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# HILBERT FUNCTIONS OF SCHEMES OF DOUBLE AND REDUCED POINTS

ENRICO CARLINI, MARIA VIRGINIA CATALISANO, ELENA GUARDO, AND ADAM VAN TUYL

**ABSTRACT.** It remains an open problem to classify the Hilbert functions of double points in  $\mathbb{P}^2$ . Given a valid Hilbert function  $H$  of a zero-dimensional scheme in  $\mathbb{P}^2$ , we show how to construct a set of fat points  $Z \subseteq \mathbb{P}^2$  of double and reduced points such that  $H_Z$ , the Hilbert function of  $Z$ , is the same as  $H$ . In other words, we show that any valid Hilbert function  $H$  of a zero-dimensional scheme is the Hilbert function of a set of positive number of double points and some reduced points. For some families of valid Hilbert functions, we are also able to show that  $H$  is the Hilbert function of only double points. In addition, we give necessary and sufficient conditions for the Hilbert function of a scheme of a double points, or double points plus one additional reduced point, to be the Hilbert function of points with support on a star configuration of lines.

## 1. INTRODUCTION

Throughout this paper,  $k$  will denote an algebraically closed field of characteristic zero. Let  $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$  be a finite set of reduced points with associated homogeneous ideal  $I_X = I_{P_1} \cap \dots \cap I_{P_s} \subseteq R = k[x_0, x_1, x_2]$ . Given positive integers  $m_1, \dots, m_s$ , we let  $Z = m_1 P_1 + \dots + m_s P_s$  denote the scheme defined by the homogeneous ideal  $I_Z = I_{P_1}^{m_1} \cap \dots \cap I_{P_s}^{m_s}$ . We refer to  $Z$  as a *set of fat points*. We call  $m_i$  the *multiplicity* of the point  $P_i$ ; when  $m_i = 2$ , we sometimes call  $P_i$  a *double point*. Given a set of fat points  $Z$ , the *support* of  $Z$  is the set  $\text{Supp}(Z) = \{P_1, \dots, P_s\}$ .

Information about the set of fat points  $Z$  is encoded into its Hilbert function. Recall that the *Hilbert function* of  $Z$  is the function  $H_Z : \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$i \mapsto \dim_k(R/I_Z)_i = \dim_k R_i - \dim_k (I_Z)_i$$

where  $R_i$ , respectively  $(I_Z)_i$ , denotes the  $i$ -th graded piece of  $R$ , respectively  $(I_Z)_i$  (see Chapter 5 of [16] for a comprehensive introduction to Hilbert functions). It is then natural to ask if one can characterize what functions are the Hilbert function of a set of fat points. A complete characterization of the Hilbert functions of reduced points (i.e., all the  $m_i = 1$ ) was first described by Geramita, Maroscia, and Roberts [12]. However, even in the case that all the fat points are double points, a characterization of the Hilbert functions remains elusive (see, for example, the surveys of Gimigliano [13] and Harbourne [14]). In this paper, we contribute to this open problem by showing that every Hilbert function of a collection of reduced points in  $\mathbb{P}^2$  is also the Hilbert function of a collection of double points and reduced points in  $\mathbb{P}^2$ . In specific cases, we can give a sufficient

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condition for a numerical function to be the Hilbert function of a scheme consisting only of double points.

To further describe our results, we introduce some additional notation. One way to study the Hilbert function of  $H_Z$  is to study its *first difference function* (sometimes called the Castelnuovo function) which is given by

$$\Delta H_Z(i) = H_Z(i) - H_Z(i-1) \text{ for all } i \geq 0, \text{ where } H_Z(-1) = 0.$$

When  $Z$  is a zero-dimensional scheme in  $\mathbb{P}^2$ , it can be shown (see Remark 2.2) that all but a finite number of values of  $\Delta H_Z(i)$  are zero. Furthermore, if  $\Delta H_Z(i) = 0$  for all  $i \geq \sigma + 1$ , and if we write  $\Delta H_Z = (h_0, \dots, h_\sigma)$  to encode all the non-zero values of  $\Delta H_Z$ , then there is an  $0 < \alpha \leq \sigma$  such that

- (a)  $h_i = i + 1$  if  $0 \leq i < \alpha$ , and
- (b)  $h_i \geq h_{i+1}$  if  $\alpha \leq i \leq \sigma$ .

We call  $\Delta H = (h_0, \dots, h_\sigma)$  a *valid Hilbert function* of a zero-dimensional scheme in  $\mathbb{P}^2$  if  $\Delta H$  satisfies conditions (a) and (b). Ideally, we want to answer the following question:

**Question 1.1.** *Let  $\Delta H = (h_0, \dots, h_\sigma)$  be a valid Hilbert function. Write  $\sum \Delta H = \sum_{i=0}^{\sigma} h_i$  as  $\sum \Delta H = 3d + r$  with  $r \in \{0, 1, 2\}$ . Does there exist a set  $Z$  of  $d$  double points and  $r$  reduced points in  $\mathbb{P}^2$  such that  $\Delta H_Z = \Delta H$ ?*

Note that a scheme  $Z$  with  $d$  double points and  $r$  reduced points in  $\mathbb{P}^2$  will have  $\deg(Z) = 3d + r$ . Furthermore, it is known that  $H_Z(i) = \deg(Z)$  for  $i \gg 0$ . This explains why we require  $\sum \Delta H = 3d + r$ . If we could answer this question, we could determine if a valid Hilbert function is the Hilbert function of a set of double points. Thus, the above question is quite difficult.

We can ask a weaker question by simply asking if any set of double points and reduced points can be constructed:

**Question 1.2.** *Let  $\Delta H = (h_0, \dots, h_\sigma)$  be a valid Hilbert function. Can one always find integers  $d$  and  $r$  where  $d$  is positive and  $r \geq 0$  with  $\sum \Delta H = 3d + r$  such that  $H$  is the Hilbert function of a set  $Z$  of  $d$  double points and  $r$  simple points in  $\mathbb{P}^2$ ?*

Note that if we allow  $d = 0$  and  $r = \sum \Delta H$ , then the above question is simply asking if  $\Delta H$  is the Hilbert function of  $r$  reduced points, which follows from Geramita, Maroscia, and Roberts [12]. We can now view Question 1.1 as asking if the  $d$  in Question 1.2 can be taken to be the maximum allowed value. Ideally, when trying to answer Question 1.2, we want to make  $d$  as large as possible.

One of the main results of this paper (Theorem 3.1) will give us a tool to answer to Question 1.2. Specifically, starting with a set of double and reduced points on a collection of general lines in  $\mathbb{P}^2$ , we describe how to “merge” three reduced points to make a new scheme with one new double point and three fewer reduced points. Moreover, this procedure does not change the Hilbert function. The results of Cooper, Harbourne, and Teitler [8] are the crucial ingredient to prove that our new configuration has the correct Hilbert function. By reiterating this process, in a controlled fashion, Construction 3.5 shows how to start from a valid Hilbert function  $\Delta H$  and create a set  $Z$  of double and

simple points with  $\Delta H = \Delta H_Z$ . Our answer to Question 1.2 is given in Theorem 3.9 where we find a  $d$ , that depends only on  $\Delta H$ , such that we can construct a set of  $d$  double points and  $(\sum \Delta H) - 3d$  reduced points whose Hilbert function is  $H$ . In fact, for all  $1 \leq d' \leq d$ , we can find a scheme of  $d'$  double points and  $(\sum \Delta H) - 3d'$  reduced points with Hilbert function  $H$  (see Corollary 3.10). Moreover, in Theorem 3.11 we give a condition on a valid Hilbert function  $H$  that guarantees that  $H$  is the Hilbert function of only double points.

We then focus on the special cases that

$$\Delta H = (\underbrace{1, 2, 3, \dots, t}_t, \underbrace{t+1, \dots, t+1}_t) \text{ or } \Delta H = (\underbrace{1, 2, 3, \dots, t}_t, \underbrace{t+1, \dots, t+1, 1}_t)$$

In these cases, our construction produces  $\binom{t+1}{2}$  double points, respectively  $\binom{t+1}{2}$  double points and one reduced points. In the first case, the support of the points are the  $\binom{t+1}{2}$  points of intersection of  $t$  general lines in  $\mathbb{P}^2$ . This fact is equivalent to the statement that the points in the support are a *star-configuration* of points in  $\mathbb{P}^2$ ; star configurations are widely studied, e.g. see [4, 5]. We prove (see Theorem 4.4) that this configuration is the *only* configuration of  $\binom{t+1}{2}$  double points in  $\mathbb{P}^2$  with  $\Delta H_Z = \Delta H$ . In the second case, we show (see Theorem 4.6) a similar result by showing again that there is only one configuration of  $\binom{t+1}{2}$  double points and one reduced point that has  $\Delta H_Z = \Delta H$ .

