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A CHARACTERIZATION OF MODULATION SPACES BY SYMPLECTIC ROTATIONS

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Abstract. This note contains a new characterization of modulation spaces $M^p_m(\mathbb{R}^n)$, $1 \leq p \leq \infty$, by symplectic rotations. Precisely, instead to measure the time-frequency content of a function by using translations and modulations of a fixed window as building blocks, we use translations and metaplectic operators corresponding to symplectic rotations. Technically, this amounts to replace, in the computation of the $M^p_m(\mathbb{R}^n)$-norm, the integral in the time-frequency plane with an integral on $\mathbb{R}^n \times U(2n,\mathbb{R})$ with respect to a suitable measure, $U(2n,\mathbb{R})$ being the group of symplectic rotations. More conceptually, we are considering a sort of polar coordinates in the time-frequency plane. To have invariance under symplectic rotations we choose a Gaussian as suitable window function. We also provide a similar (and easier) characterization with the group $U(2n,\mathbb{R})$ being reduced to the n-dimensional torus $\mathbb{T}^n$.

1. Introduction

The objective of this study is to find a new characterization of modulation spaces using symplectic rotations. Precisely, we are interested in those metaplectic operators $\widehat{S} \in Mp(n,\mathbb{R})$, such that the corresponding projection $S := \pi(\widehat{S})$ onto the symplectic group $Sp(n,\mathbb{R})$ is a symplectic rotation. Let us recall that the symplectic group $Sp(n,\mathbb{R})$ is the subgroup of $2n \times 2n$ invertible matrices $GL(2n,\mathbb{R})$, defined by

$$Sp(n,\mathbb{R}) = \left\{S \in GL(2n,\mathbb{R}) : \; SJS^T = J\right\}, \tag{1}$$

where J is the orthogonal matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

(I_n, 0_n are the $n \times n$ identity matrix and null matrix, respectively). Here we consider the subgroup

$$U(2n,\mathbb{R}) := Sp(n,\mathbb{R}) \cap O(2n,\mathbb{R}) \simeq U(n)$$

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of symplectic rotations (cf., e.g. [15, Section 2.3]), namely

$$U(2n,\mathbb{R})=\left\{\begin{pmatrix}A & -B\\ B & A\end{pmatrix}:\ AA^T+BB^T=I_n,\ AB^T=B^TA\right\}\subset Sp(n,\mathbb{R}), \tag{2}$$

endowed with the normalized Haar measure dS (the group $U(2n,\mathbb{R})$, being compact, is unimodular).

In the 80's H. Feichtinger [16] introduced modulation spaces to measure the time-frequency concentration of a function/distribution on the time-frequency space (or phase space) $\mathbb{R}^{2n}$. They are nowadays become popular among mathematicians and engineers because they have found numerous applications in signal processing [6, 19, 20], pseudodifferential and Fourier integral operators [7, 8, 9, 28, 29], partial differential equations [1, 2, 3, 4, 10, 13, 11, 11, 32, 33, 34] and quantum mechanics [12, 15].

To recall their definition, we need a few time-frequency tools. First, the translation T_x and modulation M_ξ operators are defined by

$$T_xf(t)=f(t-x),\quad M_\xi f(t)=e^{2\pi i t\cdot\xi}f(t),\quad t,x,\xi\in\mathbb{R}^n,$$

for any function f on $\mathbb{R}^n$.

The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ with respect to a function $g\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ (so-called window), given by

$$V_gf(x,\xi)=\langle f,M_\xi T_xg\rangle=\int_{\mathbb{R}^n}f(t)\overline{g(t-x)}\,e^{-2\pi i t\cdot\xi}dt,\quad x,\xi\in\mathbb{R}^n. \tag{3}$$

The short-time Fourier transform is well-defined whenever the bracket $\langle\cdot,\cdot\rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f\in\mathcal{S}'(\mathbb{R}^n)$, $g\in\mathcal{S}(\mathbb{R}^n)$, or for $f,g\in L^2(\mathbb{R}^n)$.

Let $m(x,\xi)$ be a continuous weight, v-moderate for some submultiplicative weight v (see [22, Section 11.1] for details - we will not use explicitly these properties). We also assume that m has at most polynomial growth.

Definition 1.1 (Modulation spaces). *Given $g\in\mathcal{S}(\mathbb{R}^n)$, and $1\leq p\leq\infty$, the modulation space $M^p_m(\mathbb{R}^n)$ consists of all tempered distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $V_gf\in L^p_m(\mathbb{R}^{2n})$. The norm on $M^p_m(\mathbb{R}^n)$ is*

$$\begin{aligned}\|f\|_{M^p_m}=\|V_gf\|_{L^p_m}&=\left(\int_{\mathbb{R}^{2n}}|V_gf(x,\xi)|^pm(x,\xi)^pdxd\xi\right)^{1/p}\\ &=\left(\int_{\mathbb{R}^{2n}}|\langle f,M_\xi T_xg\rangle|^pm(x,\xi)^pdxd\xi\right)^{1/p}\end{aligned} \tag{4}$$

(with obvious modifications for $p=\infty$).

The spaces $M^p_m(\mathbb{R}^n)$ are Banach spaces, and every nonzero $g \in M^1_v(\mathbb{R}^n)$ yields an equivalent norm in (4), so that their definition is independent of the choice of $g \in M^1_v(\mathbb{R}^n)$ (see [16, 22]).

We now provide an equivalent norm to (4) by using translations T_x (or modulations M_ξ) and the operators $\widehat{S}$, with $S \in U(2n,\mathbb{R})$ as follows.

