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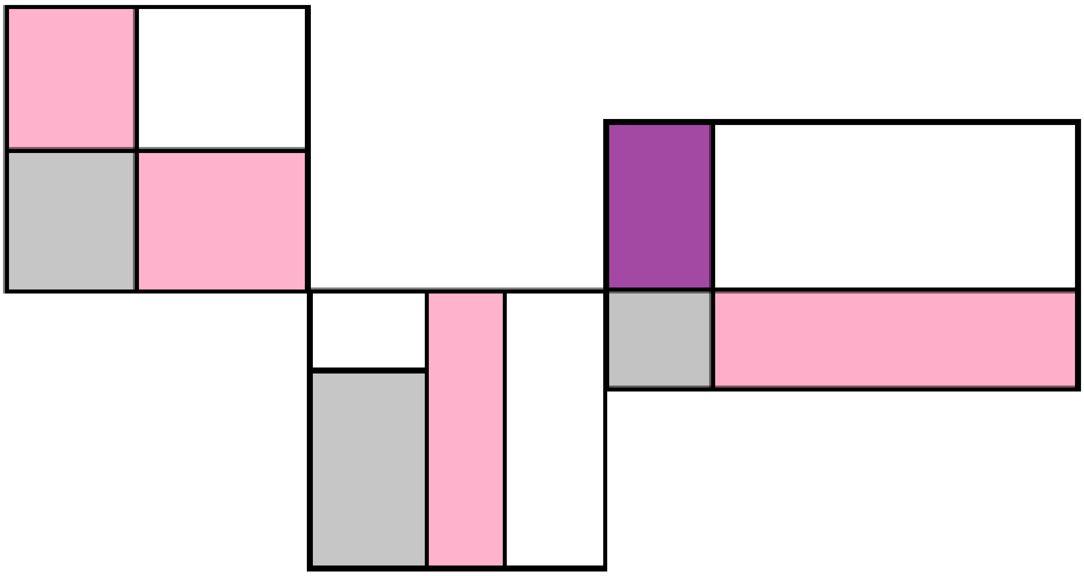
Binary operations applied to numbers



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Introduction

A binary operation is a calculus that combines two elements to obtain another elements. It seems quite simple for numbers, because we usually imagine it as a simple sum or product. However, also in the case of numbers, a binary operation can be extremely fascinating if we consider it in a generalized form. Here the reader can find several examples of generalized sums for different sets of numbers (Fibonacci, Mersenne, Fermat, q-numbers, repunits and many other numbers). These sets can form groupoid which possess different binary operators. As we will see at the end of this exposition of cases, the most relevant finding is that different integer sequences can have the same binary operator and that, consequently, can be used as different representations of the same groupoid.

Keywords: Groupoid Representations, Integer Sequences, Binary Operators, Generalized Sums, Generalized Entropies, Tsallis Entropy, q-Calculus, Abelian Groups, Fermat Numbers, Mersenne Numbers, Triangular Numbers, Repunits, Oblong Numbers

In mathematics, a binary operation is a calculation that combines two elements to obtain another element. In particular, this operation acts on a set in a manner that its two domains and its codomain are the same set. Examples of binary operations include the familiar arithmetic operations of addition and multiplication. Let us note that binary operations are the keystone of most algebraic structures: semigroups, monoids, groups, rings, fields, and vector spaces.

Here the reader can find several examples of binary operations applied to numbers. The binary operations proposed are generalizations of the sum, them the reader can find them named also as "generalized sums". Only one example is devoted to a multiplication, and it is concerning the Fibonacci Numbers.

The discussion is a collection of articles written by the author. Here the list of their arguments.

1) The additive group of q-integers (page 6) - The q-integers, that we can find in the q-calculus, are forming an additive group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems.

2) The group of the Fibonacci numbers (page 12) - These numbers are forming a group. Each number is represented by a 2x2 symmetric matrix and the operation of the group is the product of matrices. This approach allows to define the nega-Fibonacci numbers by means of the inverse of the Fibonacci matrices.

3) The generalized sum of the symmetric q-integers (page 16) - As in (1), we show that the symmetric q-integers of the q-calculus have a generalized sum which is also the generalized sum that we can find in the κ-calculus.

4) Generalized Sums Based on Transcendental Functions (page 19) - In this work, we proposed the generalized sums that we can obtain from transcendental functions. The generalized sums are operations which widespread the addition of real numbers. Using these sums, we will see that we can form some Abelian groups. The study is based on the generalized sums proposed previously. The main aim of the paper is that of popularizing the existence of groups having as their operation a generalized sum.

5) A generalized sum of the Mersenne Numbers (page 32) - We discussed these numbers to give an example of a generalized sum. Using this sum, a recurrence relation was given too.

6) The q-integers and the Mersenne numbers (page 35) - Here we show that the q-integers, the q-analogue of the integers that we can find in the q-calculus, are forming an additive group having a generalized sum similar to the sum of the Tsallis q-entropies of independent systems. The symmetric form of q-integers will be studied too. These numbers are linked to the Kaniadakis κ-calculus. A discussion is devoted to the link of the q-integers to the Mersenne numbers.

7) The group of the Fermat Numbers (page 45) - In this work we discussed the group that we can obtain if we consider the Fermat numbers with a generalized sum.

8) The generalized sums of Mersenne, Fermat, Cullen and Woodall (page 48) - Here we discussed Cullen and Woodall numbers, which are similar to Mersenne and Fermat numbers. The generalized sums are given for them. Recursive relations are given accordingly.

9) A recursive formula for Thabit numbers (page 54) - An operation of addition of these numbers is proposed. A recursive relation is given accordingly.

10) Repunits (page 57) - An operation of addition of these numbers is proposed. A recursive formula is given accordingly. Symmetric repunits are also defined.

11) Composition Operations of Generalized Entropies Applied to the Study of Numbers (page 60) - Article in international journal.

12) Binary Operators of the Groupoids of OEIS A093112 and A093069 Numbers (Carol and Kynea Numbers) (page 66) - Here we discuss the binary operators of the sets made by the OEIS sequences of integers A093112 and A093069, also called Carol and Kynea numbers. We see that these numbers are linked, through the binary operators, to the Mersenne and Fermat integers.

13) A Binary Operator Generated by Homographic (page 69) - In this work we discussed the binary operator that we can generated by homographic function. By means of this operator, that we can see as a generalized sum, we can create a group.

14) Groupoids of OEIS A002378 and A016754 Numbers (oblong and odd square numbers) (page 72) - Here we discuss the binary operators of the sets made by the OEIS sequences of integers A002378 and A016754. A002378 are defined as oblong numbers.

15) Groupoid of OEIS A001844 Numbers (centered square numbers) (page 75) - Here we discuss the binary operator of the set made by the OEIS sequence of integers A001844, defined as centered square numbers. This binary operator can be used to have a groupoid. Actually, neutral and opposite elements can be defined too, and a possible group for these numbers can be given.

16) Giuseppe Peano e i numeri di Mersenne (page 78) - Si mostra come un problema dei "Giochi Di Aritmetica E Problemi Interessanti", di Giuseppe Peano, ci porti ai numeri di Mersenne.

17) Discussion of the groupoid of Proth numbers (OEIS A080075) (page 82) - Here we show that the set of Proth numbers is a groupoid. The binary operaton between the elements of the sets is

given as a generalized compositon.

18) Groupoid of OEIS A003154 Numbers (star numbers or centered dodecagonal numbers) (page 84) - It is discussed the binary operators of the set made by the OEIS sequence of integers A003154, defined as star numbers or centered dodecagonal numbers. The binary operators can be used to have groupoids.

19) The groupoid of the Triangular Numbers and the generation of related integer sequences (page 87) - Here we discuss the binary operators of the set made by the triangular numbers, sequence A000217, in the On-Line Encyclopedia of Integer Sequences (OEIS). As we will see, by means of these binary operators we can obtain related integer sequences. Here we propose some of them. The sequences, except one, are given in OEIS.

20) The groupoids of Mersenne, Fermat, Cullen, Woodall and other Numbers and their representations by means of integer sequences (page 92) - Previous works have discussed the groupoids related to the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers. These groupoid possess different binary operators. As we can easily see, other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids. Using the On-Line Encyclopedia of Integer Sequences (OEIS), we can also identify the properties of these representations of groupoids. At the same time, we can also find integer sequences not given in OEIS and probably not yet studied.

21) Some Groupoids and their Representations by Means of Integer Sequences (page 101). Article in international journal.

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ON THE ADDITIVE GROUP OF q-INTEGERS

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Abstract Here we will show that the q-integers, that we can find in the q-calculus, are forming an additive group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems.

Keywords q-calculus, q-integers, Tsallis q-entropy.

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Introduction Many mathematicians have contributed to the calculus that today is known as the q-calculus [1-6]. As a consequence, it is known as "quantum calculus," "time-scale calculus" or "calculus of partitions" too [5]. Moreover, it is expressed by means of different notations or, as told in [5], by different "dialects". Here we will use the approach and the notation given in the book by Kac and Cheung [6].

The aim of this work is that of showing the following. The q-integers are forming a group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems. Let us start from the definition of the q-integers.

In the q-calculus, the q-difference is simply given by:

$$d_q f = f(qx) - f(x)$$

From this difference, the q-derivative is given as:

$$D_q f = \frac{f(qx) - f(x)}{qx - x}$$

The q-derivative reduces to the Newton's derivative in the limit $q \to 1$.

Let us consider the function $f(x) = x^n$. If we calculate its q-derivative, we obtain:

$$(1) \quad D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

Comparing the ordinary calculus, which is giving $(x^n)'=nx^{n-1}$, to Equation (1), we can define the "q-integer" $[n]$ by:

$$\text{(2)} \quad [n]=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1}$$

Therefore Equation (1) turns out to be:

$$D_q x^n=[n]\, x^{n-1}$$

As a consequence, the *n*-th q-derivative of $f(x)=x^n$, which is obtained by repeating *n* times the q-derivative, generates the q-factorial:

$$[n]\,!=[n][n-1]\ldots[3][2][1]$$

Form the q-factorials, we can define q-binomial coefficients:

$$\frac{[n]!}{[m]![n-m]!}$$

This means that we can use the usual Taylor formula, replacing the derivatives by the q-derivatives and the factorials by q-factorials (in a previous work, we have discussed the q-exponential and q-trigonometric functions [7]). Then, in the q-calculus, the q-integer $[n]$ acts as the integer in the ordinary calculus.

We known that the set of integers Z, which consists of the numbers ..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ..., having as operation the addition, is a group. Therefore, let us consider the set of q-integers given by (2) and investigate its group. In particular, we have to determine its operation of addition.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a*•*b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a*•*b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a*•*b*)•*c* = *a*•(*b*•*c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e*•*a* = *a*•*e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a*•*b* = *b*•*a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation).

A further requirement is the *commutativity:* For all *a*,*b* in *A*, *a•b* = *b•a*. In this case, the group is known as an Abelian group.
Therefore, to qualify a group as an Abelian group, the set and operation must satisfy five requirements which are known as the *Abelian group axioms.* A group having a not commutative operation is called a "non-abelian group" or "non-commutative group". For an Abelian group, one may choose to denote the group operation by + and the *identity element* by 0 (*neutral element*) and the inverse element as $-a$ (*opposite element*). In this case, the group is called an additive group.
First, we have to define the operation of addition. It is not the sum that we use for the integers, but it is a generalized sum which obeys the axioms of the group.
Let us start from the q-integer $[m+n]$:

$$[m+n]=\frac{q^{m+n}-1}{q-1}=\frac{1}{q-1}(q^m q^n-1+q^m-q^m)=\frac{1}{q-1}(q^m(q^n-1)+q^m-1)$$

$$[m+n]=\frac{1}{q-1}(q^m(q^n-1)+(q^m-1)+(q^n-1)+(1-q^n))=\frac{1}{q-1}((q^m-1)(q^n-1)+(q^m-1)+(q^n-1))$$

Therefore, we have:

$$(3) \quad [m+n]=[m]+[n]+(q-1)[m][n]$$

Then, we can define the generalized “sum” of the group as:

$$(4) \quad [m]\oplus[n]=[m]+[n]+(q-1)[m][n]$$

(for other examples of generalized sums see [8]):
If we use (4) as the sum, we have the closure of it, because the result of the sum is a q-integer. Moreover, this sum is commutative.
The neutral element is:

$$(5) \quad [0]=\frac{q^0-1}{q-1}=0$$

Let us determine the opposite element $[o]$, so that:

$$[o]\oplus[n]=0$$

$$0=[0]=[o]\oplus[n]=[o]+[n]+(q-1)[o][n]$$

$$-[n]=[o]+(q-1)[o][n]$$

$$(6)\quad [o]=-\frac{[n]}{1+(q-1)[n]}=-\frac{q^n-1}{(q-1)q^n}=\frac{q^{-n}-1}{q-1}=[-n]$$

The opposite element of q-integer $[n]$ is the q-integer of $-n$, that is $[-n]$.
Let us discuss the associativity of the sum.
It is necessary to have:

$$[m]\oplus([n]\oplus[l])=([m]\oplus[n])\oplus[l]$$

Let us calculate:

$$[m]\oplus([n]\oplus[l])=[m]\oplus([n]+[l]+(q-1)[n][l])$$

$$[m]\oplus([n]\oplus[l])=[m]+[n]+[l]+(q-1)[n][l]+(q-1)[m][n]+(q-1)[m][l]+(q-1)^2[m][n][l]$$

And also:

$$([m]\oplus[n])\oplus[l]=([m]+[n]+(q-1)[m][n])\oplus[l]$$

$$([m]\oplus[n])\oplus[l]=[m]+[n]+(q-1)[m][n]+[l]+(q-1)[m][l]+(q-1)[n][l]+(q-1)^2[m][n][l]$$

It is also easy to see that:

$$[m]\oplus[n]\oplus[l]=[m+n+l]$$

As we have shown, the five axioms of an Abelian group are satisfied. In this manner, using the generalized sum given by (4), we have the Abelian group of the q-integers. Let us also note that the generalized sum (4) is similar to the sum that we find in the approach to entropy proposed by Constantino Tsallis.
In 1948 [9], Claude Shannon defined the entropy *S* of a discrete random variable Ξ as the expected value of the information content: $S=\sum_i p_i I_i=-\sum_i p_i \, log_b p_i$ [10]. In this expression, *I* is the information content of Ξ, the probability of *i*-event is p_i and *b* is the base of the used logarithm. Common values of the base are 2, the Euler’s number e, and 10.
Constantino Tsallis generalized the Shannon entropy in the following manner [11]:

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right)$$

Given two independent systems *A* and *B*, for which the joint probability density satisfies:

$$p(A,B) = p(A)\,p(B)$$

the Tsallis entropy gives:

(7) $S_q(A,B) = S_q(A) + S_q(B) + (1-q)\,S_q(A)\,S_q(B)$

The parameter $(1-q)$, in a certain manner, measures the departure from the ordinary additivity, which is recovered in the limit $q \to 1$.

Actually the group on which is based the Tsallis entropy, and therefore Equation (7), is known as the “multiplicative group” [6,12-13]. As stressed in [14], the use of a group structure allows to determine a class of generalized entropies. Let us note the group of the q-integers, with addition (4), can be considered a “multiplicative group” too.

Let us conclude telling that the main result of the work here proposed is the link to the multiplicative group and the Tsallis entropy. The group of the n-integers had been studied in [15,16] too, but in these articles, a quite different expression for the generalized sum had been proposed. It is given as the “quantum sum” $[x] \oplus [y] = [x] + q^x [y]$, where the link to the Tsallis calculus is less evident.

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ON THE GROUP OF THE FIBONACCI NUMBERS

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Abstract Here we will show that the numbers of Fibonacci are forming a group. Each number is represented by a 2x2 symmetric matrix and the operation of the group is the product of matrices. This approach allows to define the negaFibonacci numbers by means of the inverse of the Fibonacci matrices.

Keywords Symmetric matrices, Fibonacci numbers, Group theory.

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The Fibonacci numbers are a sequence of integers characterized by the fact that every number, after the first two, is the sum of the two preceding ones. Therefore, we have the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …. and so on.

The recurrence relation is given by:

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$, $F_1 = 1$. Then $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, etc.

The item of Wikipedia, about the Fibonacci numbers [1], gives them also in the form:

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = M \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

However, we find also in [1] the matrices:

$$(1) \qquad M^n = \underbrace{M \cdots M}_{n} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Therefore, we have: $M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $M^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $M^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, etc.

Let us consider the group of these symmetric matrices and discuss it.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a•b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a•b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a•b*)•*c* = *a*•(*b•c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e•a* = *a•e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a•b* = *b•a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation). A further requirement is the c*ommutativity:* For all *a*,*b* in *A*, *a•b* = *b•a*. In this case, the group is known as an Abelian group.
For the set of the matrices (1), the operation is the product of the matrices. Is it *commutative*? The answer is positive.

$$M^n M^m = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} = \begin{pmatrix} (F_{n+1} F_{m+1} + F_n F_m) & (F_{n+1} F_m + F_n F_{m-1}) \\ (F_n F_{m+1} + F_{n-1} F_m) & (F_n F_m + F_{n-1} F_{m-1}) \end{pmatrix}$$

Being $F_{m+1} = F_m + F_{m-1}$ and $F_{n+1} = F_n + F_{n-1}$, we can see that the product gives a symmetric matrix:

$$M^n M^m = \begin{pmatrix} (F_{n+1} F_{m+1} + F_n F_m) & (F_n F_m + F_{n-1} F_m + F_n F_{m-1}) \\ (F_n F_m + F_n F_{m-1} + F_{n-1} F_m) & (F_n F_m + F_{n-1} F_{m-1}) \end{pmatrix}$$

And also:

$$M^n M^m = \begin{pmatrix} (F_{n+1} F_{m+1} + F_n F_m) & (F_{m+1} F_n + F_m F_{n-1}) \\ (F_{n+1} F_m + F_n F_{m-1}) & (F_n F_m + F_{n-1} F_{m-1}) \end{pmatrix} \quad (2)$$

The same for:

$$M^m M^n = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} (F_{m+1} F_{n+1} + F_m F_n) & (F_{m+1} F_n + F_m F_{n-1}) \\ (F_m F_{n+1} + F_{m-1} F_n) & (F_m F_n + F_{m-1} F_{n-1}) \end{pmatrix} \quad (3)$$

From (2) and (3):

$$M^m M^n = M^n M^m$$

We can tell that the product of two Fibonacci symmetric matrices A and B is a symmetric matrix, because A and B commute.
Let us consider the matrices again:

$$M^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

and evaluate the determinant, to obtain the Cassini identity.

Because the determinant of a matrix product of square matrices equals the product of their determinants, we have:

$$(-1)^n = F_{n+1} F_{n-1} - F_n^2 \quad (4)$$

(4) is the Cassini’s Identity.

Let us discuss the *closure*. It means that, if we have any product of two Fibonacci matrices, we have another Fibonacci matrix. Actually:

$$M^m M^n = M^{m+n} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} (F_{m+1} F_{n+1} + F_m F_n) & (F_{m+1} F_n + F_m F_{n-1}) \\ (F_m F_{n+1} + F_{m-1} F_n) & (F_m F_n + F_{m-1} F_{n-1}) \end{pmatrix} \quad (5)$$

From (5) we have other relations among Fibonacci numbers.

The *identity* element is: $M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The *inverse* element is obtained in the following manner:

$$(M^n)^{-1} M^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = M^0$$

Therefore:

$$a = \frac{F_{n-1}}{F_{n+1} F_{n-1} - F_n^2} \qquad b = \frac{F_n}{-F_{n+1} F_{n-1} + F_n^2} \qquad c = \frac{F_n}{-F_{n+1} F_{n-1} + F_n^2} \qquad d = \frac{F_{n+1}}{F_{n+1} F_{n-1} - F_n^2}$$

Let us calculate some inverses:

$$(M^1)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad (M^2)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad (M^3)^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \quad (M^4)^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \text{ etc.}$$

So we can easily see that we have here the “negaFibonacci” numbers: 0, 1, -1, 2, -3, 5, -8, 13, -21 , … etc. In [1], these numbers are given as:

$$F_{-n} = (-1)^{n+1} F_n$$

From [1], it seems that these numbers were defined by Ref.2 (in fact, I was not able to find a copy of the article mentioned by Wikipedia).

If we use the matrices, the negaFibonacci are the inverse of them.

Let us conclude considering the *associativity*, that is $(M^m M^n) M^k = M^m (M^n M^k)$

$$(M^m M^n) M^k = M^{m+n} M^k = M^{m+n+k}$$
$$M^m (M^n M^k) = M^m M^{n+k} = M^{m+n+k}$$

Here we have seen that the numbers of Fibonacci, represented by 2x2 symmetric matrices, are forming a group. The operation of the group is the product of matrices. The negaFibonacci numbers are defined by means of the inverse of the Fibonacci matrices.

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On the generalized sum of the symmetric q-integers

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Abstract Here we will show that the symmetric q-integers of the q-calculus have a generalized sum which is also the generalized sum that we can find in the κ-calculus proposed by G. Kaniadakis.

Keywords q-calculus, q-integers, Kaniadakis κ-entropy.

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Introduction In a previous work [1], we have discussed the group of the q-integers as defined by q-calculus. In the notation given in the book by Kac and Cheung [2], the q-integers are:

$$(1) \quad [n]=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1} \quad .$$

In [1], we defined the generalized sum of the group as:

$$(2) \quad [m]\oplus[n]=[m]+[n]+(q-1)[m][n]$$

As a consequence, we have that the q-integers (1) with operation (2) form a multiplicative group. The generalized sum (2) is similar to the generalized sum that we find for the Tsallis entropies of independent systems [3].

In the q-calculus [2], it is also defined the symmetric q-integer in the following form (here we use a notation different from that given in the Ref.2):

$$(3) \quad [n]_s=\frac{q^n-q^{-n}}{q-q^{-1}}$$

Repeating the approach used in [1], we can determine the group of the symmetric q-integers. Let us start from the q-integer $[m+n]_s$, which is according to (3):

$$[m+n]_s=\frac{q^{m+n}-q^{-(m+n)}}{q-q^{-1}}$$

and try to find it as a generalized sum of the q-integers $[m]_s$ and $[n]_s$.

By writing $q=\exp(\log q)$, the q-integer turns out into a hyperbolic sine:

$$\text{(4)}\quad [n]_s=\frac{q^n-q^{-n}}{q-q^{-1}}=\frac{e^{n\log q}-e^{-n\log q}}{q-q^{-1}}=2\frac{\sinh(n\log q)}{(q-q^{-1})}$$

Apart from a numerical factor, this is the form of the generalized numbers proposed by G. Kaniadakis in his κ-calculus [4-8].

