In the shadows of a hypergraph: looking for associated primes of powers of square-free monomial ideals

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In the Shadows of a hypergraph: looking for associated primes of powers of square-free monomial ideals

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Abstract The aim of this paper is to study the associated primes of powers of square-free monomial ideals. Each square-free monomial ideal corresponds uniquely to a finite simple hypergraph via the cover ideal construction, and vice versa. Let H be a finite simple hypergraph and $J(H)$ the cover ideal of H. We define the *shadows* of hypergraph, H, described as a collection of smaller hypergraphs related to H under some conditions. We then investigate how the shadows of H preserve information about the associated primes of the powers of $J(H)$. Finally, we apply our findings on shadows to study the persistence property of square-free monomial ideals and construct some examples exhibiting failure of containment.

Keywords cover ideals, associated primes, powers of ideals, hypergraphs

Mathematics Subject Classification (2010) 05C65, 13F55, 05E99, 13C99

1 Introduction

The primary decomposition of ideals in Noetherian rings is a fundamental result in commutative algebra and algebraic geometry. From a minimal primary decomposition, one can define the set of the associated primes by taking the radical of each ideal in the decomposition. Square-free monomial ideals and

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powers of ideals are central objects in combinatorial and commutative algebra and in algebraic geometry due to the connections they encode between these areas, see for instance [12]. Our goal in this paper is to investigate the associated primes of powers of square-free monomial ideals.

There are several ways to relate a square-free monomial ideal to a (finite simple) hypergraph. To serve our intent, we will associate to a hypergraph, H, the square-free monomial ideal with minimal primes corresponding to the edges of the hypergraph, and vice versa. This ideal is usually called the cover ideal of the hypergraph and denoted by $J(H)$. The associated primes of a square-free monomial ideal are easy to describe, whereas, computing the associated primes of a power can prove to be considerably more difficult. Currently, the set of the associated primes of a power of any square-free monomial ideal is far from being fully understood. There have been many attempts to address this problem. For instance, the authors of [2] give a description of the set $\mathrm{Ass}(J(H)^s)$ in terms of the coloring properties of the hypergraph H. Here we provide another approach, and seek to list the elements in $\mathrm{Ass}(J(H)^s)$.

The motivating idea is to take knowledge of associated primes of other hypergraphs, smaller than H, and to lift it to an associated prime of H. For ideals associated to a combinatorial object, one hopes to explain their behavior in terms of the original object. With this in mind, we define $\mathcal{S}(H)$ the shadow of a hypergraph, Definition 3.1, as a certain set of smaller hypergraphs related to the original one. We then show that the shadows preserve information about the associated primes of a power of the cover ideal of the hypergraph. For instance, we prove the following result.

Theorem 1 (Theorem 3.8) *Let $H = (V, E)$ be a hypergraph. If $G \in \mathcal{S}(H)$ is an odd cycle (i.e., $G = C_{2n+1}$ for some positive integer n), then $\mathfrak{p}_V \in \mathrm{Ass}(J(H)^2)$.*

Moreover, with the notation of Section 4, where $H' \in \mathcal{S}(H)$ and $\tilde{H}$ is a subhypergraph of H, the following theorem will give us an investigation of a specific case.

Theorem 2 (Theorem 4.5) *Let $(J(H')^s : m) = \mathfrak{p}$. Then, we have*

(a) $(J(H)^s : m) = \mathfrak{p}$ *if and only if* $(J(\tilde{H})^s : m) = \mathfrak{p}$*;*
(b) $(J(H)^s : m \cdot m_0) = \mathfrak{p} + (y)$*, for some monomial* $m_0 \notin \mathfrak{p}$*, if and only if* $(J(\tilde{H})^s : m) \neq \mathfrak{p}$*.*

The early results based on this novel construction are summarized in diagrams in Section 4.

Of particular interest to us are the examples of failure of persistence property. In [13] Kaiser et al. produced an example of a square-free monomial ideal, precisely the cover ideal of a graph, which fails the persistence property, i.e., the set of the associated primes could “lose” some elements from one power to the next. Based on this example and our findings on the shadows - Theorem 5.1, we construct an example of failure of persistence property for the case of a proper hypergraph, not being graph.

Organization of the article. In Section 2, we introduce the terminology and the basic results. In Section 3, we define the shadows of a hypergraph that are the new tool introduced in this paper. Then we start an investigation of the associated primes of a square-free monomial ideal in terms of the shadows of the associated hypergraph. In particular, in this section, we deal with the second power. In Section 4, under some restrictive conditions, we broaden our investigation to any power. Finally, in Section 5, we apply the results of Section 4 to the persistence property.

2 Notation and basic facts

Let $V := \{x_1, \dots, x_n\}$ and $R = K[V] = K[x_1, \dots, x_n]$ be the standard polynomial ring in n variables over a field K. A square-free monomial ideal $I \subseteq R$ always has a unique minimal primary decomposition, $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$, as an intersection of square-free prime ideals $\mathfrak{p}_i = (x_{i_1}, \dots, x_{i_s})$. For more details and a full description of the topic we refer to Section 1.3 in [8]. This property establishes a one-to-one correspondence between square-free monomial ideals and finite simple hypergraphs. First, recall that a (finite simple) **hypergraph**, H, is a pair $H = (V, E)$, where $V := \{x_1, \dots, x_n\}$ is called the set of vertices of H and E is a collection of subsets of V. In this paper, we will only consider finite simple hypergraphs, these are also called **clutters** in the literature. For a set $U = \{x_{i_1}, \dots, x_{i_s}\} \subseteq V$, we will denote by

$$\mathfrak{p}_U := (x_{i_1}, \dots, x_{i_s}) \subseteq R$$

the prime ideal generated by the variables in U, and by

$$x_U := x_{i_1} \cdots x_{i_s} \in R$$

the monomial given by the product of the variables in U.
Then, a hypergraph $H = (V, E)$ unequivocally corresponds to the square-free monomial ideal $J(H) := \bigcap_{e \in E} \mathfrak{p}_e$, called the **cover ideal** of H, and vice versa.

Let $H = (V, E)$ be a hypergraph. A subset T of V is a **vertex cover** of H if every edge $e \in E$ contains at least one element of T. A vertex cover T is a **minimal vertex cover** if no proper subset of T is a vertex cover. Minimal vertex covers are related to the minimal generators of $J(H)$. Indeed, T is a minimal vertex cover of H if and only if $x_T \in \mathcal{G}(J(H))$, the set of monomials which minimally generates $J(H)$. See [5] and [7] for a further investigation on cover ideals of hypergraphs.

In this paper we are interested in the study of the associated prime ideals of the (regular) powers of $J(H)$. Recall the following, classical, definition.

Definition 2.1 Let R be a ring and I an ideal of R. A prime ideal $\mathfrak{p} \subset R$ is called an **associated prime ideal** of I if there exists some element $\overline{m} \in R/I$ such that $\mathfrak{p} = \mathrm{Ann}(\overline{m})$, the annihilator of $\overline{m}$. Equivalently, a prime ideal $\mathfrak{p} \subset R$ is an associated prime ideal of I if there exists some element $m \in R$ such that $\mathfrak{p} = (I : m)$. The set of all associated prime ideals of I is denoted by $\mathrm{Ass}(I)$.

