interested reader.

## 5. PROOF OF THEOREM 1.3

We now assume  $V \in M^{\infty,1}(\mathbb{R}^d)$ . Therefore for an arbitrary  $\epsilon > 0$ , Lemma 3.3 allows us to write  $V = V_1 + V_2$ , with  $V_1 \in C_b^\infty(\mathbb{R}^d)$  and  $V_2 \in M^{\infty,1}(\mathbb{R}^d)$  with  $\|V_2\|_{M^{\infty,1}} \leq \epsilon$  and clearly

$$\|V_1\|_{M^{\infty,1}} \leq \|V\|_{M^{\infty,1}} + \|V_2\|_{M^{\infty,1}} \leq \|V\|_{M^{\infty,1}} + \epsilon \leq 1 + \|V\|_{M^{\infty,1}},$$

assuming, from now on,  $\epsilon \leq 1$ .

Notice that

$$\begin{aligned} e^{-i\frac{t}{n}(V_1+V_2)} &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(-i\frac{t}{n}\right)^k (V_1 + V_2)^k \\ &= 1 + i\frac{t}{n}V_1' + i\frac{t}{n}V_2', \end{aligned}$$

where we set

$$V_1' = -V_1 \sum_{k=1}^{\infty} \frac{1}{k!} \left(-i\frac{t}{n}\right)^{k-1} V_1^{k-1},$$

---

<sup>5</sup>Namely,  $\langle x - y \rangle^r \leq C_r \langle x \rangle^r \langle y \rangle^r$ , for  $x, y \in \mathbb{R}^d$ ,  $r \geq 0$ .

$$V_2' = - \sum_{k=1}^{\infty} \frac{1}{k!} \left( -i \frac{t}{n} \right)^{k-1} ((V_1 + V_2)^k - V_1^k).$$

Now, fix once for all any  $s > 2d$ . The norms of  $V_1' = V_{1,n,t}'$  and  $V_2' = V_{2,n,t}'$  can be estimated as follows for any  $t > 0$  (cf. the proof of Lemma 3.2). We have

$$(21) \quad \|V_1'\|_{M^{\infty,1}} \leq \|V_1\|_{M^{\infty,1}} e^{t\|V_1\|_{M^{\infty,1}}} \leq (1 + \|V\|_{M^{\infty,1}}) e^{t(1+\|V\|_{M^{\infty,1}})} =: C_1(t),$$

$$(22) \quad \|V_1'\|_{M_s^{\infty}} \leq \|V_1\|_{M_s^{\infty}} e^{t\|V_1\|_{M_s^{\infty}}} =: C_2(t, \epsilon).$$

Similarly, using the elementary inequality

$$(a + b)^k - a^k \leq kb(a + b)^{k-1}, \quad a, b \geq 0, \quad k \geq 1,$$

we obtain

$$(23) \quad \|V_2'\|_{M^{\infty,1}} \leq \|V_2\|_{M^{\infty,1}} e^{t(\|V_1\|_{M^{\infty,1}} + \|V_2\|_{M^{\infty,1}})} \leq \epsilon e^{t(2+\|V\|_{M^{\infty,1}})} =: \epsilon C_3(t).$$

Here  $C_1(t)$  and  $C_3(t)$  are independent of  $n$  and  $\epsilon$  and  $C_2(t, \epsilon)$  is independent of  $n$ . The approximate propagator  $E_n(t)$  thus becomes

$$\begin{aligned} E_n(t) &= \left( e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}(V_1+V_2)} \right)^n \\ &= \left( \mu(\mathcal{A}_{t/n}) \left( 1 + i\frac{t}{n}V_1' + i\frac{t}{n}V_2' \right) \right)^n, \end{aligned}$$

and similar arguments to those of the previous section yield

$$(24) \quad E_n(t) = \left[ \prod_{k=1}^n \left( I + i\frac{t}{n} \left( \sigma_{V_1'} \circ \mathcal{A}_{-k\frac{t}{n}} \right)^w + i\frac{t}{n} \left( \sigma_{V_2'} \circ \mathcal{A}_{-k\frac{t}{n}} \right)^w \right) \right] (\mu(\mathcal{A}_{t/n}))^n \\ =: [a_{n,t}^w + b_{n,t}^w] \mu(\mathcal{A}_t),$$

where we set

$$a_{n,t} = \prod_{k=1}^n \left( 1 + i\frac{t}{n} \left( \sigma_{V_1'} \circ \mathcal{A}_{-k\frac{t}{n}} \right) \right),$$

and in the latter product we mean the twisted product  $\#$  of symbols.