From (4), we can write also:

$$\frac{1}{2}(q-q^{-1})[n]_s=\sinh(n\log q)$$

Therefore:

$$[m+n]_s=\frac{q^{m+n}-q^{-(m+n)}}{q-q^{-1}}=2\frac{\sinh((m+n)\log q)}{(q-q^{-1})}$$

Using the properties:

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh x \quad ; \quad \cosh x=\sqrt{1+\sinh^2 x}$$

we obtain:

$$[m+n]_s=\frac{2}{(q-q^{-1})}[\sinh(m\log q)\cosh(n\log q)+\sinh(n\log q)\cosh(m\log q)]$$

$$[m+n]_s=[m]_s\cosh(n\log q)+[n]_s\cosh(m\log q)$$

$$[m+n]_s=[m]_s\sqrt{1+\sinh^2(n\log q)}+[n]_s\sqrt{1+\sinh^2(m\log q)}$$

Let us define: $k=(q-q^{-1})/2$ and then: $k[n]_s=\sinh(n\log q)$.

As a consequence we have the generalized sum of the symmetric q-integers as:

$$\text{(5)}\quad [m]_s\oplus[n]_s=[m]_s\sqrt{1+k^2[n]_s^2}+[n]_s\sqrt{1+k^2[m]_s^2}$$

Let us conclude stressing that (5) is also the generalized sum proposed by G. Kaniadakis in the framework of a calculus [5-8], the details of which are given in [8]. By means of (5), we can repeat the approach given in Ref.1 and study of the group of the symmetric q-integers.

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Generalized Sums Based on Transcendental Functions

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Abstract: In this work we are discussing the generalized sums that we can obtain from transcendental functions. The generalized sums are operations which widespread the addition of real numbers. Using these sums, we will see that we can form some Abelian groups. The study is based on the generalized sums proposed in his κ-calculus by Giorgio Kaniadakis, who used it in the framework of a generalized statistics, first applied to special relativity. Besides the investigation of some groups, the paper is also proposing examples which could be suitable for teaching purposes, in the framework of courses on theoretical physics, relativity and algebra applied to physics. Actually, the main aim of the paper is that of popularizing the existence of groups having as their operation a generalized sum.

Keywords: κ-calculus, generalized sum, groups, Abelian groups, hyperbolic functions, circular functions, logarithmic and exponential functions, theoretical physics, education.

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Introduction

A calculus exists, developed in the framework of a generalized statistics proposed by Giorgio Kaniadakis [1-5], which is based on deformed exponential and logarithmic functions. All the theoretical foundations and mathematical formulas of it are given in [5]. This calculus, also known as κ-calculus, has produced a series of remarkable results concerning statistics applied to many physical systems and models (see references in [5]). At the same time, it has also given new perspectives in the development economic and econometric methods [6,7].

As explained in [5], the κ-calculus turns out to be a continuous one-parameter deformation of the calculus based on the Euler exponential function. Here, we will use this calculus as a model for discussing some generalized sums based on transcendental functions. Let us note that the generalized sums are operations which widespread the addition of real numbers, and that the transcendental functions are analytic functions that do not satisfy polynomial equations, in contrast to algebraic functions [8]. It means that a transcendental function cannot be expressed by means of a finite sequence of algebraic operations such as addition, multiplication, and root extraction.

Using generalized sums, we can show that Abelian groups exist related to them. Besides the investigation of some groups, the paper is also proposing examples which could be suitable for teaching purposes, in the framework of courses of theoretical physics, relativity and algebra applied to physics. However, let us stress that the main aim of the paper is that of popularizing the existence of groups having as their operation a generalized sum.

The κ-sum

In [5], the κ-sum is defined in the following manner. Let us consider two elements *x* and *y* of reals **R**, and a parameter κ real too, which is $-1<\kappa<1$. The composition law $x \oplus y$ is given by:

$$x \oplus y = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2} \quad (1)$$

which defines a generalized sum, named κ-sum. (**R**,⊕) forms an Abelian group.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a•b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure***:** For all *a*,*b* in *A*, the result of the operation *a•b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a•b*)•*c* = *a*•(*b•c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e•a* = *a•e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a•b* = *b•a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation).

If a group is Abelian, a further requirement is the c*ommutativity:* For all *a*,*b* in *A*, *a•b* = *b•a*. Therefore, to qualify a group as an Abelian group, the set and operation must satisfy five requirements which are known as the *Abelian group axioms.* A group having a not commutative operation is called a "non-abelian group" or "non-commutative group". For an Abelian group, one may choose to denote the group operation by + and the *identity element* by 0 (*neutral element*) and the inverse element as $-a$ (*opposite element*). In this case, the group is called an additive group.

Let us note that if a function $G(x)$ exists, which is invertible $G^{-1}(G(x))=x$, we can use it as a deformation *generator* [3], to generate a consequent algebra [3,9]. We will use the *generator G* to define the *group law* $\Phi(x,y)$, such as in [10]:

$$\Phi(x,y) = G(G^{-1}(x) + G^{-1}(y))$$

or:

$$x \oplus y = G\left(G^{-1}(x) + G^{-1}(y)\right) \quad .$$

In this manner the *group law* is giving the *generalized sum* of the group.

In the case of the κ-sum, the function *G* is the hyperbolic sine:

$$x \oplus y = \frac{1}{\kappa} \sinh\left(\operatorname{arsinh}(\kappa x) + \operatorname{arsinh}(\kappa y)\right) \quad .$$

In [1,3], this sum is used for relativistic momenta.

The generalized sum from the hyperbolic sine

Actually, the κ-sum is a case of the generalized sum that we can obtain from the properties of the hyperbolic sine function, defined as:

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{1+x^2}\right)$$

The domain is the whole real line. We have that [11]:

$$\operatorname{arsinh}(x) \pm \operatorname{arsinh}(y) = \operatorname{arsinh}\left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right) \qquad (2)$$

Therefore, we have a *group law*:

$$\Phi(x, y) = \sinh\left(\operatorname{arsinh}(x) + \operatorname{arsinh}(y)\right)$$

As a consequence, the sum is:

$$x \oplus y = x\sqrt{1+y^2} + y\sqrt{1+x^2}$$

This is the same as (1), for $\kappa = 0$. The *closure* is given by the fact that the result of this operation is on the real line. The *neutral element* is 0. The *opposite element* of *x* is $-x$. Also the *commutativity* is evident.

To have a group, we need the discussion of the a*ssociativity* too, showing that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. Let us calculate $\operatorname{arsinh}((x \oplus y) \oplus z), \operatorname{arsinh}(x \oplus (y \oplus z))$; we can easily see that

$$\operatorname{arsinh}((x \oplus y) \oplus z) = \operatorname{arsinh}(x \oplus y) + \operatorname{arsinh}(z) = \operatorname{arsinh}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right) + \operatorname{arsinh}(z) \quad ,$$

which is giving, according to (2), $\operatorname{arsinh}(x)+\operatorname{arsinh}(y)+\operatorname{arsinh}(z)$.

The same for:

$$\operatorname{arsinh}(x\oplus(y\oplus z))=\operatorname{arsinh}(x)+\operatorname{arsinh}(y\oplus z)=\operatorname{arsinh}(x)+\operatorname{arsinh}(y)+\operatorname{arsinh}(z) \quad .$$

The generalized integers

Let us calculate some generalized sums in the case of integer numbers. In the case of the sum of 1 and integer *n* we have:

$$1\oplus n=\sqrt{1+n^2}+n\sqrt{2}$$

Moreover:

$$n\oplus n=2n\sqrt{1+n^2} \quad .$$

The generalized sum of integers is a real number.

To practice with the generalized addition $x\oplus y=x\sqrt{1+y^2}+y\sqrt{1+x^2}$, we can do the following. Let us assume to call 1 as α_1 and calculate:

$$\alpha_2=\alpha_1\oplus\alpha_1=1\oplus 1=\sqrt{2}+\sqrt{2}=2\sqrt{2}$$

$$\alpha_3=\alpha_1\oplus\alpha_2=1\oplus 1\oplus 1=2\sqrt{2}\sqrt{2}+\sqrt{1+8}=4+3=7$$

Then we can calculate:

$$\alpha_2\oplus\alpha_2=4\sqrt{2}\sqrt{1+8}=12\sqrt{2} \qquad \alpha_1\oplus\alpha_3=\sqrt{50}+7\sqrt{2}=12\sqrt{2}$$

So we have that:

$$\alpha_4=\alpha_2\oplus\alpha_2=\alpha_1\oplus\alpha_3=1\oplus 1\oplus 1\oplus 1=12\sqrt{2}$$

And so on. We can create a group of generalized integers α_n defined by repeating *n* times the sum:

$$\alpha_n=1\oplus 1\oplus 1\ldots\oplus 1\oplus 1$$

The group is: $(\alpha_n,\oplus)$, where n is a natural integer.

Now, let us assume to call the integer 2 as β_1 and calculate:

$$\beta_2=\beta_1\oplus\beta_1=2\oplus 2=2\sqrt{5}+2\sqrt{5}=4\sqrt{5}$$

As an exercise, we can repeat the previous calculus to obtain another group.

The generalized sum from the hyperbolic tangent

Let us consider the hyperbolic tangent. Its inverse hyperbolic function is defines as:

$$\mathrm{artanh}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

The domain is the open interval $(-1,1)$.

A property of this inverse function is the following [11]:

$$\mathrm{artanh}(x)\pm\mathrm{artanh}(y)=\mathrm{artanh}\left(\frac{x\pm y}{1\pm xy}\right)$$

Therefore, let us define the generalized sum as:

$$x\oplus y=\frac{x+y}{1+xy} \qquad (3)$$

We have a *group law*:

$$\Phi(x,y)=\tanh(\mathrm{artanh}(x)+\mathrm{artanh}(y))$$

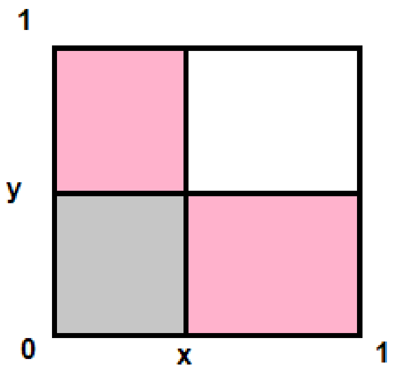
As a consequence, we have the sum defined in (3). This sum is *commutative*. The *neutral element* is 0. The *opposite element* of x is $-x$.

Let us discuss the a*ssociativity,* showing that $(x\oplus y)\oplus z=x\oplus(y\oplus z)$. Actually:

$$(x\oplus y)\oplus z=\frac{\frac{x+y}{1+xy}+z}{1+\left(\frac{x+y}{1+xy}\right)z}=\frac{x+y+z+xyz}{1+xy+xz+yz}$$

$$x\oplus(y\oplus z)=\frac{\frac{y+z}{1+yz}+x}{1+(\frac{y+z}{1+yz})x}=\frac{y+z+x+xyz}{1+yz+yx+zx}=\frac{z+y+z+xyz}{1+xy+xz+yz}$$

To show that we have a group ($(-1,1)$,⊕), it is necessary to verify the *closure*. That is, we have to see that the result of the generalized sum is in the open interval $(-1,1)$.



Let us assume $x > 0$ and $y > 0$. We need $x\oplus y<1$. And therefore:

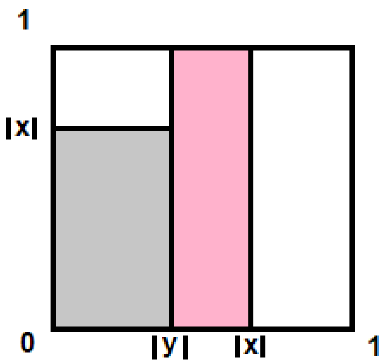
$$\frac{x+y}{1+xy}<1\Rightarrow x+y<1+xy$$

We can see that it is so, by means of a geometric approach. Let us consider a square having sides equal to 1. In this square, let us consider rectangles x X 1, 1 X y and x X y.

From the image, we can see immediately that:

$$x+y=x\times 1+1\times y=xy+x(1-y)+yx+y(1-x)<1+xy\Rightarrow xy+x(1-y)+y(1-x)<1 \quad .$$

The same happens in the case that $x < 0$ and $y < 0$. We need again $\frac{|x|+|y|}{1+|x||y|}<1$, and therefore we can repeat the previous approach.



In the case that $x > 0$ and $y < 0$, we can use the following geometry. Let us suppose $|x| > |y|$. We need to have:

$$\frac{|x|-|y|}{1-|x||y|}<1 \quad .$$

If we look at the image on the left, we have that $|x|-|y|<1-|x||y|$, the difference $|x|-|y|$ being represented by the pink rectangle. The same is true in the other case $x < 0$ and $y > 0$. This means that the axiom of the *closure* is verified and that ((-1,1),⊕) is an Abelian group.

The generalized sum (3) is used by Kaniadakis in Ref.3 for the relativistic velocity, in the following form:

$$u_1 \oplus u_2 = \frac{u_1 + u_2}{1+\kappa^2 u_1 u_2}$$

where u_1 and u_2 are dimensionless velocities.

Since we are considering the group ((-1,1),⊕), we avoid the divergence which we encounter when $y=-1/x$. In fact, in relativity, the dimensionless velocity *u=v/c* is less than 1, if we assume *c* as the speed of light.

Another manner to generate (3)

Let us consider another manner to generate the sum (3), using the following function and its inverse:

$$G(x)=\frac{1-e^x}{1+e^x} \qquad G^{-1}(x)=\ln\left(\frac{1-x}{1+x}\right)$$

For the chosen function, we need to have $-1<x<1$. A *group law* $\Phi(x,y)$ could be:

$$\Phi(x,y)=G(G^{-1}(x)+G^{-1}(y)) \quad .$$

And therefore:

$$x \oplus y=G(G^{-1}(x)+G^{-1}(y))=G\left(\ln\left(\frac{1-x}{1+x}\right)+\ln\left(\frac{1-y}{1+y}\right)\right)=G\left(\ln\left(\frac{1-x}{1+x}\frac{1-y}{1+y}\right)\right)=G(\ln Z)$$

$$G(\ln Z)=\frac{1-Z}{1+Z}=\frac{x+y}{1+xy}$$

Therefore we have again the generalized sum (3):

$$x \oplus y=\frac{x+y}{1+xy}$$

A sequence of generalized sums

Let us consider again [10] and also [12].

As previously told, we find the group law $\Phi(x,y)$ as $\Phi(x,y)=G(G^{-1}(x)+G^{-1}(y))$.

For the *additive group* law is: $\Phi(x,y)=x+y$. In this case, we can see that *G* function is:

$$G(z)=kz\,,G^{-1}(z)=k^{-1}z \quad .$$

Then: $$G(G^{-1}(x)+G^{-1}(y))=G(k^{-1}x+k^{-1}y)=k(k^{-1}x+k^{-1}y)=x+y$$

A *multiplicative group* law is given by: $\Phi(x,y)=x+y+xy$.

In [10], we find that $G(z)=\mathrm{e}^z-1$. We can easily see that $G^{-1}(z)=\ln(z+1)$, so that:

$$G(G^{-1}z)=\exp[\ln(z+1)]-1=z+1-1=z \quad .$$

In this manner, we can obtain: $G(G^{-1}x+G^{-1}y)=\exp[\ln(x+1)+\ln(y+1)]-1$ and

$$\exp[\ln(x+1)+\ln(y+1)]-1=\exp[\ln((x+1)(y+1))]-1=x+y+xy+1-1=x+y+xy$$

The neutral element is 0 and the opposite element is:

$$Opposite(x)=-\frac{x}{1+x}$$

However, we have to avoid $x=-1$, and not consider it in the group.

Let us note that the multiplicative group appears in the generalized sum of Tsallis entropy [13]. The related algebra has been investigated and discussed in detail in [6].

Recently, a multi-parametric version of this entropy has been proposed in [14]. This entropy is based on a rational group law:

$$\Phi(x,y)=\frac{x+y+axy}{1+bxy}$$

When *b* is equal to zero, we find the single-parametric Tsallis entropy.

In [12], we find mentioned the *hyperbolic group* law too:

$$\Phi(x,y)=\frac{x+y}{1+xy}$$

which was discussed by Kaniadakis in [3], for the addition of velocities in special relativity. Let us stress that another *hyperbolic group* exists, that having (1) as generalized sum, and given in the κ-calculus as the sum of momenta [3].

In [12], we find also the *Euler group* law for elliptic integrals:

$$\Phi(x,y)=\frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}$$

So that:
$$\int_0^x \frac{dt}{\sqrt{1-t^4}}+\int_0^y \frac{dt}{\sqrt{1-t^4}}=\int_0^{\Phi(x,y)} \frac{dt}{\sqrt{1-t^4}}$$

Let us conclude the discussion proposing two examples of generalized sums based on circular functions and another example of multiplicative group.

Circular functions

Let us discuss the generalized sums and the group laws, which are based on circular functions sine and tangent. For the circular sine, we consider the inverse circular functions, having the property [15]:

$$\arcsin(x)\pm\arcsin(y)=\arcsin\left(x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right) \qquad (4)$$

The group law is: $\Phi(x,y)=\sin(\arcsin(x)+\arcsin(y))$.

Again, the generalized sum is:

$$x\oplus y=x\sqrt{1-y^2}+y\sqrt{1-x^2} \qquad (5)$$

In this case we have $-1\leqslant x,y\leqslant 1$. The group ([-1,1],⊕) is Abelian.

The *closure* is given in the following manner.

Let us consider (4) and calculate the sine:

$$\sin(\arcsin(x)\pm\arcsin(y))=\sin(\arcsin(x\sqrt{1-y^2}\pm y\sqrt{1-x^2}))\quad .$$

This means that: $\sin(\arcsin(x)\pm\arcsin(y))=x\oplus y=x\sqrt{1-y^2}\pm y\sqrt{1-x^2}$ which is in interval $[-1,1]$. The *neutral element* is 0. The *opposite element* of *x* is $-x$. Also the *commutativity* is evident.

We have to discuss the a*ssociativity* too, showing that $(x\oplus y)\oplus z=x\oplus(y\oplus z)$.

Again, let us calculate $\arcsin((x\oplus y)\oplus z)$ and $\arcsin(x\oplus(y\oplus z))$, we can easily see:

$$\arcsin((x\oplus y)\oplus z)=\arcsin(x\oplus y)+\arcsin(z)=\arcsin(x\sqrt{1-y^2}+y\sqrt{1-x^2})+\arcsin(z)\quad ,$$

which is giving, according to (4), $\arcsin(x)+\arcsin(y)+\arcsin(z)$. The same for:

$$\arcsin(x\oplus(y\oplus z))=\arcsin(x)+\arcsin(y\oplus z)=\arcsin(x)+\arcsin(y)+\arcsin(z)\quad .$$

In the case of the inverse circular tangent, we have the following property to use [15]:

$$\arctan(x)\pm\arctan(y)=\arctan(\frac{x\pm y}{1\mp xy})$$

Therefore, let us define the generalized sum as:

$$x\oplus y=\frac{x+y}{1-xy} \qquad (6)$$

This sum is *commutative*. The *neutral element* is 0. The *opposite element* of *x* is $-x$.

For the a*ssociativity,* we can show that $(x\oplus y)\oplus z=x\oplus(y\oplus z)$. Actually:

$$(x\oplus y)\oplus z=\frac{\frac{x+y}{1-xy}+z}{1-(\frac{x+y}{1-xy})z}=\frac{x+y+z-xyz}{1-xy-xz-yz}$$

$$x\oplus(y\oplus z)=\frac{\frac{y+z}{1-yz}+x}{1-(\frac{y+z}{1-yz})x}=\frac{y+z+x-xyz}{1-yz-yx-zx}=\frac{z+y+z-xyz}{1-xy-xz-yz}$$

However, let us note that when we consider the sum $x \oplus y$ with $y = 1/x$, we have a divergence. This is the same as considering two angles, the sum of which being equal to 90 degrees.

A multiplicative group

Let us conclude considering the following function and its inverse:

$$G(x) = e^{-2x}(e^{2x}+1) \qquad G^{-1}(x) = \ln\left(\frac{1}{\sqrt{x-1}}\right)$$

and investigate a possible *multiplicative group* from them. For the chosen function, we need to have $1 < x$. A *group law* $\Phi(x, y)$ could be:

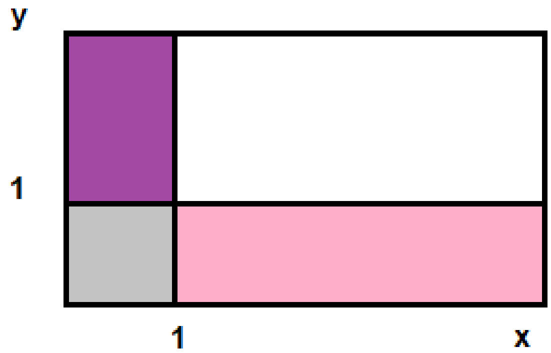
$$\Phi(x, y) = G(G^{-1}(x) + G^{-1}(y)) \quad .$$

And therefore we could imagine a *generalized sum* as:

$$x \oplus y = G(G^{-1}(x) + G^{-1}(y)) = G\left(\ln\left(\frac{1}{\sqrt{x-1}}\right) + \ln\left(\frac{1}{\sqrt{y-1}}\right)\right) = G\left(\ln\left(\frac{1}{\sqrt{x-1}}\frac{1}{\sqrt{y-1}}\right)\right) = G\left(\ln\frac{1}{Z}\right)$$

$$G\left(\ln\frac{1}{Z}\right) = e^{-2\ln Z}(e^{2\ln Z}+1) = (x-1)(y-1)\left(\frac{1}{(x-1)(y-1)}+1\right)$$

$$x \oplus y = 2 - x - y + xy = (1-x) + (1-y) + xy \qquad (7)$$



Let us consider the geometry on the left. From the rectangle $x \times y$, we can remove the colored rectangles $(x-1) \times 1$, $(y-1) \times 1$; the result is greater than 1. So it seems that have the *closure*.

Now, we need to consider the *neutral* and *opposite* elements.

As we can see from (6), the *neutral element* is not 0. In fact:

$$x \oplus 0 = 2 - x - 0 + x\,0 = (1-x) + (1-0) + x\,0 = 2 - x \neq x$$

Let us use as a *neutral element* the integer 2. $x \oplus 2 = (1-x)+(1-2)+2x = 1-x-1+2x = x$.

The *opposite element* of *x* is defined by $x \oplus Opposite(x) = 2$. We have:

$$Opposite(x) = \frac{x}{x-1} \qquad (8)$$

In this case, the *opposite element* is greater than 1 and then it is an element of the group.

Therefore, we consider 2 as the *neutral element ,* and the *opposite element* as given by (8).

