On numbers divisible by the product of their nonzero base b digits

ON NUMBERS DIVISIBLE BY THE PRODUCT OF THEIR NONZERO BASE b DIGITS

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ABSTRACT. For each integer $b \geq 3$ and every $x \geq 1$, let $\mathcal{N}_{b,0}(x)$ be the set of positive integers $n \leq x$ which are divisible by the product of their nonzero base b digits. We prove bounds of the form $x^{\rho_{b,0}+o(1)} < \#\mathcal{N}_{b,0}(x) < x^{\eta_{b,0}+o(1)}$, as $x \to +\infty$, where $\rho_{b,0}$ and $\eta_{b,0}$ are constants in $]0,1[$ depending only on b. In particular, we show that $x^{0.526} < \#\mathcal{N}_{10,0}(x) < x^{0.787}$, for all sufficiently large x. This improves the bounds $x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$, which were proved by De Koninck and Luca.

1. Introduction

Let $b \geq 2$ be an integer. Then, every positive integer n has a unique representation as

$$n = \sum_{j=0}^{\ell} d_j b^j, \quad d_0, \dots, d_\ell \in \{0, \dots, b-1\}, \quad d_\ell \neq 0,$$

where $d_0, \dots, d_\ell$ are the *base b digits* of n. Positive integers whose base b digits obey certain restrictions have been investigated by several authors. For instance, an asymptotic formula for the counting function of *b-Niven numbers*, that is, positive integers divisible by the sum of their base b digits, has been proved by De Koninck, Doyon, and Kátai [4], and (independently) by Mauduit, Pomerance, and Sárközy [9]. Also, arithmetic properties of integers with a fixed sum of their base b digits have been studied by Luca [8], Mauduit and Sárközy [10]. Moreover, prime numbers with specific restrictions on their base b digits have been investigated by Bourgain [1, 2] and Maynard [11, 12] (see [3, 7] for similar works on almost primes and squarefree numbers).

Let $p_b(n)$ be the product of the base b digits of n, and let $p_{b,0}(n)$ be the product of the nonzero base b digits of n. For all $x \geq 1$, define the sets

$$\mathcal{N}_b(x) := \big\{n \leq x : p_b(n) \mid n\big\} \quad \text{and} \quad \mathcal{N}_{b,0}(x) := \big\{n \leq x : p_{b,0}(n) \mid n\big\}.$$

Note that $\mathcal{N}_b(x) \subseteq \mathcal{N}_{b,0}(x)$ and that $n \in \mathcal{N}_b(x)$ implies that all the base b digits of n are nonzero. Furthermore, $\mathcal{N}_2(x) = \big\{2^k - 1 : k \geq 1\big\}$ and $\mathcal{N}_{2,0}(x) = \mathbb{N}$. Hence, in what follows, we will focus only on the case $b \geq 3$.

De Koninck and Luca [5] (see also [6] for the correction of a numerical error in [5]) studied $\mathcal{N}_{10}(x)$ and $\mathcal{N}_{10,0}(x)$. They proved the following bounds.

Theorem 1.1. *We have*

$$x^{0.122} < \#\mathcal{N}_{10}(x) < x^{0.863}$$

and

$$x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$$

for all sufficiently large x.

In this paper, we prove some bounds for the cardinalities of $\mathcal{N}_b(x)$ and $\mathcal{N}_{b,0}(x)$. In particular, for $b = 10$, we get the following improvement of three of the bounds of Theorem 1.1.

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Theorem 1.2. *We have*

$$\#\mathcal{N}_{10}(x) < x^{0.717}$$

and

$$x^{0.526} < \#\mathcal{N}_{10,0}(x) < x^{0.787}$$

for all sufficiently large x.

Notation. We use the Landau–Bachmann "little oh" notation o, as well as the Vinogradov symbol $\ll$. We omit the dependence on b of the implied constants. We write $P(n)$ for the greatest prime factor of an integer $n > 1$. As usual, $\pi(x)$ denotes the number of prime numbers not exceeding x. We write ν_p for the p-adic valuation.

2. Upper bounds

For every $s \geq 0$, let us define

$$\zeta_b(s) := \sum_{d=1}^{b-1} \frac{1}{d^s}.$$

We give the following upper bounds for $\#\mathcal{N}_{b,0}(x)$ and $\#\mathcal{N}_b(x)$.