We conclude our paper with some final comments related to how well our construction performs, i.e., given a known valid Hilbert function of  $t$  double points, how many double points does our procedure produce for the same valid Hilbert function. In the case that the support of points is in generic position, we derive an asymptotic estimate.

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## 2. PRELIMINARIES

We begin with a review of the relevant background; we continue to use the notation and definitions given in the introduction.

**Definition 2.1.** A sequence  $\Delta H = (h_0, h_1, \dots, h_\sigma)$  is a *valid Hilbert function of a set of points in  $\mathbb{P}^2$*  if there is an  $0 < \alpha \leq \sigma$  such that

- (a)  $h_i = i + 1$  if  $0 \leq i < \alpha$ , and
- (b)  $h_i \geq h_{i+1}$  if  $\alpha \leq i \leq \sigma$ .

Note that the indexing of  $\Delta H$  begins with 0.

*Remark 2.2.* It can be shown that  $H : \mathbb{N} \rightarrow \mathbb{N}$  is a Hilbert function of a set of points in  $\mathbb{P}^2$  if and only if  $\Delta H(i) = H(i) - H(i - 1)$  is a valid Hilbert function. More precisely, it



**Definition 2.7** ([8, Definition 1.2.5]). Let  $Z = m_1P_1 + m_2P_2 + \cdots + m_sP_s$  be a fat point scheme in  $\mathbb{P}^2$ . Fix a sequence  $\ell_1, \dots, \ell_n$  of lines in  $\mathbb{P}^2$ , not necessarily distinct.

- (a) Define the fat point schemes  $Z_0, \dots, Z_n$  by  $Z_0 = Z$  and  $Z_j = Z_{j-1} : \ell_j$  for  $1 \leq j \leq n$ . That is,  $Z_j$  is the scheme defined by  $I_{Z_{j-1}} : \langle L_j \rangle$  if  $L_j$  is the linear form defining  $\ell_j$  and  $I_{Z_j}$  is the ideal defining  $Z_j$ .
- (b) The sequence  $\ell_1, \dots, \ell_n$  *totally reduces*  $Z$  if  $Z_n = \emptyset$  is the empty scheme. This statement is equivalent to the property that for each fat point  $m_iP_i$ , there are at least  $m_i$  indices  $\{j_1, \dots, j_{m_i}\}$  such that each  $\ell_{j_k}$  passes through  $P_i$ .
- (c) We associate with  $Z$  and the sequence  $\ell_1, \dots, \ell_n$  an integer vector

$$\mathbf{d} = \mathbf{d}(Z; \ell_1, \dots, \ell_n) = (d_1, \dots, d_n),$$

where  $d_j = \deg(\ell_j \cap Z_{j-1})$ , the degree of the scheme theoretic intersection of  $\ell_j$  with  $Z_{j-1}$ . We refer to  $\mathbf{d}$  as the *reduction vector* for  $Z$  induced by the sequence  $\ell_1, \dots, \ell_n$ . We will say that  $\mathbf{d}$  is a *full reduction vector* for  $Z$  if  $\ell_1, \dots, \ell_n$  totally reduces  $Z$ .

*Remark 2.8.* If  $Z$  is a fat point scheme, and if  $P_{i_1}, \dots, P_{i_j}$  are all the points in the support of  $Z$  that lie on the line  $\ell$ , then  $\deg(\ell \cap Z) = m_{i_1} + \cdots + m_{i_j}$ , i.e., the sum of the multiplicities of the points lying on  $\ell \cap Z$ . The scheme  $Z : \ell$  is the scheme that we obtain by reducing the multiplicities of  $P_{i_1}, \dots, P_{i_j}$  by one (or removing the point if its multiplicity is 1), and leaving the other multiplicities alone.

**Example 2.9.** Consider three non-collinear points  $P_1, P_2, P_3$  and the set of fat points  $Z = 3P_1 + 3P_2 + 2P_3$ . Let  $\ell_1 = \ell_2, \ell_3$  and  $\ell_4$  be the lines through  $P_1P_2, P_1P_3$ , and  $P_2P_3$ , respectively. Then a full reduction vector for this scheme is  $(6, 4, 3, 2)$ . The pictures below show how to build this vector. For example, in Figure 1, the line  $\ell_1$  passes through  $P_1$  and  $P_2$ . Since the multiplicity of  $P_1$  is three, and the same for  $P_2$ , we have  $d_1 = 3 + 3 = 6$ . We then reduce the multiplicity of  $P_1$  and  $P_2$  by one, as in Figure 2. Then  $d_2 = 2 + 2 = 4$ . The

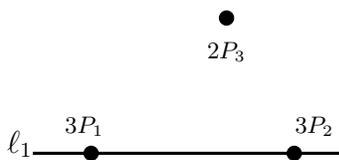


FIGURE 1.

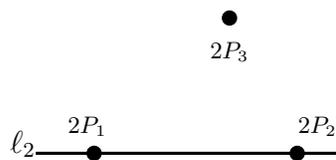


FIGURE 2.

line  $\ell_3$  in Figure 3 passes through one point of multiplicity one and one of multiplicity two, thus giving  $d_3 = 3$ . Note that when we reduce each multiplicity, the point  $P_1$  is removed. In the last step, we use the line  $\ell_4$  to get  $d_4 = 2$  as in Figure 4.

The next result is another specialization of Cooper, Harbourne and Teitler [8].

**Theorem 2.10.** *Let  $Z = Z_0$  be a fat point scheme in  $\mathbb{P}^2$  with full reduction vector  $\mathbf{d} = (d_1, \dots, d_n)$ . If  $d_1 > d_2 > \cdots > d_n$ , then  $H_Z$  only depends on  $\mathbf{d}$ .*

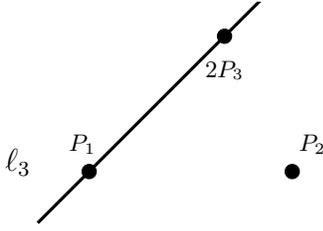


FIGURE 3.

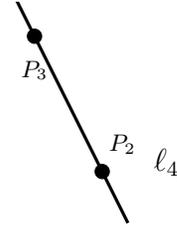


FIGURE 4.

*Proof.* From our assumption, we have  $d_i - d_{i+1} \geq 1$  for all  $i = 1, \dots, n$ . Then  $d_i - d_{i+p} \geq p$  for all  $i$ . This implies that  $\mathbf{d} = (d_1, \dots, d_n)$  is a GMS vector (see Definition 2.2.1 in [8]). Then Theorem 2.2.2 and Section 2.3 of [8] imply

$$H_Z(t) = \sum_{i=0}^{n-1} \binom{\min\{t - i + 1, d_{i+1}\}}{1},$$

that is,  $H_Z$  can be computed directly from  $\mathbf{d}$ .  $\square$

### 3. AN OPERATION THAT PRESERVES THE HILBERT FUNCTION

In this section, we first show that under certain conditions, we can degenerate a fat point scheme  $Z$  consisting of double points and reduced points to make a new fat point scheme  $Z'$  consisting of one additional double point, and three less reduced points. Furthermore, the two schemes will have the same Hilbert function. Note that degeneration techniques have been successfully used in other situations, e.g. see [6, 7].

By repeatedly applying this procedure, we can do the following. Let  $\Delta H$  be a valid Hilbert function of a set of points. Theorem 2.6 implies that there is a set of reduced points  $X$  with Hilbert function  $\Delta H$  that satisfies the hypotheses of our procedure given below. We can then remove three points from  $X$  and add a double point to make a set  $Z$  of fat points with the same Hilbert function as  $\Delta H$ . We can continue this procedure (provided the hypotheses of our procedure are still satisfied) to build sets of fat points consisting of double and reduced points that have Hilbert function  $\Delta H$ . This procedure will then allow us to give an answer to Question 1.2.