To have a group, we need to have the *associativity* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for the given sum :

$$x \oplus y = 2 - x - y + xy = (1-x)+(1-y)+xy$$

Let us evaluate:

$$(x \oplus y) \oplus z = 2 - (x \oplus y) - z + (x \oplus y) z = 2 - 2 + x + y - xy - z + 2z - xz - yz + xyz$$

$$(x \oplus y) \oplus z = x + y + z - xy - xz - yz + xyz \qquad (9)$$

And:

$$x \oplus (y \oplus z) = 2 - x - (y \oplus z) + x (y \oplus z) = 2 - x - (2 - y - z + yz) + x(2 - y - z + yz)$$

$$x \oplus (y \oplus z) = x + y + z - xy - xz - yz + xyz \qquad (10)$$

From (8) and (9), we have the *associativity*. The *commutativity* is evident.

As a conclusion, the group having elements in the set $x>1$, for the *generator* $G(x) = \mathrm{e}^{-2x}(\mathrm{e}^{2x}+1)$, has the generalized sum given by $x \oplus y = 2 - x - y + xy = (1-x)+(1-y)+xy$. The *neutral element* is 2 and the *opposite element* of *x* is $x/(x-1)$.

To conclude, let us note that the same approach can be used for many other transcendental functions, such as for algebraic functions.

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On a generalized sum of the Mersenne Numbers

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Abstract Here we discuss the Mersenne numbers to give an example of a generalized sum. Using this sum, a recurrence relation is given.

Keywords Generalized sums, Mersenne numbers.

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Of generalized sums of numbers, we have given some examples in previous works [1-3]. Here we propose the study of the Mersenne Numbers, using the same approach. About these numbers, a large literature exists (see for instance that given in [4]). The form of the numbers is that of a power of two minus 1. Among them we find the Mersenne primes. The numbers are named after Marin Mersenne (1588 – 1648), a French Minim friar, who studied them in the early 17th century.
Mersenne numbers are:

$$M_n = 2^n - 1$$

Let us consider them to give an example of generalized sum. We can start from the following calculus:

$$. \quad M_{m+n} = 2^{m+n} - 1$$

$$M_{m+n} = 2^{m+n} - 1 = 2^m 2^n - 1 - 2^m + 2^m - 2^n + 2^n - 1 + 1 = 2^m (2^n - 1) - 1 + 2^m - 2^n + 1 + 2^n - 1$$

$$M_{m+n} = (2^m - 1)(2^n - 1) + 2^m - 1 + 2^n - 1$$

Therefore, we can write the following generalized sum:

$$M_{m+n} = M_m \oplus M_n = (2^m - 1)(2^n - 1) + (2^m - 1) + (2^n - 1)$$

or:

$$(1) \qquad M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n$$

This is a generalized sum that we find in the case of the multiplicative groups (for the use of multiplicative groups in statistics and statistical mechanics see [5,6]).
Using (1), for the Mersenne numbers we can imagine the following recursive relation:

$$M_{n+1}=M_n\oplus M_1=M_n+M_1+M_n M_1$$

That is:

$$2^{n+1}-1=(2^n-1)+(2^1-1)+(2^n-1)(2^1-1)=2^n+2^{n+1}-2^n-2+1=2^{n+1}-1$$

The sum (1) is associative, so that:

$$M_m\oplus M_n\oplus M_l=M_m+M_n+M_l+M_m M_n+M_n M_l+M_m M_l+M_m M_n M_l$$

We cannot have a group of the Mersenne numbers, without considering also the opposites of them, so that:

$$0=M_n\oplus Opposite(M_n)$$

Therefore:

$$Opposite(M_n)=-\frac{M_n}{M_n+1}=M_{-n}$$

Explicitly:

$$Opposite(2^n-1)=-\frac{(2^n-1)}{(2^n-1)+1}=\frac{(-2^n+1)}{2^n}=2^{-n}-1$$

These numbers are the Mersenne numbers with a negative exponent. So we have:

$$M_{nnn}=M_n\oplus M_{-n}=M_n+M_{-n}+M_n M_{-n}$$

$$0=2^0-1=(2^n-1)+(2^{-n}-1)+(2^n-1)(2^{-n}-1)=2^n+2^{-n}-2+2^n 2^{-n}-2^{-n}-2^n+1=0$$

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The q-integers and the Mersenne numbers

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Abstract Here we will show that the q-integers, the q-analogue of the integers that we can find in the q-calculus, are forming an additive group having a generalized sum similar to the sum of the Tsallis q-entropies of independent systems. The symmetric form of q-integers will be studied too. We will see that these numbers are linked to the Kaniadakis \kappa-calculus. In the article, a final discussion will be devoted to the link of the q-integers to the Mersenne numbers. Besides the discussion of the previously mentioned numbers, the general aim of the paper is that of popularizing the existence of the q-calculus.

Keywords q-calculus, q-integers, Tsallis q-entropy, Mersenne numbers.

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Introduction Several mathematicians have contributed to a calculus that today is known as the q-calculus [1-6]. As a consequence of the many contributions, we find that it is known as "quantum calculus," or "time-scale calculus", or "calculus of partitions" too [5]. It is also called the "calculus without limits", because it is equivalent to the traditional infinitesimal calculus without the notion of limits. Besides being known with different names, the q-calculus is expressed by means of different notations or, as told in [5], by different "dialects". Here we will use the approach and the notation given in the book by Kac and Cheung [6].

The first aim of the work here proposed is that of showing the following fact. The q-integers, the q-analogue of the integers that we can find in the q-calculus, are forming a group having a generalized sum which is similar to sum of the Tsallis q-entropies of independent systems. After, we will see that the symmetric form of q-integers is linked to the Kaniadakis calculus. We will conclude the discussion considering the Mersenne numbers and their link to the q-integers. Let us stress that, besides the discussion of the previously mentioned numbers, the general aim of the paper is that of popularizing the existence of the q-calculus.

The q-integers

Let us start defining the q-integers.

In the q-calculus, the q-difference is simply given by:

$$d_q f = f(qx) - f(x)$$

From this difference, the q-derivative is given as:

$$(1) \quad D_q f = \frac{f(qx) - f(x)}{qx - x}$$

The q-derivative reduces to the Newton's derivative in the limit $q \to 1$. (1) is also known as the Jackson derivative, after Frank Hilton Jackson (1870 – 1960), the English clergyman and mathematician who worked at the beginning of the XXth century on the q-calculus.
Let us consider the function $f(x) = x^n$. If we calculate its q-derivative, we obtain:

$$(2) \quad D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

Comparing the ordinary calculus, which is giving $(x^n)' = n\, x^{n-1}$, to Equation (2), we can define the "q-integer" [*n*] by:

$$(3) \quad [n] = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}$$

Therefore Equation (2) turns out to be:

$$D_q x^n = [n]\ x^{n-1}$$

As a consequence, the *n*-th q-derivative of $f(x) = x^n$, which is obtained by repeating *n* times the q-derivative, generates the q-factorial:

$$[n]\,! = [n][n-1]\ldots[3][2][1]$$

Form the q-factorials, we can define q-binomial coefficients:

$$\frac{[n]!}{[m]![n-m]!}$$

This means that we can use the usual Taylor formula, replacing the derivatives by the q-derivatives and the factorials by q-factorials (in a previous work, we have discussed the q-exponential and q-trigonometric functions [7]). Then, in the q-calculus, the q-integer [*n*] acts as the integer in the ordinary calculus.

The group of q-integers

We known that the set of integers consisting of the numbers ..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ..., having as operation the addition, is a group. Therefore, let us consider the set of q-integers given by (3) and investigate its group. In particular, we have to determine its operation of addition.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a*•*b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a*•*b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a*•*b*)•*c* = *a*•(*b*•*c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e*•*a* = *a*•*e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a*•*b* = *b*•*a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation).

A further requirement is the c*ommutativity:* For all *a*,*b* in *A*, *a*•*b* = *b*•*a*. In this case, the group is known as an Abelian group.

Therefore, to qualify a group as an Abelian group, the set and operation must satisfy five requirements which are known as the *Abelian group axioms.* A group having a non-commutative operation is called a "non-abelian group" or "non-commutative group". For an Abelian group, one may choose to denote the group operation by + and the *identity element* by 0 (*neutral element*) and the inverse element as $-a$ (*opposite element*). In this case, the group is called an additive group.

First, we have to define the operation of addition. It is not the sum that we use for the integers, but it is a generalized sum which obeys the axioms of the group.

Let us start from the q-integer $[m+n]$:

$$[m+n]=\frac{q^{m+n}-1}{q-1}=\frac{1}{q-1}(q^m q^n-1+q^m-q^m)=\frac{1}{q-1}(q^m(q^n-1)+q^m-1)$$

$$[m+n]=\frac{1}{q-1}(q^m(q^n-1)+(q^m-1)+(q^n-1)+(1-q^n))=\frac{1}{q-1}((q^m-1)(q^n-1)+(q^m-1)+(q^n-1))$$

Therefore, we have:

$$(4)\quad [m+n]=[m]+[n]+(q-1)[m][n]$$

Then, we can define the generalized "sum" of the group as:

$$(5)\quad [m]\oplus[n]=[m]+[n]+(q-1)[m][n]$$

(for other examples of generalized sums see [8]).

If we use (5) as the sum, we have the closure of it, because the result of the sum is a q-integer. Moreover, this sum is commutative.

The neutral element is:

$$(6)\quad [0]=\frac{q^0-1}{q-1}=0$$

Let us determine the opposite element $[o]$, so that:

$$[o]\oplus[n]=0$$

$$0=[0]=[o]\oplus[n]=[o]+[n]+(q-1)[o][n]$$

$$-[n]=[o]+(q-1)[o][n]$$

$$(7)\quad [o]=-\frac{[n]}{1+(q-1)[n]}=-\frac{q^n-1}{(q-1)q^n}=\frac{q^{-n}-1}{q-1}=[-n]$$

The opposite element of q-integer $[n]$ is the q-integer of $-n$, that is $[-n]$.

Let us discuss the associativity of the sum.

It is necessary to have:

$$[m]\oplus([n]\oplus[l])=([m]\oplus[n])\oplus[l]$$

Let us calculate:

$$[m]\oplus([n]\oplus[l])=[m]\oplus([n]+[l]+(q-1)[n][l])$$

$$[m]\oplus([n]\oplus[l])=[m]+[n]+[l]+(q-1)[n][l]+(q-1)[m][n]+(q-1)[m][l]+(q-1)^2[m][n][l]$$

And also:

$$([m]\oplus[n])\oplus[l]=([m]+[n]+(q-1)[m][n])\oplus[l]$$

$$([m]\oplus[n])\oplus[l]=[m]+[n]+(q-1)[m][n]+[l]+(q-1)[m][l]+(q-1)[n][l]+(q-1)^2[m][n][l]$$

It is also easy to see that:

$$[m]\oplus[n]\oplus[l]=[m+n+l]$$

As we have shown, the five axioms of an Abelian group are satisfied. In this manner, using the generalized sum given by (5), we have the Abelian group of the q-integers.

The link to Tsallis calculus

Let us also note that the generalized sum (5) is similar to the sum that we find in the approach to entropy proposed by Constantino Tsallis.

In 1948 [9], Claude Shannon defined the entropy *S* of a discrete random variable Ξ as the expected value of the information content: $S=\sum_i p_i\, I_i = -\sum_i p_i \log_b p_i$ [10]. In this expression, *I* is the information content of Ξ, the probability of *i*-event is p_i and *b* is the base of the used logarithm. Common values of the base are 2, the Euler's number e, and 10.

Constantino Tsallis generalized the Shannon entropy in the following manner [11]:

$$S_q=\frac{1}{q-1}\left(1-\sum_i p_i^q\right)$$

Given two independent systems *A* and *B*, for which the joint probability density satisfies:

$$p(A,B)=p(A)\,p(B)$$

the Tsallis entropy gives:

$$(8)\quad S_q(A,B)=S_q(A)+S_q(B)+(1-q)S_q(A)S_q(B)$$

The sum of more than two terms of Tsallis entropies is discussed in [12].

The parameter $(1-q)$, in a certain manner, measures the departure from the ordinary additivity, which is recovered in the limit $q\rightarrow 1$.

Actually the group on which is based the Tsallis entropy, and therefore Equation (8), is known as the "multiplicative group" [7,13,14]. As stressed in [15], the use of a group structure allows to determine a class of generalized entropies. Let us note the group of the q-integers, with addition (5), can be considered a "multiplicative group" too.

Let us stress that we have a link of the multiplicative group to the Tsallis entropy. The group of the *n*-integers had been studied in [16,17] too, but in these articles, a quite different expression for the generalized sum had been proposed. It is given as the "quantum sum" $[x]\oplus[y]=[x]+q^x[y]$, where the link to the Tsallis calculus is less evident.

Symmetric q-numbers

In the previous discussion we have considered the group of the q-integers as defined by q-calculus. In [6] it is also defined the symmetric q-integer in the following form (here we use a notation different from that given in the Ref.6):

$$(9)\quad [n]_s = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Repeating the approach previously given, we can determine the group of the symmetric q-integers.

Let us start from the q-integer $[m+n]_s$, which is according to (9):

$$[m+n]_s = \frac{q^{m+n} - q^{-(m+n)}}{q - q^{-1}}$$

and try to find it as a generalized sum of the q-integers $[m]_s$ and $[n]_s$.

By writing $q = \exp(\log q)$, the q-integer turns out into a hyperbolic sine:

$$(10)\quad [n]_s = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{e^{n\log q} - e^{-n\log q}}{q - q^{-1}} = 2\frac{\sinh(n\log q)}{(q - q^{-1})}$$

Apart from a numerical factor, this is the form of the generalized numbers proposed by G. Kaniadakis in his κ-calculus [18-22].

From (10), we can write also:

$$\frac{1}{2}(q - q^{-1})[n]_s = \sinh(n\log q)$$

Therefore:

$$[m+n]_s = \frac{q^{m+n} - q^{-(m+n)}}{q - q^{-1}} = 2\frac{\sinh((m+n)\log q)}{(q - q^{-1})}$$

Using the properties:

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh x \quad ; \quad \cosh x = \sqrt{1+\sinh^2 x}$$

we obtain:

$$[m+n]_s=\frac{2}{(q-q^{-1})}[\sinh(m\log q)\cosh(n\log q)+\sinh(n\log q)\cosh(m\log q)]$$

$$[m+n]_s=[m]_s\cosh(n\log q)+[n]_s\cosh(m\log q)$$

$$[m+n]_s=[m]_s\sqrt{1+\sinh^2(n\log q)}+[n]_s\sqrt{1+\sinh^2(m\log q)}$$

Let us define: $k=(q-q^{-1})/2$ and then: $k[n]_s=\sinh(n\log q)$. As a consequence, we have the generalized sum of the symmetric q-integers as:

$$(11)\quad [m]_s\oplus[n]_s=[m]_s\sqrt{1+k^2[n]_s^2}+[n]_s\sqrt{1+k^2[m]_s^2}$$

Let us stress that (11) is also the generalized sum proposed by G. Kaniadakis in the framework of a calculus [19-22], the details of which are given in [22].

By means of (11), we can repeat the approach given previously for q-numbers (3) and study of the group of the symmetric q-integers.

The Mersenne numbers

In the case that $q=2$, we have:

$$[n]=\frac{2^n-1}{2-1}=2^n-1$$

These are the Mersenne Numbers. About these numbers, a large literature exists (see for instance that given in [23]). Among these numbers we find the Mersenne primes.

The numbers are named after Marin Mersenne (1588 – 1648), a French Minim friar, who studied them in the early 17th century.

Mersenne numbers are written as [23]:

$$M_n=2^n-1$$

Of course, because they are q-integers for $q=2$, we have the generalized sum given in (5):

$$(5')\quad [m]\oplus[n]=[m]+[n]+(2-1)[m][n]=[m]+[n]+[m][n]$$

But we can repeat the calculus as an exercise.

We can start from the number M_{m+n} and calculate.

$$M_{m+n}=2^{m+n}-1$$

$$M_{m+n}=2^{m+n}-1=2^m 2^n-1-2^m+2^m-2^n+2^n-1+1=2^m(2^n-1)-1+2^m-2^n+1+2^n-1$$

$$M_{m+n}=(2^m-1)(2^n-1)+2^m-1+2^n-1$$

Therefore, we can write the following generalized sum:

$$M_{m+n}=M_m\oplus M_n=(2^m-1)(2^n-1)+(2^m-1)+(2^n-1)$$

or:

$$(12)\qquad M_{m+n}=M_m\oplus M_n=M_m+M_n+M_m M_n$$

(12) is the same as (5’). Let us stress once more that this is a generalized sum that we can find in the case of the multiplicative groups [8].

Using (12), we can imagine for the Mersenne numbers the following recursive relation:

$$M_{n+1}=M_n\oplus M_1=M_n+M_1+M_n M_1$$

We can verify as follow:

$$2^{n+1}-1=(2^n-1)+(2^1-1)+(2^n-1)(2^1-1)=2^n+2^{n+1}-2^n-2+1=2^{n+1}-1$$

The sum (12) is associative, so that:

$$M_m\oplus M_n\oplus M_l=M_m+M_n+M_l+M_m M_n+M_n M_l+M_m M_l+M_m M_n M_l$$

We cannot have a group of the Mersenne numbers, without considering also the opposites of them, so that:

$$0=M_n\oplus Opposite(M_n)$$

Therefore:

$$Opposite(M_n)=-\frac{M_n}{M_n+1}=M_{-n}$$

Explicitly:

$$Opposite(2^n-1)=-\frac{(2^n-1)}{(2^n-1)+1}=\frac{(-2^n+1)}{2^n}=2^{-n}-1$$

These numbers are the Mersenne numbers with a negative exponent. So we have:

$$M_{n-n}=M_n\oplus M_{-n}=M_n+M_{-n}+M_n M_{-n}$$

$$0=2^0-1=(2^n-1)+(2^{-n}-1)+(2^n-1)(2^{-n}-1)=2^n+2^{-n}-2+2^n 2^{-n}-2^{-n}-2^n+1=0$$

Symmetric Mersenne

Let us consider the symmetric q-integer in the case of $q=2$.

We can define the symmetric Mersenne in the following manner:

$$(13)\quad M_n^s=[n]_s=\frac{2^n-2^{-n}}{2-2^{-1}}$$

By writing $2=\exp(\log 2)$, (13) turns out into a hyperbolic sine:

$$(14)\quad M_n^s=\frac{2^n-2^{-n}}{2-2^{-1}}=\frac{e^{n\log 2}-e^{-n\log 2}}{2-2^{-1}}=2\frac{\sinh(n\log 2)}{(2-2^{-1})}$$

Again, as previously told, apart from a numerical factor, this is the form of the generalized numbers proposed by G. Kaniadakis.

Let us define $k=(2-2^{-1})/2$; we have the generalized sum of the symmetric Mersenne as:

$$(15)\quad M_m^s\oplus M_n^s=M_m^s\sqrt{1+k^2(M_n^s)^2}+M_n^s\sqrt{1+k^2(M_m^s)^2}$$

Of course, we have again the generalized sum proposed by G. Kaniadakis.

As a conclusion we can note that, by means of the generalized sums, we have found a different approach to the Mersenne numbers too. In my opinion, it is also possible that it was the form of the Mersenne numbers that inspired the Reverend Jackson to modify the usual derivative into the definition (1) of the q-calculus.

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The group of the Fermat Numbers

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Abstract: In this work we are discussing the group that we can obtain if we consider the Fermat numbers with a generalized sum.

Keywords: generalized sum, groups, Abelian groups, transcendental functions, logarithmic and exponential functions, Fermat numbers.

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In [1] we find that there are two definitions of the Fermat numbers. We have a less common definition giving a Fermat number as $F_n = 2^n + 1$, which is obtained by setting x=1 in a Fermat polynomial of x, and the commonly encounter definition $F_n = 2^{2^n} + 1$, which is a subset of the previous assembly of numbers. Here we will consider numbers $F_n = 2^n + 1$ and - as we have recently proposed in [2] for q-integers and Mersenne numbers - investigate the set of them to find its generalized sum which defines the operation of the group.

Let us remember that a group is a set A having an operation • which is combining the elements of A. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a•b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a•b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a•b*)•*c* = *a•*(*b•c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e•a* = *a•e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a•b* = *b•a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation). A further requirement, is the c*ommutativity:* For all *a*,*b* in *A*, *a•b* = *b•a*. In this case, the group is an Abelian group. For an Abelian group, one may choose to denote the *operation* by + , the *identity element* becomes the *neutral element* and the inverse element the *opposite element*. In this case, the group is called an additive group.

The generalized sum for the Fermat numbers $F_n = 2^n + 1$ is:

$$F_m \oplus F_n = 2 - F_m - F_n + F_m F_n = (1 - F_m) + (1 - F_n) + F_m F_n \quad (1)$$

To have (1), let us evaluate:

$$F_{m+n} = 2^{m+n} + 1 = F_m \oplus F_n = 2 - F_m - F_n + F_m F_n = 2 - (2^m + 1) - (2^n + 1) + (2^m + 1)(2^n + 1)$$

$$2^{m+n} + 1 = 2 - 2^m - 2^n - 2 + 2^m 2^n + 2^n + 2^m + 1$$

This gives also the *closure* of the group.

We can provide a recurrence relation as: $F_{n+1} = 2^{n+1} + 1 = F_n \oplus F_1$

From (1), we can see that the *neutral element* is not 0. We have to use as a *neutral element* the integer 2, which is $F_0 = 2^0 + 1 = 2$ and then an element of the group. We have:

$$F_n \oplus F_0 = 2 - F_n - F_0 + F_n F_0 = F_n$$

The *opposite element* is defined by $F_n \oplus Opposite(F_n) = 2$. We have:

$$Opposite(F_n) = \frac{F_n}{F_n - 1} = 1 + 2^{-n} = F_{-n} \qquad (2)$$

Then, to have a group we need to add numbers (2) to the set of the Fermat numbers.

Therefore, we consider 2 as the *neutral element* , and the *opposite element* as given by (2).

Let us consider three Fermat numbers F_n , F_m , F_l ; to have a group we need the *associativity* of the generalized sum, so that $(F_m \oplus F_n) \oplus F_l = F_m \oplus (F_n \oplus F_l)$. Let us call $x = F_n$, $y = F_m$, $z = F_l$ and evaluate:

$$(x \oplus y) \oplus z = 2 - (x \oplus y) - z + (x \oplus y) z = 2 - 2 + x + y - xy - z + 2z - xz - yz + xyz$$

$$(x \oplus y) \oplus z = x + y + z - xy - xz - yz + xyz \qquad (3)$$

And:

$$x \oplus (y \oplus z) = 2 - x - (y \oplus z) + x (y \oplus z) = 2 - x - (2 - y - z + yz) + x (2 - y - z + yz)$$

$$x \oplus (y \oplus z) = x + y + z - xy - xz - yz + xyz \qquad (4)$$

From (3) and (4), we have the *associativity*. The *commutativity* is evident.