By definition, the hypergraph $H = (V, E)$ easily provides a description of all elements in $\mathrm{Ass}(J(H))$. Indeed, $\mathfrak{p}_U \in \mathrm{Ass}(J(H))$ if and only if $U \in E$. In order to describe the associated primes of the powers of $J(H)$, Lemma 2.11 in [2] is an essential tool. We recall that for a hypergraph $H = (V, E)$ and $U \subseteq V$ the **induced subhypergraph** of H on U is the hypergraph $H_U = (U, E(U))$ where $E(U) = \{e \in E \mid e \subseteq U\}$. Lemma 2.11 in [2] shows that, for a hypergraph $H = (V, E)$, and U subset of the vertex set V, there is a strong relation between the associated primes of the ideals $J(H)^s \subseteq R = K[V]$ and $J(H_U)^s \subseteq K[U]$, that is,

$$\mathfrak{p}_U \in \mathrm{Ass}(J(H)^s) \Leftrightarrow \mathfrak{p}_U \in \mathrm{Ass}(J(H_U)^s).$$

Thus, $\mathfrak{p}_U$ is associated to $J(H)^s$, if and only if it is associated to $J(H_U)^s \subseteq K[U]$, note that $\mathfrak{p}_U$ is the maximal ideal in $K[U]$. So, Lemma 2.11 in [2] ensures that in a certain sense it is enough to look if the maximal ideal is an associated prime.

Remark 2.2 An immediate consequence of Lemma 2.11 [2] is a first, well known, step in the description of the elements in $\mathrm{Ass}(J(H)^s)$. For any hypergraph H, $\mathfrak{p}_e \in \mathrm{Ass}(J(H)^s)$ for each integer $s \geq 1$ and each edge e of H.

Remark 2.3 An other consequence of Lemma 2.11 [2] will be useful in Section 4. For a hypergraph $H = (V, E)$ and $F \subseteq U \subseteq V$, since $(H_U)_F = H_F$, we have

$$\mathfrak{p}_F \in \mathrm{Ass}(J(H)^s) \Leftrightarrow \mathfrak{p}_F \in \mathrm{Ass}(J(H_U)^s),$$

where $J(H)^s \subseteq R = K[V]$ and $J(H_U)^s \subseteq K[U]$.

In the literature there are only few results explicitly describing the elements in $\mathrm{Ass}(J(H)^s)$. Most of them deal with the case that H is a graph, i.e., the edges all have cardinality 2. If H is a graph, we will often denote it by the letter G. For instance, see proposition below, the authors of [3] describe the set $\mathrm{Ass}(J(G)^2)$. They prove that the new primes match the (minimal) odd cycles of G. Recall that in a graph $G = (V, E)$ a set of distinct vertices $C = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\} \subseteq V$ is called an n-**cycle** (or cycle of length n) if $\{x_{i_j}, x_{i_{j+1}}\} \in E$ for each $j \in \{1, \ldots, n\}$ and $x_{i_{n+1}} := x_{i_1}$. We call C an odd (even) cycle if n is odd (even). The vertices x_{i_j}, $x_{i_{j+1}}$ connected by an edge $\{x_{i_j}, x_{i_{j+1}}\}$ are called **adjacent** vertices in C. A **chord** of C is an edge of G which joins two non-adjacent vertices. If C has no chord, we shall call it **chordless**. Corollary 3.4 in [3] characterizes the elements in $\mathrm{Ass}(J(G)^2)$, where G is a finite graph. The authors show that a prime ideal $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_s})$ is in $\mathrm{Ass}(J(G)^2)$ if and only if:

(a) $s = 2$ and $\mathfrak{p} \in \mathrm{Ass}(J(G))$; or
(b) s is odd, and after re-indexing, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$ is a chordless cycle of G.

3 Introducing the shadows

In this section, we introduce the definition of the shadows of a hypergraph, give some illustrative examples and present some early results obtained from our novel construction.

Definition 3.1 Let $H = (V, E)$ be a hypergraph. We say that a hypergraph $H' = (V', E')$ is a **shadow** of H if

(a) $V' \subseteq V$; and
(b) $|E| = |E'|$ (same cardinalities) and $e \cap V' \in E'$ for each $e \in E$.

The condition $|E| = |E'|$ in the above definition could look very restrictive, but it is necessary to our purposes. We will show why in Example 3.13, after developing some background.

We denote by $\mathcal{S}(H)$ the set of all the shadows of H. Note that two different elements in $\mathcal{S}(H)$ have different vertex sets. Thus $H' = (V', E') \in \mathcal{S}(H)$ will be also called the shadow of H on V'. By definition, H is always a shadow of itself on the vertex set V; we refer to this as the trivial shadow. However, not every subset of V produces a shadow of H, as we show in the following example.

Example 3.2 Consider the hypergraph H on the vertex set $V := \{x_1, \ldots, x_5\}$ with the edge set $E = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_4, x_5\}\}$. Then, the set $\mathcal{S}(H)$ contains non-trivial elements, namely, shadows on the vertex sets $V_1 := \{x_1, x_2, x_4\}$, $V_2 := \{x_1, x_3, x_4\}$, $V_3 := \{x_1, x_2, x_3, x_4\}$, $V_4 := \{x_1, x_3, x_4, x_5\}$ and $V_5 := \{x_1, x_2, x_4, x_5\}$. Indeed, we have

$$(V_1, \{\{x_1, x_2\}, \{x_2, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H), \text{ and}$$
$$(V_2, \{\{x_1, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H).$$

Both of these shadows are graphs, more precisely they are 3-cycles. Additionally, we also have the following shadows

$$(V_3, \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H),$$
$$(V_4, \{\{x_1, x_3\}, \{x_3, x_4\}, \{x_1, x_4, x_5\}\}) \in \mathcal{S}(H) \text{ and}$$
$$(V_5, \{\{x_1, x_2\}, \{x_2, x_4\}, \{x_1, x_4, x_5\}\}) \in \mathcal{S}(H).$$

Furthermore, for instance, H has no shadow on the set $V_6 := \{x_1, x_2, x_3\}$ since we get

$$(V_6, \{\{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_1\}\}),$$

and this fails to be a simple hypergraph.
The hypergraph H (in two different representations) and its shadows are showed in Figure 1 and Figure 2, where an edge $\{a, b, v_1, \ldots, v_m\}$ is depicted as the segment $a \bullet\!\overset{v_1 \ldots v_m}{\overline{\qquad}}\!\bullet b$

In the following example, we show a hypergraph which only has trivial shadow.

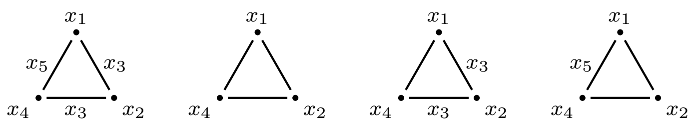

Fig. 1: The hypergraph H and its shadows on V_1, V_3 and V_5.

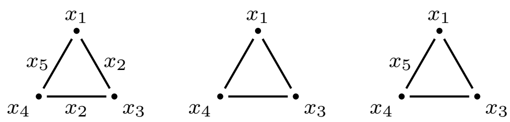

Fig. 2: An other representation of the hypergraph H and its shadows on V_2 and V_4.