Theorem 2.1. *Let $b \geq 3$ be an integer. We have*

$$\#\mathcal{N}_{b,0}(x) < x^{\eta_{b,0}+o(1)},$$

as $x \to +\infty$, where

$$\eta_{b,0} := 1 + \frac{1}{(1+s_{b,0})\log b} \log\left(\frac{1+\zeta_b(s_{b,0})}{b}\right) \in \,]0,1[$$

and $s_{b,0}$ is the unique solution of the equation

$$\frac{(1+s)\zeta_b'(s)}{1+\zeta_b(s)} - \log\left(\frac{1+\zeta_b(s)}{b}\right) = 0 \tag{1}$$

over the positive real numbers.

Theorem 2.2. *Let $b \geq 3$ be an integer. We have*

$$\#\mathcal{N}_b(x) < x^{\eta_b+o(1)},$$

as $x \to +\infty$, where $\eta_3 := \log 2/\log 3$,

$$\eta_b := 1 + \frac{1}{(1+s_b)\log b} \log\left(\frac{\zeta_b(s_b)}{b}\right), \quad b \geq 4,$$

and s_b is the unique solution of the equation

$$\frac{(1+s)\zeta_b'(s)}{\zeta_b(s)} - \log\left(\frac{\zeta_b(s)}{b}\right) = 0 \tag{2}$$

over the positive real numbers.

We remark that for $b = 3$ the bound of Theorem 2.2 is obvious. Indeed, it is an easy consequence of the fact that all the base 3 digits of each $n \in \mathcal{N}_3(x)$ are equal to 1 or 2. We included it just for completeness.

Using the PARI/GP [13] computer algebra system, the author computed $s_{10,0} = 1.286985\ldots$ and $s_{10} = 1.392189\ldots$, which in turn give $\eta_{10,0} = 0.7869364\ldots$ and $\eta_{10} = 0.7167170\ldots$ Hence, the upper bounds of Theorem 1.2 follow.

Proof of Theorem 2.1. First, we shall prove that Equation (1) has a unique positive solution. For $s \geq 0$, let

$$F_b(s) := \frac{(1+s)\zeta_b'(s)}{1+\zeta_b(s)} - \log\left(\frac{1+\zeta_b(s)}{b}\right).$$

Since $b \geq 3$, we have

$$F_b(0) = -\frac{\log((b-1)!)}{b} < 0 \quad \text{and} \quad \lim_{s \to +\infty} F_b(s) = \log\left(\frac{b}{2}\right) > 0. \tag{3}$$

Furthermore, a bit of computation shows that

$$F_b'(s) = \frac{(1+s)\left((1+\zeta_b(s))\zeta_b''(s) - (\zeta_b'(s))^2\right)}{(1+\zeta_b(s))^2} > 0, \tag{4}$$

for all $s \geq 0$, since, by Cauchy–Schwarz inequality, we have

$$(\zeta_b'(s))^2 = \left(-\sum_{d=1}^{b-1} (\log d) d^{-s}\right)^2 < \left(\sum_{d=1}^{b-1} d^{-s}\right)\left(\sum_{d=1}^{b-1} (\log d)^2 d^{-s}\right) = \zeta_b(s)\zeta_b''(s). \tag{5}$$

At this point, by (3) and (4), it follows that Equation (1) has a unique positive solution.

Let us assume $x \geq 1$ sufficiently large, and let $\alpha \in \,]0,1[$ be a constant (depending on b) to be determined later. Also, let P_b be the greatest prime number less than b, and define the set

$$\mathcal{N}_b'(x) := \left\{n \leq x : d \mid n \text{ for some } d > x^\alpha \text{ with } P(d) \leq P_b\right\}.$$

Suppose $n \in \mathcal{N}_b'(x)$. Then there exists $d > x^\alpha$ with $P(d) \leq P_b$ such that $d \mid n$. Clearly, for any fixed d, there are at most x/d possible values for n. Moreover, setting

$$\mathcal{S}(t) := \left\{d \leq t : P(d) \leq P_b\right\},$$

it follows easily that $\#\mathcal{S}(t) \ll (\log t)^{\pi(P_b)}$ for all $t > 2$. Therefore, we have

$$\#\mathcal{N}_b'(x) \leq \sum_{x^\alpha < d \leq x} \frac{x}{d} = x\left(\left.\frac{\#\mathcal{S}(t)}{t}\right|_{t=x^\alpha}^{x} + \int_{t=x^\alpha}^{x} \frac{\#\mathcal{S}(t)}{t^2}\,\mathrm{d}t\right) \ll (\log x)^{\pi(P_b)}\left(1 + x^{1-\alpha}\right),$$

and consequently

$$\#\mathcal{N}_b'(x) < x^{1-\alpha+o(1)}, \tag{6}$$

as $x \to +\infty$.