We have already considered the generalized sum (1) in a recent work [3].

In [3], we consider some functions $G(x)$, having inverses so that $G^{-1}(G(x))=x$, which are *generators* of *group law* [4-6]:

$$\Phi(x,y)=G(G^{-1}(x)+G^{-1}(y))$$

The *group law* is giving the *generalized sum* of the group $x\oplus y=G(G^{-1}(x)+G^{-1}(y))$.

In [3] we considered the following generator and inverse:

$$G(x)=\mathrm{e}^{-2x}(\mathrm{e}^{2x}+1) \qquad G^{-1}(x)=\ln(\frac{1}{\sqrt{x-1}}) \quad (5)$$

and investigate a possible group from them. The *group law* $\Phi(x,y)$ gives the generalized sum:

$$x\oplus y=G(G^{-1}(x)+G^{-1}(y))=G(\ln(\frac{1}{\sqrt{x-1}})+\ln(\frac{1}{\sqrt{y-1}}))=G(\ln(\frac{1}{\sqrt{x-1}}\frac{1}{\sqrt{y-1}}))=G(\ln\frac{1}{Z})$$

$$G(\ln\frac{1}{Z})=\mathrm{e}^{-2\ln Z}(e^{2\ln Z}+1)=(x-1)(y-1)(\frac{1}{(x-1)(y-1)}+1)$$

$$x\oplus y=2-x-y+xy=(1-x)+(1-y)+xy \qquad (6)$$

And (6) is the generalized sum (1) proposed for the Fermat numbers.

Let us also note that, if we use (5), we need $x>1$. And this is a condition satisfied by the Fermat numbers and their opposites (2).

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On the generalized sums of Mersenne, Fermat, Cullen and Woodall Numbers

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Abstract Recently we have discussed Mersenne and Fermat numbers using generalized sums. Here we discuss Cullen and Woodall numbers, which are similar to Mersenne and Fermat numbers. The generalized sums are given for them. Recursive relations are given accordingly.

Keywords Generalized sums, Mersenne numbers.

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In two recent papers we have discussed some properties of the Mersenne numbers [1,2] and of the Fermat numbers [3], using an approach based on the generalized sums [4-8].
In [2], in particular, the generalized sum of the Mersenne numbers and the group based on this sum is proposed. Mersenne numbers are $M_n = 2^n - 1$. These numbers form a group with the following generalized sum:

$$(1) \quad M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n$$

Using (1), for the Mersenne numbers we can imagine the following recursive relation:

$$M_{n+1} = M_n \oplus M_1 = M_n + M_1 + M_n M_1$$

Being $M_1 = 1$:

$$M_{n+1} = 2 M_n + 1$$

With a Fortran program, we have 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, 4194303, 8388607, 16777215, 33554431, 67108863, 134217727, 268435455, 536870911, 1073741823, 2147483647, 4294967295, 8589934591, 17179869183, 34359738367, 68719476735, 137438953471, 274877906943, 549755813887, 1099511627775, 2199023255551, 4398046511103, 8796093022207, 17592186044415, 35184372088831, 70368744177663, 140737488355327, 281474976710655, 562949953421311, 1125899906842623, in agreement to http://oeis.org/A000225 for the first 32 numbers.

The sum (1) is associative. The neutral element is $M_0=2^0-1=0$ and the opposites of the numbers are given by $0=M_n \oplus Opposite(M_n)$.

$$Opposite(M_n)=-\frac{M_n}{M_n+1}=M_{-n}$$

Explicitly:

$$Opposite(2^n-1)=2^{-n}-1$$

These numbers are the Mersenne numbers with a negative exponent. If we use them, we can have a group associated to the Mersenne numbers.

Fermat numbers are $F_n=2^n+1$ [9]. These numbers have the following generalized sum [3]:

$$(2) \quad F_m \oplus F_n=(1-F_m)+(1-F_n)+F_m F_n$$

Using (2), for the Fermat numbers we can imagine the following recursive relation:

$$F_{n+1}=F_n \oplus F_1=(1-F_n)+(1-F_1)+F_n F_1$$

Since $F_1=2^1+1=3$:

$$F_{n+1}=F_n \oplus F_1=(1-F_n)-2+3F_n=2F_n-1$$

Using a Fortran program we have: 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, 4194305, 8388609, 16777217, 33554433, 67108865, 134217729, 268435457, 536870913, 1073741825, 2147483649, 4294967297, 8589934593, 17179869185, 34359738369, 68719476737, 137438953473, 274877906945, 549755813889, 1099511627777, 2199023255553, 4398046511105, 8796093022209, 17592186044417, 35184372088833, 70368744177665, 140737488355329, 281474976710657, 562949953421313, 1125899906842625, in agreement to http://oeis.org/A000051 for the first 32 numbers.

The sum (2) is associative. The neutral element is $F_0=2^0+1=2$ and the opposites of the numbers are $Opposite(F_n)=F_{-n}$ [3].

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Similar to the Fermat numbers, we have the Cullen numbers. The Woodall numbers are similar to the Mersenne numbers [8,9]. Let us find the generalized sums of them.
The Cullen numbers are:

$$C_n = 2^n n + 1$$

Let us find the generalized sum, as we did in [2,3]:

$$C_{m+n} = 2^{m+n}(m+n)+1$$

$$C_{m+n} = 2^m 2^n n + 2^m 2^n m + 2^m - 2^m + 2^n - 2^n + 1 = 2^m(2^n n+1) + 2^n(2^m m+1) - 2^n - 2^m + 1$$

$$C_{m+n} = 2^m C_n + 2^n C_m - 2^n - 2^m + 1 = 2^m(C_n - 1) + 2^n(C_m - 1) + 1$$

Let us write the generalized sum as:

$$(3) \quad C_m \oplus C_n = 2^m(C_n - 1) + 2^n(C_m - 1) + 1$$

The neutral element of this sum is $C_0 = 2^0 0 + 1 = 1$. Using (3):

$$C_m \oplus C_0 = 2^m(C_0 - 1) + 2^0(C_m - 1) + 1 = C_m$$

We have $C_1 = 2^1 1 + 1 = 3$. Recursive relation is:

$$C_{n+1} = C_n \oplus C_1 = 2^n(C_1 - 1) + 2^1(C_n - 1) + 1 = 2^{n+1} + 2(C_n - 1) + 1$$

So we have: 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, 1048577, 2228225, 4718593, 9961473, 20971521, 44040193, 92274689, 192937985, 402653185, 838860801, 1744830465, 3623878657, 7516192769, 15569256449, 32212254721, 66571993089, 137438953473, 283467841537, 584115552257, 1202590842881, 2473901162497, 5085241278465, 10445360463873, 21440476741633, 43980465111041, in agreement to http://oeis.org/A002064.
Of the Cullen numbers, we can also give another form of the generalized sum (3):

$$C_m \oplus C_n = 2^m \frac{m}{m}(C_n - 1) + 2^n \frac{n}{n}(C_m - 1) + 1 + \frac{(C_n - 1)}{m} - \frac{(C_n - 1)}{m} + \frac{(C_m - 1)}{n} - \frac{(C_m - 1)}{n}$$

$$C_m \oplus C_n = \frac{C_m}{m}(C_n - 1) + \frac{C_n}{n}(C_m - 1) + 1 - \frac{(C_n - 1)}{m} - \frac{(C_m - 1)}{n}$$

$$(4) \quad C_m \oplus C_n = \frac{1}{m}(C_m - 1)(C_n - 1) + \frac{1}{n}(C_n - 1)(C_m - 1) + 1$$

The recursive relation assumes the form:

$$C_{n+1} = C_n \oplus C_1 = \frac{1}{1}(C_1 - 1)(C_n - 1) + \frac{1}{n}(C_n - 1)(C_1 - 1) + 1$$

$$C_{n+1} = C_n \oplus C_1 = 2(C_n - 1) + \frac{2}{n}(C_n - 1) + 1$$

In the case that we use the generalized sum (4), we have to remember that when m or n are equal to zero, we need to assume $(C_m - 1)/m = 1$, $(C_n - 1)/n = 1$.

In this manner:

$$C_m \oplus C_0 = \frac{1}{m}(C_m - 1)(C_0 - 1) + (C_m - 1) + 1 = C_m$$

since $C_0 = 1$.

The Woodall numbers are:

$$W_n = 2^n n - 1 \quad .$$

Let us find the generalized sum:

$$W_{m+n} = 2^{m+n}(m+n) - 1$$

$$W_{m+n} = 2^m 2^n n + 2^m 2^n m + 2^m - 2^m + 2^n - 2^n - 1 = 2^m (2^n n - 1) + 2^n (2^m m - 1) + 2^n + 2^m - 1$$

$$W_{m+n} = 2^m W_n + 2^n W_m + 2^n + 2^m - 1 = 2^m (W_n + 1) + 2^n (W_m + 1) - 1$$

Let us write the generalized sum as:

$$(5) \quad W_m \oplus W_n = 2^m (W_n + 1) + 2^n (W_m + 1) - 1$$

The neutral element of this sum is $W_0 = 2^0 0 - 1 = -1$. Using (5):

$$W_m \oplus W_0 = 2^m (W_0 + 1) + 2^0 (W_m + 1) - 1 = W_m$$

We have $W_1 = 2^1 1 - 1 = 1$. The recursive relation is:

$$W_{n+1} = W_n \oplus W_1 = 2^n (W_1 + 1) + 2^1 (W_n + 1) - 1 = 2^{n+1} + 2(W_n + 1) - 1$$

So we have: 1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, 2228223, 4718591, 9961471, 20971519, 44040191, 92274687, 192937983, 402653183, 838860799, 1744830463, 3623878655, 7516192767, 15569256447, 32212254719, 66571993087, 137438953471, 283467841535, 584115552255, 1202590842879, 2473901162495, 5085241278463, 10445360463871, 21440476741631 43980465111039, in agreement to http://oeis.org/A003261.

Again, we can give another form of the generalized sum (5):

$$W_m \oplus W_n = 2^m \frac{m}{m}(W_n+1) + 2^n \frac{n}{n}(W_m+1) - 1 + \frac{(W_n+1)}{m} - \frac{(W_n+1)}{m} + \frac{(W_m+1)}{n} - \frac{(W_m+1)}{n}$$

$$W_m \oplus W_n = \frac{W_m}{m}(W_n+1) + \frac{W_n}{n}(W_m+1) - 1 + \frac{(W_n+1)}{m} + \frac{(W_m+1)}{n}$$

$$(6) \quad W_m \oplus W_n = \frac{1}{m}(W_m+1)(W_n+1) + \frac{1}{n}(W_n+1)(W_m+1) - 1$$

The recursive relation assumes the form:

$$W_{n+1} = W_n \oplus W_1 = \frac{1}{n}(W_n+1)(W_1+1) + (W_n+1)(W_1+1) - 1$$

$$W_{n+1} = W_n \oplus W_1 = \frac{2}{n}(W_n+1) + 2(W_n+1) - 1$$

In the case that we use the generalized sum (6), we have to remember that when m or n are equal to zero, we need to assume $(W_m+1)/m=1$, $(W_n+1)/n=1$.

In this manner:

$$W_m \oplus W_0 = \frac{1}{m}(W_m+1)(W_0+1) + (W_m+1) - 1 = W_m$$

since $W_0 = -1$.

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A recursive formula for Thabit numbers

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Abstract Here we discuss the Thabit numbers. An operation of addition of these numbers is proposed. A recursive relation is given accordingly.

Keywords Thabit numbers

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In recent papers we have discussed some properties of the Mersenne numbers [1,2] and of the Fermat numbers [3], using an approach based on generalized operations of addition [4-8]. In [9], we have discussed the Cullen and Woodall numbers too (for references on these numbers, see [10-13]). Here we consider the Thabit numbers [14]. These numbers are given as $T_n = 3*2^n - 1$, where the asterisk represents the ordinary multiplication.

Let us consider the following operation:

$$T_{m+n} = T_m \oplus T_n$$

Therefore

$$T_{m+n} = 3*2^{m+n} - 1 = 3*2^{m+n} - 1 + 2^n - 2^n = 2^n(3*2^m - 1) - 1 + 2^n = 2^n T_m + 2^n - 1$$

$$T_{m+n} = 2^n \frac{3}{3} T_m + 2^n - 1 - \frac{T_m}{3} + \frac{T_m}{3} = \frac{1}{3} T_m T_n + \frac{3}{3} 2^n - \frac{1}{3} + \frac{1}{3} - 1 + \frac{T_m}{3}$$

So we have:

$$(1) \quad T_m \oplus T_n = \frac{1}{3} T_m T_n + \frac{1}{3} T_m + \frac{1}{3} T_n - \frac{2}{3} = \frac{1}{3}(T_m + T_n + T_m T_n - 2)$$

Using (1), we can see that the neutral element is $T_0 = 2$, so that:

$$T_m \oplus T_0 = \frac{1}{3}(T_m + T_0 + T_m T_0 - 2) = \frac{1}{3}(3 T_m) = T_m$$

The recursive relation is given accordingly to (1), starting from $T_1=5$:

$$T_{n+1}=T_n\oplus T_1=\frac{1}{3}(T_n+T_1+T_nT_1-2)=\frac{1}{3}(T_n+5+5T_n-2)=\frac{1}{3}(6T_n+3)=2T_n+1$$

With a Fortran program (double precision), we have **5**, **11**, **23**, **47**, 95, **191**, **383**, 767, 1535, 3071, **6143**, 12287, 24575, 49151, 98303, 196607, 393215, **786431**, 1572863, 3145727, 6291455, 12582911, 25165823, 50331647, 100663295, 201326591, 402653183, 805306367, 1610612735, 3221225471, 6442450943, 12884901887, 25769803775, **51539607551**, 103079215103, 206158430207, 412316860415, **824633720831**, 1649267441663, 3298534883327, 6597069766655, 13194139533311, **26388279066623**, 52776558133247, 105553116266495, 211106232532991, 422212465065983, 844424930131967, 1688849860263935, 3377699720527871, 6755399441055743. In bold characters, the prime numbers as from http://oeis.org/A007505.

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On Repunits

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Abstract Here we discuss the repunits. An operation of addition of these numbers is proposed. A recursive formula is given accordingly. Symmetric repunits are also defined.

Keywords Repunits

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As explained in [1], the term "repunit" was coined by Beiler in a book of 1966 [2], for the numbers defined as:

$$R_n = \frac{10^n - 1}{10 - 1}$$

The sequence of repunits starts with 1, 11, 111, 1111, 11111, 111111, ... (sequence A002275 in the OEIS, https://oeis.org/A002275). As we can easily see, these numbers are linked to q-integers and Mersenne numbers [3-7]. A q-integer is defined as [3]:

$$[n] = \frac{q^n - 1}{q - 1}$$

so we have the Mersenne numbers for q=2. The repunits are the q-integers for q=10 :

$$[n]_{q=10} = \frac{10^n - 1}{10 - 1}$$

We can use the same approach for the repunits of that proposed in [4-6]. Let us consider the following operation (generalized sum):

$$R_{m+n} = R_m \oplus R_n$$

defined in the following manner:

(1) $R_m \oplus R_n = R_m + R_n + (10-1) R_m R_n$

This is the addition of the q-units as given in [4,5]. The neutral element for (1) is $R_0 = 0$, so that: $R_m \oplus R_0 = R_m + R_0 + (10-1) R_m R_0 = R_m$.

The recursive relation for the repunits, given according to (1) and starting from $R_1 = 1$, is:

$$R_m \oplus R_1 = R_m + R_1 + (10-1) R_m R_1$$

That is: 11, 111, 1111, 11111, 111111, 1111111, 11111111, and so on.

In [8], we have discussed the symmetric q-integers, which are defined as [3]:

$$[n]_s = \frac{q^n - q^{-n}}{q - q^{-1}}$$

We can define the "symmetric" repunits as:

$$R_{n,s} = \frac{10^n - 10^{-n}}{10 - 10^{-1}} = 2 \frac{\sinh(n \ \ln 10)}{10 - 10^{-1}}$$

The sequence is: 1, 10.1, 101.01, 1010.101, 10101.0101, 101010.10101, etc.

In this case, the addition is defined [8]:

$$R_{m,s} \oplus R_{n,s} = R_{m,s} \cosh(n \ \ln 10) + R_{n,s} \cosh(m \ \ln 10)$$

or

$$R_{m,s} \oplus R_{n,s} = R_{m,s} \sqrt{1 + k^2 (R_{n,s})^2} + R_{n,s} \sqrt{1 + k^2 (R_{m,s})^2}$$

where $k = \frac{1}{2} \left(10 - \frac{1}{10}\right)$. Let us note that $R_{1,s} = \frac{10 - 10^{-1}}{10 - 10^{-1}} = 1$.

The recursive formula for the symmetric repunits is:

$$R_{n+1,s} = R_{n,s} \oplus R_{1,s} = R_{n,s} \sqrt{1 + k^2} + \sqrt{1 + k^2 (R_{n,s})^2}$$

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Composition Operations of Generalized Entropies Applied to the Study of Numbers

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Abstract: The generalized entropies of C. Tsallis and G. Kaniadakis have composition operations, which can be applied to the study of numbers. Here we will discuss these composition rules and use them to study some famous sequences of numbers (Mersenne, Fermat, Cullen, Woodall and Thabit numbers). We will also consider the sequence of the repunits, which can be seen as a specific case of q-integers.

Keywords: Generalized Entropies, Q-Calculus, Abelian Groups, Hyperbolic Functions, Fermat Numbers, Mersenne Numbers, Thabit Numbers, Repunits

Introduction

In some recent works [1-3], we have discussed the generalized entropies of C. Tsallis [4] and G. Kaniadakis [5,6], with the aim of applying them to image processing and image segmentation. What is quite attractive of these entropies is the fact that they are non-additive. It means that they are following rules of composition, which are different from the usual operation of addition. Moreover, these composition rules contain indices, which are useful to have a specific segmentation of images, or even to drive a gray-level image transition among the textures of the image [7].

The Tsallis composition rule is defined in [8] as a pseudo-additivity. However, as these rules are concerning generalized entropies, we could call them "generalized sums". Actually, I used this locution in a discussion about the rules of composition that we can obtain from the transcendental functions [9]. The approach, given in [9], is using a method based on the generators of algebras [10-12]. Here I show that we can use the generalized sums, as those that we can obtain from Tsallis and Kaniadakis generalized statistics, to the study of the sequences of numbers. In particular, we will discuss the composition rules that can be applied to famous sequences of numbers, such as Fermat, Mersenne and Thabit numbers. We will also consider the sequence of the repunits, which are a specific case of q-integers.

Let us start remembering the composition rules of Kaniadakis and Tsallis entropies.

The generalized sum of Kaniadakis statistics

In [6], a generalized sum is defined in the following manner. Let us consider two elements x and y of reals **R**, and a parameter κ real too. The composition law $x \oplus y$ is given by:

$$x \oplus y = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2} \qquad (1)$$

which defines a generalized sum, named κ-sum. Reals R and operation (1) form an Abelian group.

Let us remember that a group is a set A and an operation •. The operation combines any two elements a,b to form another element of the group denoted a•b.

To qualify (A,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all a,b in A, the result of the operation a•b is also in A. *Associativity:* For all a,b and c in A, it holds (a•b)•c = a•(b•c). *Identity element*: An element e exists in A, such that for all elements a in A, it is e•a = a•e = a. *Inverse element:* For each a in A, there exists an element b in A such that a•b = b•a = e, where e is the identity (the notation is inherited from the multiplicative operation).

If a group is Abelian, a further requirement is the *commutativity*. For all a,b in A, a•b = b•a. Therefore, to qualify a group as an Abelian group, the set and operation must satisfy five requirements, which are known as the *Abelian group axioms*. A group having a not commutative operation is a "non-Abelian group" or a "non-commutative group". For an Abelian group, one may choose to denote the group operation by + and the *identity element* by 0 (*neutral element*) and the inverse element of a as $-a$ (*opposite element*). In this case, the group is called an additive group.

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We can obtain the operation of a group by means of functions. Actually, if a function $G(x)$ exists, which is invertible $G^{-1}(G(x)) = x$, we can use it as a *generator*, to generate an algebra [10]. In [11], G is used to define the *group law* $\Phi(x, y)$, such as: $\Phi(x, y) = G(G^{-1}(x) + G^{-1}(y))$. $\Phi(x, y)$ is the operation $x \oplus y$. In the case of the κ -sum, the function G is the hyperbolic sine, so that [9]:

$$x \oplus y = \frac{1}{\kappa} sinh(arsinh(\kappa x) + arsinh(\kappa y)).$$

This operation is used in Kaniadakis generalized statistics for the sum of relativistic momenta.
A property of the hyperbolic sine function is that:

$$arsinh(x) = ln\left(x + \sqrt{1 + x^2}\right)$$

The domain is the whole real line. We have [13]:

$$arsinh(x) \pm arsinh(y) = arsinh\left(x\sqrt{1 + y^2} \pm y\sqrt{1 + x^2}\right) \quad (2)$$

Therefore, we have that the *group law* is given as [9]: $\Phi(x, y) = sinh(arsinh(x) + arsinh(y))$.
Therefore, the generalized sum is:

$$x \oplus y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}$$

This is the same as (1), for parameter $\kappa = 0$. The *closure* is given by the fact that the result of this operation is on the real line. The *neutral element* is 0. The *opposite element* of x is $-x$. Also the *commutativity* is evident.
To have a group, we need the discussion of the *associativity* too, showing therefore that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. Let us calculate $arsinh((x \oplus y) \oplus z)$ and $arsinh(x \oplus (y \oplus z))$; we can easily see that [9]: $arsinh((x \oplus y) \oplus z) = arsinh(x \oplus (y \oplus z)) = arsinh(x) + arsinh(y) + arsinh(z)$.

Other groups
Of course, other generalized sums can be obtained. Let us consider, for instance, the hyperbolic tangent. Its inverse hyperbolic function is defined as:

$$artanh(x) = \frac{1}{2} ln\left(\frac{1 + x}{1 - x}\right)$$

The domain is the open interval $(-1,1)$. A property of this function is the following [13]:

$$artanh(x) \pm artanh(y) = artanh\left(\frac{x \pm y}{1 \pm xy}\right)$$

We have a *group law* given by $\Phi(x, y) = tanh(artanh(x) + artanh(y))$. Therefore, we can obtain the generalized sum as [9]:

$$x \oplus y = (x + y)/(1 + xy) \quad (3)$$

This sum is *commutative*. The *neutral element* is 0. The *opposite element* of x is $-x$. We have the *associativity*, that is $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ [9].
To show that we have a group of (-1,1) and operation (3) it is necessary to verify the *closure*. That is, we have to see that the result of the sum is in the open interval (-1,1). This is discussed in [9].
The generalized sum (3) is used by Kaniadakis in Ref.14 for the relativistic velocity.
Another group is obtained from function $G(x) = e^{-2x}(e^{2x} - 1)$ and its inverse $G^{-1}(x) = ln\left(\frac{1}{\sqrt{x-1}}\right)$.
The *group law* $\Phi(x, y) = G(G^{-1}(x) + G^{-1}(y))$ gives [9]:

$$x \oplus y = (1 - x) + (1 - y) + xy \quad (4)$$

We will show in a following section that the same rule of additivity exists for the Fermat numbers.