Theorem 1.2. *Consider the Gaussian function* $\varphi(t) = 2^{d/4} e^{-\pi |t|^2}$.
(i) For $1 \le p < \infty$ *and* $f \in M^p_m(\mathbb{R}^n)$*, we have*

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times U(2n,\mathbb{R})} |x|^n |\langle f, \widehat{S} T_x \varphi \rangle|^p m(S(x,0)^T)^p \, dx dS \right)^{\frac{1}{p}}, \tag{5}$$

where dx *is the Lebesgue measure on* $\mathbb{R}^n$ *and* dS *the Haar measure on* $U(2n,\mathbb{R})$*.*
Similarly,

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times U(2n,\mathbb{R})} |\xi|^n |\langle f, \widehat{S} M_\xi \varphi \rangle|^p m(S(0,\xi)^T)^p \, d\xi dS \right)^{\frac{1}{p}}, \tag{6}$$

with $d\xi$ *being the Lebesgue measure on* $\mathbb{R}^n$ *and* dS *the Haar measure on* $U(2n,\mathbb{R})$*.*
(ii) For $p = \infty$, $f \in M^\infty_m(\mathbb{R}^n)$*, it occurs*

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{S \in U(2n,\mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S} T_x \varphi \rangle| m(S(x,0)^T) \tag{7}$$

and, similarly,

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{S \in U(2n,\mathbb{R})} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S} M_\xi \varphi \rangle| m(S(0,\xi)^T). \tag{8}$$

The interpretation of the integral (5) above is as follows. The metaplectic operator $\widehat{S}$ produces a time-frequency rotation of the shifted Gaussian $T_x\varphi$. In this way, the operator

$$f \mapsto \langle f, \widehat{S} T_x \varphi \rangle$$

detects the time-frequency content of f in an oblique strip, see Figure 1. All the contributions are then added together with a weight $|x|^n$ which takes into account the underlapping of the strips as $|x| \to +\infty$ and the overlapping as $x \to 0$.

Formulas (6), (7) and (8) have similar meanings.

Observe that in dimension $n = 1$, $U(2,\mathbb{R}) \simeq U(1)$ and the above formula is essentially a transition to polar coordinates with $|x|$ being the Jacobian.

Comparing (4) and (5) we observe that in (5) the modulation operator M_ξ is replaced by the metaplectic operator $\widehat{S}$ and the integral on the phase space $\mathbb{R}^{2n}$ has become an integral on the cartesian product $\mathbb{R}^n \times U(2n,\mathbb{R})$. The integration parameters (x,ξ) of (4) live in $\mathbb{R}^{2n}$, with $\dim \mathbb{R}^{2n} = 2n$, whereas the parameters (x,S) of (5) live in $\mathbb{R}^n \times U(2n,\mathbb{R})$. Recall that $\dim U(2n,\mathbb{R}) = n^2$ [15]; this suggests that a formula similar to (5) should hold when $U(2n,\mathbb{R})$ is reduced to

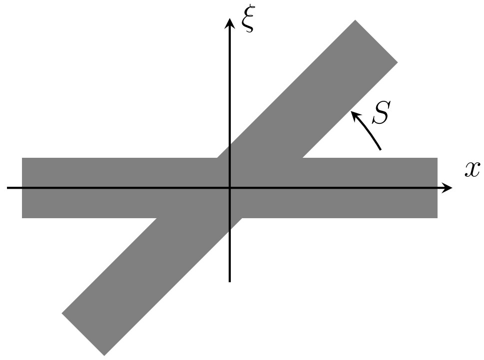

Figure 1. The time-frequency content of f in the oblique strip is detected by the operator $f \mapsto \langle f, \widehat{S} T_x \varphi \rangle$.

a suitable subgroup $K \subset U(2n, \mathbb{R})$ of dimension n. This is indeed the case (and easier to see), as shown in the subsequent Theorem 1.3.

Consider the n-dimensional torus

$$\mathbb{T}^n = \left\{ S = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \,:\, \theta_1, \dots, \theta_n \in \mathbb{R} \right\} \subset U(n) \tag{9}$$

with the Haar measure $dS = d\theta_1 \dots d\theta_n$. The torus is isomorphic to a subgroup $K \subset U(2n, \mathbb{R})$, via the isomorphism ι in formula (16) below (see the subsequent section).

We exhibit the following characterization for M^p-spaces.

Theorem 1.3. *Let φ be the Gaussian of Theorem 1.2.*

(i) For $1 \leq p < \infty$, $f \in M^p_m(\mathbb{R}^n)$, we have

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \dots x_n| |\langle f, \widehat{S} T_x \varphi \rangle|^p m(S(x,0)^T)^p dx dS \right)^{\frac{1}{p}}, \tag{10}$$

and, similarly,

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times \mathbb{T}^n} |\xi_1 \dots \xi_n| |\langle f, \widehat{S} M_\xi \varphi \rangle|^p m(S(0,\xi)^T)^p d\xi dS \right)^{\frac{1}{p}}. \tag{11}$$

(ii) For $p = \infty$,

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S} T_x \varphi \rangle| m(S(x,0)^T) \tag{12}$$

and

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S} M_\xi \varphi \rangle| m(S(0,\xi)^T). \tag{13}$$

The above results for the groups $U(2n,\mathbb{R})$ and $\mathbb{T}^n$ can be interpreted, in a sense, as two extreme cases, and it would be interesting to find, more generally, for which compact subgroups $K \subset U(2n,\mathbb{R})$ similar characterizations hold. We conjecture that they should be precisely the subgroups $K \subset U(2n,\mathbb{R})$ such that every orbit for their action on $\mathbb{R}^{2n}$ intersects $\{0\} \times \mathbb{R}^n$ (up to subsets of measure zero), with a corresponding weighted measure on $\mathbb{R}^n \times K$ to be determined.