We now state and prove the main step in our procedure.

**Theorem 3.1.** *Let  $\ell_1, \dots, \ell_n$  be  $n$  lines in  $\mathbb{P}^2$  such that no three lines meet at a point. Let  $P_{i,j} = \ell_i \cap \ell_j$  for  $1 \leq i < j \leq n$ . Suppose that  $Z$  is a set of double points and reduced points in  $\mathbb{P}^2$  that satisfies the following conditions:*

- (a)  $\text{Supp}(Z) \subseteq \bigcup_{i=1}^n \ell_i$ , i.e., all the points in the support lie on the lines  $\ell_i$ .
- (b) If  $2P$  is a double point of  $Z$ , then  $P = P_{i,j}$  for some  $i < j$ , i.e., all double points of  $Z$  lie at an intersection point of two  $\ell_p$ 's.
- (c) If  $Q$  is a reduced point of  $Z$  and  $Q \in \ell_i$ , then  $Q \neq \ell_i \cap \ell_p$  for  $p \neq i$ , i.e., the reduced points do not lie at an intersection point.
- (d) If

$$d_j = \deg(Z_{j-1} \cap \ell_j)$$

- for  $j = 1, \dots, n$ , then  $d_1 > d_2 > \dots > d_n$ , where  $Z_j = Z_{j-1} : \ell_j$  and we set  $Z_0 = Z$ .  
 (e) There exist  $i, j$  with  $i < j$  such that  $Z$  contains two reduced points  $Q_1, Q_2 \in \ell_i$  and a reduced point  $R \in \ell_j$ , but  $2P_{i,j}$  is not a double point of  $Z$ .

Let  $Z'$  be the set of double and reduced points obtained by adding the double point  $2P_{i,j}$  to  $Z$ , and removing the reduced points  $\{R, Q_1, Q_2\}$ . Then  $Z$  and  $Z'$  have the same Hilbert function.

*Proof.* We begin by observing that  $\text{Supp}(Z')$  is also contained in  $\bigcup_{i=1}^n \ell_i$  by our construction since the only point we added to the support is  $P_{i,j}$ . This observation and (a) imply that the lines  $\ell_1, \dots, \ell_n$  totally reduce both  $Z$  and  $Z'$ . Indeed, if  $Q$  is a reduced point of  $Z$ , respectively  $Z'$ , then it lies on some distinct  $\ell_i$  by (c). If  $2P$  is a double point of  $Z$ , or  $Z'$ , then  $P = \ell_i \cap \ell_j$  for some  $i < j$  by (b) (or by the construction of  $Z'$ ), so there are at least two  $\ell_p$ 's that pass through  $2P$ . It follows from the equivalent statement in Definition 2.7 that the  $\ell_i$ 's totally reduce  $Z$  and  $Z'$ .

To finish the proof, we claim it is enough to show that

$$\deg(Z_{j-1} \cap \ell_j) = \deg(Z'_{j-1} \cap \ell_j) \text{ for all } 1 \leq j \leq n.$$

Indeed, if this fact is true, then part (d) and Theorem 2.10 imply that the Hilbert function of  $Z$  and  $Z'$  are the same.

To verify the claim, we first observe that our change from  $Z$  to  $Z'$  only effects the points on the lines  $\ell_i$  and  $\ell_j$ , and consequently, could only effect the value of  $\deg(Z'_{p-1} \cap \ell_p)$  for  $p = i$  and  $j$ . In the computation of  $\deg(Z_{i-1} \cap \ell_i)$  we get a contribution of two from each reduced point  $Q_1$  and  $Q_2$ . Those two points do not contribute to  $\deg((Z'_{i-1} \cap \ell_i)$  since we have removed them, but the fat point  $2P_{i,j}$  (which is not in  $Z$ ) contributes two to the degree. The other points of  $Z_{i-1}$  and  $Z'_{i-1}$  on  $\ell_i$  remain the same, so they contribute equally to the degree. So  $\deg(Z_{i-1} \cap \ell_i) = \deg(Z'_{i-1} \cap \ell_i)$ .

When we compute  $\deg(Z_{j-1} \cap \ell_j)$  we get a contribution of one from  $R$ . This point does not contribute to  $\deg(Z'_{j-1} \cap \ell_j)$  since it was removed. We, however, get a contribution of one from  $P_{i,j}$ . (The multiplicity of  $P_{i,j}$  was dropped from two to one when we formed  $Z'_i$ .) As we mentioned above, the other points on  $\ell_j$  contribute the same. So, again we have  $\deg(Z_{j-1} \cap \ell_j) = \deg(Z'_{j-1} \cap \ell_j)$ . This completes the proof.  $\square$

*Remark 3.2.* We note that the hypothesis of Theorem 3.1 are sufficient conditions which allow us, for example, to apply the results of [8]. Consider condition (d) on the degrees  $d_i$ . If some of the inequalities do not hold, then the conclusion might be false. Let  $Z$  be a set of five points supported on the union of the lines  $\ell$  and  $\ell'$ , namely two points on the former and three on the latter. Thus  $\Delta H_Z = (1, 2, 2)$ . If we set  $\ell_1 = \ell$  and  $\ell_2 = \ell'$ , we get  $d_1 = 2$  and  $d_2 = 3$ , and condition (c) is not satisfied. Indeed, the resulting set  $Z'$  is such that  $\Delta H_{Z'} = (1, 2, 1, 1)$ . Hence the two Hilbert functions are not equal.

*Remark 3.3.* There are also counterexamples when the degrees  $d_i$  are not all distinct. Consider, for example, the complete intersection of a cubic  $\ell_1 \cup \ell_2 \cup \ell_3$  with the union of two distinct lines. This set of six points lie on a conic, but applying our construction leads to a scheme of one double point and three simple points not lying on a conic. That is,

$d_1 = d_2 = d_3 = 2$ , and our construction does not preserve the Hilbert function in degree two.

*Remark 3.4.* When we construct  $Z'$ , then  $Z'$  will also satisfy the hypotheses (a) – (d) of Theorem 3.1. If  $Z'$  also satisfies (e), then we can add another double point, and so on, until hypothesis (e) is no longer satisfied.

We expand upon the above remark. Given a valid Hilbert function  $\Delta H$  with  $\Delta H^* = (h_1^*, \dots, h_\alpha^*)$ , we present a construction based upon Theorem 3.1 which will allow us to produce a set  $Z$  of double and simple points such that  $\Delta H = \Delta H_Z$ . The rough idea behind our construction is to start with a set of reduced points with the correct Hilbert function, and then, in a controlled fashion, repeatedly replace three reduced points with a double point, and use Theorem 3.1 to show that the Hilbert function does not change after each iteration. Below, we will use the notation  $n_+ = \max\{n, 0\}$ .

### Construction 3.5.

**INPUT:** A valid Hilbert function  $\Delta H$  with  $\Delta H^* = (h_1^*, \dots, h_\alpha^*)$ .

**OUTPUT:** A scheme  $Z$  of double points and reduced points in  $\mathbb{P}^2$  with  $H_Z = H$ .

**STEP 0.** Let  $\ell_1, \dots, \ell_\alpha$  and  $P_{i,j}$  be as in Theorem 3.1. Let  $Z_0$  be a set reduced points of  $\mathbb{P}^2$  with  $H_{Z_0} = H$  as constructed in Theorem 2.6 with  $Z_0 \subseteq \bigcup_{i=1}^\alpha \ell_i$  such that  $|Z_0 \cap \ell_i| = h_i^*$ . Continue to STEP 1.

For  $n \geq 1$ :

**STEP  $n$ .** Set  $\mathbf{h}_n = ((h_n^* - (n-1))_+, \dots, (h_\alpha^* - (n-1))_+)$  and

$$s_n = \#\{k \mid n+1 \leq k \leq \alpha \text{ and } h_k^* \geq n\}.$$

If  $s_n = 0$ , then return  $Z_{n-1}$ . Otherwise, let

$$t_n = \min \left\{ \left\lfloor \frac{(h_n^* - (n-1))_+}{2} \right\rfloor, s_n \right\}.$$

Remove  $2t_n$  points on  $\ell_n$  and one point on each  $\ell_j$  for  $j = n+1, \dots, n+t_n$  from  $Z_{n-1}$ , and add the double points  $2P_{n,j}$  where  $P_{n,j} = \ell_n \cap \ell_j$ , for  $j = n+1, \dots, n+t_n$ . Let  $Z_n$  denote the resulting scheme. Continue to STEP  $n+1$ .