Example 3.3 Let H be the hypergraph on the vertex set $V = \{x_1, \ldots, x_5\}$ with edge set $E = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_1\}, \{x_5, x_1, x_2\}\}$. In this case, the set $\mathcal{S}(H)$ has only one element, namely H. Indeed, notice that each edge of H consists of three vertices with "consecutive" indexes. Since any subset of V with two elements is contained in some edge, then H has no shadow on any set $V' \subsetneq V$. For instance, H has no shadow on the subset V' obtained from V by removing x_1 since $\{x_4, x_5\} \subset \{x_3, x_4, x_5\}$.

One has $J(H')$ is an ideal of $K[V']$ and there is a natural inclusion from $K[V']$ into $K[V]$. The ideal generated by the image of $J(H')$ under this map, i.e., the ideal generated by $\mathcal{G}(J(H')) \subseteq K[V]$, is called **cone ideal** of $J(H')$ in $K[V]$. The next lemma provides a connection between the monomial generators of $J(H)$ and $J(H')$ for a shadow H' of H.

Lemma 3.4 *Let $H = (V, E)$ be a hypergraph and $H' = (V', E') \in \mathcal{S}(H)$ a shadow of H. Then $\mathcal{G}(J(H')) \subseteq \mathcal{G}(J(H))$.*

Proof The ideal $J(H')$ is generated by monomials x_U where U is a minimal vertex cover of H'. By the definition of shadow, U is also a minimal vertex cover of H, and U does not involve the variables in $V \setminus V'$. □

Remark 3.5 From Lemma 3.4, we have $J(H') = K[V'] \cap J(H)$. Thus, each element m in $J(H')$ also belongs to $J(H)$.

As a consequence of Lemma 3.4, we get the following result.

Lemma 3.6 *If $(J(H')^s : m) = \mathfrak{p} \neq (1)$ for some prime ideal $\mathfrak{p}$, then $m \notin J(H)^s$.*

Proof Suppose that $m \in J(H)^s$, then $m = m_1 \cdots m_s M$ where the m_i's are monomial minimal generators of $J(H)$. Since m only contains the variables in V', each m_i will also have this property. That means, $m_i \in J(H')$ for all $i \in \{1, 2, \ldots, s\}$. Hence $m \in J(H')^s$, which contradicts $(J(H')^s : m) \neq (1)$. □

The next results show the first evidences that our construction really serves our purpose. We strongly use the classification in Corollary 3.4 in [3] and assume the existence of a graph $G \in \mathcal{S}(H)$. Then, we show that $J(H)^2$ only has associated primes inherited from $J(G)^2$. The following lemma can be deduced from Corollary 3.4 in [3]. We also include a proof for the convenience of the reader.

Lemma 3.7 *Let $C_{2n+1} = (V, E)$ be a $(2n+1)$-cycle. Then $(J(C_{2n+1})^2 : x_V) = \mathfrak{p}_V$.*

Proof A minimal cover of C_{2n+1} involves at least $n+1$ vertices, this implies that $J(C_{2n+1})^2$ does not contain any elements of degree $2n+1$ and in particular, $x_V \notin J(C_{2n+1})^2$. Note that $x_1 \in (J(C_{2n+1})^2 : x_V)$, indeed $x_1 \cdot x_V = x_{\{1,2,4,\ldots,2n\}} \cdot x_{\{1,3,5,\ldots,2n+1\}} \in J(C_{2n+1})^2$. Analogously, we get $x_i x_V \in J(C_{2n+1})^2$ for each $x_i \in V$. □

Theorem 3.8 *Let $H = (V, E)$ be a hypergraph. If $G \in \mathcal{S}(H)$ is an odd cycle (i.e., $G = C_{2n+1}$ for some positive integer n), then $\mathfrak{p}_V \in \mathrm{Ass}(J(H)^2)$.*

Proof Let $E = (e_1, \ldots, e_k)$. Since $G = (V', E') \in \mathcal{S}(H)$, the edges of G are given by $\{e'_1, \ldots, e'_k\}$ where $e'_i = e_i \cap V'$. By hypothesis, G is an odd cycle, so $k = 2n+1$ for some positive integer n. Without loss of generality, we relabel the vertices of G so that

$$e'_i = \begin{cases} \{x_i, x_{i+1}\}, & \text{if } 1 \le i \le 2n, \\ \{x_{2n+1}, x_1\}, & \text{if } i = 2n+1. \end{cases}$$

From Corollary 3.4 in [3], we know that $\mathfrak{p}_{V'} \in \mathrm{Ass}(J(G)^2)$, and by Lemma 3.7 we have $(J(G)^2 : x_{V'}) = \mathfrak{p}'_V$, where $x_{V'} = \prod_{i=1}^{2n+1} x_i$. Then, we claim that $(J(H)^2 : x_{V'}) = \mathfrak{p}_V$. If $x_j \in V'$, $x_j x_{V'} \in J(G)^2 \subseteq J(H)^2$. So $x_j \in (J(H)^2 : x_{V'})$. Moreover, if $y_j \in V \setminus V'$, then there exists an edge $e_i \in E$ such that $y_j \in e_i$. Without loss of generality, one can assume that $i = 1$. Thus we have that

$$y_j x_{V'} = y_j x_1 x_2 \cdots x_{2n+1} = (y_j x_3 x_5 \cdots x_{2n+1})(x_1 x_2 x_4 \cdots x_{2n}).$$

The right hand side of the above equality is in $J(H)^2$ since it is the product of two vertex covers of H. Thus, $y_j \in (J(H)^2 : x_{V'})$. Finally, $x_{V'} \notin J(H)^2$ since $x_{V'} \notin J(H')^2$. □

Example 3.9 Let H be the hypergraph in Example 3.2. Since, for instance, the shadow of H on $\{x_1, x_2, x_4\}$ is an odd cycle, we can state that

$$\mathfrak{p}_V = (x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}(J(H)^2).$$

Now we show that Theorem 3.8 works in a more general setting. We need some further notation. Let $H = (V, E)$ be a hypergraph and let $G = (V', E') \in \mathcal{S}(H)$ be a graph. Set $e' := e \cap V'$ for any $e \in E$. Then, for a subset $U \subset V'$, we denote by

$$\widehat{U} := \bigcup_{e' \subseteq U} e \subseteq V.$$

The set $\widehat{U}$ is a subset of V containing all the vertices in e, for any e in correspondence to an edge e' that is contained in U.

Corollary 3.10 *Let H be a hypergraph and H' a shadow of H. If C_{2n+1} is an odd cycle that is a subhypergraph of H', then $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{Ass}(J(H)^2)$.*

Proof Say $H' = (V', E')$. We take the subhypergraph $\tilde{H} := H_{\widehat{C}_{2n+1}}$ of H on the vertex set $\widehat{C}_{2n+1}$. Notice that $\tilde{H}$ has a shadow on C_{2n+1}. That is the odd cycle C_{2n+1}. Thus, from Corollary 3.4 in [3] and Theorem 3.8, $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{Ass}(J(\tilde{H})^2)$. Moreover, from Lemma 2.11 [2], we have $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{Ass}(J(H)^2)$. □

Corollary 3.11 *Let H be a hypergraph and $\tilde{H}$ a subhypergraph of H. If an odd cycle $C_{2n+1} \in \mathcal{S}(\tilde{H})$, then $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{Ass}(J(H)^2)$.*

Example 3.12 Let $H = (V, E)$ (see Figure 3) be the hypergraph with the vertex set

$$V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$$

and the edge set

$$E = \{\{x_1, x_2, x_6\}, \{x_2, x_3, x_6\}, \{x_3, x_4, x_8\}, \{x_4, x_5, x_6\}, \{x_1, x_5, x_7\}\}.$$

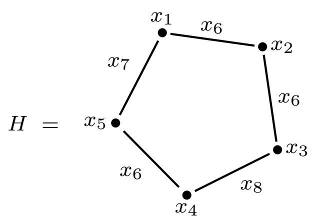

Fig. 3: A representation of the hypergraph H.