Another problem which is worth investigating is the study of discrete versions of the above characterizations via coorbit theory [17].

The paper is organized as follows: in Section 2 we collected some preliminary results, whereas Section 3 is devoted to the proof of Theorems 1.2 and 1.3. In Section 4 we rephrase more explicitly Theorem 1.3 in terms of the partial fractional Fourier transform.

2. Notation and Preliminaries

Notation. We write $x \cdot y$ for the scalar product on $\mathbb{R}^n$ and $|t|^2 = t \cdot t$, for $t, x, y \in \mathbb{R}^n$. For expressions $A, B \geq 0$, we use the notation $A \lesssim B$ to represent the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^n)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $\mathcal{S}(\mathbb{R}^n)$.

Metaplectic Operators. The metaplectic representation μ of $Mp(n,\mathbb{R})$, the two-sheeted cover of the symplectic group $Sp(n,\mathbb{R})$, defined in (1) arises as intertwining operator between the standard Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^d$ and the representation that is obtained from it by composing ρ with the action of $Sp(n,\mathbb{R})$ by automorphisms on $\mathbb{H}^d$ (see, e.g., [15, 21, 23]). Let us recall the main points of a direct construction.

The symplectic group $Sp(n,\mathbb{R})$ is generated by the so-called free symplectic matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}), \quad \det B \neq 0.$$

To each such a matrix the associated generating function is defined by

$$W(x,x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'.$$

Conversely, to every polynomial of the type

$$W(x,x') = \frac{1}{2}Px \cdot x - Lx \cdot x' + \frac{1}{2}Qx' \cdot x'$$

with

$$P = P^T, Q = Q^T$$

and

$$\det L \neq 0$$

it can be associated a free symplectic matrix, namely

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}.$$

Given S_W as above and $m \in \mathbb{Z}$ such that

$$m\pi \equiv \arg \det L \mod 2\pi,$$

the related operator $\widehat{S}_{W,m}$ is defined by setting, for $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{S}_{W,m}\psi(x) = \frac{1}{i^{n/2}}\Delta(W)\int_{\mathbb{R}^n} e^{2\pi i W(x,x')}\psi(x')dx' \tag{14}$$

(with $i^{n/2} = e^{i\pi n/4}$) where

$$\Delta(W) = i^m\sqrt{|\det L|}.$$

The operator $\widehat{S}_{W,m}$ is named *quadratic Fourier transform* associated to the free symplectic matrix S_W (as a remark, for integral representations of metaplectic operators that do not arise from free symplectic matrices see [14, 24]). The class modulo 4 of the integer m is called *Maslov index* of $\widehat{S}_{W,m}$. Observe that if m is one choice of Maslov index, then $m+2$ is another equally good choice: hence to each function W we associate two operators, namely $\widehat{S}_{W,m}$ and $\widehat{S}_{W,m+2} = -\widehat{S}_{W,m}$.

The quadratic Fourier transform corresponding to the choices $S_W = J$ and $m = 0$ is denoted by $\widehat{J}$. The generating function of J is simply $W(x,x') = -x\cdot x'$. It follows that

$$\widehat{J}\psi(x) = \frac{1}{i^{n/2}}\int_{\mathbb{R}^n} e^{-2\pi i x\cdot x'}\psi(x')dx' = \frac{1}{i^{n/2}}\mathcal{F}\psi(x) \tag{15}$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{F}$ is the usual unitary Fourier transform.

The quadratic Fourier transforms $\widehat{S}_{W,m}$ form a subset of the group $\mathcal{U}(L^2(\mathbb{R}^n))$ of unitary operators acting on $L^2(\mathbb{R}^n)$, which is mapped into itself by the operation of inversion and they generate a subgroup of $\mathcal{U}(L^2(\mathbb{R}^n))$ which is, by definition, the metaplectic group $Mp(n,\mathbb{R})$. The elements of $Mp(n,\mathbb{R})$ are called metaplectic operators.

Hence, every $\widehat{S} \in Mp(n,\mathbb{R})$ is, by definition, a product

$$\widehat{S}_{W_1,m_1}\ldots\widehat{S}_{W_k,m_k}$$

of metaplectic operators associated to free symplectic matrices.

Indeed, it can be proved that every $\widehat{S} \in Mp(n,\mathbb{R})$ can be written as a product of exactly two quadratic Fourier transforms: $\widehat{S} = \widehat{S}_{W,m}\widehat{S}_{W',m'}$. Now, it can be shown that the mapping $\widehat{S}_{W,m} \longmapsto S_W$ extends to a group homomorphism $\pi : Mp(n,\mathbb{R}) \to Sp(n,\mathbb{R})$, which is in fact a double covering.

We also observe that each metaplectic operator is, by construction, a unitary operator in $L^2(\mathbb{R}^n)$, but also an automorphism of $\mathcal{S}(\mathbb{R}^n)$ and of $\mathcal{S}'(\mathbb{R}^n)$.

We are interested in its restriction $\widehat{S} = \pi(S)$, with $S \in U(2n,\mathbb{R})$, the symplectic rotations in (2).

Observe that $U(n) := U(n,\mathbb{C})$, the complex unitary group (the group of $n \times n$ invertible complex matrices V satisfying $VV^* = V^*V = I_n$) is isomorphic to $U(2n,\mathbb{R})$. The isomorphism ι is the mapping $\iota : U(n) \to U(2n,\mathbb{R})$ given by

$$\iota(A+iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{16}$$

for details see [15, Chapter 2.3].