Now suppose $n \in \mathcal{N}_{b,0}''(x) := \mathcal{N}_{b,0}(x) \setminus \mathcal{N}_b'(x)$. Put $N := \lfloor \log x / \log b \rfloor + 1$, so that n has at most N base b digits. For each $d \in \{1, \ldots, b-1\}$, let N_d be the number of base b digits of n which are equal to d. Also, let $N_0 := N - (N_1 + \cdots + N_{b-1})$. Hence, $N_0, \ldots, N_{b-1}$ are nonnegative integers such that $N_0 + \cdots + N_{b-1} = N$. Furthermore,

$$p_{b,0}(n) = 1^{N_1} \cdots (b-1)^{N_{b-1}} \leq x^\alpha < b^{\alpha N}.$$

Let $\beta > 0$ be a constant (depending on b) to be determined later. For fixed $N_0, \ldots, N_{b-1}$, by elementary combinatorics, the number of possible values for n is at most

$$\frac{N!}{N_0! \cdots N_{b-1}!}.$$

Hence, summing over all possible values for $N_0, \ldots, N_{b-1}$, we get

$$\begin{aligned}
\#\mathcal{N}_{b,0}''(x) &\leq \sum_{\substack{N_0+\cdots+N_{b-1}=N \\ 1^{N_1}\cdots(b-1)^{N_{b-1}} \leq b^{\alpha N}}} \frac{N!}{N_0! \cdots N_{b-1}!} \\
&\leq \sum_{N_0+\cdots+N_{b-1}=N} \frac{N!}{N_0! \cdots N_{b-1}!}\left(\frac{b^{\alpha N}}{1^{N_1} \cdots (b-1)^{N_{b-1}}}\right)^\beta \\
&= \left(b^{\alpha\beta}(1+\zeta_b(\beta))\right)^N,
\end{aligned}$$

where we employed the multinomial theorem. Therefore, since $N \leq \log x/\log b + 1$, we have

$$\#\mathcal{N}''_{b,0}(x) < x^{\gamma+o(1)}, \tag{7}$$

as $x \to +\infty$, where

$$\gamma := \alpha\beta + \frac{\log(1+\zeta_b(\beta))}{\log b}. \tag{8}$$

At this point, in light of (6) and (7), we shall choose α and β so that $\max\{1-\alpha, \gamma\}$ is minimal. It is easy to see that this requires $1-\alpha = \gamma$, which in turn gives

$$\alpha = -\frac{1}{(1+\beta)\log b}\log\left(\frac{1+\zeta_b(\beta)}{b}\right).$$

Note that this choice indeed satisfies $\alpha \in \,]0,1[$, as required in our previous arguments. Hence, we have to choose β in order to minimize

$$\gamma = 1 + \frac{1}{(1+\beta)\log b}\log\left(\frac{1+\zeta_b(\beta)}{b}\right).$$

Since

$$\frac{\partial\gamma}{\partial\beta} = \frac{F_b(\beta)}{(1+\beta)^2\log b},$$

by the previous considerations on $F_b(s)$, we get that γ is minimal for $\beta = s_{b,0}$. Thus, we make this choice for β, so that $1-\alpha = \gamma = \eta_{b,0}$. Finally, putting together (6) and (7), we obtain

$$\#\mathcal{N}_{b,0}(x) < x^{1-\alpha+o(1)} + x^{\gamma+o(1)} < x^{\eta_{b,0}+o(1)}$$

as $x \to +\infty$. The proof is complete.

Proof of Theorem 2.2. The proof of Theorem 2.2 proceeds similarly to the one of Theorem 2.1. We highlight just the main differences. First, we shall prove that, for $b \geq 4$, Equation (2) has a unique positive solution. For $s \geq 0$, define

$$G_b(s) := \frac{(1+s)\zeta_b'(s)}{\zeta_b(s)} - \log\left(\frac{\zeta_b(s)}{b}\right).$$

Since $b \geq 4$, we have

$$G_b(0) = -\log\left(\left(1-\frac{1}{b}\right)(b-1)!^{1/(b-1)}\right) < 0 \quad \text{and} \quad \lim_{s\to+\infty} G_b(s) = \log b > 0. \tag{9}$$

Furthermore, a bit of computation shows that

$$G_b'(s) = \frac{(1+s)(\zeta_b(s)\zeta_b''(s) - (\zeta_b'(s))^2)}{(\zeta_b(s))^2} > 0, \tag{10}$$

for all $s \geq 0$, since (5). Therefore, by (9) and (10), Equation (2) has a unique positive solution. Note also that $G_3(0) > 0$, so that $G_3(s) > 0$ for all $s \geq 0$.