*Proof.* We verify that Construction 3.5 produces the stated output. First, note that the construction will stop at (or before) STEP  $\alpha$  because  $s_\alpha = 0$ .

We now show that the scheme  $Z_n$  of double and reduced points constructed at STEP  $n$  has the property that  $H_{Z_n} = H$ . We proceed by induction on  $n$ . If  $n = 0$ , then the set of reduced points  $Z_0$  has the property  $H_{Z_0} = H$  by Theorem 2.6.

So, now assume that  $n \geq 1$ , and that  $Z_{n-1}$  satisfies the induction hypothesis. We first observe that the value  $(h_i^* - (n-1))_+$  in the tuple  $\mathbf{h}_n = ((h_n^* - (n-1))_+, \dots, (h_\alpha^* - (n-1))_+)$  counts the minimal possible number of reduced points on  $\ell_i$  for  $i = n, \dots, \alpha$  after constructing  $Z_{n-1}$ . This is because in each of STEP 1 through  $n-1$ , we used at most one reduced point on  $\ell_i$  when constructing  $Z_j$  with  $0 \leq j \leq n-1$ .

If  $s_n = 0$ , then our procedure terminates with  $Z_{n-1}$ , which, by induction, is a set of double and reduced points with  $H_{Z_{n-1}} = H$ . Now suppose that  $s_n > 0$ , i.e., there exists

a  $n + 1 \leq k \leq \alpha$  such that  $h_k^* \geq n$ , or equivalently,  $h_k^* - (n - 1) \geq 1$ . In fact, because  $h_1^* > h_2^* > \dots > h_n^* > \dots > h_k^* > \dots > h_\alpha^*$ , we can assume  $h_{n+j}^* - (n - 1) \geq 1$  for  $j = 1, \dots, s_n$ , and also  $h_n^* - (n - 1) \geq 2$ . Consequently,

$$t_n = \min \left\{ \left\lfloor \frac{(h_n^* - (n - 1))_+}{2} \right\rfloor, s_n \right\} \geq 1.$$

From our description of the entries of  $\mathbf{h}_n$ , it follows that we can find  $2t_n$  reduced points on  $\ell_n$  and a reduced point on each of the lines  $\ell_{n+1}, \dots, \ell_{n+t_n}$  in  $Z_{n-1}$ . For each  $j = 1, \dots, t_n$ , we remove two reduced points from  $\ell_n$  and one reduced point from  $\ell_{n+j}$ , and replace these three reduced points with the double  $2P_{n,n+j}$  where  $P_{n,n+j} = \ell_n \cap \ell_{n+j}$ . By repeatedly applying Theorem 3.1, each time we add a new double point and remove the reduced points, the new scheme has the same Hilbert function. Consequently, the scheme  $Z_n$  produced by STEP  $n$  satisfies  $H_{Z_n} = H_{Z_{n-1}} = H$ , as desired.  $\square$

**Example 3.6.** We illustrate Construction 3.5 with the valid Hilbert function  $\Delta H = (1, 2, 3, 4, 2)$ . In this case  $\Delta H^* = (5, 4, 2, 1)$ . Fix four general lines  $\ell_1, \ell_2, \ell_3$ , and  $\ell_4$ , i.e., no three of the lines meet at a point. If we place five points on  $\ell_1$ , four points on  $\ell_2$ , two points on  $\ell_3$ , and one point on  $\ell_4$ , as in the Figure 5, then the set of reduced points  $Z_0$  has Hilbert function  $\Delta H_X = \Delta H$  (by Theorem 2.6). The construction of  $Z_0$  is STEP 0 of Construction 3.5.

For STEP 1, we let  $\mathbf{h}_1 = (5, 4, 2, 1)$  and  $s_1 = 3$ . Since  $s_1 \neq 0$ , we let  $t_1 = \min\{\lfloor \frac{5}{2} \rfloor, 3\} = 2$ . We remove  $2 \cdot 2 = 4$  points from  $\ell_1$ , and 1 point from  $\ell_2$  and 1 point from  $\ell_3$ , and we add the double points  $2P_{1,2}$  and  $2P_{1,3}$  to make the scheme  $Z_1$  as in Figure 6. The double points are denoted with a 2 in the figure. Note that by Theorem 3.1 this scheme has the same Hilbert function as  $Z_0$ . Roughly speaking, we are “merging” two points on  $\ell_1$  with a third point on  $\ell_2$  (or  $\ell_3$ ) to make the double point  $2P_{1,2}$  (or  $2P_{1,3}$ ).

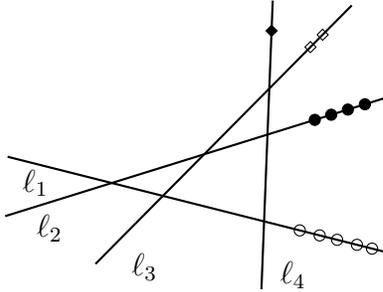


FIGURE 5. The set  $Z_0$  with  $H_{Z_0} = H$

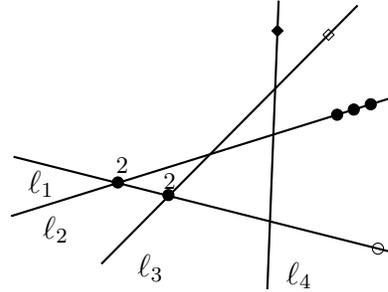


FIGURE 6. The scheme  $Z_1$  with  $H_{Z_1} = H$

Moving to STEP 2, we set  $\mathbf{h}_2 = ((4 - 1)_+, (2 - 1)_+, (1 - 1)_+) = (3, 1, 0)$  and  $s_2 = 1$ . (Note the  $j$ -th entry of  $\mathbf{h}_2$  is a lower bound on the number of reduced points on  $\ell_{j+1}$  for  $j = 2, 3, 4$ .) Because  $s_2 \neq 0$ , we let  $t_2 = \min\{\lfloor \frac{3}{2} \rfloor, 1\} = 1$ . We remove two points from  $\ell_2$  and one point from  $\ell_3$  from  $Z_1$ , but add the double point  $2P_{2,3}$  to form the scheme  $Z_2$ . Again, Theorem 3.1 implies that this construction of double and reduced points has the same Hilbert function as  $Z_1$ , and consequently,  $Z$ . See Figure 7 for an illustration of  $Z_2$ .

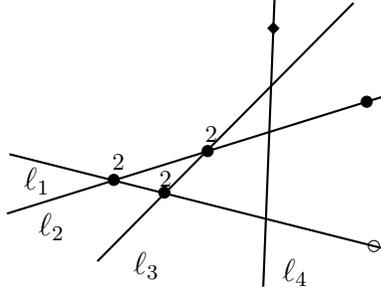


FIGURE 7. The scheme  $Z_2$   
with  $H_{Z_2} = H$

Finally, at STEP 3 we have  $\mathbf{h}_3 = ((2 - 2)_+, (1 - 2)_+) = (0, 0)$  and  $s_3 = 0$ . Our construction terminates and returns the scheme  $Z_2$ , which consists of three double points and three reduced points, and  $\Delta H_{Z_2} = \Delta H = (1, 2, 3, 4, 2)$ .

**Example 3.7.** Construction 3.5 may not produce a set of just double points, even if  $\Delta H$  is known to be the valid Hilbert function of a set of double points. For example, consider the scheme  $Z$  of two double points. We have  $\Delta H_Z = (1, 2, 2, 1)$ . So,  $\Delta H = (1, 2, 2, 1)$  is a valid Hilbert function of a set of double points. However, Construction 3.5 cannot be used to detect this fact. In particular, since  $\Delta H^* = (4, 2)$  Construction 3.5 returns only one double point (and two simple points on  $\ell_1$ , and an other simple point on  $\ell_2$ ).

**Example 3.8.** Consider the valid Hilbert function  $\Delta H = (\underbrace{1, \dots, 1}_{\sigma+1})$ . Construction 3.5 applied to this  $\Delta H$  stops at the beginning of STEP 1 by producing  $\sigma + 1$  reduced points on a line  $\ell_1$ . A valid Hilbert function of this type is the only time Construction 3.5 terminates at STEP 1. Note that it can be shown that if  $\Delta H = (1, \dots, 1)$ , then the only zero-dimensional scheme  $Z$  with  $\Delta H_Z = \Delta H$  is precisely a set of reduced points on a line. If  $\Delta H = (1, 2, \dots)$ , then Construction 3.5 will produce a scheme  $Z$  with at least one double point.

