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THE PLATEAU PROBLEM IN THE CALCULUS OF VARIATIONS

Abstract. This is a survey paper written for a course held for the Ph. D. program in Pure and Applied Mathematics at Politecnico di Torino during autumn 2018. The course has been dedicated to an overview of the main techniques for solving the Plateau problem, that is to find a surface with minimal area that spans a given boundary curve in the space. This problem dates back to the physical experiments of Plateau who tried to understand the possible configurations of soap films. From the mathematical point of view, the problem is very hard and a lot of possible formulations are available: perhaps still today none of these answers is the answer to the original formulation by Plateau. In this paper, first of all we will briefly introduce the problem showing that, at least in the smooth case, if the first variation of the area vanishes then the surface must have zero mean curvature. Then we will describe how the classical solution by Douglas and Radó works, and we will pass to modern formulations of the problem in the context of Geometric Measure Theory: sets of finite perimeter, currents, and minimal sets.

1. Introduction

The original formulation of the problem might be the following one: *given a closed curve* Γ *in the space find a surface with minimal area spanning* Γ. The Italian mathematician J.-L. Lagrange (1736-1813) was the first, around the year 1760, that investigated the problem, but today this problem is known as *Plateau problem* since the Belgian physicist J. Plateau (1801-1833), in the middle of the 19th century, devised many illustrative soap films experiments putting wires in a soap solution. The connection

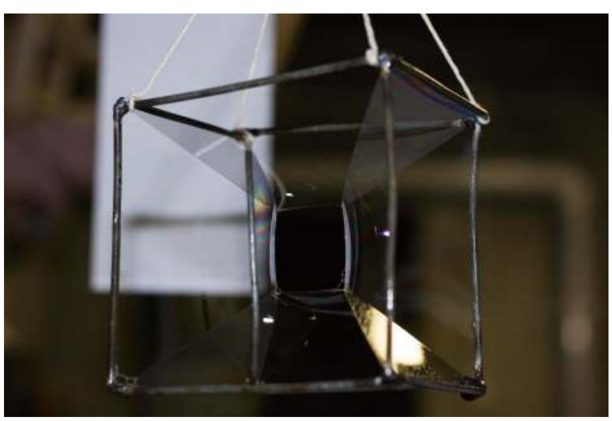

Figure 1: A soap film created by the edges of a cube.

between soap films and minimal surfaces was established by C.F. Gauss (1777-1855) who worked, in 1830, on capillarity problems. Precisely, he found that at the equilibrium any liquid surface is a minimizer of the potential energy caused by the molecular forces. For soap films such an energy is proportional to the area. In other words, soap films can be viewed as physical models of stable minimal surfaces. Motivated by experiments, Plateau conjectured that every closed curve (without double points) spans a surface which minimizes the area, as every closed wire seemed to span some soap film. The aim of this survey is to present some of the established solutions of the Plateau problem. We will also take into account the generalization to higher dimension and/or higher codimension: find a d-dimensional surface with minimal d-dimensional volume spanning a $(d-1)$-dimensional boundary Γ. In this problem there are a lot of ingredients that need to be clarified. For instance, we have to say what *surface* means, that is at which level of generality we might work. Next, for a given notion of surface what do we mean by *d-dimensional volume*? Again, what does it mean *spanning a given boundary?* Depending on the meaning of these objects, the Plateau problem will admit a suitable framework and, possibly, a solution. Here we are interested only in the existence of solutions for the Plateau problem; we will not enter in details about uniqueness.

2. Minimal surfaces equation and first examples

In this section we deal only with the smooth case. Precisely, we review some facts of smooth differential geometry, we recall how to compute the area of a smooth surface, and we prove that a smooth surface with minimal area has zero mean curvature everywhere. The equation of minimal surfaces, namely

$$\mathbf{H} = 0$$

($\mathbf{H}$ stands for the mean curvature), is the Euler-Lagrange equation of the area functional. Lagrange found this equation in 1762, but without explaining the geometrical meaning; four years later the French mathematician J.-B. Meusnier (1754-1793) realized that the Euler-Lagrange equation of the area functional says that the mean curvature vanishes at any point.