Let $\alpha \in \,]0,1[$ be a constant (depending on b) to be determined later, and define $\mathcal{N}_b'(x)$ as in the proof of Theorem 2.1. Hence, by the previous arguments, the bound (6) holds.

Suppose $n \in \mathcal{N}_b''(x) := \mathcal{N}_b(x) \setminus \mathcal{N}_b'(x)$. This time, put $N := \lfloor \log n/\log b\rfloor + 1$ (instead of $N := \lfloor \log x/\log b \rfloor + 1$), so that n has exactly N base b digits. For each $d \in \{1,\dots,b-1\}$, let N_d be the number of base b digits of n which are equal to d. Note that, since $p_b(n) \mid n$, we have $p_b(n) \neq 0$, that is, all the base b digits of n are nonzero. Hence, $N_1,\dots,N_{b-1}$ are nonnegative integers such that $N_1 + \cdots + N_{b-1} = N$. Furthermore,

$$p_b(n) = 1^{N_1}\cdots(b-1)^{N_{b-1}} \leq x^\alpha < b^{\alpha N}.$$

Let $\beta > 0$ be a constant (depending on b) to be determined later. Summing over all possible values for $N_1, \dots, N_{b-1}$ and N, we get

$$\begin{aligned}
\#\mathcal{N}_b''(x) &\leq \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} \sum_{\substack{N_1 + \cdots + N_{b-1} = N \\ 1^{N_1} \cdots (b-1)^{N_{b-1}} \leq b^{\alpha N}}} \frac{N!}{N_1! \cdots N_{b-1}!} \\
&\leq \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} \sum_{N_0 + \cdots + N_{b-1} = N} \frac{N!}{N_1! \cdots N_{b-1}!} \left(\frac{b^{\alpha N}}{1^{N_1} \cdots (b-1)^{N_{b-1}}} \right)^{\beta} \\
&= \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} (b^{\alpha\beta} \zeta_b(\beta))^N \ll (b^{\alpha\beta} \zeta_b(\beta))^{\log x / \log b},
\end{aligned}$$

and consequently

$$\#\mathcal{N}_b''(x) < x^{\delta + o(1)}, \tag{11}$$

as $x \to +\infty$, where

$$\delta := \alpha\beta + \frac{\log \zeta_b(\beta)}{\log b}. \tag{12}$$

At this point, in light of (6) and (11), we shall choose α and β so that $\max\{1-\alpha, \delta\}$ is minimal. This requires $1 - \alpha = \delta$, which in turn yields

$$\alpha = -\frac{1}{(1+\beta)\log b} \log\left(\frac{\zeta_b(\beta)}{b} \right).$$

Note that this choice indeed satisfies $\alpha \in]0, 1[$, as required in our previous arguments. Hence, we have to minimize

$$\delta = 1 + \frac{1}{(1+\beta)\log b} \log\left(\frac{\zeta_b(\beta)}{b} \right).$$

We have

$$\frac{\partial \delta}{\partial \beta} = \frac{G_b(\beta)}{(1+\beta)^2 \log b}.$$

Hence, by the previous considerations on $G_b(s)$, for $b = 3$ we have to choose $\beta = 0$, while if $b \geq 4$ we have to choose $\beta = s_b$. Making this choice, we get $1 - \alpha = \delta = \eta_b$. Finally, putting together (6) and (11), we obtain

$$\#\mathcal{N}_b(x) < x^{1-\alpha+o(1)} + x^{\delta+o(1)} < x^{\eta_b + o(1)}$$

as $x \to +\infty$. The proof is complete.