As noted in the last remark, if  $\Delta H = (1, 2, \dots)$ , then our procedure produces a scheme with at least one double point. We are actually interested in producing a scheme with the largest possible number of double points. We can determine the number of double points produced by Construction 3.5 directly from  $\Delta H$ , as shown in the next result which gives an answer to Question 1.2.

**Theorem 3.9.** *Let  $\Delta H$  be a valid Hilbert function with  $\Delta H^* = (h_1^*, \dots, h_\alpha^*)$ . Set*

$$d = \sum_{i=1}^{\alpha-1} \min \left\{ \left\lfloor \frac{(h_i^* - (i-1))_+}{2} \right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^* \geq i\} \right\}.$$

*Then there is a set of fat points  $Z$  of  $d$  double points and  $r = (\sum \Delta H) - 3d$  reduced points with  $\Delta H_Z = \Delta H$ .*

*Proof.* We first note that  $s_i$  in Construction 3.5 equals  $\#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^* \geq i\}$ . Also, we have  $s_1 \geq s_2 \geq \dots \geq s_{\alpha-1} \geq s_\alpha = 0$ . Let  $j = \max\{i \mid s_i \neq 0\}$ . So, for

$i = 1, \dots, j,$

$$\min \left\{ \left\lfloor \frac{(h_i^* - (i-1))_+}{2} \right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^* \geq i\} \right\}.$$

is the number of double points added to  $Z_{i-1}$  at STEP  $i$ . For  $i = j+1, \dots, \alpha$ , Construction 3.5 has terminated, and no new double points are added. This fact is captured in the summation via the fact that all the summands

$$\min \left\{ \left\lfloor \frac{(h_i^* - (i-1))_+}{2} \right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^* \geq i\} \right\} = 0$$

for  $i = j+1, \dots, \alpha$ . □

We record some corollaries:

**Corollary 3.10.** *Let  $\Delta H = (1, 2, \dots, \alpha, h_\alpha, \dots, h_\sigma)$  be a valid Hilbert function, and let  $d$  be as in Theorem 3.9.*

- (i) *The integer  $d$  satisfies  $\min \{ \lfloor \frac{\sigma+1}{2} \rfloor, \alpha - 1 \} \leq d \leq \binom{\alpha}{2}$ .*
- (ii) *For every integer  $1 \leq e \leq d$ , there exists a scheme  $Z$  of  $e$  double points and  $(\sum \Delta H) - 3e$  reduced points with  $H_Z = H$ .*
- (iii) *Let  $e = \min \{ \lfloor \frac{\sigma+1}{2} \rfloor, \alpha - 1 \}$ . Then there exists a scheme  $Z$  with  $e$  double points and  $(\sum \Delta H) - 3e$  reduced points.*

*Proof.* For (i), the support of each double point produced by Construction 3.5 has the form  $\ell_i \cap \ell_j$ . Since there are only  $\alpha$  lines  $\ell_1, \dots, \ell_\alpha$  used in this construction, the outputted scheme can have at most  $\binom{\alpha}{2}$  double points, thus giving the upper bound. For the lower bound, note that in the computation of  $d$  using Theorem 3.9, when the index is  $i = 1$ , we have  $h_1^* = \sigma + 1$  and  $\alpha - 1 = \#\{2 \leq k \leq \alpha \text{ and } h_k^* \geq 1\}$ , i.e., the lower bound is the first term in the sum.

The proof of (iii) will follow from (i) and (ii) since  $e \leq d$ . For (ii), notice that Construction 3.5 adds one double point at a time, and when it finishes, we have  $d$  double points. Since  $e \leq d$ , we use this procedure again, but changing our stopping criterion so the procedure terminates when we have  $e$  double points. □

Recall that one of the fundamental problems about Hilbert functions of double points in  $\mathbb{P}^2$  is to classify what functions are the Hilbert functions of double (or more generally, fat) points. We can now contribute to this problem by identifying some new functions as the Hilbert functions of double points.

**Theorem 3.11.** *Let  $\Delta H$  be a valid Hilbert function, and let  $\Delta H^* = (h_1^*, \dots, h_\alpha^*)$ . Suppose that*

$$\frac{(h_i^* - (i-1))_+}{2} = \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^* \geq i\} \text{ for } i = 1, \dots, \alpha - 1.$$

*Then  $\Delta H$  is Hilbert function of a set of double points in  $\mathbb{P}^2$ .*

*Proof.* We prove this by showing that Construction 3.5 terminates with a scheme of only double points. At STEP 1, observe that the vector  $\mathbf{h}_1 = (h_1^*, \dots, h_\alpha^*)$  has the property

that  $h_i^*$  is the exact number of reduced points on  $\ell_i$ . The hypotheses imply that when Construction 3.5 executes STEP 1, the value of  $t_1$  is given by

$$t_1 = \frac{(h_1^*)_+}{2} = s_1.$$

But this means that all the reduced points on  $\ell_1$  and one reduced point on each of  $\ell_{1+1}, \dots, \ell_{1+s_1=\alpha}$  are removed to form  $2t_1$  double points. In particular, no reduced points are left on  $\ell_1$  and  $\mathbf{h}_2 = ((h_2^* - 1)_+, \dots, (h_\alpha^* - 1)_+)$  has the property that  $(h_i - 1)_+^*$  is exactly the number of reduced points remaining on  $\ell_i$ . (In the general, case,  $(h_i^* - 1)_+$  is only a lower bound for the number of reduced points that remain).

Proceeding by induction on  $n$ , note that at STEP  $n$ , the tuple  $\mathbf{h}_n = ((h_n^* - (n - 1))_+, \dots, (h_\alpha^* - (n - 1))_+)$  counts exactly the number of reduced points on  $\ell_i$  for  $i = n, \dots, \alpha$  after constructing  $Z_{n-1}$ . This is because in each of STEP 1 through  $n - 1$ , we used at exactly one reduced point on  $\ell_i$  when constructing  $Z_j$  with  $0 \leq j \leq n - 1$ . The hypotheses imply when Construction 3.5 executes STEP  $n$  we get

$$t_n = \frac{(h_n^* - (n - 1))_+}{2} = s_n.$$

So, all the remaining reduced points on  $\ell_n$  and one reduced point on each of  $\ell_{n+1}, \dots, \ell_{n+s_n}$  are removed to form  $2t_n$  double points. In particular, no reduced points are left on  $\ell_n$  and  $\mathbf{h}_{n+1} = ((h_{n+1}^* - n)_+, \dots, (h_\alpha^* - (n))_+)$  has the property that  $(h_i - n)_+^*$  is the exactly the number of reduced points remaining on  $\ell_i$ .

When the algorithm terminates,  $\mathbf{h}_n = (0, \dots, 0)$ , that is, no reduced points remain.  $\square$

**Example 3.12.** Consider the valid Hilbert function

$$\Delta H = (1, 2, 3, 4, 5, 6, 2, 2, 1, 1) \Leftrightarrow \Delta H^* = (10, 7, 4, 3, 2, 1).$$

Then  $\Delta H^*$  satisfies the hypothesis of Theorem 3.11 since

$$\begin{aligned} \frac{h_1^*}{2} &= 5 = \#\{k \mid 2 \leq k \leq 6 \text{ and } h_k^* \geq 1\} \\ \frac{(h_2^* - 1)_+}{2} &= 3 = \#\{k \mid 3 \leq k \leq 6 \text{ and } h_k^* \geq 2\} \\ \frac{(h_3^* - 2)_+}{2} &= 1 = \#\{k \mid 4 \leq k \leq 6 \text{ and } h_k^* \geq 3\} \\ \frac{(h_i^* - (i - 1))_+}{2} &= 0 = \#\{k \mid i + 1 \leq k \leq 6 \text{ and } h_k^* \geq i\} \text{ for } i \geq 4. \end{aligned}$$

Then  $\Delta H$  is the Hilbert function of  $(\sum \Delta H)/3 = 9$  double points. Indeed, if  $\ell_1, \dots, \ell_6$  are six general lines such that no three meet at a point, Construction 3.5 will produce the scheme of nine double points

$$Z = 2P_{1,2} + 2P_{1,3} + 2P_{1,4} + 2P_{1,5} + 2P_{1,6} + 2P_{2,3} + 2P_{2,4} + 2P_{2,5} + 2P_{3,4}$$

where  $P_{i,j} = \ell_i \cap \ell_j$  with Hilbert function  $H_Z = H$ .