We present here some results related to the group $U(2n,\mathbb{R})$, which will be used in the sequel to attain the characterization of Theorem 1.2. First, we recall a well-known result, see for instance [22, Lemma 9.4.3]:

Lemma 2.1. *For $f, g \in L^2(\mathbb{R}^n)$ and $S \in Sp(n,\mathbb{R})$, the STFT $V_g f$ satisfies*

$$|V_{\widehat{S}g}(\widehat{S}f)(x,\xi)| = |V_g f(S^{-1}(x,\xi))|, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{17}$$

This second issue is contained in [5], we sketch the proof for the sake of consistency.

Lemma 2.2. *For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $S \in U(2n,\mathbb{R})$, the STFT $V_\varphi(\widehat{S}\psi)$ is a Schwartz function, with seminorms uniformly bounded when $S \in U(2n,\mathbb{R})$.*

Proof. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the STFT V_φ is a continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$ (see [16]). Hence, it is enough to show that

$$\{\hat{S}\varphi : \ S \in U(2n,\mathbb{R})\}$$

is a bounded subset of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, i.e., every Schwartz seminorm is bounded on it. Since the group $U(2n,\mathbb{R})$ is compact, it is sufficient to show that every seminorm is locally bounded, that is, we can limit ourselves to consider S in a sufficiently small neighbourhood for any fixed $S_0 \in U(2n,\mathbb{R})$. Equivalently, we can consider S of the form $S = S_1 J^{-1} S_0$ where S_1 belongs to a enough small neighbourhood of J in $U(2n,\mathbb{R})$. Using the representation of metaplectic operators recalled at the beginning of this section, we can write

$$\begin{aligned} \hat{S}\varphi(x) &= \pm \widehat{S}_1[\widehat{J}^{-1}\widehat{S}_0\varphi](x) \\ &= c\sqrt{|\det L|}\int_{\mathbb{R}^n} e^{2\pi i(\frac{1}{2}Px\cdot x - Lx\cdot y + \frac{1}{2}Qy\cdot y)} \underbrace{[\widehat{J}^{-1}\widehat{S}_0\varphi]}_{\in \mathcal{S}(\mathbb{R}^n)}(y)dy \end{aligned}$$

where $|c| = 1$ and, we might say, $\|P\| < \epsilon$, $\|Q\| < \epsilon$, $\|L - I\| < \epsilon$. If $\epsilon < 1$, it is straightforward to check that $\hat{S}\varphi$ belongs to a bounded subset of $\mathcal{S}(\mathbb{R}^n)$, as desired. □

Lemma 2.3. *Let $B = (b_{i,j})_{i,j=1,\dots n}$ be the $n \times n$ submatrix in* (2)*. The subset $\Sigma \subset U(2n, \mathbb{R})$ obtained by setting $b_{i,1} = 0$, $i = 1, \dots n$ (i.e., the first column of B is set to zero), is a submanifold of codimension n.*

Proof. We have to verify that the coordinates $b_{1,1}, \dots, b_{n,1}$ are independent on the subset Σ, namely the projection

$$(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \to \mathbb{R}^n$$

has rank n on Σ.

Let us first show that for every $S_0 \in \Sigma$ there exists a $U(2n, \mathbb{R})$-valued smooth function $S(b_1, \dots, b_n)$, defined in a neighbourhood of $0 \in \mathbb{R}^n$, such that $S(0) = S_0$ and the first column "of its submatrix B" is precisely $(b_1, \dots, b_n)^T$.

Let $S_0 = A + iB = (V_1, \dots, V_n) \in \Sigma$, with V_j being a $n \times 1$ complex vector, $j = 1, \dots, n$, so that by assumption $(b_{i,1})_{i=1,\dots,n} = \operatorname{Im} V_1 = 0$. We consider any smooth function $V_1(b_1, \dots, b_n)$, defined in a neighbourhood of $0 \in \mathbb{R}^n$, valued in the unit sphere of $\mathbb{C}^n$, such that

$$\operatorname{Im} V_1(b_1, \dots, b_n) = (b_1, \dots, b_n)^T, \quad V_1(0) = V_1.$$

Then, we apply the Gram-Schmidt orthonormalization procedure in $\mathbb{C}^n$ to the set of vectors $(V_1(b_1, \dots, b_n), V_2, \dots, V_n)$. This provides the desired $U(n)$-valued function $S(b_1, \dots, b_n)$. In particular $S(0) = S_0$.

Now, the composition of the mapping

$$(b_1, \dots, b_n) \mapsto S(b_1, \dots, b_n)$$

followed by the projection $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \to \mathbb{R}^n$ is therefore the identity mapping in a neighbourhood of 0 and has rank n. Hence the same is true for the projection $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \to \mathbb{R}^n$ at S_0. □

Lemma 2.4. *For every $\epsilon > 0$, define the ($x-$independent) function*

$$\chi_\epsilon(x, \xi) = \frac{1}{\epsilon^n} \mathbb{1}_Q \left(\frac{\xi}{\epsilon} \right), \tag{18}$$

where

$$Q = \left[-\frac{1}{2}, \frac{1}{2} \right]^n \subset \mathbb{R}^n \quad \text{and} \quad \mathbb{1}_Q = \begin{cases} 1, & \xi \in Q \\ 0, & \xi \notin Q \end{cases}$$

and

$$\tilde{\chi}_\epsilon(z) = \frac{\chi_\epsilon(z)}{\int_{U(2n,\mathbb{R})} \chi_\epsilon(Sz)\, dS}, \quad z \in \mathbb{R}^{2n}. \tag{19}$$

Then we have

$$\int_{U(2n,\mathbb{R})} \tilde{\chi}_\epsilon(Sz)\, dS = 1, \quad \forall z \in \mathbb{R}^{2n} \tag{20}$$

and

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(x,\xi)\Phi(x,\xi)dxd\xi = C\int_{\mathbb{R}^n}|x|^n\Phi(x,0)\,dx, \tag{21}$$

for some $C>0$ and for every continuous function Φ on $\mathbb{R}^{2n}$ with a rapid decay at infinity.