The term  $a_{n,t}^w$  can be estimated as in the proof of Theorem 1.1; in particular, using (22), we get

$$(25) \quad \|a_{n,t}\|_{M_s^{\infty}} \leq C(t, \epsilon)$$

cf. (17).

In order to estimate the  $M^{\infty,1}$  norm of the remainder  $b_{n,t}$ , it is useful the following result, which can be easily proved by induction on  $n$ .

**Lemma 5.1.** *Let  $A$  be a Banach algebra. For any  $u_1, \dots, u_n, v_1, \dots, v_n \in A$ , with  $\|u_i\| \leq R$  and  $\|v_i\| \leq S$  for any  $i = 1, \dots, n$  and some  $R, S > 0$ , and setting  $w_k = u_k + v_k$ , we have*

$$\prod_{k=1}^n (u_k + v_k) = u_1 u_2 \dots u_n + z_n,$$

where

$$z_n = v_1 w_2 \dots w_n + u_1 v_2 w_3 \dots w_n + \dots + u_1 u_2 \dots u_{n-2} v_{n-1} w_n + u_1 u_2 \dots u_{n-1} v_n,$$

and therefore

$$\|z_n\| \leq nS(R+S)^{n-1}.$$

Setting

$$u_k = 1 + i \frac{t}{n} \left( \sigma_{V_1'} \circ \mathcal{A}_{-k \frac{t}{n}} \right), \quad v_k = i \frac{t}{n} \left( \sigma_{V_2'} \circ \mathcal{A}_{-k \frac{t}{n}} \right), \quad k = 1, \dots, n,$$

and applying Lemma 3.1 with  $T = t$ , and (21) and (23), we get

$$\begin{aligned} \|u_k\|_{M^{\infty,1}} &= \left\| 1 + i \frac{t}{n} \left( \sigma_{V_1'} \circ \mathcal{A}_{-k \frac{t}{n}} \right) \right\|_{M^{\infty,1}} \leq 1 + \frac{t}{n} \left\| \sigma_{V_1'} \circ \mathcal{A}_{-k \frac{t}{n}} \right\|_{M^{\infty,1}} \leq 1 + \frac{t}{n} C(t), \\ \|v_k\|_{M^{\infty,1}} &= \frac{t}{n} \left\| \sigma_{V_2'} \circ \mathcal{A}_{-k \frac{t}{n}} \right\|_{M^{\infty,1}} \leq \frac{t}{n} C(t) \epsilon, \end{aligned}$$

for some locally bounded constant  $C(t) > 0$  independent of  $n$  and  $\epsilon$ . Therefore, by Lemma 5.1,

$$(26) \quad \|b_{n,t}\|_{M^{\infty,1}} \leq n \frac{t}{n} C(t) \epsilon \left( 1 + 2 \frac{t}{n} C(t) \right)^{n-1} \leq \epsilon t C(t) e^{2tC(t)}.$$

Following the pathway of the proof of Theorem 1.1, we write  $E_n(t)$  as an integral operator with kernel

$$\begin{aligned} e_{n,t}(x, y) &= c(t) |\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} (\widetilde{a}_{n,t} + \widetilde{b}_{n,t})(x, y) \\ &= c(t) |\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} k_{n,t}(x, y), \end{aligned}$$

that is  $k_{n,t} = \widetilde{a}_{n,t} + \widetilde{b}_{n,t}$ , and the Trotter formula (6) combined with Proposition 2.6 imply that  $k_{n,t} \rightarrow k_t$  in  $\mathcal{S}'(\mathbb{R}^{2d})$ , where the distribution  $k_t$  is conveniently introduced to rephrase the integral kernel  $u_t$  of the propagator  $U(t) = e^{-it(H_0+V)}$  as

$$u_t(x, y) = c(t) |\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} k_t(x, y).$$

By repeating this argument with  $V_2 = 0$  (hence  $\widetilde{b}_{n,t} = 0$  and  $k_{n,t} = \widetilde{a}_{n,t}$ ) we see that  $\widetilde{a}_{n,t}$  converges in  $\mathcal{S}'(\mathbb{R}^{2d})$  as well, hence  $\widetilde{b}_{n,t}$  converges in  $\mathcal{S}'(\mathbb{R}^{2d})$  by difference. Therefore, for any non-zero  $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$  the functions  $V_\Psi \widetilde{a}_{n,t}$  and  $V_\Psi \widetilde{b}_{n,t}$  converge pointwise in  $\mathbb{R}^{4d}$ .