3. Lower bound

Theorem 3.1. *Let $b \geq 3$ be an integer. We have*

$$\#\mathcal{N}_{b,0}(x) > x^{\rho_{b,0} + o(1)}, \tag{13}$$

as $x \to +\infty$, where

$$\rho_{b,0} := \sup_{\alpha_0, \dots, \alpha_{b-1}} \frac{\left(\sum_{d=1}^{b-1} \alpha_d\right) \log\left(\sum_{d=1}^{b-1} \alpha_d\right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d}{\left(1 + \sum_{d=1}^{b-1} \alpha_d\right) \log b} \tag{14}$$

with $\alpha_0, \dots, \alpha_{b-1} \geq 0$ satisfying the conditions

$$\begin{cases} \alpha_d = 0 & \text{if } d > 1 \text{ and } p \mid d,\ p \nmid b \text{ for some prime } p, \\ \sum_{d=2}^{b-1} \alpha_d \nu_p(d) \leq 1 & \text{for all primes } p \mid b, \end{cases} \tag{15}$$

and with the convention $0 \cdot \log 0 := 0$.

We remark that if b is a prime number then the bound of Theorem 3.1 is obvious. Indeed, the primality of b implies $\alpha_d = 0$ for each $d \in \{2, \dots, b-1\}$, so that

$$\rho_{b,0} = \sup_{\alpha_0, \alpha_1 \geq 0} \frac{(\alpha_0 + \alpha_1)\log(\alpha_0 + \alpha_1) - \alpha_0 \log \alpha_0 - \alpha_1 \log \alpha_1}{(1 + \alpha_0 + \alpha_1)\log b} = \frac{\log 2}{\log b},$$

and the bound is

$$\#\mathcal{N}_{b,0}(x) > x^{\log 2/\log b + o(1)}, \tag{16}$$

as $x \to +\infty$. However, the bound (16) follows just by considering that $\mathcal{N}_{b,0}(x)$ contains all positive integers having their base b digits in $\{0, 1\}$.

If b is not a prime number, then Theorem 3.1 gives a better bound than (16). In particular, for $b = 10$, conditions (15) become

$$\begin{cases} \alpha_3 = \alpha_6 = \alpha_7 = \alpha_9 = 0, \\ \alpha_2 + 2\alpha_4 + 3\alpha_8 \leq 1, \\ \alpha_5 \leq 1, \end{cases} \tag{17}$$

and the right-hand side of (14) can be maximized under the constrains given by (17) using the method of Lagrange multipliers. This gives $\rho_{10,0} > 0.526$, for the choice

$$\alpha_0 = \alpha_1 = 1.331, \quad \alpha_2 = 0.476, \quad \alpha_4 = 0.170, \quad \alpha_5 = 1, \quad \alpha_8 = 0.060\,.$$

Hence, the lower bound for $\#\mathcal{N}_{10,0}(x)$ of Theorem 1.2 follows.

3.1. **Proof of Theorem 3.1.** Let us assume $x \geq 1$ sufficiently large, and let $\alpha_0, \dots, \alpha_{b-1} \geq 0$ be constants (depending on b) to be determined later. Define

$$s := \left\lfloor \frac{\log x}{(1 + \alpha_0 + \dots + \alpha_{b-1})\log b} \right\rfloor.$$

Also, let $N_d := \lfloor \alpha_d s \rfloor$ for each $d \in \{0, \dots, b-1\}$, and put $N := N_0 + \dots + N_{b-1}$.

Now suppose m is a positive integer with at most N base b digits, and such that exactly N_d of its base b digits are equal to d, for each $d \in \{1, \dots, b-1\}$. Moreover, put $n := b^s m$. Clearly, $n \leq b^{s+N} \leq x$ and $b^s \mid n$. Then, imposing the conditions (15), we get that

$$p_{b,0}(n) = 1^{N_1} \cdots (b-1)^{N_{b-1}} \mid b^s \mid n,$$

so that $n \in \mathcal{N}_{b,0}(x)$. By elementary combinatorics and by using Stirling's formula, the number of possible values for m is

$$\begin{aligned} \frac{N!}{N_0! \cdots N_{b-1}!} &= \frac{(\lfloor \alpha_0 s \rfloor + \dots + \lfloor \alpha_{b-1} s \rfloor)!}{\lfloor \alpha_0 s \rfloor! \cdots \lfloor \alpha_{b-1} s \rfloor!} \\ &= \exp\left(s \left(\left(\sum_{d=1}^{b-1} \alpha_d \right) \log \left(\sum_{d=1}^{b-1} \alpha_d \right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d + o(1) \right) \right), \end{aligned}$$

as $s \to +\infty$. Hence, lower bound (13) follows. The proof is complete.

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