Proof. We will show in a moment that, for $z=(x,\xi)\in\mathbb{R}^{2n}$,

$$\int_{U(2n,\mathbb{R})}\chi_\epsilon(Sz)\,dS \gtrsim \min\{\epsilon^{-n},|z|^{-n}\} \tag{22}$$

(with the convention, at $z=0$, that $\min\{\epsilon^{-n},+\infty\}=\epsilon^{-n}$). In particular, $\int_{U(2n,\mathbb{R})}\chi_\epsilon(Sz)\,dS\neq 0$, for every $z\in\mathbb{R}^{2n}$. Formula (20) then follows, because

$$\begin{aligned}\int_{U(2n,\mathbb{R})}\tilde{\chi}_\epsilon(Sz)\,dS &= \int_{U(2n,\mathbb{R})}\frac{\chi_\epsilon(Sz)}{\int_{U(2n,\mathbb{R})}\chi_\epsilon(USz)\,dU}dS\\ &= \int_{U(2n,\mathbb{R})}\frac{\chi_\epsilon(Sz)}{\int_{U(2n,\mathbb{R})}\chi_\epsilon(Uz)\,dU}dS = 1\end{aligned}$$

for every $z\in\mathbb{R}^{2n}$, since the Haar measure is right invariant.

Let us now prove (22). For $z=0$ we have

$$\int_{U(2n,\mathbb{R})}\chi_\epsilon(Sz)\,dS = \frac{1}{\epsilon^n}\int_{U(2n,\mathbb{R})}dS = \frac{C_0}{\epsilon^n},$$

with $C_0 = meas(U(2n,\mathbb{R}))>0$. Consider now $z\neq 0$. Observe that the function

$$\Psi_\epsilon(z) := \int_{U(2n,\mathbb{R})}\chi_\epsilon(Sz)\,dS$$

is constant on the orbits of $U(2n,\mathbb{R})$ in $\mathbb{R}^{2n}$, so that we can suppose

$$z=(x,0),\quad x=(x_1,0,\dots,0),\quad x_1=|x|=|z|>0.$$

Now, by the definition of χ_ϵ and Ψ_ϵ,

$$\Psi_\epsilon(z)=\epsilon^{-n}\,meas\left\{S=\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in U(2n,\mathbb{R}):\,|b_{i,1}|<\frac{\epsilon}{2|z|},\ i=1,\dots,n\right\}, \tag{23}$$

where $(b_{i,1})_{i=1,\dots,n}$, is the first column of the matrix $B=(b_{i,j})_{i,j=1,\dots n}$.

Define, for $\mu>0$,

$$f(\mu)=meas\left\{S=\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in U(2n,\mathbb{R}):\,|b_{i,1}|<\mu,\ i=1,\dots,n\right\}.$$

Observe that $f(\mu)$ is non-decreasing and constant for $\mu\geq 1$. Moreover, from Lemma 2.3 we know that by setting $b_{i,1}=0$, $i=1,\dots,n$, in $U(2n,\mathbb{R})$, we get a submanifold Σ of codimension n, and the function $f(\mu)$ is the measure of a tubular neighbourhood of Σ in $U(2n,\mathbb{R})$. Hence we have the asymptotic behaviour

$$\mu^{-n} f(\mu) \to C_0 > 0, \quad \text{as} \quad \mu \to 0^+ \tag{24}$$

and in particular

$$f(\mu) \gtrsim \min\{1, \mu^n\}. \tag{25}$$

We then infer

$$\Psi_\epsilon(z) = \epsilon^{-n} f\left(\frac{\epsilon}{2|z|}\right) \to \frac{C_1}{|z|^n}, \quad \text{as } \epsilon \to 0^+ \tag{26}$$

locally uniformly in $\mathbb{R}^{2n} \setminus \{0\}$, with $C_1 = 2^{-n} C_0$, and

$$\Psi_\epsilon(z) \gtrsim \epsilon^{-n} \min\left\{1, \left(\frac{\epsilon}{|z|}\right)^n\right\} = \min\{\epsilon^{-n}, |z|^{-n}\}, \tag{27}$$

which is (22).

Let us finally prove (21). We are interested in the limit $\epsilon \to 0^+$, so we can assume $\epsilon \leq 1$. Consider a continuous function Φ on $\mathbb{R}^{2n}$ with rapid decay at infinity. By definition of $\tilde{\chi}_\epsilon(z)$ in (19) we have

$$\tilde{\chi}_\epsilon(x,\xi) = \frac{\epsilon^{-n}}{\Psi_\epsilon(x,\xi)} \mathbb{1}_{[-\epsilon/2,\epsilon/2]^n}(\xi)$$

so that, by (27),

$$|\tilde{\chi}_\epsilon(x,\xi)\Phi(x,\xi)| \lesssim \epsilon^{-n}(1+|x|^n)\mathbb{1}_{[-\epsilon/2,\epsilon/2]^n}(\xi)|\Phi(x,\xi)| \in L^1(\mathbb{R}^{2n})$$