The following corollary is used in the next section.

**Corollary 3.13.** *Let  $\Delta H$  be a valid Hilbert function with  $\Delta H^* = (h_1^*, \dots, h_\alpha^*)$ . If*

$$\frac{h_i^* - (i - 1)}{2} = \alpha - i \text{ for } i = 1, \dots, \alpha,$$

*then there exists a scheme  $Z$  in  $\mathbb{P}^2$  of only double points with  $\Delta H = \Delta H_Z$ .*

*Proof.* We have

$$\Delta H^* = \underbrace{(2\alpha - 2, \dots, \alpha + 1, \alpha, \alpha - 1)}_\alpha.$$

Now apply Theorem 3.11. □

#### 4. SPECIAL CONFIGURATIONS

In this section, we will examine two valid Hilbert functions:

$$\Delta H_1 = \underbrace{(1, 2, 3, \dots, t)}_t, \underbrace{(t + 1, \dots, t + 1)}_t, \text{ and } \Delta H_2 = \underbrace{(1, 2, 3, \dots, t)}_t, \underbrace{(t + 1, \dots, t + 1, 1)}_t.$$

In the first case, we will show that only the sets  $Z$  of  $\binom{t}{2}$  double points with support on a star configuration have  $\Delta H_Z = \Delta H_1$ . In the second case, we will show that the only sets  $Z$  of  $\binom{t}{2}$  double points and one simple point that have  $\Delta H_Z = \Delta H_2$  are sets of double points with support on a star configuration plus one extra point on one of the lines that define the star configuration.

We start by collecting together some required tools. The first result we need is the following theorem (see [3] or [9]).

**Theorem 4.1.** *Let  $X \subset \mathbb{P}^2$  be a zero-dimensional subscheme, and assume that there is a  $t$  such that  $\Delta H_X(t - 1) = \Delta H_X(t) = d$ . Then the degree  $t$  components of  $I_X$  have a GCD, say  $F$ , of degree  $d$ . Furthermore, the subscheme  $W$  of  $X$  lying on the curve defined by  $F$  (i.e.,  $I_W$  is the saturation of the ideal  $(I_X, F)$ ) has Hilbert function whose first difference is given by the truncation  $\Delta H_W(i) = \min\{\Delta H_X(i), d\}$ .*

Now we recall some well known results about star configurations. Given any linear form  $L \in R$ , we let  $\ell$  denote the corresponding line in  $\mathbb{P}^2$ . Let  $\ell_1, \dots, \ell_{t+1}$  be a set of  $t + 1$  distinct lines in  $\mathbb{P}^2$  that are three-wise linearly independent (general linear forms). In other words, no three lines meet at a point. A *star configuration* of  $\binom{t+1}{2}$  points in  $\mathbb{P}^2$  is formed from all pairwise intersections of the  $t + 1$  linear forms.

Geramita, Harbourne, and Migliore have computed the Hilbert function of double points whose support is a star configuration. Specifically,

**Theorem 4.2** ([10, Theorem 3.2]). *In  $\mathbb{P}^2$ , let  $t$  be a positive integer, let  $X$  be a star configuration of  $\binom{t+1}{2}$  points, and let  $Z = 2X$  be a set of double points whose support is  $X$ . Then the first difference of the Hilbert function of  $Z$  is*

$$\Delta H_Z(i) = \begin{cases} i + 1 & \text{if } 0 \leq i \leq t \\ t + 1 & \text{if } t + 1 \leq i \leq 2t - 1 \\ 0 & \text{if } i \geq 2t. \end{cases}$$

*Remark 4.3.* We point out that the above result can be generalized to  $\mathbb{P}^n$ .

4.1. **Double points on a star configuration.** We consider the valid Hilbert function

$$\Delta H = \underbrace{(1, 2, 3, \dots, t)}_t, \underbrace{(t+1, \dots, t+1)}_t.$$

By Corollary 3.13 with  $t = \alpha - 1$ , we already know in this case that Construction 3.5 gives a set of only double points. In this subsection, we will prove that the configuration produced by our construction is a set of double points lying on star configuration, and furthermore, this is the only set of double points whose first difference is equal to  $\Delta H$ . Since for  $t \leq 2$  all is trivial, we will assume that  $t \geq 3$ .

**Theorem 4.4.** *Let  $t \geq 3$  be an integer, and let  $Z \subset \mathbb{P}^2$  be a set of  $\binom{t+1}{2}$  double points. Then the sequence*

$$(4.1) \quad \Delta H = \underbrace{(1, 2, 3, \dots, t)}_t, \underbrace{(t+1, \dots, t+1)}_t$$

*is the first difference of the Hilbert function of  $Z$  if and only if  $Z$  is a set of double points whose support is a star configuration.*

*Proof.* ( $\Leftarrow$ ) This follows from Theorem 4.2.

( $\Rightarrow$ ) Suppose that  $Z$  is a set of  $\binom{t+1}{2}$  double points such that the first difference of Hilbert function is given by (4.1), i.e.,

$$(4.2) \quad \Delta H_Z = \begin{array}{cccccccc} & & & & \bullet & \bullet & \bullet & \dots & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ & & & & \vdots & & & & & \\ & & & & \bullet & \dots & \bullet & \bullet & \bullet & \dots & \bullet \\ & & & \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \dots & \bullet \\ & & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \dots & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \dots & \bullet \\ & & 0 & 1 & 2 & 3 & & t & & & 2t-1 \end{array} \quad t+1 \text{ rows}$$

We want to prove that the support of  $Z$  must be a star configuration constructed from  $t+1$  general lines.

Let  $I_Z$  be the ideal of  $Z$ . We shall sometimes refer to  $(I_Z)_t$  as *the linear system of all the plane curves of degree  $t$  containing  $Z$* , since this is what the forms in  $(I_Z)_t$  correspond to from a geometrical point of view. From (4.1) we note that the smallest degree in a minimal set of generators of  $I_Z$  is  $t+1$ , the largest degree is  $2t$ , and there is only one curve, say  $\mathcal{C} = \{F = 0\}$ , in the linear system defined by  $(I_Z)_{t+1}$ . Moreover  $I_Z$  does not have new minimal generators until the degree  $2t$ .

Because  $(I_Z)_{2t}$  contains all the forms of type  $F \cdot (x, y, z)^{t-1}$ , then the linear system  $(I_Z)_{2t}$  cannot be composed with a pencil, see [18, pg. 26]. We recall that a linear system is composed with a pencil if any of its elements is of the type  $\phi_1 \cdot \phi_2 \cdots \phi_n$  where the forms  $\phi_i$  are of the type  $c_1\psi_1 + c_2\psi_2$  for scalars  $c_i$  and forms  $\psi_i$ . Moreover, since  $I_Z$  is generated in degrees  $\leq 2t$ , the base locus of  $(I_Z)_{2t}$  is exactly  $Z$ . Hence,  $(I_Z)_{2t}$  has no fixed components. By Bertini's Theorem (for example, see [15]), the general curve of

the linear system  $(I_Z)_{2t}$  is integral (irreducible and reduced). Thus we may assume that  $(I_Z)_{2t} = (G_1, \dots, G_r)$  where  $r = \dim_k(I_Z)_{2t}$  and each  $\mathcal{G}_i = \{G_i = 0\}$  is an integral curve.