The q-integers
Let us see how we can apply the previously discussed generalized sums to the numbers. Let us start from the q-integers of the q-calculus.

Many mathematicians have contributed to this calculus [15-20]. Consequently, the q-calculus is also known as "quantum calculus" and "time-scale calculus", or "calculus of partitions" too [19]. Moreover, it is expressed by means of different notations or, as told in [19], by different "dialects". Here we will use the notation given in the book by Kac and Cheung [20].

As discussed in [21], the q-integers are forming a group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems. The symmetric q-integers are linked to the Kaniadakis calculus.
The "q-integer" $[n]$ is defines by:

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}$$

First, we have to define the operation of addition by composing two q-integers. This operation is not the sum that we use for the natural integers of course, but it is a generalized sum.

Let us start from the q-integer $[m + n]$ and calculate as in [21]. We have:

$$[m+n] = [m] + [n] + (q-1)[m][n]$$

Then, we can define the generalized "sum" of the group as:

$$[m] \oplus [n] = [m] + [n] + (q-1)[m][n] \qquad (5)$$

If we use (5) as the sum, we have the closure of it, because the result of the sum is a q-integer. Moreover, this sum is commutative. The neutral element is $[0] = 0$. The opposite element of $[n]$ is equal to $[-n]$ [21].

The generalized sum (5) is associative [21]. We have also that: $[m] \oplus [n] \oplus [l] = [m+n+l]$.

Therefore, the five axioms of an Abelian group are satisfied. In this manner, using the generalized sum given by (5), we have the Abelian group of the q-integers.

What is important for the present discussion is the fact that the generalized sum (5) is similar to the sum that we find in the approach to entropy proposed by Constantino Tsallis [4], for his generalized entropy. For two independent systems *A* and *B*, the Tsallis entropy is given by:

$$S_q(A,B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$$

In this formula the parameter $(1-q)$, in a certain manner, measures the departure from the ordinary additivity, which is recovered in the limit $q \to 1$. The group on which is based the Tsallis entropy, and therefore the generalized sum given above, is known in literature as the "multiplicative group". As told in [12], the use of a group structure allows determining a class of generalized entropies.

Symmetric q-numbers

In the previous section, we have considered the group of the q-integers as defined by q-calculus. In [20], it is also defined the symmetric q-integer in the following form (here we use a notation different from that given in the Ref.20):

$$[n]_s = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Repeating the approach previously given, we can determine the group of the symmetric q-integers. Let us start from the q-integer $[m+n]_s$, which is:

$$[m+n]_s = \frac{q^{m+n} - q^{-(m+n)}}{q - q^{-1}} \qquad (6)$$

and try to find it as a generalized sum of the q-integers $[m]_s$ and $[n]_s$ [21].

By writing $q = exp(lnq)$, the q-integer turns out into a hyperbolic sine:

$$[n]_s = \frac{q^n - q^{-n}}{q - q^{-1}} =$$

$$\frac{e^{n\,lnq} - e^{-n\,lnq}}{q - q^{-1}} = 2\frac{sinh(n\,lnq)}{(q - q^{-1})} \qquad (7)$$

from (6), after some passages using (7), we find [21]:

$$[m+n]_s = [m]_s\sqrt{1 + sinh^2(n\,lnq)} + [n]_s\sqrt{1 + sinh^2(m\,lnq)}$$

Let us define: $k = (q - q^{-1})/2$ and then: $k[n]_s = sinh(n\,lnq)$. Therefore, we have the generalized sum of the symmetric q-integers as:

$$[m]_s \oplus [n]_s = [m]_s\sqrt{1 + k^2[n]_s^2} + [n]_s\sqrt{1 + k^2[m]_s^2} \qquad (8)$$

Let us stress that (8) is also the generalized sum (1) proposed by G. Kaniadakis (see also the discussion in [22]).

The Mersenne numbers

In the case that $q = 2$, we have:

$$[n] = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

These are the Mersenne Numbers.

About these numbers, a large literature exists (see for instance that given in [23]). Among the Mersenne integers, we find the Mersenne primes. The numbers are named after Marin Mersenne (1588 – 1648), a French Minim friar, who studied them in the early 17th century. Let us call M_n the Mersenne number. Of course, we have the generalized sum for the q-numbers as given in (5):

$$[m] \oplus [n] = [m] + [n] + [m][n]$$

However, we can repeat the calculus starting from $M_{m+n} = 2^{m+n} - 1$. After some passages we obtain $M_{m+n} = (2^m - 1)(2^n - 1) + 2^m - 1 + 2^n - 1$, that is:

$$M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n \qquad (9)$$

Let us stress that we have a generalized sum of the form of those of the groups known as "multiplicative groups".

Using (9), for the Mersenne numbers we can imagine the following recursive relation:

$$M_{n+1} = M_n \oplus M_1 = M_n + M_1 + M_n M_1$$

The sum (9) is associative.
We cannot have a group of the Mersenne numbers, without considering also the opposites O of them, so that: $0 = M_n \oplus O(M_n)$. Therefore:

$$O(M_n) = \frac{-M_n}{M_n + 1} = M_{-n}$$

Explicitly: $O(2^n - 1) = 2^{-n} - 1$. These numbers are the Mersenne numbers with a negative exponent.

Symmetric Mersenne
Let us consider again the symmetric q-integer in the case of $q = 2$.
We can define the symmetric Mersenne in the following manner:

$$M_n^s = [n]_s = \frac{2^n - 2^{-n}}{2 - 2^{-1}} \quad (10)$$

By writing $2 = exp(ln2)$, (10) turns out into a hyperbolic sine:

$$M_n^s = \frac{e^{n\, ln2} - e^{-n\, ln2}}{2 - 2^{-1}} = 2\frac{sinh(n\, ln2)}{(2 - 2^{-1})}$$

Let us define $k = (2 - 2^{-1})/2$. We have the generalized sum of the symmetric Mersenne as:

$$M_m^s \oplus M_n^s = M_m^s\sqrt{1 + k^2(M_n^s)^2} + M_n^s\sqrt{1 + k^2(M_m^s)^2} \quad (11)$$

Of course, we have again the generalized sum proposed by G. Kaniadakis.

Fermat, Cullen and Woodall Numbers
As seen before, the Mersenne numbers $M_n = 2^n - 1$ are forming a group with the following generalized sum:

$$M_m \oplus M_n = M_m + M_n + M_m M_n$$

Using this composition rule, we can have the following recursive relation:

$$M_{n+1} = M_n \oplus M_1 = M_n + M_1 + M_n M_1 = 2M_n + 1$$

Numbers are 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, and so on, in agreement to the sequence given in http://oeis.org/A000225.

Another famous sequence of integers is that of the Fermat numbers. Fermat numbers are defined as $F_n = 2^n + 1$ [24]. These numbers have the following generalized sum [25]:

$$F_m \oplus F_n = (1 - F_m) + (1 - F_n) + F_m F_n \quad (12)$$

which is the same of the *group law* (4) that we have discussed in [25] and in a previous section of this work.

From (12), we can give the following recursive relation for the Fermat numbers: $F_{n+1} = F_n \oplus F_1$.
Starting from $F_1 = 2^1 + 1 = 3$, we have: 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, and so on in agreement to http://oeis.org/A000051.

The sum is associative. The neutral element is $F_0 = 2^0 + 1 = 2$ and the opposites O of the Fermat numbers are given by $O(F_n) = F_{-n}$ [25].

Similar to the Fermat numbers, we have the Cullen numbers. The Woodall numbers are similar to the Mersenne numbers [26,27]. Let us find the generalized sums of them, as detailed in [28].
The Cullen numbers are $C_n = 2^n n + 1$.
The generalized sum is [28]:

$$C_m \oplus C_n = 2^m(C_n - 1) + 2^n(C_m - 1) + 1 \quad (13)$$

The neutral element of this sum is $C_0 = 2^0 0 + 1 = 1$.
Using (13):

$$C_m \oplus C_0 = 2^m(C_0 - 1) + 2^0(C_m - 1) + 1 = C_m$$

We have $C_1 = 2^1 1 + 1 = 3$. Recursive relation is:

$$C_{n+1} = 2^{n+1} + 2(C_n - 1) + 1$$

So we have: 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, 1048577, and so on, in agreement to http://oeis.org/A002064.

Of the Cullen numbers, we can also give another form of the generalized sum [28]:

$$C_m \oplus C_n = \frac{1}{m}(C_m - 1)(C_n - 1) + \frac{1}{n}(C_n - 1)(C_m - 1) + 1 \quad (14)$$

The recursive relation assumes the form: $C_{n+1} = 2(C_n - 1) + \frac{2}{n}(C_n - 1) + 1$. In the case that we use the generalized sum (14), we have to remember that when m or n are equal to zero, we need to assume $(C_m - 1)/m = 1$, $(C_n - 1)/n = 1$.
The Woodall numbers are defined as: $W_n = 2^n n - 1$.
Let us write the generalized sum as [28]:

$$W_m \oplus W_n = 2^m(W_n + 1) + 2^n(W_m + 1) - 1 \quad (15)$$

The neutral element of this sum is $W_0 = 2^0 0 - 1 = -1$. Using (15):

$$W_m \oplus W_0 = 2^m(W_0+1) + 2^0(W_m+1) - 1 = W_m$$

We have $W_1 = 2^1 1 - 1 = 1$. Recursive relation is:

$$W_{n+1} = 2^{n+1} + 2(W_n+1) - 1$$

So we have: 1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, and so on, in agreement to http://oeis.org/A003261.

Also for the Woodall numbers, we can give another form of the generalized sum (15), as shown in [28].

Thabit numbers

Let us consider the Thabit numbers [29]. These numbers are given as $T_n = 3 * 2^n - 1$, where the asterisk represents the ordinary multiplication. The operation of addition between Thabit numbers is [30]:

$$T_m \oplus T_n = (T_m + T_n + T_m T_n - 2)/3 \qquad (16)$$

Using (16), we can see that the neutral element is $T_0 = 2$, so that:

$$T_m \oplus T_0 = \frac{1}{3}(T_m + T_0 + T_m T_0 - 2) = T_m$$

The recursive relation is given accordingly to (16), starting from $T_1 = 5$:

$$T_{n+1} = T_n \oplus T_1 = 2T_n + 1$$

We have **5**, **11**, **23**, **47**, 95, **191**, **383**, 767, 1535, 3071, **6143**, 12287, 24575, 49151, and so on. In bold characters, the prime numbers as from http://oeis.org/A007505.

Repunits

As explained in [31], the term "repunit" was coined by Beiler in a book of 1966 [32], for the numbers defined as:

$$R_n = \frac{10^n - 1}{10 - 1}$$

The sequence of repunits starts with 1, 11, 111, 1111, 11111, 111111, … (sequence A002275 in the OEIS, https://oeis.org/A002275). As we can easily see, these numbers are linked to q-integers and Mersenne numbers [33]. The repunits are the q-integers for q=10 :

$$[n]_{q=10} = \frac{10^n - 1}{10 - 1}$$

We can use the same approach for the repunits of that given for the q-numbers. Let us consider the following operation (generalized sum): $R_{m+n} = R_m \oplus R_n$. It is defined in the following manner:

$$R_m \oplus R_n = R_m + R_n + (10-1) R_m R_n \qquad (17)$$

The recursive relation for the repunits, given according to (17) and starting from $R_1 = 1$, is:

$$R_m \oplus R_1 = R_m + R_1 + (10-1) R_m R_1$$

That is: 11, 111, 1111, 11111, 111111, 1111111, 11111111, and so on.

As we have considered the symmetric q-integers, we can define the "symmetric" repunits as [33]:

$$R_{n,s} = \frac{10^n - 10^{-n}}{10 - 10^{-1}} = 2\frac{sinh(n\, ln10)}{10 - 10^{-1}}$$

The sequence is: 1, 10.1, 101.01, 1010.101, 10101.0101, 101010.10101, etc.

In this case, the addition is the same as that for the symmetric q-numbers:

$$R_{m,s} \oplus R_{n,s} = R_{m,s}\sqrt{1 + k^2 \left(R_{n,s}\right)^2} + R_{n,s}\sqrt{1 + k^2 \left(R_{m,s}\right)^2}$$

Here $k = \frac{1}{2}\left(10 - \frac{1}{10}\right)$. Let us note that $R_{1,s} = 1$.

The recursive formula for the symmetric repunits is:

$$R_{n+1,s} = R_{n,s} \oplus R_{1,s} = R_{n,s}\sqrt{1 + k^2} + \sqrt{1 + k^2 \left(R_{n,s}\right)^2}$$

Conclusion

In this work, we have discussed the composition operations of generalized entropies (Tsallis and Kaniadakis). These operations can be obtained from some group laws based on functions and their inverses. The group laws can be defined as "generalized sums" because the generalized entropies are motivating them.

The approach using the group operations can be applied to the study of numbers. We have discussed the composition rules for some famous sequences of numbers (Mersenne, Fermat, Cullen, Woodall and Thabit numbers). We have also considered the sequence of the repunits, which can be seen as a specific case of the q-integers.

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Binary Operators of the Groupoids of OEIS A093112 and A093069 Numbers (Carol and Kynea Numbers)

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Here we discuss the binary operators of the sets made by the OEIS sequences of integers A093112 and A093069, also called Carol and Kynea numbers. We will see that these numbers are linked, through the binary operators, to the Mersenne and Fermat integers.

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As told in [1], there are at least three definitions of "groupoid", which are currently in use. The first type of groupoid that we can find is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S.
Here, we consider the groupoids of the sets of the numbers given by OEIS sequences A093112 and A093069 [2,3], which are also called by Cletus Emmanuel, the Carol and Kynea numbers.

An A093112 number (Carol number) is an integer having the following form [2]:

$$C_n = 4^n - 2^{n+1} - 1 = (2^n - 1)^2 - 2$$

We can find the first numbers of the sequence A093112 in the OEIS [2]. That is: -1, 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, 16769023, 67092479, 268402687, and so on.
An A093069 number (Kynea numbers) is defined as [3]:

$$K_n = (2^n + 1)^2 - 2$$

So we have [3]: 7, 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, 16785407, 67125247, 268468223, 1073807359, 4295098367, 17180131327, 68720001023, and so on.
As we did in some previous discussions (see for instance [4,5]), we can find a binary operator, which satisfy the closure, of given sets of numbers. In [4], we considered the groupoids of Mersenne, Fermat, Cullen and Woodall numbers.

Here how to find the operator for A093112 numbers. Let us use:

$$(C_m + 2)^{1/2} = (2^m - 1) = M_m \quad ; \quad (C_n + 2)^{1/2} = (2^n - 1) = M_n \quad ; \quad (C_{m+n} + 2)^{1/2} = (2^{m+n} - 1) = M_{m+n}$$

which are Mersenne numbers [4]. So we have the binary operator [4]:

$$(2^{m+n} - 1) = M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n = (2^m - 1)(2^n - 1) + (2^m - 1) + (2^n - 1)$$

Therefore, since $(C_{m+n} + 2)^{1/2} = (2^{m+n} - 1) = M_{m+n}$:

$$(C_{m+n}+2)^{1/2}=(C_m+2)^{1/2}(C_n+2)^{1/2}+(C_m+2)^{1/2}+(C_n+2)^{1/2}$$

We can find the binary operator for the Carol numbers as:

$$C_{m+n}=-2+[(C_m+2)^{1/2}(C_n+2)^{1/2}+(C_m+2)^{1/2}+(C_n+2)^{1/2}]^2 \ =$$

$$-2+(C_m+2)(C_n+2)+(C_m+2)+(C_n+2)+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

So we have:

$$C_{m+n}=6+C_m C_n+3C_m+3C_n+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

Therefore, the binary operator is defined as:

$$C_m \oplus C_n=6+C_m C_n+3C_m+3C_n+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

From this binary operation, we can have the recurrence relation: $C_{n+1}=C_n \oplus C_1$.

That is:

$$C_{n+1}=6+C_1 C_n+3C_1+3C_n+2(C_1+2)(C_n+2)^{1/2}+2(C_1+2)^{1/2}(C_n+2)+2(C_1+2)^{1/2}(C_n+2)^{1/2}$$

From $C_1=-1$, we have: 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, 16769023, 67092479, 268402687, and so on.

Let us consider the Kynea numbers.

As we did before for the Carol numbers, let us use the following approach:

$$(K_m+2)^{1/2}=(2^m+1)=F_m \quad ; \quad (K_n+2)^{1/2}=(2^n+1)=F_n \quad ; \quad (K_{m+n}+2)^{1/2}=(2^{m+n}+1)=F_{m+n}$$

which are Fermat numbers [4]. So we have the binary operator [4]:

$$(2^{m+n}+1)=F_{m+n}=F_m \oplus F_n=(1-F_m)+(1-F_n)+F_m F_n=2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)$$

$$(K_{m+n}+2)^{1/2}=(2^{m+n}+1)=F_{m+n}=2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)$$

$$K_{m+n}=-2+[2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)]^2=-2+[2+(K_m+2)^{1/2}(K_n+2)^{1/2}-(K_m+2)^{1/2}-(K_n+2)^{1/2}]^2$$

Then:

$$K_{m+n}=2+(K_m+2)(K_n+2)+(K_m+2)+(K_n+2) \ + \ 4(K_m+2)^{1/2}(K_n+2)^{1/2}-4(K_m+2)^{1/2}-4(K_n+2)^{1/2}$$

$$-2(K_m+2)(K_n+2)^{1/2}-2(K_m+2)^{1/2}(K_n+2)+2(K_m+2)^{1/2}(K_n+2)^{1/2}$$

Therefore, the binary operator is defined as:

$$K_m \oplus K_n = 10 + K_m K_n + 3K_m + 3K_n \;\;+\;\; 4(K_m+2)^{1/2}(K_n+2)^{1/2} - 4(K_m+2)^{1/2} - 4(K_n+2)^{1/2}$$

$$-2(K_m+2)(K_n+2)^{1/2} - 2(K_m+2)^{1/2}(K_n+2) + 2(K_m+2)^{1/2}(K_n+2)^{1/2}$$

Recurrence is given by:

$$K_{n+1} = K_n \oplus K_1 = 10 + K_n K_1 + 3K_n + 3K_1 \;\;+\;\; 4(K_1+2)^{1/2}(K_n+2)^{1/2} - 4(K_1+2)^{1/2} - 4(K_n+2)^{1/2}$$

$$-2(K_1+2)(K_n+2)^{1/2} - 2(K_1+2)^{1/2}(K_n+2) + 2(K_1+2)^{1/2}(K_n+2)^{1/2}$$

Then, starting from $K_1 = 7$, we have 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, 16785407, 67125247, 268468223, and so on.

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A Binary Operator Generated by Homographic Function

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Abstract: In this work we are discussing the binary operator that we can generated by homographic function. By means of this operator, that we can see as a generalized sum, we can create a group.

Keywords: generalized sum, binary operator, groups, Abelian groups.

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In recent works we have shown some generalized sums and related groups, which are based on transcendental functions. Let us note that the generalized sums are binary operators which widespread the addition of real numbers. Besides the investigation of groups generated by transcendental functions, we have also considered groups involving generalized integers [1-5]. Here we consider the group having a binary operator generated by homographic function.

Let us consider two elements x and y of reals **R,** and a related binary operation. We indicate this composition law by the notation $x \oplus y$, a generalized sum so that (**R**,⊕) is giving a group.

Let us remember that a group is a set A having an operation ⊕ which is combining the elements of A. That is, the operation combines any two elements a,b to form another element of the group denoted $a\oplus b$. To qualify $(A,\oplus)$ as a group, the set and operation must satisfy the following requirements. *Closure***:** For all a,b in A, the result of the operation $a\oplus b$ is also in A. *Associativity:* For all a,b and c in A, it holds $(a\oplus b)\oplus c = a\oplus(b\oplus c)$. *Neutral (or identity) element*: An element e exists in A, such that for all elements a in A, it is $e\oplus a = a\oplus e = a$. *Opposite (or inverse) element:* For each a in A, there exists an element b in A such that $a\oplus b = b\oplus a = e$, where e is the identity. If a group is Abelian, a further requirement is the c*ommutativity:* For all a,b in A, $a\oplus b = b\oplus a$.

To have a given generalized sum, we follow an approach based on a "generation" [1,6-8].

Let us have a function $G(x)$, which is invertible $G^{-1}(G(x))=x$. A deformation *generator* can define the *group law* $\Phi(x,y)$ [6,7]:

$$\Phi(x,y)=G(G^{-1}(x)+G^{-1}(y)) \quad \text{or} \quad x\oplus y=G(G^{-1}(x)+G^{-1}(y)) \ .$$

In this manner the *group law* is giving the *generalized sum,* as we can call the binary operator of the group.

Let us consider the homographic function.

$$G(x)=\frac{x+1}{x-1} \ ; \ G^{-1}(x)=\frac{x+1}{x-1} \ ; \ G^{-1}(G(x))=\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}=\frac{x+1+x-1}{x+1-x+1}=x$$

$$x\oplus y=G(G^{-1}(x)+G^{-1}(y))=G(\frac{x+1}{x-1}+\frac{y+1}{y-1})=G(\frac{2xy-2}{(x-1)(y-1)})=\frac{\frac{2xy-2}{(x-1)(y-1)}+1}{\frac{2xy-2}{(x-1)(y-1)}-1}$$

So that: $x\oplus y=\dfrac{3xy-x-y-1}{xy+x+y-3}$ (1).

(1) is the generalized sum based on the homographic function..

To have a finite value of (1), we need $xy+x+y-3\neq 0$. That is: $y\neq(3-x)/(1+x)$ (*). In this manner we have the closure on finite values.

Let us consider the neutral element *e* of this sum. It is different from zero, as we can easily see if we use 0 in the generalized sum:

$$x\oplus 0=\frac{3x0-x-0-1}{x0+x+0-3}=\frac{-x-1}{x-3}=\frac{x+1}{3-x}$$

Let us note that $x\oplus 0$ is the number that we have in the condition (*). For this reason, let us also avoid 0, from the element of the set (**).