for $0 < \epsilon \leq 1$. Fubini's Theorem then allows one to look at the first integral in (21) as an iterated integral

$$I_\epsilon := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x,\xi)\Phi(x,\xi) d\xi \right) dx$$

and we apply the dominated convergence theorem to the integral with respect to the x variable as follows. Setting

$$\Upsilon_\epsilon(x) := \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x,\xi)\Phi(x,\xi)d\xi = \epsilon^{-n} \int_{[-\epsilon/2,\epsilon/2]^n} \frac{1}{\Psi_\epsilon(x,\xi)} \Phi(x,\xi)\, d\xi,$$

by (26) we have, for every fixed $x \neq 0$,

$$\Upsilon_\epsilon(x) \to C|x|^n \Phi(x,0);$$

for some constant $C > 0$. On the other hand $\Upsilon_\epsilon(x)$ is dominated, using (27), by

$$(1+|x|)^n \sup_{\xi \in \mathbb{R}^n} |\Phi(x,\xi)| \in L^1(\mathbb{R}^n).$$

Hence

$$\lim_{\epsilon \to 0^+} I_\epsilon = \int_{\mathbb{R}^n} \lim_{\epsilon \to 0^+} \Upsilon_\epsilon(x) dx = C \int_{\mathbb{R}^n} |x|^n \Phi(x,0)\, dx.$$

This concludes the proof. □

Remark 2.5. *Observe that there are no conditions on the derivatives of the function* Φ *in* (21).

3. Proofs of the main results

In what follows we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. (i) **First Step.** Let us start with showing that formula (5) is true for any function ψ in the Schwartz class $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Using the Gaussian $\varphi(t) = 2^{d/4} e^{-\pi|t|^2}$ as window function, we compute the M^p_m-norm of ψ as in (4) and then use Lemma 2.4 so that

$$\begin{aligned}\|\psi\|^p_{M^p_m} &= \int_{\mathbb{R}^{2n}} |V_\varphi \psi(z)|^p m(z)^p \, dz = \int_{\mathbb{R}^{2n}} \int_{U(2n,\mathbb{R})} \tilde{\chi}_\epsilon(Sz) |V_\varphi \psi(z)|^p m(z)^p \, dS dz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n,\mathbb{R})} \tilde{\chi}_\epsilon(z) |V_\varphi \psi(S^{-1}z)|^p m(S^{-1}z)^p \, dS dz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n,\mathbb{R})} \tilde{\chi}_\epsilon(z) |V_{\widehat{S}\varphi} \widehat{S}\psi(z)|^p m(S^{-1}z)^p \, dS dz\end{aligned}$$

where in the last equality we used Lemma 2.1. Observe that, since S is unitary and φ is a Gaussian, $\widehat{S}\varphi = c\varphi$, for some phase factor $c \in \mathbb{C}$, with $|c| = 1$ (see [15, Proposition 252]) and this phase factor is killed by the modulus obtaining $|V_{\widehat{S}\varphi}\widehat{S}\psi(z)| = |V_\varphi \widehat{S}\psi(z)|$. Continuing the above computation we infer

$$\|\psi\|^p_{M^p_m} = \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(z) \int_{U(2n,\mathbb{R})} |V_\varphi \widehat{S}\psi(z)|^p m(S^{-1}z)^p \, dS dz.$$

Set

$$\Phi(z) = \int_{U(2n,\mathbb{R})} |V_\varphi \widehat{S}\psi(z)|^p m(S^{-1}z)^p \, dS.$$

The dominated convergence theorem guarantees that Φ is continuous on $\mathbb{R}^{2n}$ and moreover Φ has rapid decay at infinity. This follows from Lemma 2.2 (recall that m is continuous and has at most polynomial growth).

Letting $\epsilon \to 0^+$ and using (21) we obtain

$$\begin{aligned}\|\psi\|^p_{M^p_m} &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |V_\varphi \widehat{S}\psi(x,0)|^p m(S^{-1}(x,0)^T)^p \, dS dx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |\langle \widehat{S}\psi, T_x\varphi\rangle|^p m(S^{-1}(x,0)^T)^p \, dS dx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |\langle \psi, \widehat{S} T_x\varphi\rangle|^p m(S(x,0)^T)^p \, dS dx.\end{aligned}$$

The last equality is due to $\langle \widehat{S}\psi, T_x\varphi\rangle = \langle \psi, \widehat{S}^{-1}T_x\varphi\rangle$ and the invariance of the Haar measure of $U(2n,\mathbb{R})$ with respect to the change of variable $S \to S^{-1}$.

Second Step. Consider $f \in M^p_m(\mathbb{R}^n)$, $1 \le p < \infty$. Using the density of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in $M^p_m(\mathbb{R}^n)$ (cf. e.g., [22, Chapter 12]), there exists a sequence $\{\psi_k\}_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\psi_k \to f$ in $M^p_m(\mathbb{R}^n)$. This implies that $\psi_k \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\langle \psi_k, \widehat{S}T_x\varphi\rangle \to \langle \psi, \widehat{S}T_x\varphi\rangle$$

pointwise for every $x \in \mathbb{R}^n$, $S \in U(2n,\mathbb{R})$. Let us define, for every $f \in M^p_m(\mathbb{R}^n)$,

$$|||f||| = \left(\int_{\mathbb{R}^n\times U(2n,\mathbb{R})} |x|^n |\langle f, \widehat{S}T_x\varphi\rangle|^p m(S(x,0)^T)^p dx dS\right)^{\frac{1}{p}}. \tag{28}$$