We need a technical lemma at this point.

**Lemma 5.2.** *Let  $F_n$  and  $G_n$  be two sequences of complex-valued functions on  $\mathbb{R}^{2d}$  such that  $F_n \rightarrow F$ ,  $G_n \rightarrow G$  pointwise, and assume  $|F_n| \leq H \in L^1(\mathbb{R}^{2d})$  and  $\|G_n\|_{L^1} \leq \epsilon$  for any  $n \in \mathbb{N}$ . Then,*

$$\limsup_{n \rightarrow \infty} \|F_n + G_n - (F + G)\|_{L^1} \leq 2\epsilon.$$

*Proof.* First, notice that  $\|G\|_{L^1} \leq \epsilon$  by Fatou's lemma. Now,

$$\|F_n + G_n - (F + G)\|_{L^1} \leq \|F_n - F\|_{L^1} + \|G_n - G\|_{L^1},$$

where the first term on the right-hand side goes to zero by dominated convergence, while for the other one we have  $\|G_n - G\|_{L^1} \leq 2\epsilon$ . The desired conclusion is then immediate.  $\square$

For any fixed  $z \in \mathbb{R}^{2d}$ , set  $F_n(\zeta) = V_\Psi \widetilde{a}_{n,t}(z, \zeta)$  and  $G_n(\zeta) = V_\Psi \widetilde{b}_{n,t}(z, \zeta)$ .

By Lemma 3.1 and (25) we have

$$\sup_{\zeta \in \mathbb{R}^{2d}} \langle \zeta \rangle^s |F_n(\zeta)| \lesssim \|\widetilde{a}_{n,t}\|_{M_s^\infty} \lesssim \|a_{n,t}\|_{M_s^\infty} \leq C(t, \epsilon).$$

Similarly, by Lemma 3.1 and (26),

$$\|G_n\|_{L^1} \lesssim \|\widetilde{b}_{n,t}\|_{M^{\infty,1}} \lesssim \|b_{n,t}\|_{M^{\infty,1}} \leq \epsilon C(t).$$

These estimates yield two results: on the one hand, the first claim of Theorem 1.3 is proved. On the other hand, the assumptions of Lemma 5.2 are satisfied: we have  $(F_n + G_n)(\zeta) = V_\Psi k_{n,t}(z, \zeta)$  and  $(F + G)(\zeta) = V_\Psi k_t(z, \zeta)$ , and therefore we obtain

$$\limsup_{n \rightarrow \infty} \|\mathcal{F}[(k_{n,t} - k_t) \overline{T_z \Psi}]\|_{L^1} \leq 2\epsilon C(t).$$

Since  $\epsilon$  can be made arbitrarily small and the left-hand side is independent of  $\epsilon$ , we conclude that

$$\lim_{n \rightarrow \infty} \|\mathcal{F}[(k_{n,t} - k_t) \overline{T_z \Psi}]\|_{L^1} = 0,$$

in particular  $k_{n,t} \rightarrow k_t$  in  $(\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^{2d})$ .

Finally, with the help of a suitable bump function  $\Theta$  as in the preceding section, for any fixed  $z \in \mathbb{R}^{2d}$  we deduce

$$\|\mathcal{F}[(e_{n,t} - u_t) \overline{T_z \Psi}]\|_{L^1} \leq |\det B_t|^{-1/2} \|\mathcal{F}[(T_z \Theta e^{2\pi i \Phi_t})]\|_{L^1} \|\mathcal{F}[(k_{n,t} - k_t) \overline{T_z \Psi}]\|_{L^1},$$

and thus

$$\|\mathcal{F}[(e_{n,t} - u_t) \overline{T_z \Psi}]\|_{L^1} \rightarrow 0.$$

This gives  $e_{n,t} \rightarrow u_t$  in  $(\mathcal{F}L^1)_{\text{loc}}(\mathbb{R}^{2d})$  and therefore uniformly on compact subsets of  $\mathbb{R}^{2d}$ .

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