The neutral element *e* is equal to -1 : $x\oplus(-1)=\frac{-3x-x+1-1}{-x+x-1-3}=\frac{-4x}{-4}=x$

The opposite element of x is $1/x$, so that: $x\oplus(1/x)=\frac{3x/x-x-1/x-1}{x/x+x+1/x-3}=-1$

For the a*ssociativity,* we can show that $x\oplus(y\oplus z)=(x\oplus y)\oplus z$. Actually:

$$x\oplus(y\oplus z)=\frac{3x(y\oplus z)-x-(y\oplus z)-1}{x(y\oplus z)+x+(y\oplus z)-3} \;;\; (x\oplus y)\oplus z=\frac{3(x\oplus y)z-(x\oplus y)-z-1}{(x\oplus y)z+(x\oplus y)+z-3}$$

$$x\oplus(y\oplus z)=\frac{8xyz-4xy-4xz-4yz+4}{4xyz-4x-4y-4z+8}=(x\oplus y)\oplus z$$

Using the binary operator (1), and conditions (*),(**), we can define the neutral element. We have also that the binary operator possesses the associative property. In this manner (1) is a generalized sum of a group.

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Groupoids of OEIS A002378 and A016754 Numbers
(oblong and odd square numbers)

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Here we discuss the binary operators of the sets made by the OEIS sequences of integers A002378 and A016754. A002378 are defined as oblong numbers.

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Let us use the definition of the first type of groupoid given in [1]: it is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. Here, we consider the groupoids of the sets of the numbers given by OEIS sequences A002378 and A016754 [2,3], which are told as oblong and odd squares (centered octagonal) numbers.

An A002378 number is also known as a promic, pronic, or heteromecic number (formerly M1581 N0616). It is an integer having the following form [2]:

$$O_n = n(n+1)$$

OEIS gives: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, and so on.
Ref. 2 tells that $4O_n+1$ are the odd squares A016754 numbers.
An A016754 number (odd square number) is defined as [3]:

$$o_n = (2n+1)^2$$

So we have [3]: 1, 9, 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, and so on.
As we did in some previous discussions (see for instance [4,5]), we can find a binary operator, which satisfy the closure, of given sets of numbers. In [4], we considered the groupoids of Mersenne, Fermat, Cullen and Woodall numbers. Here, we follow the same approach as in [6], for Carol and Kynea numbers.

Here how to find the operator for A002378 numbers. Let us use:

$$(n(n+1)+1/4)^{1/2} = (n^2+n+1/4)^{1/2} = ((n+1/2)^2)^{1/2}$$

So we define:

$$(O_m+1/4)^{1/2}=(m+1/2)=A_m \quad ; \quad (O_n+1/4)^{1/2}=(n+1/2)=A_n \quad ;$$

$$(O_{m+n}+1/4)^{1/2}=(m+n+1/2)=A_{m+n}$$

We use numbers A_m to help us in the calculation. We have for them the binary operator:

$$A_{m+n}=A_m\oplus A_n=A_m+A_n-1/2=(m+1/2)+(n+1/2)-1/2=m+n+1/2$$

Therefore: $(O_{m+n}+1/4)^{1/2}=A_{m+n} \;=\; A_m\oplus A_n=A_m+A_n-1/2$

$$(O_{m+n}+1/4)^{1/2}=(O_m+1/4)^{1/2}+(O_n+1/4)^{1/2}-1/2$$

We can find the binary operator for the Oblong numbers as:

$$O_{m+n}+1/4=O_m+O_n+3/4+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

$$O_{m+n}=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

So we have the binary operator defined as:

$$O_m\oplus O_n=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

Associativity:

$$O_m\oplus(O_n\oplus O_p)=O_m\oplus O_{n+p}=O_{m+n+p} \quad ; \quad (O_m\oplus O_n)\oplus O_p=O_{m+n}\oplus O_p=O_{m+n+p}$$

From this binary operation, we can have the recursive relation: $O_{n+1}=O_n\oplus O_1$.

From $O_1=2$, we have: 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, and so on.

Let us consider the odd square numbers.
Here how to find the operator for A016754 numbers: $o_n=(2n+1)^2$. Let us use:

$$o_m^{1/2}=(2m+1)=A_m \quad ; \quad o_n^{1/2}=(2n+1)=A_n \quad ; \quad o_{m+n}^{1/2}=(2(m+n)+1)=A_{m+n}$$

We use numbers A_m to help us in the calculation. So we have the binary operator:

$$A_{m+n}=A_m\oplus A_n=A_m+A_n-1=(2m+1)+(2n+1)-1=2(m+n)+1$$

Therefore: $o_{m+n}^{1/2}=A_{m+n} \;=\; A_m\oplus A_n=A_m+A_n-1$

We can find the binary operator for the odd square numbers as:

$$o_{m+n} = o_m + o_n + 1 + 2 o_m^{1/2} o_n^{1/2} - 2 o_m^{1/2} - 2 o_n^{1/2}$$

So we have the binary operator defined as:

$$o_m \oplus o_n = o_m + o_n + 1 + 2 o_m^{1/2} o_n^{1/2} - 2 o_m^{1/2} - 2 o_n^{1/2}$$

Associativity:

$$o_m \oplus (o_n \oplus o_p) = o_m \oplus o_{n+p} = o_{m+n+p} \quad ; \quad (o_m \oplus o_n) \oplus o_p = o_{m+n} \oplus o_p = o_{m+n+p}$$

From this binary operation, we can have the recursive relation: $o_{n+1} = o_n \oplus o_1$.

From $o_1 = 9$, we have: 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, 1681, 1849, and so on.

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Groupoid of OEIS A001844 Numbers (centered square numbers)

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Here we discuss the binary operator of the set made by the OEIS sequence of integers A001844, defined as centered square numbers. This binary operator can be used to have a groupoid. Actually, neutral and opposite elements can be defined too, and a possible group for these numbers can be given.

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A groupoid is an algebraic structure made by a set with a binary operator [1]. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this operation is associative and we have a neutral element and opposite elements into the set, then the groupoid becomes a group.
Here, we consider the set of numbers given by OEIS sequence A001844 (centered square numbers). The numbers have the following form [2]:

$$C_n = n^2 + (n+1)^2 = 2n(n+1) + 1$$

As we did in some previous discussions (see for instance [3-5]), we can find a binary operator, which satisfy the closure. Let us follow the same approach as in [5], for Carol and Kynea numbers.

We have: $C_n = n^2 + (n+1)^2 = 2n(n+1) + 1 = 2(n+1/2)^2 + 1/2$.

Let us use numbers $A_{n:}$, so that: $(n+1/2)^2 = (A_n)^2$.

$$[(C_m - 1/2)/2]^{1/2} = (m+1/2) = A_m \quad ; \quad [(C_n - 1/2)/2]^{1/2} = (n+1/2) = A_n \quad ;$$

$$[(C_{m+n} - 1/2)/2]^{1/2} = (m+n+1/2) = A_{m+n}$$

Numbers A_m can help us in the calculation (the same numbers we used in [6]).
For them, the binary operator is:

$$A_{m+n} = A_m \oplus A_n = A_m + A_n - 1/2 = (m+1/2) + (n+1/2) - 1/2 = m+n+1/2$$

Therefore:

$$[(C_{m+n}-1/2)/2]^{1/2}=(m+n+1/2)=A_{m+n} \quad = \quad A_m \oplus A_n = A_m + A_n - 1/2$$

Consequently, for the centered square numbers, the binary operator of a generalized sum is coming from:

$$C_{m+n}=C_m+C_n+2(C_m-1/2)^{1/2}(C_n-1/2)^{1/2}-\sqrt{2}(C_m-1/2)^{1/2}-\sqrt{2}(C_n-1/2)^{1/2}$$

The generalized sum is given as:

$$C_m \oplus C_n=C_m+C_n+2(C_m-1/2)^{1/2}(C_n-1/2)^{1/2}-\sqrt{2}(C_m-1/2)^{1/2}-\sqrt{2}(C_n-1/2)^{1/2} \quad (1)$$

From (1), we have the recursive relation: $C_{n+1}=C_n \oplus C_1$. Starting with number $C_1=5$, we have: 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925, and so on. The same as http://oeis.org/A001844.

The binary operator is associative: $C_{m+n+p}=C_{m+n} \oplus C_p = C_m \oplus C_{n+p}$

Using (1), we can see that we can have a neutral element: $C_0=1$.

$$C_{m+0}=C_m \oplus C_0=C_m+C_0+2(C_m-1/2)^{1/2}(C_0-1/2)^{1/2}-\sqrt{2}(C_m-1/2)^{1/2}-\sqrt{2}(C_0-1/2)^{1/2}$$

$$C_m+1+2(C_m-1/2)^{1/2}(1/2)^{1/2}-\sqrt{2}(C_m-1/2)^{1/2}-\sqrt{2}(1/2)^{1/2}=C_m$$

Since we have a neutral element, we could try to find the opposite element so that:

$$C_{m-m}=C_m \oplus Opp(C_m)=C_m \oplus C_{-m}=C_0$$

In [2], we have the relation: $a(-m)=a(m-1)$ (*), where $a(m)=C_m$.

Let us consider, in the framework given above, the meaning of (*).

Let us define $X=\sqrt{(Opp(C_m)-1/2)}$ and evaluate

$$C_0=C_m \oplus Opp(C_m)=C_m+X^2+1/2+2X(C_m-1/2)^{1/2}-\sqrt{2}(C_m-1/2)^{1/2}-\sqrt{2}X=1 \quad (**)$$

This is an equation of the form $X^2+BX+K=0$, where coefficients are:

$$B=2(C_m-1/2)^{1/2}-\sqrt{2} \quad \text{and} \quad K=C_m+1/2-1-\sqrt{2}(C_m-1/2)^{1/2} \quad .$$

Solutions are given by $X=(-B\pm\sqrt{(B^2-4K)})/2$, where $\sqrt{(B^2-4K)}=\sqrt{2}$.

For $X=(-B-\sqrt{2})/2$, the opposite element turns out to be: $Opp(C_m)=C_m$. We have therefore that the composition (**) of an element with itself produces the identity. So each element turns out to be self-inverse.

For $X=(-B+\sqrt{2})/2$, the opposite element is given as:

$$Opp(C_m)=X^2+1/2=C_{m-1}$$

and therefore we find relation (*). However, in the binary operation (**), X is negative.

As a consequence, when we use C_{m-1} in the generalized sum (1), if we assume the negative value of root $X=-\sqrt{(C_{m-1}-1/2)}$, we obtain the neutral element, so that $C_m \oplus C_{-m}=C_0$. In the case that we use the positive root, the generalized sum gives $C_m \oplus C_{m-1}=C_{2m-1}$. The reason is that, to find (1), we could use a positive or negative sign in front of the square root:

$$\pm[(C_{m-1}-1/2)/2]^{1/2}=(m-1+1/2)=A_{m-1}$$

If we want to use the binary operator for a groupoid, it is enough to use the positive value of the root.

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Giuseppe Peano e i numeri di Mersenne

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Si mostra come un problema dei "Giochi Di Aritmetica E Problemi Interessanti", di Giuseppe Peano, ci porti ai numeri di Mersenne.

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Giuseppe Peano è stato un grande matematico piemontese, noto anche per aver inventato il "latino sine flexione", una lingua ausiliaria internazionale derivata dal Latino classico.

Peano nacque il 27 agosto 1858, a Spinetta presso Cuneo. Studiò a Torino presso il Liceo classico Cavour, ottenendo la licenza liceale nel 1876 [1]. Assistente di Angelo Genocchi all'Università di Torino, divenne professore di calcolo infinitesimale presso lo stesso ateneo a partire dal 1890 [1]. Morì nella sua villa a Cavoretto, il 20 aprile del 1932.

Tra i suoi primi risultati scientifici troviamo il teorema dell'esistenza della soluzione di equazioni differenziali ordinarie ed il primo esempio di una curva che riempie una superficie, la cosiddetta Curva di Peano (in effetti, un frattale). Con questa curva, Peano mostrò come la definizione di curva sia una questione delicata. Una curva piana viene vista, in modo intuitivo, come un oggetto monodimensionale in un piano bidimensionale, e quindi come incapace di riempirlo. La curva di Peano è invece capace di riempire lo spazio delimitato da un quadrato. "Da questo lavoro partì la revisione del concetto di curva, che fu ridefinito da Camille Jordan (1838 – 1932) (curva secondo Jordan)." [2]. Peano fu "anche uno dei padri del calcolo vettoriale insieme a Tullio Levi-Civita" [2]. Introdusse anche il "resto di Peano" nella formula di Taylor e la misura di Peano-Jordan [3].

Dopo essersi dedicato al calcolo infinitesimale, Peano passò all'aritmetica e alla logica. Diede "una definizione assiomatica dei numeri naturali, i famosi Assiomi di Peano, i quali vennero ripresi da Russell e Whitehead nei loro Principia Mathematica per sviluppare la teoria dei tipi." [2]. Bertrand Russell disse di Peano: «Provai una grande ammirazione per lui quando lo incontrai per la prima volta al Congresso di Filosofia del 1900, che fu dominato dall'esattezza della sua mente.» [2].

L'assioma è, nel linguaggio comune, una verità o un principio che si ammette senza discussione, evidente di per sé. In matematica, l'assioma "è in genere sinonimo di postulato, da cui tuttavia si distingue, specialmente in logica matematica" [4]. In questo caso, con gli assiomi "si vuole indicare un sistema formale di proprietà che costituiscono una definizione implicita dell'ente o dell'espressione cui si riferiscono, a prescindere quindi dalla loro evidenza, dal momento che non hanno la pretesa di essere verità assolutamente valide" [4]. Gli assiomi di Peano sono dati, in tal modo, per definire i numeri naturali.

In [5], gli assiomi vengono proposti nella seguente maniera.

• zero è un numero naturale;

• se *n* è un numero naturale, anche il successore di *n* è un numero naturale;

• se i successori di due numeri naturali sono uguali, allora i due numeri sono uguali;

• zero non è successore di alcun numero naturale;

• se A è un insieme di numeri naturali che contiene lo zero e il successore di ogni numero appartenente a esso, allora A coincide con tutto l'insieme dei numeri naturali.

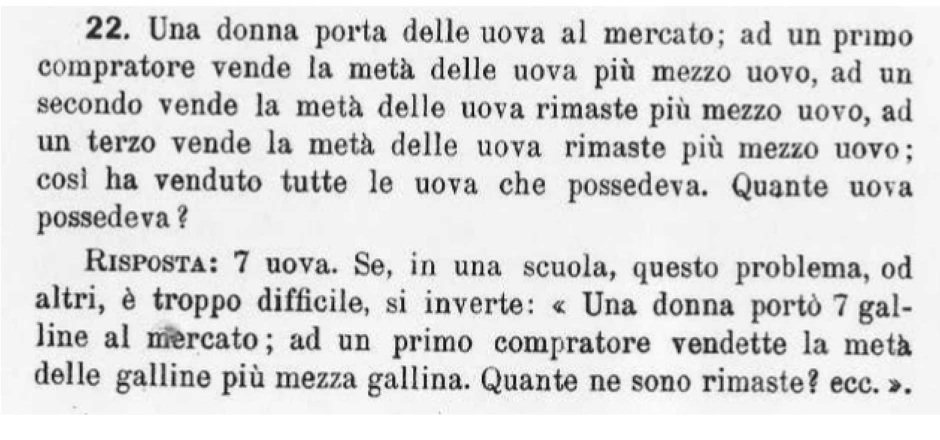
L'ultimo assioma è noto come *principio di induzione matematica.* In [5] è riformulato in modo equivalente come segue: «se P è una proprietà concernente i numeri naturali soddisfatta da zero e tale che, se è soddisfatta da un dato numero naturale, lo è anche dal suo successore, allora P è soddisfatta da ogni numero naturale». Nella formulazione originaria degli assiomi, Peano definì i numeri naturali a partire da 1 e non da 0.

Il sistema che si è ottenuto con questi assiomi è unico, a meno di isomorfismi [6].

Dopo la definizione assiomatica dei principi di aritmetica e geometria, Peano passò poi alla logica matematica, come spiegato nella prefazione al libro "Giochi Di Aritmetica E Problemi Interessanti", libro che Peano scrisse nel 1925 [7]. Egli diede vita a Torino, nell'ultimo decennio del secolo, ad una pionieristica "scuola" di logica. Di questa scuola, il "Formulario di Matematica" ne era stato "la realizzazione più compiuta e coerente. Peano stesso fu a lungo, prima di Russell, leader riconosciuto e influente nel campo della logica" [7]. Da ultimo Peano si dedicò anche allo studio comparato delle lingue. Come sottolinea Umberto Bottazzini, autore della prefazione [7], nell'ampio spettro delle ricerche matematiche e logiche, rientra naturalmente il volumetto sui Giochi Di Aritmetica E Problemi Interessanti, che si lega anche all'interesse di Peano per l'insegnamento delle matematiche alle scuole elementari.

Il volumetto [7] comincia così "In tutti i tempi, e presso tutti i popoli, si insegnavano dei giochi per rendere dilettevole o meno noiosa l'aritmetica. Saggiamente questi giochi si trovano nei nuovi programmi delle scuole elementari. Credo far cosa utile agli insegnati col pubblicarne alcuni".

In verità l'aritmetica non è mai noiosa, se piace, ed anche i problemi di Peano per i bambini sono tutt'altro che banali. Eccone uno.



22. Una donna porta delle uova al mercato; ad un primo compratore vende la metà delle uova più mezzo uovo, ad un secondo vende la metà delle uova rimaste più mezzo uovo, ad un terzo vende la metà delle uova rimaste più mezzo uovo; così ha venduto tutte le uova che possedeva. Quante uova possedeva?

RISPOSTA: 7 uova. Se, in una scuola, questo problema, od altri, è troppo difficile, si inverte: « Una donna portò 7 galline al mercato; ad un primo compratore vendette la metà delle galline più mezza gallina. Quante ne sono rimaste? ecc. ».

Usiamo qualche formula. Sia M_n il numero di uova che resta dopo ciascuna vendita. Esso è legato al numero di uova prima della vendita nella seguente maniera:

$$M_{n+1}-\frac{M_{n+1}}{2}-\frac{1}{2}=M_n \qquad (1)$$

Quindi, se partiamo da 7, abbiamo $7-7/2-1/2=7/2-1/2=3$ (prima vendita). Poi: $3-3/2-1/2=3/2-1/2=1$ (seconda vendita). Infine: $1-1/2-1/2=1/2-1/2=0$. Oppure riscriviamo la relazione (1), come:

$$\frac{M_{n+1}}{2}-\frac{1}{2}=M_n$$

da cui:

$$M_{n+1}=2M_n+1 \qquad (2)$$

E quindi, se partiamo da 0, con la (2), abbiamo 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, 4194303, 8388607, 16777215, 33554431, 67108863, 134217727, 268435455, 536870911, 1073741823, 2147483647, 4294967295 e così via. Ma questi sono i numeri che troviamo nella sequenza OEIS A000225 di numeri interi [8]. Detti talvolta interi di Mersenne, tra di essi troviamo i primi di Mersenne.

Ecco perché il numero delle uova l'ho indicato con la lettera *M* maiuscola.

Al sito [9], troviamo l'indovinello matematico concernente le uova ed i numeri di Mersenne formulato come "Tre Signore vanno al mercato. La prima compra da un contadino metà delle sue uova, più un mezzo uovo. La seconda compra metà delle uova rimaste più un mezzo uovo. La terza acquista l'unico uovo che è rimasto. Quante uova aveva il contadino all'inizio?".

Il problema delle uova è anche definito come il "No broken eggs puzzle" [10], perché in effetti, nessun uovo è spaccato a metà, come logica vuole.

I numeri di Mersenne sono scritti come:

$$M_n=2^n-1$$

In [11], ne ho studiato la somma "generalizzata", somma che generalizza la somma aritmetica usuale (si veda la discussione in [12]), in modo che da due numeri di Mersenne, opportunamente combinati, si abbia un terzo numero di Mersenne. Partiamo dal numero:

$$M_{m+n}=2^{m+n}-1$$

Si ha che [11]:

$$M_{m+n}=2^{m+n}-1=(2^m-1)(2^n-1)+2^m-1+2^n-1=M_m+M_n+M_mM_n$$

La somma generalizzata è quindi:

$$M_m\oplus M_n=M_m+M_n+M_mM_n \qquad (3)$$

Prendiamo ora questa somma e riscriviamola così:

$$M_1\oplus M_n=M_1+M_n+M_1M_n$$

Sappiamo che il risultato è pari a M_{n+1} . Inoltre $M_1=2^1-1=1$. Si ha quindi la relazione di "successione" per i numeri di Mersenne:

$$M_{n+1} = M_1 \oplus M_n = M_1 + M_n + M_1 M_n = 2 M_n + 1$$

e questa è la relazione (2) del problema di Peano. Ed è anche un isomorfismo degli interi.

C'è una differenza tra la (2) e la (3). La (3), se scritta come $M_m \oplus M_0 = M_m + M_0 + M_m M_0 = M_m$, non mi produce la successione, ma mi indica che esiste un elemento neutro nel semi-gruppo dei numeri di Mersenne, e che questo elemento neutro è lo zero.

In conclusione: mai sottovalutare i problemini per le elementari, specie se formulati da un grande matematico.

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Discussion of the groupoid of Proth numbers (OEIS A080075)

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Abstract Here we show that the set of Proth numbers is a groupoid. The binary operation between the elements of the sets is given as a generalized composition.

Keywords: Groupoids, Integers, Proth numbers. DOI: 10.5281/zenodo.3339313

A groupoid is a set with a binary operation [1]. Let us consider the set of the Proth numbers and find the binary operator which is rendering the set a groupoid.

The Proth are integers given by:

$$P_{k,n}=k\,2^n+1$$

Integers k and n are given so that k is odd and $2^n>k$. Details on the Proth numbers are given in OEIS A080075.

Let us consider $P_{k,m+n}=k2^{m+n}+1$. This number is given as:

$$P_{k,m+n}=\frac{1}{k}(P_{k,m}P_{k,n}-P_{k,m}-P_{k,n}+1)+1$$

$$P_{k,m+n}=\frac{1}{k}((k\,2^m+1)(k2^n+1)-(k\,2^m+1)-(k\,2^n+1)+1)+1$$

$$P_{k,m+n}=\frac{1}{k}(k^2\,2^{m+n}+k\,2^m+k\,2^n+1-k\,2^m-1-k\,2^n-1+1)+1=k\,2^{m+n}+1$$

Let us define the binary operator as:

$$P_{k,m}\oplus P_{k,n}=\frac{1}{k}(P_{k,m}P_{k,n}-P_{k,m}-P_{k,n}+1)+1 \quad (*)$$

This is the same approach we used in some previous works (see [2] and references therein), where we discussed the binary operation of Mersenne, Fermat and other integers, in the framework of the generalized algebras [3,4].

Let us consider k=1. We obtain the following numbers (Fermat numbers [2]):

$$P_{1,n}=2^n+1 \quad \text{, with n=1,2,3,…}$$

That is: 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, and so on. If k=3, we have:

$$P_{3,n}=3\cdot 2^n+1 \quad \text{, with n=2,3,4,…}$$

This relation is giving: 13, 25, 49, 97, 193, 385, 769, 1537, 3073, … . Then we have:

$P_{5,n}=5\cdot 2^n+1$, with n=3,4,5,... ; $P_{7,n}=7\cdot 2^n+1$, with n=3,4,5,...