For each curve  $\mathcal{G}_i$ , consider the intersection  $\mathcal{G}_i \cap \mathcal{C}$ . Since each  $\mathcal{G}_i$  is integral and  $\deg \mathcal{G}_i > \deg \mathcal{C}$ , we have  $\deg(\mathcal{G}_i \cap \mathcal{C}) = \deg \mathcal{G}_i \cdot \deg \mathcal{C}$ . Now each point of  $Z$  is a double point of both  $\mathcal{G}_i$  and  $\mathcal{C}$ . So, the degree of the scheme  $\mathcal{G}_i \cap \mathcal{C}$  at each point of  $Z$  is at least 4. Hence  $\deg(\mathcal{G}_i \cap \mathcal{C}) \geq 4 \binom{t+1}{2} = 2t(t+1) = \deg \mathcal{G}_i \cdot \deg \mathcal{C} = \deg(\mathcal{G}_i \cap \mathcal{C})$ . It follows that  $\mathcal{G}_i$  and  $\mathcal{C}$  only intersect at the points of  $Z$ , and that for each point  $P$  in the support of  $Z$ , the degree of the scheme  $\mathcal{G}_i \cap \mathcal{C}$  at  $P$  is exactly 4.

Now observe that the curve  $\mathcal{C}$  is not integral. In fact,  $\mathcal{C}$  has  $\binom{t+1}{2}$  double points, but an integral curve of degree  $t+1$  has at most  $\binom{t}{2}$  double points.

We claim that  $\mathcal{C}$  totally reduces by distinct lines, that is, if  $a\mathcal{L}$  is an irreducible component of  $\mathcal{C}$  (i.e., the polynomial defining  $\mathcal{L}$  is irreducible) of multiplicity  $a$ , we will show that  $\mathcal{L}$  is a line and  $a = 1$ .

Let  $P_1$  be a general point on  $\mathcal{L}$ . Since  $F$  vanishes on  $P_1$ , the first difference of the Hilbert function of  $Z + P_1$  is the following

$$\Delta H_{Z+P_1}(i) = \begin{cases} i+1 & \text{if } 0 \leq i \leq t \\ t+1 & \text{if } t+1 \leq i \leq 2t-1 \\ 1 & \text{if } i = 2t \\ 0 & \text{if } i > 2t. \end{cases}$$

To see why this is the case, when we add a point to  $Z$ , we have to add a point to the diagram in (4.2). There are only two places to put a point and maintain a valid Hilbert function: 1) put a point where  $i = t+1$  or 2) put a point where  $i = 2t$ . In the first case we get  $\Delta H_{Z+P_1}(t+1) = t+2$ , and so we do not have curves in the linear system defined by  $(I_Z)_{t+1}$ . But this is a contradiction since  $P_1$  is a point of  $\mathcal{C}$ . So 2) must hold. We will prove that  $\mathcal{L}$  is a common component for the curves of the linear system  $(I_{Z+P_1})_{2t}$ .

Recall that each  $\mathcal{G}_i$  intersects  $\mathcal{L}$  only at the points of  $Z$ . If  $d$  denotes the degree of  $\mathcal{L}$ , then the degree of  $\mathcal{G}_i \cap a\mathcal{L}$  is  $2tad$ . Now, consider a curve  $\mathcal{T} = \{T = 0\}$  with  $T \in (I_{Z+P_1})_{2t}$ . We have that  $\deg(\mathcal{T} \cap a\mathcal{L}) \geq 2tad + 1$ . However  $\deg \mathcal{T} = 2t$  and  $\deg a\mathcal{L} = ad$ . So, by Bezout's Theorem,  $\mathcal{L}$  is a common component for every curve of the linear system  $(I_{Z+P_1})_{2t}$ .

Now look at  $Z + P_1 + P_2$  where  $P_2$  is another general point on  $\mathcal{L}$ . Since  $\mathcal{L}$  is a common component for every curve of  $(I_{Z+P_1})_{2t}$ , then the first difference of the Hilbert function of  $Z + P_1 + P_2$  is given by

$$\Delta H_{Z+P_1+P_2}(i) = \begin{cases} i+1 & \text{if } 0 \leq i \leq t \\ t+1 & \text{if } t+1 \leq i \leq 2t-1 \\ 1 & \text{if } i = 2t \\ 1 & \text{if } i = 2t+1 \\ 0 & \text{if } i > 2t+1. \end{cases}$$

So, using Theorem 4.1 with  $d = 1$ , we get that  $Z + P_1 + P_2$  has a subscheme of degree  $2t+2$  lying on a line  $\ell$ . But  $Z$  imposes independent conditions to the curves of degree  $2t-1$ , hence  $Z$  has at most  $t$  double points with support on a line, and so  $P_1, P_2 \in \ell$  and  $\deg(\mathcal{C} \cap \ell) = 2t+2$ . Since  $2t+2 > t+1 = \deg \mathcal{C} \cdot \deg \ell$ , then the line  $\ell$  is a component of  $\mathcal{C}$ . Now observe that  $P_1$  and  $P_2$  are generic points on  $\mathcal{L}$ , so  $\mathcal{L}$  must be the line  $\ell$ .

It follows that every irreducible component of  $\mathcal{C}$  is a line, and thus

$$\mathcal{C} = a_1\ell_1 + \cdots + a_v\ell_v,$$

where the  $\ell_i$  are lines and  $a_1 + \cdots + a_v = t + 1$ . Now we will show that  $a_i = 1$  for all  $i$ , that is,  $\mathcal{C}$  is a union of  $t + 1$  distinct lines. First observe that no  $a_i$  can be bigger than 2. Indeed, if  $a_i > 2$ , then the curve  $\mathcal{C} \setminus \ell_i$  would be a curve of degree  $t$  containing  $Z$ ; this contradicts the fact that  $\mathcal{C}$  is the curve of minimal degree containing  $Z$ . Hence, by this observation, or simply by recalling that  $\deg(\mathcal{G}_i \cap \mathcal{C})$  in every  $P \in Z$  is exactly 4, the irreducible components of  $\mathcal{C}$  are simple or double lines. After relabeling, we can assume

$$\mathcal{C} = 2\ell_1 + \cdots + 2\ell_s + \ell_{s+1} + \cdots + \ell_{2s+r},$$

where  $2s + r = t + 1$ .

We observe that the points of  $Z$  lying on the simple lines can lie only on the intersection with other simple lines. So there are at most  $\binom{r}{2}$  such points. Since  $\deg(\mathcal{G}_i \cap \mathcal{C}) = 4$  for every  $P \in Z$ , the points of  $Z$  on the double lines cannot lie on the intersections with other lines. Moreover, since  $Z$  imposes independent conditions to the curves of  $(I_Z)_{2t}$ , on each line  $\mathcal{L}_i$  we have at most  $t$  double points, and so the number of points of  $Z$  on the double lines is at most  $st$ . It follows that at most  $st + \binom{r}{2}$  points of  $Z$  lie on  $\mathcal{C}$ , that is,

$$|Z| = \binom{t+1}{2} \leq st + \binom{r}{2}.$$

Because  $t + 1 = 2s + r$ , we have

$$\binom{2s+r}{2} \leq s(2s+r-1) + \binom{r}{2},$$

and from here we get  $rs = 0$ . If  $r = 0$ , then we get

$$\mathcal{C} = 2\ell_1 + \cdots + 2\ell_s \quad \text{where} \quad 2s = t + 1,$$

and  $Z$  must have  $t$  points on each line  $\ell_i$ . By Bezout's Theorem, it follows that the curves of degree  $2t - 1$  through  $Z$  have the lines  $\ell_i$  as fixed components. Removing these  $s$  lines from  $Z$  we remain with a scheme  $Z'$  of  $|Z| = \binom{t+1}{2}$  simple points and we get

$$\dim_k(I_Z)_{2t-1} = \dim_k(I_{Z'})_{2t-1-s} \geq \binom{2t-1-s+2}{2} - \binom{t+1}{2} = \frac{5t^2 - 4t - 1}{8}.$$

But  $(I_Z)_{2t-1}$  is not defective, hence

$$\dim_k(I_Z)_{2t-1} = \binom{2t-1+2}{2} - 3\binom{t+1}{2} = \frac{t^2-t}{2}.$$

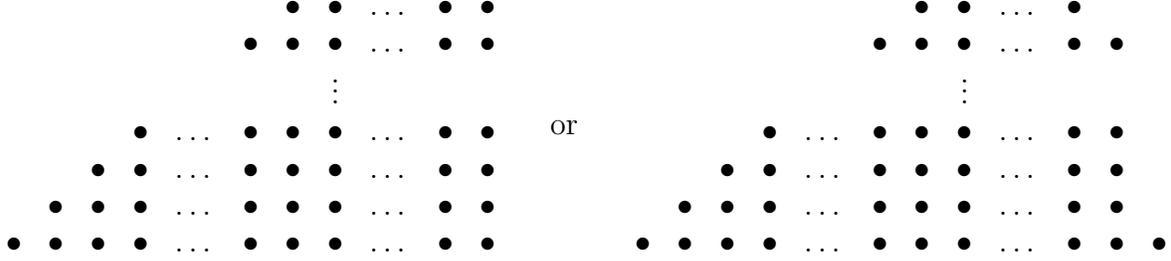
But  $\frac{t^2-t}{2} \not\geq \frac{5t^2-4t-1}{8}$  for any  $t$ , so we get a contradiction. Therefore,  $s = 0$  and  $\mathcal{C}$  is a union of  $t + 1$  distinct lines. It follows that the support of  $Z$  is a star configuration of  $t + 1$  lines.  $\square$