2.1. A review on differential geometry

First of all, we review some basic facts of differential geometry for smooth surfaces; for details we refer to Do Carmo [7]. For us, a *d-dimensional surface in* $\mathbb{R}^n$ $(0 < d < n)$ is the image of a smooth map $\mathbf{X}\colon D \to \mathbb{R}^n$, where D is open in $\mathbb{R}^d$, $\mathbf{X}$ is a homoemorphism between D and $\mathbf{X}(D)$, and $\nabla\mathbf{X}$ has rank d. We will refer to *local coordinates* as the coordinates $u_1, \dots, u_d \in D$. Usually, the geometric properties of $S = \mathbf{X}(D)$ do not depend on $\mathbf{X}$ (think to the area of S), and for this reason $\mathbf{X}$ is often called a *parametrization of S.* Since the rank of $\nabla\mathbf{X}$ is d, the tangent vectors

$$\partial_1\mathbf{X}, \dots, \partial_d\mathbf{X}$$

are linearly independent everywhere, hence we can well define the *tangent space to S at p*, denoted by $\mathrm{Tan}(S,p)$, as the d-dimensional vector space generated by

$$\partial_1\mathbf{X}(\mathbf{X}^{-1}(p)),\dots,\partial_d\mathbf{X}(\mathbf{X}^{-1}(p)).$$

In order to define the mean curvature of S, we restrict to the case of hypersurfaces, namely $d=n-1$. Fix $p\in S$. First of all we have to choose a normal direction to S at p. Let us take a unit vector $\mathbf{n}(p)$ in such a way the matrix

$$[\partial_1\mathbf{X}|\partial_2\mathbf{X}|\cdots|\partial_{n-1}\mathbf{X}|\mathbf{n}]$$

has positive determinant at p.

We denote by $\mathrm{Nor}(S,p)$ the one dimensional vector space generated by $\mathbf{n}(p)$. Hence, close to p the surface S is the graph of a smooth function

$$f\colon \mathrm{Tan}(S,p)\to\mathrm{Nor}(S,p)$$

Notice that a change of the direction of $\mathbf{n}$ gives a change of sign of f. Let us denote by $\mathbf{A}_p$ the Hessian of f at p. The linear map

$$\mathbf{A}_p\colon \mathrm{Tan}(S,p)\to\mathrm{Tan}(S,p)$$

is self-adjoint hence it admits $n-1$ real eigenvalues $\lambda_1,\dots,\lambda_{n-1}$ which are called *principal curvatures of S at p*. We can define the *mean curvature of S at p* as

$$\mathbf{H}(p):=\mathrm{tr}\mathbf{A}_p=\lambda_1+\cdots+\lambda_{n-1}.$$

There is an important relation between $\mathbf{A}$ and $\mathbf{n}$: it turns out that

$$-\mathrm{d}\mathbf{n}(p)=\mathbf{A}_p.$$

For the special case $n=3$ it is usual to take

$$\mathbf{n}=\frac{\partial_1\mathbf{X}\wedge\partial_2\mathbf{X}}{|\partial_1\mathbf{X}\wedge\partial_2\mathbf{X}|},$$

where $\wedge$ is the standard vector product in $\mathbb{R}^3$. It is possibile to prove that $\mathbf{H}$ has the following expression in term of $\mathbf{X}$:

$$\mathbf{H}=\frac{eG-2Ff+gE}{EG-F^2}$$

where

$$E=|\partial_1\mathbf{X}|^2,\quad F=\langle\partial_1\mathbf{X},\partial_2\mathbf{X}\rangle,\quad G=|\partial_2\mathbf{X}|^2,$$

and

$$e=\langle\mathbf{n},\partial^2_{11}\mathbf{X}\rangle,\quad f=\langle\mathbf{n},\partial^2_{12}\mathbf{X}\rangle,\quad g=\langle\mathbf{n},\partial^2_{22}\mathbf{X}\rangle$$

and $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^3$. In the theory of surfaces the coefficients E,F,G are usually known as coefficients of the *first fundamental form*, while the coefficients e, f, g are usually known as coefficients of the *second fundamental form*. We conclude this section with a remark on conformal coordinates. We say that $\mathbf{X}$ is *conformal* if

$$E = G, \quad F = 0.$$

In this case, it can be shown that the formula for $\mathbf{H}$ can be simplified, and it gives

$$\Delta\mathbf{X} = 2E\mathbf{H}, \tag{1}$$

where $\Delta\mathbf{X} = (\Delta\mathbf{X}^1, \Delta\mathbf{X}^2, \Delta\mathbf{X}^3)$. A question naturally arises: is it always true that any surface can be reparametrized conformally? The answer is positive if $\mathbf{X}$ is smooth enough (for instance Hölder continuous). Let us also mention that the existence of conformal reparametrizations is much harder in higher dimension* and holds true under restrictive assumptions on the surface.