The shadow of H on the vertex set $V' = \{x_1, x_2, x_3, x_4, x_5\}$ is

$$H' = (V', \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}) \in \mathcal{S}(H).$$

We see that H' is a graph, precisely it is an odd cycle of length 5, see Figure 4. By Theorem 3.8, we have that

$$\mathfrak{p}_V = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \operatorname{Ass}(J(H)^2).$$

Now, we take the shadow of H on the vertex set $V'' = \{x_1, x_3, x_5, x_6, x_8\}$. The shadow of H on V'' is

$$H'' = (V'', \{\{x_1, x_6\}, \{x_3, x_6\}, \{x_3, x_8\}, \{x_5, x_6\}, \{x_1, x_5\}\}) \in \mathcal{S}(H).$$

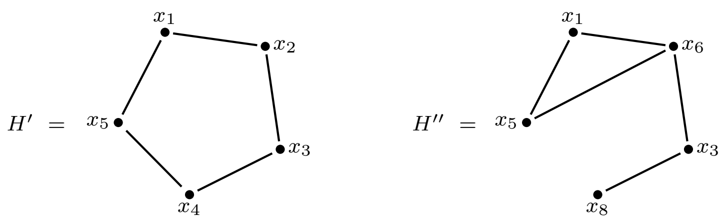

Fig. 4: The shadows of H on V' and V''.

Note that H'', see Figure 4, has a subhypergraph that is a cycle of length 3, $C_3 = \{\{x_1, x_6\}, \{x_5, x_6\}, \{x_1, x_5\}\}$. By Corollary 3.10, this cycle produces an element in $\mathrm{Ass}(J(H)^2)$. So, we get

$$\mathfrak{p}_{\widehat{C}_3} = (x_1, x_2, x_4, x_5, x_6, x_7) \in \mathrm{Ass}(J(H)^2).$$

In the following example we show that condition (b) in Definition 3.1 is strictly necessary for the validity of Theorem 3.8.

Example 3.13 Let $H = (V, E)$ be the hypergraph with the vertex set

$$V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$$

and the edge set

$$\begin{aligned} E = \{&\{x_1, x_2, x_6, x_8\}, \{x_2, x_3, x_8, x_6\}, \{x_3, x_4, x_7, x_9\}, \\ &\{x_4, x_5, x_6, x_8\}, \{x_1, x_5, x_7, x_9\}, \{x_8, x_9\}, \{x_6, x_7\}\}. \end{aligned}$$

A Macaulay2 computation [10] shows that

$$\mathrm{Ass}(J(H)^2) = \{\mathfrak{p}_e \mid e \in E\}.$$

Ignoring the rule $|E| = |E'|$ in the condition (b) of Definition 3.1, we get on the vertex set $V' = \{x_1, x_2, x_3, x_4, x_5\}$ the hypergraph

$$H' = (V', \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}) \in \mathcal{S}(H).$$

One has H' is a graph, in particular it is an odd cycle and by Lemma 3.7, we have

$$\mathfrak{p}_{V'} = (x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}(J(H')^2).$$

So, without the condition (b) in Definition 3.1, Theorem 3.8 does not hold.

In the last part of this section we prove that, under some suitable hypothesis, all the associated primes of $J(H)^2$ come from some non-trivial shadow (we will see in Proposition 3.15). We need an auxiliary lemma.

Lemma 3.14 *Let $H = (V, E)$ be a hypergraph, and suppose that $(J(H)^s : m) = \mathfrak{p}_V$ for some monomial m. Let $V' \subsetneq V$ be a proper subset such that $e_i \cap e_j \subseteq V'$ for each $e_i, e_j \in E$, $i \neq j$. Then y^{s-1} does not divide m for each $y \in V \setminus V'$.*

Proof Let y be an element in $V \setminus V'$. We write $m = y^a m'$, where, unless to rename, $y \in e_1$ and y does not divide m'. If $a \geq s$ then, since $ym \in J(H)^s$, we get $ym = m_1 \cdots m_s M$, where m_j corresponds to a minimal vertex cover of H for $j \in \{1, \ldots, s\}$. Thus y divides M and $m = m_1 \cdots m_s(M/y)$. This contradicts $m \notin J(H)^s$. Therefore, we can assume $a = s - 1$. We work by induction on $r = |e_1 \setminus V'|$. If $r = 1$, i.e., $e_1 = (e_1 \cap V') \cup \{y\}$, then from $ym \in J(H)^s$, we get $ym = (ym_1) \cdots (ym_s)M$, where ym_j are minimal vertex covers of H. Note that, for each $x_j \in e_1 \cap V'$, we can see that x_j does not divide $m_1, \ldots, m_s$ (these are minimal vertex covers) and x_j does not divide M (otherwise we can just delete y and get $m \in J(H)^s$). This implies that $m \notin (\mathfrak{p}_{e_1})^s$. To get a contradiction, we just take some $z \notin e_1$ and remember that by hypothesis $zm \in J(H)^s$ but $zm \notin (\mathfrak{p}_{e_1})^s$. If $r > 1$, i.e., $e_1 = (e_1 \cap V') \cup \{y_1, \ldots, y_r\}$, then just note that $V'' = V' \cup \{y_1, \ldots, \widehat{y_i}, \ldots, y_r\}$ satisfies the hypothesis of the theorem and $e_1 = (e_1 \cap V'') \cup \{y_i\}$. □

Proposition 3.15 *Let $H = (V, E)$ be a hypergraph and $H' = (V', E') \in \mathcal{S}(H)$ a shadow of H. Assume that $e_i \cap e_j \subseteq V'$ for each $e_i, e_j \in E$, where $i \neq j$. If $\mathfrak{p}_V \in \operatorname{Ass}(J(H)^2)$, then $\mathfrak{p}_{V'} \in \operatorname{Ass}(J(H')^2)$.*

Proof By the definition of associated primes, there exists a monomial $m \in K[V]$ such that $(J(H)^2 : m) = \mathfrak{p}_V$. Say $V' = \{x_1, \ldots, x_a\}$ and $V \setminus V' = \{y_1, \ldots, y_b\}$. By Lemma 3.14 y_j does not divide m for $j = 1, \ldots, b$. Then $m \in K[V']$ and therefore $(J(H')^2 : m) = \mathfrak{p}_{V'}$. □

4 A first case

In this section we investigate the relations between a hypergraph and its shadows in a particular case of study. Precisely, we consider shadows that only differ from the starting hypergraph by one edge and one vertex.

Throughout this section, we shall use the following notation.

Notation 4.1 *Let $H = (V, E)$ be a hypergraph and $H' = (X, E')$ a shadow of H such that*

(a) $X = \{x_1, \ldots, x_n\}$ and $V = X \cup \{y\}$; and
(b) y only belongs to one edge, say $e_y \in E$.