*Remark 4.5.* It is easy to see that if  $\Delta H_Z$  is of type (4.1), Construction 3.5 gives a set of double points on a star configuration. For instance, if  $\Delta H = (1, 2, 3, 4, 4, 4)$ , then  $\Delta H^* = (6, 5, 4, 3)$ . Step 0 and the final step of Construction 3.5 are given in Figure 8,



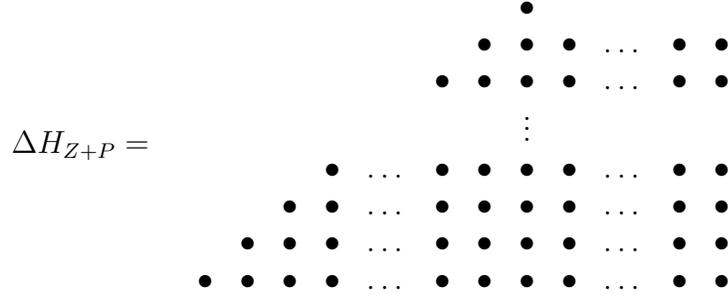
Consider only the scheme  $Z$ . The first difference of the Hilbert function of  $Z$  can only have one of the following two forms:

(4.4)



To see why, note that  $\Delta H_Z$  is constructed from  $\Delta H_{Z+P}$  by removing exactly one point. The two cases represent the only two ways to remove a point from (4.3) and still have a valid Hilbert function.

If  $\Delta H_Z$  is of the first type, then by Theorem 4.1, the support of  $Z$  is a star configuration and the curve  $\mathcal{C}$  is given by the product of the  $t + 1$  lines of the star configuration. If  $P$  does not lie on a line of the star configuration, then  $P \notin \mathcal{C}$ , and thus  $(I_{Z+P})_{t+1} = 0$ . Hence the first difference Hilbert function of  $Z + P$  would have type



which is different from (4.3). So  $P$  lies on a line of  $\mathcal{C}$ , and the conclusion follows.

It suffices to show that the second case cannot occur. So assume for a contradiction that  $\Delta H_Z$  is given by the second diagram in (4.4). Note that the smallest degree in a minimal set of generators of  $I_Z$  is  $t + 1$  and there is only one curve, say  $\mathcal{C} = \{F = 0\}$ , in the linear system defined by  $(I_Z)_{t+1}$ .

Now we will show that the linear system  $(I_{Z+P})_{2t}$  has no fixed components. Suppose for a contradiction that  $\mathcal{T}$  is a fixed irreducible component of  $(I_{Z+P})_{2t}$  and let  $Q \in \mathcal{T}$  be a generic point. Since  $\dim_k(I_{Z+P+Q})_{2t} = \dim_k(I_{Z+P})_{2t}$  we have

$$\Delta H_{Z+P+Q}(i) = \begin{cases} i + 1 & \text{if } i \leq t \\ t + 1 & \text{if } t + 1 \leq i \leq 2t - 1 \\ 1 & \text{if } i = 2t \\ 1 & \text{if } i = 2t + 1 \\ 0 & \text{if } i \geq 2t + 2. \end{cases}$$

By Theorem 4.1 with  $d = 1$ , we have that  $Z + P + Q$  has a subscheme  $W$  of degree  $2t + 2$  lying on a line  $\ell$ . But  $W$  cannot be contained in  $Z$ , since  $(I_Z)_{2t}$  cannot have a scheme of degree  $2t + 2$  on a line. So the scheme  $W$  is the intersection of  $\ell$  with  $t$  points of  $Z$  plus



$\dim_k(I_Q)_1 = 2$ . Since  $X$  imposes independent conditions to the curve of degree  $t - 1$  we have

$$\dim_k(I_X)_{t-1} = \binom{t-1+2}{2} - t - 1 = \binom{t}{2} - 1,$$

which is the expected dimension of  $(I_{Z+P})_{2t-1}$ , and we are done.

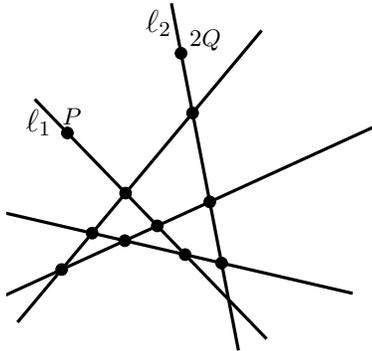


FIGURE 10. All but one double point on a star configuration plus one double point and one simple point

## 5. FINAL REMARKS

In this paper we presented an algorithm that, given a valid Hilbert function  $H$  for a zero-dimensional scheme, will produce a scheme consisting of double and simple points having Hilbert function  $H$ .

We know that for some special  $H$  (for example, see Theorem 3.11) our algorithm will produce a set consisting of *only* double points. Furthermore, in Section 4 we showed that for one family of valid Hilbert functions  $H$ , not only does our algorithm produce a scheme with the maximal possible number of double points, our algorithm produces the only possible configuration of double points with this Hilbert function.

However, there are  $H$  for which our algorithm does not perform well. Consider, for example, when  $H$  is the Hilbert function of double points with collinear support; our algorithm will produce a set with just *one* double point! Thus it is natural to ask how well our algorithm performs.

The major obstacle in answering this question is the following: given a Hilbert function  $H$  of a degree  $3t$  zero-dimensional scheme, we do not know the maximal number of double points that the scheme can possess. Of course  $t$  gives an upper bound, but this bound might not be sharp. Ideally we would like to compare this unknown number with the number of double points that our algorithm produces for  $H$  and possibly make some asymptotic estimate.

This problem will be the object of further investigations, but we can already present an interesting result. Consider the scheme consisting of  $t$ , generic double points and let  $H$  be its Hilbert function. It is well known (e.g., see [2]) that, with the exceptions  $t = 2$  and

$t = 5$ ,  $H(i) = \min \left\{ \binom{i+2}{2}, 3t \right\}$ . The following result describes the asymptotic behavior of our algorithm for this  $H$ .

**Proposition 5.1.** *Let  $H$  be the Hilbert function of  $t$  generic double points, and let  $s(t)$  be the number of double points produced by our algorithm with input  $H$ . Then*

$$\lim_{t \rightarrow +\infty} \frac{s(t)}{t} = \frac{3}{4}.$$

*Proof.* For each positive integer  $t$ , we choose  $b$  and  $0 \leq \epsilon \leq b + 1$  such that  $3t = \binom{b+2}{2} + \epsilon$ ; note that  $b$  is uniquely determined by  $t$  and viceversa. We consider the case  $b$  odd, and a similar argument applies in the case  $b$  even. With this notation we have that, for  $t \geq 6$ ,

$$\Delta H = (1, 2, \dots, b + 1, \epsilon).$$

Moreover, it is easy to see that, applying our algorithm to  $\Delta H$  produces the same result when applying our algorithm to the length  $b + 1$  sequence

$$\Delta H_1 = (1, 2, \dots, \frac{b+1}{2}, \underbrace{\frac{b+3}{2}, \dots, \frac{b+3}{2}}_{\frac{b+1}{2}}).$$

As shown in Section 4, we obtain a set of

$$s(b) = \frac{1}{8}(b + 1)(b + 3)$$

double points. By a change of variables and using the bound  $\epsilon \leq b + 1$  we get

$$\lim_{t \rightarrow +\infty} \frac{s(t)}{t} = \lim_{d \rightarrow +\infty} \frac{s(b)}{\frac{1}{3} \binom{b+2}{2} + \frac{\epsilon}{3}} = \frac{3}{4},$$

and this completes the proof. □

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