By Fatou's Lemma, for any $f \in M^p_m(\mathbb{R}^n)$:

$$|||f|||^p \le \liminf_{k\to\infty} |||\psi_k|||^p \lesssim \liminf_{k\to\infty} \|\psi_k\|^p_{M^p_m} = \|f\|^p_{M^p_m}. \tag{29}$$

It is easy to check that $|||f|||$ is a seminorm on $M^p_m(\mathbb{R}^n)$. Applying (29) to the difference $f - \psi_k$ we obtain $|||f - \psi_k||| \to 0$ and hence $|||\psi_k||| \to |||f|||$. By assumption we also have $\|\psi_k\|_{M^p_m} \to \|f\|_{M^p_m}$, and the desired norm equivalence in (5) then extents from $\mathcal{S}(\mathbb{R}^n)$ to $M^p_m(\mathbb{R}^n)$.

Third Step. We will show that (6) easily follows from (5). By the definition of the symplectic group (1), for any $S \in U(2n,\mathbb{R})$,

$$J^{-1}S = (S^T)^{-1}J^{-1} = SJ^{-1}$$

for $S^{-1} = S^T$. On the other hand, for any $f \in M^p_m(\mathbb{R}^n)$, $\|f\|_{M^p_m} \asymp \|\hat{f}\|_{M^p_{\tilde{m}}}$, with $\tilde{m}(z) = m(J^{-1}z)$; see [16]. Using (15),

$$\begin{aligned}|\langle \hat{f}, \widehat{S}T_x\varphi\rangle| &= |\langle f, \widehat{J^{-1}}\widehat{S}T_x\varphi\rangle| = |\langle f, \widehat{S}\mathcal{F}^{-1}T_x\varphi\rangle| \\ &= |\langle f, \widehat{S}M_x\mathcal{F}^{-1}\varphi\rangle| = |\langle f, \widehat{S}M_x\varphi\rangle|\end{aligned}$$

since the Gaussian is an eigenvector of $\mathcal{F}^{-1}$ with eigenvalue equal to 1. Moreover

$$\tilde{m}(S(x,0)^T) = m(J^{-1}S(x,0)^T) = m(SJ^{-1}(x,0)^T) = m(S(0,x)^T).$$

Hence (6) follows from (5).

(ii) Case $p = \infty$. Observe that any $z \in \mathbb{R}^{2n}$ can be written as

$$z = S^{-1}(x,0)^T,$$

for some $x \in \mathbb{R}^n$, $S \in U(2n, \mathbb{R})$, so that, for any $f \in M^\infty_m(\mathbb{R}^n)$,

$$\begin{aligned}\|f\|_{M^\infty_m(\mathbb{R}^n)} &= \sup_{z\in\mathbb{R}^{2n}} |V_\varphi f(z)| m(z) \asymp \sup_{S\in U(2n,\mathbb{R})} \sup_{x\in\mathbb{R}^n} \left|V_\varphi f(S^{-1}(x,0)^T)\right| m(S^{-1}(x,0)^T)\\ &= \sup_{S\in U(2n,\mathbb{R})} \sup_{x\in\mathbb{R}^n} |V_\varphi(\widehat{S}f)(x,0)| m(S^{-1}(x,0)^T)\\ &= \sup_{S\in U(2n,\mathbb{R})} \sup_{x\in\mathbb{R}^n} |\langle \widehat{S}f, T_x\varphi\rangle| m(S^{-1}(x,0)^T)\\ &= \sup_{S\in U(2n,\mathbb{R})} \sup_{x\in\mathbb{R}^n} |\langle f, \widehat{S}T_x\varphi\rangle| m(S(x,0)^T),\end{aligned}$$

which gives (7). Formula (8) follows as above. □

We now prove the similar result, with the group $U(2n, \mathbb{R})$ replaced by the subgroup $\mathbb{T}^n$ (up to isomorphisms).

Proof of Theorem 1.3. (i) We could follow a similar pattern to the proof of Theorem 1.2, replacing the group $U(2n, \mathbb{R})$ by $\mathbb{T}^n$. The preparation of Lemma 2.3 would be no longer necessary. Lemma 2.4 would require some small adjustments. On the other hand a more direct argument can be given. Namely, writing $z_j = (x_j, \xi_j)$ in complex notation as $r_j e^{i\theta_j}$, and setting $r = (r_1, \dots, r_n)$, $\theta = (\theta_1, \dots, \theta_n)$ we have

$$\begin{aligned}\|f\|^p_{M^p_m} &= \int_{\mathbb{R}^{2n}} |V_\varphi f(z)|^p m(z)^p \, dz\\ &= \int_{\mathbb{R}^n_+ \times [0,2\pi]^n} r_1 \cdots r_n |V_\varphi f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p m(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})^p \, dr \, d\theta.\end{aligned}$$

With S as in (9) and using Lemma 2.1, therefore we have

$$\begin{aligned}\|f\|^p_{M^p_m} &\asymp \int_{\mathbb{R}^n\times\mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi f(S(x,0)^T)|^p m(S(x,0)^T)^p \, dx \, dS\\ &= \int_{\mathbb{R}^n\times\mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi(\widehat{S}^{-1} f)(x,0)|^p m(S(x,0)^T)^p \, dx \, dS\\ &= \int_{\mathbb{R}^n\times\mathbb{T}^n} |x_1 \cdots x_n| |\langle \widehat{S}^{-1} f, T_x\varphi\rangle|^p m(S(x,0)^T)^p \, dx \, dS,\end{aligned}$$

which is (10). The characterization (11) has the same proof as the corresponding formula (6).