2.2. Area formula

We pass now to the definition of area and the *area formula*. Let S be a d-dimensional surface in $\mathbb{R}^n$ parametrized by $\mathbf{X}\colon D \to \mathbb{R}^n$. First, consider the simple case $d = 2$ and $n = 3$. In this case we know that the area of S is given by

$$\mathbf{A}(S) = \int_D |\partial_1\mathbf{X} \wedge \partial_2\mathbf{X}|\, du.$$

Notice now that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are linearly independent then

$$\det\left(\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{pmatrix}\begin{pmatrix} \mathbf{v}_1 & \mathbf{w}_1 \\ \mathbf{v}_2 & \mathbf{w}_2 \\ \mathbf{v}_3 & \mathbf{w}_3 \end{pmatrix}\right) = |\mathbf{v}|^2|\mathbf{w}|^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2 = |\mathbf{v} \wedge \mathbf{w}|^2.$$

This means that

$$|\partial_1\mathbf{X} \wedge \partial_2\mathbf{X}| = \sqrt{\det((\nabla\mathbf{X})^T\nabla\mathbf{X})}.$$

Therefore, the area of S is also given by

$$\mathbf{A}(S) = \int_D \sqrt{\det((\nabla\mathbf{X})^T\nabla\mathbf{X})}\, du.$$

This formula makes sense for general d and n and indeed it holds true:

$$\mathbf{A}(\mathbf{X}(D)) = \int_D \sqrt{\det((\nabla\mathbf{X})^T\nabla\mathbf{X})}\, du.$$

We need a further generalization of this formula. Precisely, if $f\colon S \to \mathbb{R}^n$ is smooth and injective we have

$$\mathbf{A}(f(S)) = \int_S \sqrt{\det((df(p))^T df(p))}\, ds \tag{2}$$

*In higher dimension we say that $\mathbf{X}$ is conformal if $|\partial_i\mathbf{X}| = |\partial_j\mathbf{X}|$ for any i, j and $\langle\partial_i\mathbf{X}, \partial_j\mathbf{X}\rangle = 0$ for any $i \neq j$.

where $df(p)\colon \mathrm{Tan}(S,p) \to \mathbb{R}^n$ is the differential of f as a map between surfaces, that is

$$df(p)(v) = \nabla(f \circ \mathbf{X})v, \quad \forall v \in \mathrm{Tan}(S,p).$$

2.3. The first variation of the area

Using the area formula it is possibile to compute the first variation of the area. In other words, given a smooth hypersurface S in $\mathbb{R}^n$ we want to find a formula for the quantity

$$\frac{d}{dt}\mathbf{A}(S_t)_{|t=0}$$

where S_t is a one-parameter family of hypersurfaces in $\mathbb{R}^n$ such that $S_0 = S$. The idea is to choose a suitable family of variations of S. Precisely, we choose a smooth normal vector field η to S (S is assumed to be orientable, which means that such η exists). Then, if $\mathbf{n}$ is a choice of unit normal vector field on S, it is $\eta = \varphi\mathbf{n}$ for some function $\varphi\colon S \to \mathbb{R}$. Let

$$S_t = \{p + t\eta(p) : p \in S\}.$$

Clearly S_t is a smooth surface only if t is small enough. Of course, the surface S_t can be parametrized by $\psi_t\colon S \to S_t$ given by

$$\psi_t(p) = p + t\eta(p).$$

Since for t small ψ_t is smooth and injective we can apply area formula (2). We need to compute $d\psi_t$. Formally, we have

$$d\psi_t = dp + t\varphi d\mathbf{n} + t\mathbf{n}d\varphi.$$

We are going to find the matrix which represents $d\psi_t$. Choose an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ in $\mathrm{Tan}(S,p)$ and take the orthonormal basis $\{e_1, \ldots, e_{n-1}, \mathbf{n}\}$ in $\mathbb{R}^n$. With respect to this choice of bases, the linear map $d\psi_t$ is represented by the $n \times (n-1)$ matrix

$$M = \begin{pmatrix} I_{(n-1)\times(n-1)} - t\varphi\mathbf{A} \\ \hline t(\nabla\varphi)^T \end{pmatrix}.$$