After renaming, say $e_y = \{x_1, \ldots, x_t, y\}$. We set $e := e'_y = \{x_1, \ldots, x_t\}$, then we have $H' = \{X, (E \setminus \{e_y\}) \cup \{e\}\}$. Moreover, to shorten the notation, $\tilde{H}$ will denote the subhypergraph of H on X. We denote by $\mathfrak{p}_e$ and $\mathfrak{p}_{e_y}$ the prime ideals generated by the variables in e and e_y respectively.

We remark that, in this setting, the hypergraphs $\tilde{H}$ and H' share the vertex set X. Moreover, they share the same edges except for e. We will abuse notation: given a subset $F \subseteq X \subseteq V$, we will write $\mathfrak{p}_F$ to denote both the ideals in $K[X]$ and in $K[V]$.

Here, we anticipate the results of this section. In the first part of the section, we investigate the relation linking associated primes of $J(\tilde{H})^s$ and $J(H')^s$

with the elements in $\mathrm{Ass}(J(H)^s)$. We have seen in Lemma 2.11 [2] that if $\mathfrak{p} \in \mathrm{Ass}(J(\tilde{H})^s)$ then $\mathfrak{p} \in \mathrm{Ass}(J(H)^s)$. What about the associated prime of $J(H')^s$? We will show that if $\mathfrak{p} \in \mathrm{Ass}(J(H')^s)$, then either $\mathfrak{p}+(y) \in \mathrm{Ass}(J(H)^s)$ or $\mathfrak{p} \in \mathrm{Ass}(J(H)^s)$. This depends on a further condition of a monomial m such that $(J(H')^s : m) = \mathfrak{p}$. The diagram in Figure 5 summarizes these results.

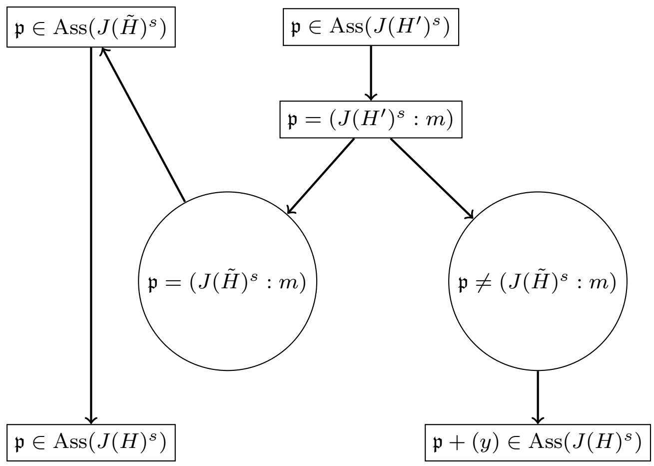

Fig. 5: The chart depicts the steps we follow in the first part of the section.

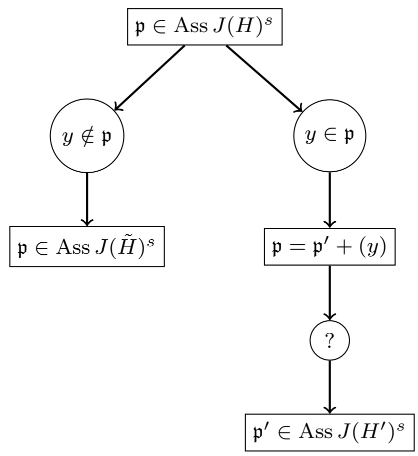

Fig. 6: The chart depicts the steps we follow in the second part of the section. The question mark means that some additional conditions are necessary for that implication.

In the second part of the section - see Figure 6, we will reverse the investigation. Starting from a prime associated to $J(H)^s$, we will look for which conditions allow us to find a relation with an element in $J(\tilde{H})^s$ or $J(H')^s$. Precisely, if $\mathfrak{p} \in \mathrm{Ass}(J(H)^s)$ and $y \notin \mathfrak{p}$ then $\mathfrak{p} \in \mathrm{Ass}(J(\tilde{H})^s)$. Moreover, if $\mathfrak{p} = (y) + \mathfrak{p}'$, it seems natural to ask if $\mathfrak{p}' \in \mathrm{Ass}(J(H')^s)$, which we positively answer under an extra (restrictive) condition. We will show in the next section, see Example 5.3, that not all the primes $(y) + \mathfrak{p}'$ associated to $J(H)^s$ come from a prime $\mathfrak{p}'$ in the shadow.

We start with an auxiliary result.

Lemma 4.2 *Let $m \in \mathcal{G}(J(H))$ be a monomial minimal generator of $J(H)$. If $y|m$, then $x_i \not| m$ for all $x_i \in e$.*

Proof In our setting, y only belongs to the edge $e_y = \{x_1, \ldots, x_t, y\}$. Since m is a minimal vertex cover of H, if $x_i \in e = \{x_1, \ldots, x_t\}$ divides m, then $\dfrac{m}{y}$ is also a vertex cover. This contradicts the minimality of m. □

In order to relate the associated primes of $J(H')^s$ to the associated primes of $J(H)^s$, the following proposition will be crucial.

Proposition 4.3 *Let $(J(H')^s : m) = \mathfrak{p}_F$ be a prime ideal, for some $F \subseteq X$. Then we have,*

$$(J(H)^s : m) = \mathfrak{p}_F + \mathfrak{q},$$

where $\mathfrak{q} \subseteq (y)$. In other words, no monomial only involving the variables in $X \setminus F$ belongs to $(J(H)^s : m)$.

Proof Say $F := \{x_{i_1}, \ldots, x_{i_k}\}$ and $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$. Recall that $e = \{x_1, \ldots, x_t\}$. First we show that $\mathfrak{p}_F \subseteq (J(H)^s : m) \subsetneq (1)$. From Lemma 3.6 we have $m \notin J(H)^s$ and then $(J(H)^s : m) \neq (1)$. By hypothesis, for each $x_j \in F$ we have $x_j m \in J(H')^s$ i.e. $m = m_1 \cdots m_s M$ for some monomials $m_i \in J(H') \subseteq K[X]$. But these monomials, see Remark 3.5 also belongs to $J(H)$. Hence, $x_j m \in J(H)^s$ and $\mathfrak{p}_F \subseteq (J(H)^s : m) \subseteq K[V]$.

In order to conclude the proof, take any monomial $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t}$ in variables in $X \setminus F$. Suppose that $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t} m = m_1 \cdots m_s M \in J(H)^s$, where the m_j's are minimal generators of $J(H)$ in the variables in X. The monomials $m_j \in J(H')$ and then $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t} \in (J(H')^s : m) = \mathfrak{p}_F$, which is a contradiction. □

Lemma 4.4 *Let $(J(H)^s : m) = \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then $(J(\tilde{H})^s : m) = \mathfrak{p}$.*

Proof Say $\mathfrak{p} = \mathfrak{p}_F$ for some $F \subseteq X$. First note that $m \notin J(\tilde{H})^s$. Indeed, if $m = m_1 \cdots m_s \cdot M \in J(\tilde{H})^s$ with $m_1, \ldots, m_s$ minimal vertex covers of $\tilde{H}$, then $y^s m \in J(H)^s$. This contradicts $(J(H)^s : m) = \mathfrak{p}$. We claim that $(J(\tilde{H})^s : m) \supseteq \mathfrak{p}$. Indeed, if $x_j \in F$, then $x_j m \in J(H)^s \subseteq J(\tilde{H})^s$. In order to obtain the assertion, we take a monomial $T \notin \mathfrak{p}_F$ and assume that $Tm \in J(\tilde{H})^s$. Again from $Tm = m_1 \cdots m_s \cdot M \in J(\tilde{H})^s$ with $m_1, \ldots, m_s$ minimal vertex covers of $\tilde{H}$, we get $Ty^s \in (J(H)^s : m)$ which contradicts the hypothesis. □

Theorem 4.5 *Let $(J(H')^s : m) = \mathfrak{p}$. Then, we have*

(a) $(J(H)^s : m) = \mathfrak{p}$ if and only if $(J(\tilde{H})^s : m) = \mathfrak{p}$;

(b) $(J(H)^s : m \cdot m_0) = \mathfrak{p} + (y)$, *for some monomial* $m_0 \notin \mathfrak{p}$, *if and only if* $(J(\tilde{H})^s : m) \neq \mathfrak{p}$.