$P_{9,n}=9\cdot 2^n+1$, with n=4,5,6,... ; $P_{11,n}=11\cdot 2^n+1$, with n=4,5,6,...

$P_{13,n}=13\cdot 2^n+1$, with n=4,5,6,... ; $P_{15,n}=15\cdot 2^n+1$, with n=4,5,6,...

$P_{17,n}=17\cdot 2^n+1$, with n=5,6,7,... ; $P_{19,n}=19\cdot 2^n+1$, with n=5,6,7,...

and so on. For k=5, we have: 41, 81, 161, 321, 641, 1281, 2561, 5121, ...

Let us note that all these sets of numbers are groupoids as well.

Together, all these sets are giving the sequence in https://oeis.org/A080075/list

From https://oeis.org/A080075, we have that the Proth numbers can be obtained from other sequences, in the two following manners: 1) a(n) = A116882(n+1)+1, obtained by Klaus Brockhaus, Georgi Guninski and M. F. Hasler, Aug 16, 2010, 2) a(n) = A157892(n)*2^A157893(n) + 1, by M. F. Hasler, Aug 16, 2010.

From the binary operator (*), we can obtain a recurrence formula from the binary operator, in the following manner. Let us use $P_{k,n+1}=k\,2^{n+1}+1$ and $P_{k,1}=k\,2^1+1=2k+1$. We have:

$$P_{k,n+1}=P_{k,n}\oplus P_{k,1}=\frac{1}{k}(P_{k,n}P_{k,1}-P_{k,n}-P_{k,1}+1)+1$$

$$P_{k,n+1}=\frac{1}{k}(P_{k,n}(2k+1)-P_{k,n}-(2k+1)+1)+1=\frac{1}{k}(2kP_{k,n}-2k)+1=2\,P_{k,n}-1$$

The binary operation (*) is commutative and associative, so that:

$$(P_{k,m}\oplus P_{k,n})\oplus P_{k,o}=P_{k,m}\oplus(P_{k,n}\oplus P_{k,o}) \quad .$$

Let us note that we could repeat the same approach for numbers of the form $T_{k,n}=k\cdot 2^n-1$, In the case of k=3, we have the Thabit numbers [5]. We obtain a groupoid with binary operator:

$$T_{k,m}\oplus T_{k,n}=\frac{1}{k}(T_{k,m}T_{k,n}+T_{k,m}+T_{k,n}+1)-1$$

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Groupoid of OEIS A003154 Numbers
(star numbers or centered dodecagonal numbers)

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Here we discuss the binary operators of the set made by the OEIS sequence of integers A003154, defined as star numbers or centered dodecagonal numbers. The binary operators can be used to have groupoids.

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In [1] we can find discussed the Star Numbers. These numbers are representing the cells in a generalized Chinese checkers board (or "centered" hexagram). To the star numbers are linked some sequences of integers [1,2]. In [3], these numbers are also defined as centered dodecagonal numbers. As illustrated by Omar E. Pol, these number shave a classic representation in the form of stars, but they can also be represented by n-1 concentric hexagons around a central element. In general, centered polygonal numbers are those numbers represented by a central dot, surrounded by polygonal layers with a constant number of sides. Here we consider the Oeis A003154 numbers as a groupoid.
A groupoid is an algebraic structure made by a set with a binary operator [4]. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this operation is associative and we have a neutral element and opposite elements into the set, then the groupoid becomes a group. So let us consider OEIS A003154 numbers.
The numbers have the following form [3]:

$$S_n = 6n(n-1)+1 = 6n^2-6n+1$$

As we did in some previous discussions (see for instance [5]), we can find a binary operator, which satisfy the closure. Let us follow the same approach as in [6-8].

We have: $S_n = 6n^2-6n+1 = 6(n-1)^2+6(n-1)+1$

Let us use numbers $A_{n:}$, so that: $A_n = (n-1)$. We have that:

$$A_{n+m} = (n-1)+(m-1)+1 = (n+m-1)$$

So we can define a binary operation such as: $A_{n+m} = A_n \oplus A_m = A_n + A_m + 1$.

We have that: $S_n=6A_n^2+6A_n+1$; $A_n=-\frac{1}{2}\pm\frac{1}{12}(12+24S_n)^{1/2}=-\frac{1}{2}\pm\frac{1}{6}(3+6S_n)^{1/2}$ (1)

Let us consider in (1) the positive sign:

$$A_{n+m}=A_n+A_m+1=-\frac{1}{2}+\frac{1}{6}(3+6S_n)^{1/2}-\frac{1}{2}+\frac{1}{6}(3+6S_m)^{1/2}+1$$

Then it must be:

$$A_{n+m}=-\frac{1}{2}+\frac{1}{6}(3+6S_{n+m})^{1/2} \quad = \quad \frac{1}{6}(3+6S_n)^{1/2}+\frac{1}{6}(3+6S_m)^{1/2}$$

$$\frac{1}{6}(3+6S_{n+m})^{1/2}=\frac{1}{6}(3+6S_n)^{1/2}+\frac{1}{6}(3+6S_m)^{1/2}+\frac{1}{2}$$

So we have:

$$(3+6S_{n+m})=((3+6S_n)^{1/2}+(3+6S_m)^{1/2}+3)^2 \quad =$$

$$(3+6S_n)+(3+6S_m)+9+6(3+6S_n)^{1/2}+6(3+6S_m)^{1/2}+2(3+6S_n)^{1/2}(3+6S_m)^{1/2}$$

Then:

$$S_{n+m}=S_n+S_m+2+(3+6S_n)^{1/2}+(3+6S_m)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2}$$

The generalized sum for the star numbers is given as:

$$S_n\oplus S_m=S_n+S_m+2+(3+6S_n)^{1/2}+(3+6S_m)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2} \quad (2)$$

From (1), we have the recursive relation: $S_{n+1}=S_n\oplus S_1$. Starting from number $S_1=1$, we have: 13, 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, and so on. The same as http://oeis.org/A003154 .
The recursive relation is:

$$S_{n+1}=S_n+1+2+(3+6S_n)^{1/2}+3+(3+6S_n)^{1/2}$$

$$S_{n+1}=S_n+6+2(3+6S_n)^{1/2}$$

The square root:

$$(3+6S_n)^{1/2}$$

gives the sequence: 3, 9, 15, 21, 27, 33, 39, 45, etc.

Let us consider in (1) the negative sign:

$$A_{n+m}=A_n+A_m+1=-\frac{1}{2}-\frac{1}{6}(3+6S_n)^{1/2}-\frac{1}{2}-\frac{1}{6}(3+6S_m)^{1/2}+1$$

We have:

$$S_n\oplus S_m=S_n+S_m+2-(3+6S_n)^{1/2}-(3+6S_m)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2} \quad (3)$$

From (3), with the number $S_1=1$ we have the relation: $S_n=S_n\oplus S_1$. Therefore, S_1 is a neutral element, as we can easily see:

$$S_n+1+2-(3+6S_n)^{1/2}-(3+6)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6)^{1/2} =$$

$$S_n-(3+6S_n)^{1/2}+(3+6S_n)^{1/2}=S_n$$

Using (3) and starting from number $S_2=13$, we have: 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, and so on. Again, it is same as http://oeis.org/A003154 .

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The groupoid of the Triangular Numbers and the generation of related integer sequences

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Here we discuss the binary operators of the set made by the triangular numbers, sequence A000217, in the On-Line Encyclopedia of Integer Sequences (OEIS). As we will see, by means of these binary operators we can obtain related integer sequences. Here we propose some of them. The sequences, except one, are given in OEIS.

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In [1], we find defined the triangular numbers as those which are counting dots arranged in equilateral triangles. Then, the n-th triangular number is the number of dots in the triangle with n dots on a side. It is equal to the sum of the natural numbers from 1 to n:

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The triangular numbers are forming the sequence A000217 in OEIS, the On-Line Encyclopedia of Integer Sequences [2,3].
Some properties of triangular numbers are given in [1] and [4]. One of the properties that we find in [1] is:

$$T_{n+m} = T_n + T_m + nm \quad (1)$$

Actually, we have another manner to write T_{n+m}, if we consider OEIS A000217 as a groupoid.
A groupoid is an algebraic structure made by a set with a binary operator [5]. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this operation is associative and we have a neutral element and opposite elements into the set, then the groupoid becomes a group. So, let us consider OEIS A000217 numbers and find binary operators between them.

As we did in some previous discussions (see for instance [6]), we can find a binary operator, which is satisfying the closure. We can follow the same approach as in [7-10].
We have:

$$2T_n = n^2 + n = (n+1)^2 - (n+1)$$

Let us use numbers $A_{n:}$, so that: $A_n = (n+1)$. Then:

$$A_{n+m} = (n+1) + (m+1) - 1$$

So we can define a binary operation such as: $A_{n+m} = A_n \oplus A_m = A_n + A_m - 1$.

Moreover, we have that: $2T_n = A_n^2 - A_n$; $A_n = \frac{1}{2} \pm \frac{1}{2}(1+8T_n)^{1/2}$ (2)

Let us consider in (2) the positive sign:

$$A_{n+m} = A_n + A_m - 1 = \frac{1}{2}(1+8T_n)^{1/2} + \frac{1}{2}(1+8T_m)^{1/2}$$

$$A_{n+m} = \frac{1}{2} + \frac{1}{2}(1+8T_{n+m})^{1/2}$$

So we have:

$$(1+8T_{n+m}) = [-1 + (1+8T_n)^{1/2} + (1+8T_m)^{1/2}]^2 =$$

$$(1+8T_n) + (1+8T_m) + 1 - 2(1+8T_n)^{1/2} - 2(1+8T_m)^{1/2} + 2(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

Then:

$$T_{n+m} = T_n + T_m + \frac{1}{4} - \frac{1}{4}(1+8S_n)^{1/2} - \frac{1}{4}(1+8T_m)^{1/2} + \frac{1}{4}(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

The binary operator, that is, the generalized sum for the triangular numbers is given as:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}[1 - (1+8T_n)^{1/2} - (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}] \quad (3)$$

Using (3) and (1), we have the following identity:

$$4nm = 1-(1+8T_n)^{1/2}-(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

From the generalized sum (3), we have the recursive relation: $T_{n+1}=T_n \oplus T_1$.

Starting from number $T_1=1$, the generated sequence is 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on.
The recursive relation can be written, in this case with $T_1=1$, as:

$$T_{n+1}=T_n+1+\frac{1}{4}[-2-(1+8T_n)^{1/2}+3(1+8T_n)^{1/2}]$$

$$T_{n+1}=T_n+1+\frac{1}{2}[-1+(1+8T_n)^{1/2}]$$

Moreover, we have that $(1+8T_n)^{1/2}$ is the sequence of the odd numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, and so on.

Let us consider again (3), that is:

$$T_n \oplus T_m = T_n+T_m+\frac{1}{4}[1-(1+8T_n)^{1/2}-(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}]$$

in the form $T_{n+1}=T_n \oplus T_1$, but here we change the values of T_1 . Here in the following the sequences that we generate.

$T_1=0$, sequence 0, 0, 0, 0, 0, 0, … .

$T_1=1$, sequence 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on. And this is OEIS A000217, the sequence of triangular numbers.

$T_1=3$, sequence 10, 21, 36, 55, 78, 105, 136, 171, 210, 253, 300, 351, 406, 465, 528, 595, 666, 741, 820, 903, … . Searching this sequence in OEIS, we can easily find that it is A014105, that is, the Second Hexagonal Numbers: $H_n=n(2n+1)$.

$T_1=4$, sequence 12, 24, 40, 60, 84, 112, 144, 180, 220, 264, 312, 364, 420, 480, 544, 612, 684, 760, 840, 924, … OEIS A046092 (four times triangular numbers).

$T_1=6$, sequence 21, 45, 78, 120, 171, 231, 300, 378, 465, 561, 666, 780, 903, 1035, 1176, 1326, 1485, 1653, 1830, 2016, … OEIS A081266 (Staggered diagonal of triangular spiral in

A051682).

$T_1 = 7$, sequence 23, 48, 82, 125, 177, 238, 308, 387, 475, 572, 678, 793, 917, 1050, 1192, 1343, 1503, 1672, 1850, 2037, … OEIS A062725.

$T_1 = 10$, sequence 36, 78, 136, 210, 300, 406, 528, 666, 820, 990, 1176, 1378, 1596, 1830, 2080, 2346, 2628, 2926, 3240, 3570, … OEIS A033585, that is, numbers: $2n(4n+1)$.

$T_1 = 11$, sequence 38, 81, 140, 215, 306, 413, 536, 675, 830, 1001, 1188, 1391, 1610, 1845, 2096, 2363, 2646, 2945, 3260, 3591, … OEIS A139276, that is, numbers $n(8n+3)$.

Of course, we can continue and obtain further sequences.

Let us remember that, in (2), we can consider the negative sign too. Then we have another binary operation:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}[1 + (1+8T_n)^{1/2} + (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}]$$

Again, let us consider $T_{n+1} = T_n \oplus T_1$ as we did before.

$T_1 = 0$, sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, and so on. OEIS A000217, the sequence of triangular numbers.

$T_1 = 1$, sequence 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, … OEIS A000384, Hexagonal numbers $H_n = n(2n-1)$.

$T_1 = 3$, sequence 15, 36, 66, 105, 153, 210, 276, 351, 435, 528, 630, 741, 861, 990, 1128, 1275, 1431, 1596, 1770, 1953, … OEIS A062741, three times pentagonal numbers $3n(3n-1)/2$.

$T_1 = 4$, sequence 17, 39, 70, 110, 159, 217, 284, 360, 445, 539, 642, 754, 875, 1005, 1144, 1292, 1449, 1615, 1790, 1974, … OEIS A022266, numbers $n(9n-1)/2$.

$T_1 = 6$, sequence 28, 66, 120, 190, 276, 378, 496, 630, 780, 946, 1128, 1326, 1540, 1770, 2016, 2278, 2556, 2850, 3160, 3486, … OEIS A014635, numbers $2n(4n-1)$.

$T_1 = 7$, sequence 30, 69, 124, 195, 282, 385, 504, 639, 790, 957, 1140, 1339, 1554, 1785, 2032, 2295, 2574, 2869, 3180, 3507, … OEIS A139274, numbers $n(8n-1)$.

$T_1 = 10$, sequence 45, 105, 190, 300, 435, 595, 780, 990, 1225, 1485, 1770, 2080, 2415, 2775, 3160, 3570, 4005, 4465, 4950, 5460 …. This sequence is not present in OEIS.

$T_1 = 11$, sequence 47, 108, 194, 305, 441, 602, 788, 999, 1235, 1496, 1782, 2093, 2429, 2790, 3176, 3587, 4023, 4484, 4970, 5481, …. OEIS A178572, numbers with ordered partitions that have periods of length 5.

Of course, the approach here proposed can used for the generation of further integer sequences, using the binary operators given in the previous works [6-10]. It is possible that, among the generated sequences, news sequences are produced too.

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The groupoids of Mersenne, Fermat, Cullen, Woodall and other Numbers and their representations by means of integer sequences

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In some previous works, we have discussed the groupoids related to the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers. These groupoid possess different binary operators. As we can easily see, other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids. Using the On-Line Encyclopedia of Integer Sequences (OEIS), we can also identify the properties of these representations of groupoids. At the same time, we can also find integer sequences not given in OEIS and probably not yet studied.

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A groupoid is an algebraic structure made by a set with a binary operator [1]. The only restriction on the operator is closure. This property means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this binary operation is associative and we have a neutral element and opposite elements into the set, the groupoid becomes a group.
Groupoids are interesting also for the study of integer numbers. As shown in some previous works [2-7], the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers are groupoid possessing different binary operators. Here we show that other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids. That is, we can obtain different integer sequences by means of the recurrence relations generated by the considered binary operations.
In [7], we started the search for different representations for the groupoid of Triangular Numbers. Here we continue this search, using the binary operators obtained in the previous analyses. In particular, we will see the representations linked to Mersenne, Fermat, Cullen, Woodall, Carol and Kynea, and Oblong numbers. The binary operators of these numbers have been already discussed in previous works. The results concerning the Triangular numbers are also reported.
Using the On-Line Encyclopedia of Integer Sequences (OEIS), we can easily identify the several representations of groupoids. At the same time, we can also find integer sequences not given in OEIS and probably not yet studied.

Mersenne numbers
We discussed the binary operator of the set of Mersenne numbers in [8,9].
The numbers are given as $M_n = 2^n - 1$. The binary operator is:

$$M_{n+m} = M_n \oplus M_m = M_n + M_m + M_n M_m \quad (1)$$

As shown in [9], this binary operation is a specific case of the binary operator of q-integers, which can be linked to the generalized sum of Tsallis entropy [10,11].
The binary operator can be used to have a recurrence relation:

$$M_{n+1} = M_n \oplus M_1 \quad (2)$$

Here in the following, let us show the sequences that we can generate from (1) and (2).
We use OEIS, the On-Line Encyclopedia of Integer Sequences, to give more details on them.

$M_1 = 0$, sequence 0, 0, 0, 0, 0, 0, … .

$M_1 = 1$, sequence 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, and so on. The Mersenne numbers $2^n - 1$. This sequence is OEIS A000225. (OEIS tells that these numbers are sometimes called Mersenne numbers, "although that name is usually reserved for A001348").

$M_1 = 2$, sequence 8, 26, 80, 242, 728, 2186, 6560, 19682, 59048, 177146, 531440, 1594322, 4782968, 14348906, 43046720, 129140162, 387420488, and so on (OEIS A024023, $a_n = 3^n - 1$).

$M_1 = 3$, sequence 15, 63, 255, 1023, 4095, 16383, 65535, 262143, 1048575, 4194303, 16777215, 67108863, 268435455, and so on (OEIS A046092, $a_n = 4^n - 1$).

And we can continue: $M_1 = 4$, OEIS A024049, $a_n = 5^n - 1$; $M_1 = 5$, OEIS A024062, $a_n = 6^n - 1$; $M_1 = 6$ OEIS A024075, $a_n = 7^n - 1$, and so on.
An interesting sequence is $M_1 = 9$, A002283, $a_n = 10^n - 1$. Dividing this sequence by 9, we have the repunits A002275, $a_n = (10^n - 1)/9$. The generalized sum of the repunits is given in [12].

Fermat numbers
The group of Fermat numbers has been discussed in [13]. As explained in [14], there are two definitions of the Fermat numbers. "The less common is a number of the form $2^n + 1$ obtained by setting x=1 in a Fermat polynomial, the first few of which are 3, 5, 9, 17, 33, ... (OEIS A000051)" [14]. We used this definition.

$$F_n = 2^n + 1$$

$$F_{n+m} = F_n \oplus F_m = (1 - F_n) + (1 - F_m) + F_n F_m \quad (3)$$

The binary operator can be used to have a recurrence relation: $F_{n+1} = F_n \oplus F_1$ (4)

Sequences can generate from (3) and (4).

$F_1 = 0$, sequence 2, 0, 2, 0, 2, 0, … . ;

$F_1 = 1$, sequence 1, 1, 1, 1, 1, 1, …. .

$F_1 = 2$, sequence 2, 2, 2, 2, 2, 2, … .

$F_1 = 3$, sequence 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, and so on, the Fermat numbers. (OEIS A000051, $a_n = 2^n + 1$.

$F_1 = 4$, sequence A034472, $a_n = 3^n + 1$; $F_1 = 5$, sequence A052539, a(n) = 4^n + 1, (using the notation of OEIS). Continuing with 6, we have A034474, a(n) = 5^n + 1. For 7, we have A062394, a(n) = 6^n + 1. And so on.

Cullen and Woodall numbers
These numbers had been studied in [15].
Let us consider the Cullen numbers.

$$C_n = n 2^n + 1$$

$$C_{n+m} = C_n \oplus C_m = \left(\frac{1}{n} + \frac{1}{m}\right)(C_n - 1)(C_m - 1) + 1 \quad (5)$$

$$C_{n+1} = C_n \oplus C_1 \quad (6)$$

$C_1 = 1$, sequence 1, 1, 1, 1, 1, 1, 1, and so on.

$C_1 = 2$, sequence 3, 4, 5, 6, 7, 8, 9, and so on.

$C_1 = 3$, sequence 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, 1048577, 2228225, 4718593, 9961473, 20971521, 44040193, and so on.

OEIS A002064, Cullen numbers: n*2^n + 1.

$C_1=4$, sequence 19, 82, 325, 1216, 4375, 15310, 52489, 177148, 590491, 1948618, 6377293, 20726200, 66961567, 215233606, 688747537, and so on. OEIS A050914, a(n) = n*3^n + 1.

$C_1=5$ sequence A050915, a(n) = n*4^n + 1. And so on.

Let us mention the case $C_1=11$ which is giving A064748, a(n) = n*10^n + 1. That is: 201, 3001, 40001, 500001, 6000001, 70000001, 800000001, and so on.

Woodall numbers are $W_n=n2^n-1$, and the binary operator is:

$$W_{n+m}=W_n \oplus W_m=\left(\frac{1}{n}+\frac{1}{m}\right)(W_n+1)(W_m+1)-1 \qquad (7)$$

$$W_{n+1}=W_n \oplus W_1 \qquad (8)$$

$W_1=0$, sequence 1, 2, 3, 4, 5, 6, 7, and so on.

$W_1=1$, sequence 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, 2228223, 4718591, 9961471, 20971519, 44040191, and so on. A003261, Woodall (or Riesel) numbers: n*2^n - 1.

$W_1=2$, sequence A060352, a(n) = n*3^n - 1.

$W_1=3$, sequence A060416, a(n) = n*4^n – 1. And so on.

Let us mention the case $W_1=9$, which is giving A064756, a(n) = n*10^n – 1, that is, 199, 2999, 39999, 499999, 5999999, 69999999, 799999999, and so on.

Carol and Kynea Numbers
These numbers have been studied in [3].
Carol number is:

$$C_n=(2^n-1)^2-2$$

The binary operator is:

$$C_n \oplus C_m = 6 + C_n C_m + 3(C_n + C_m) + 2(C_n+2)(C_m+2)^{1/2} + 2(C_m+2)(C_n+2)^{1/2} + 2(C_n+2)^{1/2}(C_m+2)^{1/2} \quad (9)$$

$$C_{n+1} = C_n \oplus C_1 \quad (10)$$

Here we have square roots, so we can obtain integer sequences only in some cases.

$C_1 = -1$, sequence A093112, a(n) = (2^n-1)^2 – 2, that is 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, … As told in [16], Cletus Emmanuel called these numbers as "Carol numbers".