(ii) The M^∞ case uses the same argument as in the proofs of (7) and (8), with the group $U(2n, \mathbb{R})$ replaced by $\mathbb{T}^n$. □

4. Integral representations for the torus in terms of the fractional Fourier transform

Observe that the symplectic matrix in $U(2n,\mathbb{R})$ corresponding to the complex matrix $S\in\mathbb{T}^n$ in (9) via the isomorphism ι in (16) is given by

$$\iota(S)=\begin{pmatrix} A & -B\\ B & A\end{pmatrix}$$

with

$$A=\operatorname{diag}[\cos\theta_1,\dots,\cos\theta_n]\quad B=\operatorname{diag}[\sin\theta_1,\dots,\sin\theta_n].$$

Consider the case $\theta_i\neq k\pi$, $k\in\mathbb{Z}$, $i=1,\dots,n$. The matrix $\iota(S)$ is a free symplectic matrix and the related metaplectic operator possesses the integral representation (14). Since

$$AB^{-1}=B^{-1}A=\operatorname{diag}\left[\frac{\cos\theta_1}{\sin\theta_1},\dots,\frac{\cos\theta_n}{\sin\theta_n}\right],$$

the polynomial $W(x,x')$ becomes

$$W(x_1,\dots,x_n,x_1',\dots,x_n')=\sum_{i=1}^{n}\frac{1}{2\sin\theta_i}(\cos\theta_i x_i^2-2x_ix_i'+\cos\theta_i x_i'^2) \tag{30}$$

and

$$\Delta(W)=\frac{c}{\sqrt{|\sin\theta_1\cdots\sin\theta_n|}}.$$

for some phase factor $c\in\mathbb{C}$, with $|c|=1$. Hence we obtain, for $\psi\in\mathcal{S}(\mathbb{R}^n)$,

$$\widehat{\iota(S)}\psi(x)=\frac{c}{\sqrt{|\sin\theta_1\cdots\sin\theta_n|}}\int_{\mathbb{R}^n}e^{2\pi iW(x,x')}\psi(x')dx', \tag{31}$$

with $W(x,x')$ in (30). From (31) we deduce that $\widehat{\iota(S)}$ can be written as the composition of the operators

$$\widehat{\iota(S)}=\pm\widehat{\iota(S_1)}\cdots\widehat{\iota(S_n)}, \tag{32}$$

where, for some phase factor c,

$$\widehat{\iota(S_i)}\psi(x)=\frac{c}{\sqrt{|\sin\theta_i|}}\int_{\mathbb{R}}e^{\frac{\pi i}{\sin\theta_i}(\cos\theta_i x_i^2-2x_ix_i'+\cos\theta_i x_i'^2)}\psi(x_1',\dots,x_i',\dots,x_n')dx_i'.$$

Indeed if $\theta_i=\pi/2$, then $\widehat{\iota(S_i)}=\pm\widehat{J}$ is the Fourier transform with respect to the variable x_i. Otherwise, $\widehat{\iota(S_i)}=\pm\mathcal{F}_{\theta_i}$, the θ_i-angle partial fractional Fourier transform (again referred to the variable x_i).

Alternatively, the same conclusion (32) can be drawn by writing

$$S=\operatorname{diag}[e^{i\theta_1}\dots,e^{i\theta_n}]=\operatorname{diag}[e^{i\theta_1},1,\dots 1]\cdots\operatorname{diag}[1,\dots 1,e^{i\theta_n}] \tag{33}$$

that is

$$S=S_1\cdots S_i\cdots S_n,$$

with

$$S_i = \mathrm{diag}\,[1, \dots 1, e^{i\theta_i}, 1, \dots 1], \quad i = 1, \dots, n$$

so that

$$\widehat{\iota(S)} = \iota(S_1)\widehat{\dots \iota}(S_1) = \pm\widehat{\iota(S_1)} \cdots \widehat{\iota(S_n)}.$$

If $\theta_i = 2k\pi$ for some $k \in \mathbb{Z}$, $\widehat{\iota(S_i)} = \pm I$ with I the identity operator. If $\theta_i = (2k+1)\pi$ for some $k \in \mathbb{Z}$, $\widehat{\iota(S_i)}\psi(x) = \pm\psi(x_1, \dots, -x_i, \dots, x_n)$.

Hence using the θ_i-angle partial fractional Fourier transform $\mathcal{F}_{\theta_i} = \pm\widehat{\iota(S_i)}$ we can rephrase Theorem 1.3 as follows.

Theorem 4.1. *Let φ be the Gaussian of Theorem 1.2.*
(i) For $1 \le p < \infty$, $f \in M^p_m(\mathbb{R}^n)$, we have

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times [0,2\pi]^n} |x_1 \dots x_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi\rangle|^p m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})^p dx d\theta\right)^{\frac{1}{p}},$$

and

$$\|f\|_{M^p_m(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times [0,2\pi]^n} |\xi_1 \dots \xi_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi\rangle|^p m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n})^p d\xi d\theta\right)^{\frac{1}{p}}.$$

(ii) For $p = \infty$,

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{\theta \in [0,2\pi]^n} \sup_{x \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi\rangle| m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})$$

and

$$\|f\|_{M^\infty_m(\mathbb{R}^n)} \asymp \sup_{\theta \in [0,2\pi]^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi\rangle| m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n}).$$

We observe that this result could also be obtained by writing $\|f\|_{M^p_m(\mathbb{R}^n)}$ in terms of the weighted L^p norm of the Bargmann transform of f and using the covariance property of the Bargmann transform; the papers [18, 30] and specially [31] are relevant in this connection.

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