Proof Note that $y \notin \mathfrak{p}$, so one has the implication in (a) follows from Lemma 4.4. Set $\mathfrak{p}_F := \mathfrak{p} = (J(\tilde{H})^s : m)$ and say $X \setminus F = \{x_{\ell_1}, \ldots, x_{\ell_r}\}$. By Proposition 4.3, we have $(J(H)^s : m) = \mathfrak{p} + \mathfrak{q}$ where either $\mathfrak{q} = (0)$ or $\mathfrak{q}$ is minimally generated by monomials $y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r}$ for some $a > 0$ and $a_1, \ldots, a_r \geq 0$. We claim that $\mathfrak{q} = (0)$. Indeed, if $T := y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \in \mathfrak{q}$, we get $\dfrac{T}{y^a} \in (J(\tilde{H})^s : m) = \mathfrak{p}_F$ which contradicts the hypothesis.

Now we prove item (b). With the notation as above, we have $(J(H)^s : m) = \mathfrak{p} + \mathfrak{q}$. First we assume that $(J(\tilde{H})^s : m) \neq \mathfrak{p}$. Then $\mathfrak{q}$ is not the zero ideal. Consider the non-empty set

$$\{b \in \mathbb{N} \mid y^b \text{ divides } M \text{ for some } M \in \mathfrak{q}\},$$

and let a be its minimum element. Let $T := y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \in \mathfrak{q}$ be a monomial minimal generator in $\mathfrak{q}$. We collect some relevant facts:

- $a > 0$, by Proposition 4.3;
- $m\dfrac{T}{y} \notin J(H)^s$, by the minimality of T;
- $x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \cdot m\dfrac{T}{y} \notin J(H)^s$, by the minimality of a;
- $y \cdot m\dfrac{T}{y} = mT \in J(H)^s$.

Then, we get $\left(J(H)^s : m\dfrac{T}{y}\right) = \mathfrak{p} + (y)$, and $\mathfrak{p} + (y) \in \operatorname{Ass}(J(H)^s)$.
Vice versa, assume $(J(H)^s : m \cdot m_0) = \mathfrak{p} + (y)$, for some monomial $m_0 \notin \mathfrak{p}$. So, we have $ymm_0 \in J(H)^s$ and say $ymm_0 = ym_1 \cdot m_2 \cdots m_s \cdot M \in J(H)^s$ with $ym_1, \ldots, m_s$ corresponding to minimal vertex covers of H. Then, we get $mm_0 = m_1 \cdot m_2 \cdots m_s \cdot M \in J(\tilde{H})^s$, i.e., $m_0 \in (J(\tilde{H})^s : m)$. Since m_0 does not involve the variables in $\mathfrak{p}$, we get a contradiction. □

In particular, the next result shows that item (a) in Theorem 4.5 is always satisfied if $\mathfrak{p}_e \not\subseteq \mathfrak{p}$.

Proposition 4.6 *Let* $(J(H')^s : m) = \mathfrak{p}$. *If* $\mathfrak{p}_e \not\subseteq \mathfrak{p}$, *then* $(J(H)^s : m) = \mathfrak{p}$.

Proof Say $\mathfrak{p} = \mathfrak{p}_F$ with $F := \{x_{i_1}, \ldots, x_{i_k}\}$ and $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$. By Proposition 4.3 we have $(J(H) : m) = \mathfrak{p} + \mathfrak{q}$ where $\mathfrak{q}$ is an ideal minimally generated by monomials which are not only in variables $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$; i.e., a minimal generator of $\mathfrak{q}$ is a monomial $y^b x_{\ell_1}^{a_1} \cdots x_{\ell_r}^{a_r}$ for some $a_1, \ldots, a_r \geq 0$ and $b > 0$. Assume on the contrary that $\mathfrak{q} \neq 0$. Take any minimal generator in $\mathfrak{q}$, say $T := y^b x_{\ell_1}^{a_1} \cdots x_{\ell_r}^{a_r}$. Then $m \cdot T = m_1 \cdots m_s \cdot M \in J(H)^s$ where the m_i's are minimal vertex covers of H. Note that y does not divide M. Otherwise, we get $x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} y^{b-1} \in (J(H)^s : m)$, contradicting the minimality of T. Then we can write (after relabeling, $m_i = ym_i'$ for $i = 1, \ldots, b$) $m \cdot T =$

$(ym'_1)\cdots(ym'_b)\cdot m_{b+1}\cdots m_s\cdot M\in J(H)^s$. Say $x_1\in\mathfrak{p}_e$ and $x_1\notin\mathfrak{p}$, then we get

$$m\cdot T\frac{x_1^b}{y^b}=(x_1m'_1)\cdots(x_1m'_b)\cdot m_{b+1}\cdots m_s\cdot M\in J(H)^s.$$

Additionally, $m\cdot T\frac{x_1^b}{y^b}$ only contains variables of X. Then $T\frac{x_1^b}{y^b}\in(J(H'):m)=\mathfrak{p}$. By Proposition 4.3, this is a contradiction since $T\frac{x_1^b}{y^b}$ only contains variables not in $\mathfrak{p}$. □

Recall that by Lemma 2.11 [2], a prime associated to $J(H)^s$ either belongs to $\mathrm{Ass}(J(\tilde{H})^s)$ or it contains the variable y. This is summarized in the following statement.

Corollary 4.7 *We have*

$$\mathrm{Ass}(J(H)^s)=\mathrm{Ass}(J(\tilde{H})^s)\cup\mathcal{A},$$

where if $\mathfrak{p}\in\mathcal{A}$*, then* $y\in\mathfrak{p}$.

Question 4.8 Do the elements in $\mathcal{A}$, mentioned in Corollary 4.7, all come from the shadows? More precisely, if $\mathfrak{p}=\mathfrak{p}'+(y)\in\mathrm{Ass}(J(H)^s)$, then is there $\mathfrak{p}'\in\mathrm{Ass}(J(H')^s)$?

We will show in the next section, see Example 5.3, that such question has in general a negative answer. But, in the next theorem, we positively answer this question under a suitable condition.