$C_1 = 2$, sequence 62, 674, 6398, 58562, 529982, 4778594, 43033598, 387381122, 3486666302, 31380705314, and so on. Not given in OEIS.

$C_1 = 7$, sequence 223, 3967, 65023, 1046527, 16769023, 268402687, 4294836223, 68718952447, 1099509530623, 17592177655807, and so on. Not given in OEIS.

Let us consider the Kynea numbers.

$$K_n = (2^n + 1)^2 - 2$$

The binary operator is:

$$K_n \oplus K_m = -2 + [2 + (K_m+2)^{1/2}(K_n+2)^{1/2} - (K_m+2)^{1/2} - (K_n+2)^{1/2}]^2 \quad (11)$$

$$K_{n+1} = K_n \oplus K_1 \quad (10)$$

Here we have square roots, so we can obtain integer sequences only in some cases.

$K_1 = -1$, sequence -1, -1, -1, -1, -1, and so on.

$K_1 = 2$, sequence 2, 2, 2, 2, 2, 2, and so on.

$K_1 = 7$, sequence A093069, a(n) = (2^n + 1)^2 – 2, that is 7, 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, and so on. As told in [17], Cletus Emmanuel calls these "Kynea numbers" [17].

$K_1 = 14$, sequence 98, 782, 6722, 59534, 532898, 4787342, 43059842, 387459854, 3486902498, 31381413902, and so on. Not given in OEIS.

Oblong numbers

These numbers are discussed in [4]. The oblong number is defined as: $O_n = n(n+1)$. It is given by OEIS A002378. An oblong number is also known as a promic, pronic, or heteromecic number. OEIS gives the list: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, and so on.

The binary operator is:

$$O_m \oplus O_n = O_m + O_n + 1/2 + 2(O_m + 1/4)^{1/2}(O_n + 1/4)^{1/2} - (O_m + 1/4)^{1/2} - (O_n + 1/4)^{1/2}$$

Again, as we did before we have:

$O_1 = 0$, sequence 0, 0, 0, 0, 0, and so on.

$O_1 = 2$, sequence OEIS A002378, as given above.

$O_1 = 6$, sequence, A002943, a(n) = 2*n*(2*n+1).

$O_1 = 12$, sequence A045945, Hexagonal matchstick numbers: a(n) = 3*n*(3*n+1).

$O_1 = 20$, sequence 72, 156, 272, 420, 600, 812, 1056, 1332, 1640, 1980, and so on. Not given in OEIS.

Of course, we can repeat the same approach for the odd squares (A016754) numbers. Their binary operator is given in [4]. Also for the centered square numbers and the star numbers, we have the binary operators [5,6], so we can find the related representations by means of integer sequences too. As previously told, among the generated sequences, news sequences are produced that can be interesting for further investigation of integer sequences.

Triangular numbers

These numbers are really interesting. The numbers are of the form (OEIS A000217):

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

I have discussed them in [7]. For these numbers we can give two binary operators.
For the convenience of the reader, I show the results that we can obtain.
The first binary operator is [7]:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}[1 - (1+8T_n)^{1/2} - (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}]$$

Again we consider $T_{n+1}=T_n \oplus T_1$, and change the value of T_1 . Here in the following the sequences that we generate.

$T_1=0$, sequence 0, 0, 0, 0, 0, 0, … .

$T_1=1$, sequence 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on. And this is OEIS A000217, the sequence of triangular numbers.

$T_1=3$, sequence 10, 21, 36, 55, 78, 105, 136, 171, 210, 253, 300, 351, 406, 465, 528, 595, 666, 741, 820, 903, … . Searching this sequence in OEIS, we can easily find that it is A014105, that is, the Second Hexagonal Numbers: $H_n=n(2n+1)$.

$T_1=4$, sequence 12, 24, 40, 60, 84, 112, 144, 180, 220, 264, 312, 364, 420, 480, 544, 612, 684, 760, 840, 924, … OEIS A046092 (four times triangular numbers).

$T_1=6$, sequence 21, 45, 78, 120, 171, 231, 300, 378, 465, 561, 666, 780, 903, 1035, 1176, 1326, 1485, 1653, 1830, 2016, … OEIS A081266 (Staggered diagonal of triangular spiral in A051682).

$T_1=7$, sequence 23, 48, 82, 125, 177, 238, 308, 387, 475, 572, 678, 793, 917, 1050, 1192, 1343, 1503, 1672, 1850, 2037, … OEIS A062725.

$T_1=10$, sequence 36, 78, 136, 210, 300, 406, 528, 666, 820, 990, 1176, 1378, 1596, 1830, 2080, 2346, 2628, 2926, 3240, 3570, … OEIS A033585, that is, numbers: $2n(4n+1)$.

$T_1=11$, sequence 38, 81, 140, 215, 306, 413, 536, 675, 830, 1001, 1188, 1391, 1610, 1845, 2096, 2363, 2646, 2945, 3260, 3591, … OEIS A139276, that is, numbers $n(8n+3)$.

Of course, we can continue and obtain further sequences.
As previously told, we have a second binary operator for the triangular numbers [7]. It is the following:

$$T_n \oplus T_m=T_n+T_m+\frac{1}{4}\left[1+(1+8T_n)^{1/2}+(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}\right]$$

Again, let us consider $T_{n+1}=T_n \oplus T_1$ as we did before.

$T_1=0$, sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, and so on. OEIS A000217, the sequence of triangular numbers.

$T_1=1$, sequence 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, … OEIS A000384, Hexagonal numbers $H_n=n(2n-1)$.

$T_1=3$, sequence 15, 36, 66, 105, 153, 210, 276, 351, 435, 528, 630, 741, 861, 990, 1128, 1275, 1431, 1596, 1770, 1953, … OEIS A062741, three times pentagonal numbers $3n(3n-1)/2$.

$T_1=4$, sequence 17, 39, 70, 110, 159, 217, 284, 360, 445, 539, 642, 754, 875, 1005, 1144, 1292, 1449, 1615, 1790, 1974, … OEIS A022266, numbers $n(9n-1)/2$.

$T_1=6$, sequence 28, 66, 120, 190, 276, 378, 496, 630, 780, 946, 1128, 1326, 1540, 1770, 2016, 2278, 2556, 2850, 3160, 3486, … OEIS A014635, numbers $2n(4n-1)$.

$T_1=7$, sequence 30, 69, 124, 195, 282, 385, 504, 639, 790, 957, 1140, 1339, 1554, 1785, 2032, 2295, 2574, 2869, 3180, 3507, … OEIS A139274, numbers $n(8n-1)$.

$T_1=10$, sequence 45, 105, 190, 300, 435, 595, 780, 990, 1225, 1485, 1770, 2080, 2415, 2775, 3160, 3570, 4005, 4465, 4950, 5460 …. This sequence is not present in OEIS.

$T_1=11$, sequence 47, 108, 194, 305, 441, 602, 788, 999, 1235, 1496, 1782, 2093, 2429, 2790, 3176, 3587, 4023, 4484, 4970, 5481, …. OEIS A178572, numbers with ordered partitions that have periods of length 5.

Using the On-Line Encyclopedia of Integer Sequences (OEIS), we have seen that quite different sequences can have the same binary operators. We have also found integer sequences not given in OEIS and that need to be studied.

Conclusion

Groupoids are related to the integer sequences. These groupoid possess different binary operators. As we have shown, other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids.

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Some Groupoids and their Representations by Means of Integer Sequences

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Abstract: In some previous works, we have discussed the groupoids related to the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers. These groupoids possess different binary operators. As we can easily see, other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids. Using the On-Line Encyclopedia of Integer Sequences (OEIS), we are able to identify the properties of these representations of groupoids. At the same time, we can also find integer sequences not given in OEIS and probably not yet studied.

Keywords: Groupoid Representations, Integer Sequences, Binary Operators, Generalized Sums, Generalized Entropies, Tsallis Entropy, Q-Calculus, Abelian Groups, Fermat Numbers, Mersenne Numbers, Triangular Numbers, Repunits, Oblong Numbers

Introduction

A groupoid is an algebraic structure made by a set with a binary operator [1]. The only restriction on the operator is closure. This property means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this binary operation is associative and we have a neutral element and opposite elements into the set, the groupoid becomes a group.

Groupoids are interesting also for the study of integer numbers. As shown in some previous works [2-7], the integer sequences of Mersenne, Fermat, Cullen, Woodall and other numbers are groupoid possessing different binary operators. Here we show that other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids. That is, we can obtain different integer sequences by means of the recurrence relations generated by the considered binary operations.

In [7], we started the search for different representations for the groupoid of Triangular Numbers. Here we generalize this search, using the binary operators obtained in the previous analyses. In particular, we will see the representations linked to Mersenne, Fermat, Cullen, Woodall, Carol and Kynea, and Oblong numbers. The binary operators of these numbers have been already discussed in previous works. The results concerning the Triangular numbers are also reported.

Using the On-Line Encyclopedia of Integer Sequences (OEIS), we are able to identify the several representations of groupoids. At the same time, we can also find integer sequences not given in OEIS and probably not yet studied.

Mersenne numbers

We discussed the binary operator of the set of Mersenne numbers in [8,9]. The numbers are given as$M_n = 2^n - 1$. The binary operator is:

$$M_{n+m} = M_n \oplus M_m = M_n + M_m + M_n M_m \quad (1)$$

As shown in [9], this binary operation is a specific case of the binary operator of q-integers, which can be linked to the generalized sum of Tsallis entropy [10,11].

The binary operator can be used to have a recurrence relation:

$$M_{n+1} = M_n \oplus M_1 \quad (2)$$

Here in the following, let us show the sequences that we can generate from (1) and (2).

We use OEIS, the On-Line Encyclopedia of Integer Sequences, to give more details on them.

$M_1 = 0$, sequence 0, 0, 0, 0, 0, 0, … .

$M_1 = 1$, sequence 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, and so on. The Mersenne numbers $2^n - 1$. This sequence is OEIS A000225. (OEIS tells that these numbers are sometimes called Mersenne numbers, "although that name is usually reserved for A001348").

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$M_1 = 2$, sequence 8, 26, 80, 242, 728, 2186, 6560, 19682, 59048, 177146, 531440, 1594322, 4782968, 14348906, 43046720, 129140162, 387420488, and so on (OEIS A024023, $a_n = 3^n - 1$).

$M_1 = 3$, sequence 15, 63, 255, 1023, 4095, 16383, 65535, 262143, 1048575, 4194303, 16777215, 67108863, 268435455, and so on (OEIS A046092, $a_n = 4^n - 1$).

And we can continue: $M_1 = 4$, OEIS A024049, $a_n = 5^n - 1$; $M_1 = 5$, OEIS A024062, $a_n = 6^n - 1$; $M_1 = 6$ OEIS A024075, $a_n = 7^n - 1$, and so on. An interesting sequence is $M_1 = 9$, A002283, $a_n = 10^n - 1$. Dividing this sequence by 9, we have the repunits A002275, $a_n = (10^n - 1)/9$. The generalized sum of the repunits is given in [12].

Fermat numbers

The group of Fermat numbers has been discussed in [13]. As explained in [14], there are two definitions of the Fermat numbers. "The less common is a number of the form $2^n + 1$ obtained by setting x=1 in a Fermat polynomial, the first few of which are 3, 5, 9, 17, 33, ... (OEIS A000051)" [14]. We used this definition.

$$F_n = 2^n + 1$$

$$F_{n+m} = F_n \oplus F_m = (1 - F_n) + (1 - F_m) + F_n F_m \quad (3)$$

The binary operator can be used to have a recurrence relation: $F_{n+1} = F_n \oplus F_1$ (4)

Sequences can generate from (3) and (4).

$F_1 = 0$, sequence 2, 0, 2, 0, 2, 0, ... ;

$F_1 = 1$, sequence 1, 1, 1, 1, 1, 1, ...

$F_1 = 2$, sequence 2, 2, 2, 2, 2, 2, ...

$F_1 = 3$, sequence 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, and so on, the Fermat numbers. (OEIS A000051, $a_n = 2^n + 1$.

$F_1 = 4$, sequence A034472,$a_n = 3^n + 1$; for $F_1 = 5$, sequence A052539, $a_n = 4^n + 1$. Continuing with 6, we have A034474, $a_n = 5^n + 1$. For 7, we have A062394, $a_n = 6^n + 1$. And so on.

Cullen and Woodall numbers

These numbers had been studied in [15].

Let us consider the Cullen numbers, $C_n = n2^n + 1$. We have the binary operator:

$$C_{n+m} = C_n \oplus C_m = \left(\frac{1}{n} + \frac{1}{m}\right)(C_n - 1)(C_m - 1) + 1 \quad (5)$$

$$C_{n+1} = C_n \oplus C_1 \quad (6)$$

$C_1 = 1$, sequence 1, 1, 1, 1, 1, 1, 1, and so on.

$C_1 = 2$, sequence 3, 4, 5, 6, 7, 8, 9, and so on.

$C_1 = 3$, sequence 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, 1048577, 2228225, 4718593, 9961473, 20971521, 44040193, and so on. OEIS A002064, Cullen numbers: $a_n = n2^n + 1$.

$C_1 = 4$, sequence 19, 82, 325, 1216, 4375, 15310, 52489, 177148, 590491, 1948618, 6377293, 20726200, 66961567, 215233606, 688747537, and so on. OEIS A050914, $a_n = n3^n + 1$. For $C_1 = 5$ sequence A050915, $a_n = n4^n + 1$. And so on. Let us mention the case $C_1 = 11$which is giving A064748, $a_n = n10^n + 1$. That is: 201, 3001, 40001, 500001, 6000001, 70000001, 800000001, and so on.

Woodall numbers are $W_n = n2^n - 1$, and the binary operator is:

$$W_{n+m} = W_n \oplus W_m = \left(\frac{1}{n} + \frac{1}{m}\right)(W_n + 1)(W_m + 1) - 1 \quad (7)$$

$$W_{n+1} = W_n \oplus W_1 \quad (8)$$

$W_1 = 0$, sequence 1, 2, 3, 4, 5, 6, 7, and so on.

$W_1 = 1$, sequence 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, 2228223, 4718591, 9961471, 20971519, 44040191, and so on. A003261, Woodall (or Riesel) numbers:$a_n = n2^n - 1$.

$W_1 = 2$, sequence A060352, $a_n = n3^n - 1$. For $W_1 = 3$, we have sequence A060416, $a_n = n4^n - 1$. And so on. Let us mention the case $W_1 = 9$, which is giving A064756, $a_n = n10^n - 1$, that is, 199, 2999, 39999, 499999, 5999999, 69999999, 799999999, and so on.

Carol and Kynea Numbers

These numbers have been studied in [3]. Carol number is: $C_n = (2^n - 1)^2 - 2$. The binary operator $C_n \oplus C_m$is given in [3]:

$$C_m \oplus C_n = 6 + C_m C_n + 3C_m + 3C_n + a + b + c$$

where $a = 2(C_m + 2)(C_n + 2)^{1/2}$,$b = 2(C_m + 2)^{1/2}(C_n + 2)$,$c = 2(C_m + 2)^{1/2}(C_n + 2)^{1/2}$.
We can use again $C_{n+1} = C_n \oplus C_1$. Since the binary operator contains square roots, we can obtain integer

sequences only in some cases.

$C_1 = -1$, sequence A093112, $a_n = (2^n - 1)^2 - 2$, that is 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, … As told in [16], Cletus Emmanuel called these numbers as "Carol numbers".

$C_1 = 2$, sequence 62, 674, 6398, 58562, 529982, 4778594, 43033598, 387381122, 3486666302, 31380705314, and so on. Not given in OEIS.

$C_1 = 7$, sequence 223, 3967, 65023, 1046527, 16769023, 268402687, 4294836223, 68718952447, 1099509530623, 17592177655807, and so on. Not given in OEIS.

Let us consider the Kynea numbers.

$$K_n = (2^n + 1)^2 - 2$$

The binary operator $K_n \oplus K_m$ is given in [3]. We use again $K_{n+1} = K_n \oplus K_1$. Again, we have square roots, so we can obtain integer sequences only in some cases.

$K_1 = -1$, sequence -1, -1, -1, -1, -1, and so on.

$K_1 = 2$, sequence 2, 2, 2, 2, 2, 2, and so on.

$K_1 = 7$, sequence A093069 , $a_n = (2^n + 1)^2 - 2$, that is 7, 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, and so on. As told in [17], Cletus Emmanuel calls these "Kynea numbers" [17].

$K_1 = 14$, sequence 98, 782, 6722, 59534, 532898, 4787342, 43059842, 387459854, 3486902498, 31381413902, and so on. Not given in OEIS.

Oblong numbers
These numbers are discussed in [4]. The oblong number is defined as: $O_n = n(n + 1)$. It is given by OEIS A002378. An oblong number is also known as a promic, pronic, or heteromecic number. OEIS gives the list: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, and so on.
The binary operator $O_n \oplus O_m$ is:

$$O_m \oplus O_n = \frac{1}{2} + O_m + O_n + a + b$$

where $a = 2(O_m + 1/4)^{1/2}(O_n + 1/4)^{1/2}$, $b = -(O_m + 1/4)^{1/2} - (O_n + 1/4)^2$. Again, as we did before we have:

$O_1 = 0$, sequence 0, 0, 0, 0, 0, and so on.

$O_1 = 2$, sequence OEIS A002378, as given above.

$O_1 = 6$, sequence, A002943, $a_n = 2n(2n + 1)$.

$O_1 = 12$, sequence A045945, Hexagonal matchstick numbers: $a_n = 3n(3n + 1)$.

$O_1 = 20$, sequence 72, 156, 272, 420, 600, 812, 1056, 1332, 1640, 1980, and so on. Not given in OEIS. It is $a_n = 4n(4n + 1)$. And we can continue.

Of course, we can repeat the same approach for the odd squares (A016754) numbers. Their binary operator is given in [4]. Also for the centered square numbers and the star numbers, we have the binary operators [5,6], so we can find the related representations by means of integer sequences too. As previously told, among the generated sequences, news sequences are produced that can be interesting for further investigation of integer sequences.

Triangular numbers
These numbers are really interesting. The numbers are of the form (OEIS A000217):

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

I have discussed them in [7]. For these numbers we can give two binary operators. For the convenience of the reader, I show the results that we can obtain.
The first binary operator is [7]:

$$T_n \oplus T_m = T_n + T_m + \tfrac{1}{4}\left[1 - (1 + 8T_n)^{1/2} - (1 + 8T_m)^{1/2} + (1 + 8T_n)^{1/2}(1 + 8T_m)^{1/2}\right]$$

Again we consider $T_{n+1} = T_n \oplus T_1$, and change the value of T_1. Here in the following the sequences that we generate.

$T_1 = 0$, sequence 0, 0, 0, 0, 0, 0, … .

$T_1 = 1$, sequence 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on. And this is OEIS A000217, the sequence of triangular numbers.

$T_1 = 3$, sequence 10, 21, 36, 55, 78, 105, 136, 171, 210, 253, 300, 351, 406, 465, 528, 595, 666, 741, 820, 903, … . Searching this sequence in OEIS, we can easily find that it is A014105, that is, the Second Hexagonal Numbers: $H_n = n(2n + 1)$.

$T_1 = 4$, sequence 12, 24, 40, 60, 84, 112, 144, 180, 220, 264, 312, 364, 420, 480, 544, 612, 684, 760, 840, 924, … OEIS A046092 (four times triangular numbers).

$T_1 = 6$, sequence 21, 45, 78, 120, 171, 231, 300, 378, 465, 561, 666, 780, 903, 1035, 1176, 1326, 1485, 1653, 1830, 2016, … OEIS A081266 (Staggered diagonal of triangular spiral in A051682).

$T_1 = 7$, sequence 23, 48, 82, 125, 177, 238, 308, 387, 475, 572, 678, 793, 917, 1050, 1192, 1343, 1503, 1672, 1850, 2037, … OEIS A062725.

$T_1 = 10$, sequence 36, 78, 136, 210, 300, 406, 528, 666, 820, 990, 1176, 1378, 1596, 1830, 2080, 2346, 2628, 2926, 3240, 3570, … OEIS A033585, that is, numbers: $2n(4n+1)$.

$T_1 = 11$, sequence 38, 81, 140, 215, 306, 413, 536, 675, 830, 1001, 1188, 1391, 1610, 1845, 2096, 2363, 2646, 2945, 3260, 3591, … OEIS A139276, that is, numbers $n(8n+3)$.

Of course, we can continue and obtain further sequences.

As previously told, we have a second binary operator for the triangular numbers [7]. It is the following:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}\left[1 + (1+8T_n)^{1/2} + (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}\right]$$

Again, let us consider $T_{n+1} = T_n \oplus T_1$ as we did before.

$T_1 = 0$, sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, and so on. OEIS A000217, the sequence of triangular numbers.

$T_1 = 1$, sequence 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, … OEIS A000384, Hexagonal numbers $H_n = n(2n-1)$.

$T_1 = 3$, sequence 15, 36, 66, 105, 153, 210, 276, 351, 435, 528, 630, 741, 861, 990, 1128, 1275, 1431, 1596, 1770, 1953, … OEIS A062741, three times pentagonal numbers $3n(3n-1)/2$.

$T_1 = 4$, sequence 17, 39, 70, 110, 159, 217, 284, 360, 445, 539, 642, 754, 875, 1005, 1144, 1292, 1449, 1615, 1790, 1974, … OEIS A022266, numbers $n(9n-1)/2$.

$T_1 = 6$, sequence 28, 66, 120, 190, 276, 378, 496, 630, 780, 946, 1128, 1326, 1540, 1770, 2016, 2278, 2556, 2850, 3160, 3486, … OEIS A014635, numbers $2n(4n-1)$.

$T_1 = 7$, sequence 30, 69, 124, 195, 282, 385, 504, 639, 790, 957, 1140, 1339, 1554, 1785, 2032, 2295, 2574, 2869, 3180, 3507, … OEIS A139274, numbers $n(8n-1)$.

$T_1 = 10$, sequence 45, 105, 190, 300, 435, 595, 780, 990, 1225, 1485, 1770, 2080, 2415, 2775, 3160, 3570, 4005, 4465, 4950, 5460 …. This sequence is not present in OEIS.

$T_1 = 11$, sequence 47, 108, 194, 305, 441, 602, 788, 999, 1235, 1496, 1782, 2093, 2429, 2790, 3176, 3587, 4023, 4484, 4970, 5481, …. OEIS A178572, numbers with ordered partitions that have periods of length 5.

Using the On-Line Encyclopedia of Integer Sequences (OEIS), we have seen that quite different sequences can have the same binary operators. We have also found integer sequences not given in OEIS and that need to be studied.

Conclusion

Groupoids are related to the integer sequences. These groupoid possess different binary operators. As we have shown, other integer sequences can have the same binary operators, and therefore can be used to represent the related groupoids.

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