Theorem 4.9 *Let* $\mathfrak{p}=\mathfrak{p}'+(y)\in\mathrm{Ass}(J(H)^s)$*. If* $\mathfrak{p}\notin\mathrm{Ass}(J(H)^s:y)$*, then* $\mathfrak{p}'\in\mathrm{Ass}(J(H')^s)$*.*

Proof Take the short exact sequence

$$0\to\frac{K[V]}{(J(H)^s:y)}\to\frac{K[V]}{J(H)^s}\to\frac{K[V]}{J(H)^s+(y)}\to 0.$$

By Theorem 6.3 in [11] we have that

$$\mathrm{Ass}(J(H)^s)\subseteq\mathrm{Ass}(J(H)^s:y)\ \cup\ \mathrm{Ass}(J(H)^s+(y)).$$

Denoted by J' the cone ideal of $J(H')^s$ in the ring $K[V]$, we note that

$$K[V]/J(H)^s+(y)=K[V]/J'+(y).$$

Since, by hypothesis $\mathfrak{p}\in\mathrm{Ass}(J(H)^s+(y))$, then $\mathfrak{p}\in\mathrm{Ass}(J'+(y))$, i.e. $\mathfrak{p}'\in\mathrm{Ass}(J(H')^s)$. □

Remark 4.10 Question 4.8 has a positive answer if $(J(H)^s:m)=\mathfrak{p}+(y)$ for some $m\in K[X]$.

In the next examples we show how to describe all the associated prime ideals of $J(H)^s$ from $\mathrm{Ass}(J(\tilde{H})^s)$ and $\mathrm{Ass}(J(H')^s)$.

Example 4.11 Let H be the hypergraph on the vertex set $V = \{x_1, \dots, x_5, y\}$ and the edge set

$$E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}, \{x_1, x_3, y\}\}.$$

Set $X := \{x_1, x_2, x_3, x_4, x_5\}$. Then the shadow of H on X is

$$H' = (X, \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}, \{x_1, x_3\}\}).$$

Moreover, the subhypergraph of H on X is

$$\tilde{H} = (X, \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}).$$

The software Macaulay2 allows to compute the sets $\mathrm{Ass}(J(H')^3) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_1, x_5), (x_1, x_3)\} \cup \{(x_1, x_2, x_3)\}$ and $\mathrm{Ass}(J(\tilde{H})^3) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_1, x_5)\} \cup \{(x_1, x_2, x_3, x_4, x_5)\}$.
From Theorem 4.5, we know that $(x_1, x_2, x_3, y) \in \mathrm{Ass}(J(H)^3)$. Moreover, one can check that $\mathrm{Ass}(J(H)^3) = \mathrm{Ass}(J(\tilde{H})^3) \cup \{(x_1, x_3, y), (x_1, x_2, x_3, y)\}$.

Example 4.12 Let H be the hypergraph on the vertex set $V = \{x_1, \dots, x_5, y\}$ given by

$$H = (V, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5, y\}\}).$$

Set $X := \{x_1, x_2, x_3, x_4, x_5\}$; then the shadow of H on X is

$$H' = \big(X, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5\}\}\big).$$

Moreover, the subhypergraph of H on X is

$$\tilde{H} = \big(X, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}\}\big).$$

Using Macaulay2, we compute that

$$\begin{aligned}\mathrm{Ass}(J(H')^2) = \{&(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5), (x_4, x_5)\}\cup \\ &\cup\{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_2, x_4, x_5)\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ass}(J(H')^3) = \{&(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5), (x_4, x_5)\}\cup \\ &\cup\{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_2, x_4, x_5)\}\cup \\ &\cup\{(x_1, x_2, x_3, x_4, x_5)\}.\end{aligned}$$

We also know that $\mathrm{Ass}(J(\tilde{H})^3)$ and $\mathrm{Ass}(J(\tilde{H})^2)$ share the same elements, precisely

$$\begin{aligned}&\{(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5)\} \\ &\cup\{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_1, x_2, x_3, x_4, x_5)\}.\end{aligned}$$

Then, from Proposition 3.15 and Theorem 4.5, we have

$$\mathrm{Ass}(J(H)^2) = \mathrm{Ass}(J(\tilde{H})^2) \cup \{(x_4, x_5, y), (x_2, x_4, x_5, y)\}.$$

What about $\mathrm{Ass}(J(H)^3)$? The element $(x_1, x_2, x_3, x_4, x_5)$ appears both in $\mathrm{Ass}(J(\tilde{H})^3)$ and $\mathrm{Ass}(J(H')^3)$ and it contains (x_4, x_5). One can check that

$$\mathrm{Ass}(J(H')^3 : m) = (x_1, x_2, x_3, x_4, x_5)$$

and

$$\mathrm{Ass}(J(\tilde{H})^3 : m) = (x_1, x_2, x_3, x_4, x_5)$$

where $m := x_1 x_2^2 x_3 x_4^2 x_5^2$.

Thus, by Theorem 4.5, $(x_1, x_2, x_3, x_4, x_5, y) \notin \mathrm{Ass}(J(H)^3)$ and one can check that

$$\mathrm{Ass}(J(H)^3) = \mathrm{Ass}(J(\tilde{H})^3) \cup \{(x_4, x_5, y), (x_2, x_4, x_5, y)\}.$$

Example 4.13 Let H be the hypergraph on the vertex set $V := \{x_1, \ldots, x_5, y\}$, given by

$$H = (V, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5, y\}, \{x_2, x_3, x_4, x_5\}\}).$$

Let H' be the shadow of H on the vertex set $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $\tilde{H}$ the subhypergraph of H on X. Then

$$H' = (X, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_3, x_4, x_5\}\})$$

and

$$\tilde{H} = (X, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3, x_4, x_5\}\}).$$

A Macaulay2 computation shows that the elements in $\mathrm{Ass}(J(H')^2$ are

$$\{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5)\}$$

and also

$$\mathrm{Ass}(J(\tilde{H})^2) = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5)\}.$$

Note that the element $(x_1, x_2, x_3, x_4, x_5)$ belongs to both the sets $\mathrm{Ass}(J(\tilde{H})^3)$ and $\mathrm{Ass}(J(H')^3)$, and it contains (x_1, x_5). One can check that

$$(J(H')^2 : x_1 x_2 x_3 x_4 x_5) = (x_1, x_2, x_3, x_4, x_5)$$

but $x_1 x_2 x_3 x_4 x_5 \in J(\tilde{H})^2$. Thus

$$\mathrm{Ass}(J(H)^2) = \mathrm{Ass}(J(\tilde{H})^2) \cup \{(x_1, x_5, y), (x_1, x_2, x_3, x_4, x_5, y)\}.$$

5 An application to the persistence property

In this section, we apply the results of Section 4 to the persistence problem. A square-free monomial ideal I is said to have the persistence property if $\mathrm{Ass}(I^s) \subseteq \mathrm{Ass}(I^{s+1})$ for any integer $s > 0$. The authors of [13] describe an example of a cover ideal of a graph failing the persistence property. We show how to construct, starting from a hypergraph whose cover ideal fails the persistence property, a new hypergraph whose cover ideal fails such property. We use the notation introduced in Section 4.

Theorem 5.1 *Let $H = (V, E)$ be a hypergraph where $V = X \cup \{y\}$ such that*

(a) there exists only one edge $e_y \in E$ containing y;
(b) H has a shadow on X, say $H' = (X, E') \in \mathcal{S}(H)$.

Suppose that $J(H')$ fails the persistence property and let $\mathfrak{p}' \in \mathrm{Ass}(J(H')^s)$ and $\mathfrak{p}' \notin \mathrm{Ass}(J(H')^{s+1})$ for some $s > 0$. Set $\tilde{H} := H_X$ the subhypergraph of H on X. If the following conditions hold,

(c) $\mathfrak{p}' \notin \mathrm{Ass}(J(\tilde{H})^s)$; and
(d) $\mathfrak{p}' + (y) \notin \mathrm{Ass}(J(H)^{s+1} : y)$,

then $J(H)$ fails the persistence property.

Proof By hypothesis we have $\mathfrak{p}' \in \mathrm{Ass}(J(H')^s)$ and $\mathfrak{p}' \notin \mathrm{Ass}(J(\tilde{H})^s)$. So by Theorem 4.5, one gets that

$$\mathfrak{p}' + (y) \in \mathrm{Ass}(J(H)^s).$$

Moreover, the hypothesis also ensures that $\mathfrak{p}' \in \mathrm{Ass}(J(H')^{s+1})$ and $\mathfrak{p}' + (y) \notin \mathrm{Ass}(J(H)^{s+1} : y)$. Thus, by Theorem 4.9 we have $\mathfrak{p}' + (y) \notin \mathrm{Ass}(J(H)^{s+1})$. $\square$

Example 5.2 In [13], Theorem 11 provides an example of a graph failing the persistence property. The graph, denoted by H_4, has the vertex set on $X := \{x_1, \dots, x_{12}\}$ and the edge set

$$\begin{aligned} E := \{&\{x_1, x_2\}, \{x_1, x_5\}, \{x_1, x_9\}, \{x_1, x_{12}\}, \{x_2, x_3\}, \{x_2, x_6\}, \{x_2, x_{10}\},\\ &\{x_3, x_4\}, \{x_3, x_7\}, \{x_3, x_{11}\}, \{x_4, x_8\}, \{x_4, x_9\}, \{x_4, x_{12}\}, \{x_5, x_6\},\\ &\{x_5, x_8\}, \{x_5, x_9\}, \{x_6, x_7\}, \{x_6, x_{10}\}, \{x_7, x_8\}, \{x_7, x_{11}\}, \{x_8, x_{12}\},\\ &\{x_9, x_{10}\}, \{x_{10}, x_{11}\}, \{x_{11}, x_{12}\}\}. \end{aligned}$$

The persistence property fails since $\mathrm{Ass}(J(H_4)^3) \not\subseteq \mathrm{Ass}(J(H_4)^4)$. In particular, $\mathfrak{p}_X \in \mathrm{Ass}(J(H_4)^3) \setminus \mathrm{Ass}(J(H_4)^4)$.We consider now the hypergraph H (see Figure 7) on vertex set $V := X \cup \{y\}$, constructed from H_4 by adding the variable "y" only to the edge $\{x_1, x_2\}$:

$$H = (X \cup \{y\}, (E \setminus \{\{x_1, x_2\}\}) \cup \{\{x_1, x_2, y\}\}).$$

With this construction, H_4 is the shadow of H on the set X. Moreover, the subhypergraph of H on X is $\tilde{H} = (X, E \setminus \{\{x_1, x_2\}\})$.

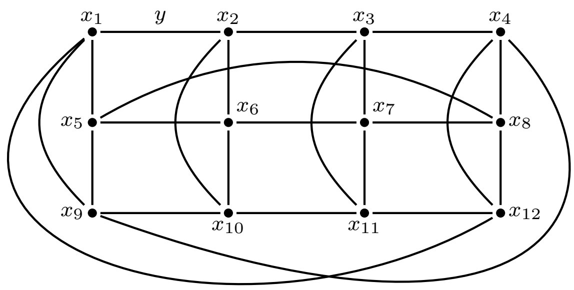

Fig. 7: The hypergraph H constructed from H_4 by adding the vertex y.

By Theorem 5.1, H fails the persistence property and

$$\mathfrak{p}_V \in \operatorname{Ass}(J(H)^3) \setminus \operatorname{Ass}(J(H)^4).$$

One can check that, by using Macaulay2, actually $\mathfrak{p}_X \notin \operatorname{Ass}(J(\tilde{H})^3)$ and also $\mathfrak{p}_V \notin \operatorname{Ass}(J(H)^4 : y)$.

Example 5.3 Take the hypergraph H' on the vertex set $X = \{x_1, \ldots, x_{12}, x_{13}\}$ and the edge set

$$\begin{aligned} E = \{&\{x_1, x_2, x_{13}\}, \{x_1, x_5\}, \{x_1, x_9, x_{13}\}, \{x_1, x_{12}, x_{13}\}, \{x_2, x_3, x_{13}\}, \\ &\{x_2, x_6, x_{13}\}, \{x_2, x_{10}, x_{13}\}, \{x_3, x_4, x_{13}\}, \{x_3, x_7, x_{13}\}, \{x_3, x_{11}, x_{13}\}, \\ &\{x_4, x_8, x_{13}\}, \{x_4, x_9, x_{13}\}, \{x_4, x_{12}, x_{13}\}, \{x_5, x_6, x_{13}\}, \{x_5, x_8, x_{13}\}, \\ &\{x_5, x_9, x_{13}\}, \{x_6, x_7, x_{13}\}, \{x_6, x_{10}, x_{13}\}, \{x_7, x_8, x_{13}\}, \{x_7, x_{11}, x_{13}\}, \\ &\{x_8, x_{12}, x_{13}\}, \{x_9, x_{10}, x_{13}\}, \{x_{10}, x_{11}, x_{13}\}, \{x_{11}, x_{12}, x_{13}\}\}. \end{aligned}$$

It was constructed from H_4, see example 5.2, by adding a new variable "x_{13}" to all the edges but $\{x_1, x_5\}$. Consider now the hypergraph H on vertex set $V := X \cup \{y\}$, constructed from H' by adding the variable "y" only to the edge $\{x_1, x_5\}$:

$$H = (X \cup \{y\}, (E \setminus \{\{x_1, x_5\}\}) \cup \{\{x_1, x_5, y\}\}).$$

With this construction, H' is the shadow of H on the set X. A computation with Macaulay2 shows that $\mathfrak{p}_V \in J(H)^4$, but $\mathfrak{p}_X \notin J(H')^4$. Indeed, we found two minimal monomials m_1, m_2 such that $\mathfrak{p}_V = (J(H)^4 : m_1) = (J(H)^4 : m_2)$ that are

$$\begin{aligned} m_1 &:= x_1^2 x_2^3 x_3^3 x_4^2 x_5^2 x_6^3 x_7^2 x_8^3 x_9^3 x_{10}^2 x_{11}^3 x_{12}^3 y, \\ m_2 &:= x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 x_7^2 x_8^2 x_9^2 x_{10}^2 x_{11}^2 x_{12}^2 x_{13} y. \end{aligned}$$

Both are divisible by y.
Then the conditions in the statement of Theorem 5.1 are not satisfied. By Theorem 4.9, we get $\mathfrak{p}_V \in \operatorname{Ass}(J(H)^4 : y)$. Using Macaulay2, one can check that even if the hypergraph H' fails the persistence property, and in particular $\mathfrak{p}_X \in \operatorname{Ass}(J(H')^3) \setminus \operatorname{Ass}(J(H')^4)$, we have $\operatorname{Ass}(J(H)^3) \subseteq \operatorname{Ass}(J(H)^4)$.

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