Hence we get

$$M^T M = I_{(n-1)\times(n-1)} - t\varphi(\mathbf{A}^T + \mathbf{A}) + O(t^2).$$

Remember that at first order $\det(I+X) \sim 1 + \mathrm{tr}X$, from which

$$\sqrt{\det(M^T M)} = \sqrt{1 - 2t\varphi\mathrm{tr}\mathbf{A} + O(t^2)} = 1 - t\varphi\mathbf{H} + O(t^2).$$

Applying formula (2) we deduce that

$$\mathbf{A}(S_t) = \int_S 1 - t\varphi\mathbf{H} + O(t^2)\, ds = \mathrm{Area}(S) - t\int_S \varphi\mathbf{H}\, ds + O(t^2).$$

Finally, we get

$$\frac{d}{dt}\mathbf{A}(S_t)_{|t=0} = -\int_S \varphi \mathbf{H}\, ds. \tag{3}$$

We can now deduce an important conclusion from (3). Indeed, by the arbitrariness of φ we can say that if S minimizes the area among a class of surfaces for which S_t produces admissible variations, then it must be $\mathbf{H} = 0$ everywhere, which is the *equation of minimal surfaces*. In literature actually minimal surface means simply that $\mathbf{H}$ vanishes, or in other words S is a critical point of the area functional, and not necessarily a minimizer.

2.4. Some examples

We discuss some explicit examples of minimal surfaces, that is surfaces satisfying $\mathbf{H} = 0$ everywhere. A first, obvious, example is the flat surface: this is also the unique solution of the Plateau problem when the boundary curve is a planar curve. A less obvious example is the *catenoid*. We ask for a minimal surface which is also a revolution surface. If we let

$$\mathbf{X}(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av), \quad (u,v) \in (0,2\pi) \times \mathbb{R}, \quad a > 0,$$

we find the surface generated by rotating the catenary

$$y = a\cosh\left(\frac{z}{a}\right)$$

around the z-axis. This surface, called catenoid, is a minimal surface. In order to see this, first notice that the coordinates u, v are conformal:

$$E = G = a^2\cosh^2 v, \quad F = 0.$$

Hence, we can apply (1) and we obtain, by direct computation,

$$\mathbf{H} = \frac{2}{a^2\cosh^2 v}\Delta\mathbf{X} = 0.$$

It is possible to prove that the catenoid is the unique minimal surface of revolution (see Do Carmo [7] for details). The catenoid appears also as a solution of the soap film bounded by two coaxial rings sufficiently close (see figure 2). The catenoid as a solution of the Plateau problem with boundary given by two coaxial rings is not the unique solution. There always exists the so called *Goldschmidt solution* (existence proved by K. Goldschmidt (1807-1851) in 1831) to this Plateau problem: it consists in two plane discs with no intermediate surface. If the two rings are close enough then the Goldschmidt solution is a local minimizer and the minimizing catenoid is the absolute minimizer, while at some distance the catenoid becomes unstable and so if the two rings are far enough the Goldschmidt solution is the absolute minimizer and the

Figure 2: A catenoid as a soap film that spans two coaxial rings.

catenoid is a local minimizer. We conclude with another example of minimal surface which is the *elicoid*. Let

$$\mathbf{X}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, av), \quad (u,v) \in (0,2\pi)\times\mathbb{R}, \quad a > 0.$$

This surface, called elicoid, is a minimal surface. Indeed the same considerations for the catenoid hold in this case:

$$E = G = a^2\cosh^2 v, \quad F = 0, \quad \mathbf{H} = \frac{2}{a^2\cosh^2 v}\Delta\mathbf{X} = 0.$$

It is possible to prove that the elicoid is the unique (other than the plane) ruled minimal surface (see Do Carmo [7] for details).

2.5. Further remarks on the equation $\mathbf{H} = 0$

We can ask, in general, for solutions of the equation $\mathbf{H} = 0$ with prescribed boundary conditions. From the point of view of PDE's this problem is very hard since the minimal surfaces equation arises from a functional with linear growth in the derivative. In order to see this, let us restrict to the case of 2-dimensional graphs. Let $\Omega \subset \mathbb{R}^2$ open bounded with smooth boundary and let $u\colon \Omega \to \mathbb{R}$ smooth enough. The area of the graph of u is given by

$$\mathbf{A}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx.$$

We can set the Plateau problem in this case simply asking for minimizers of $A(u)$ when u is fixed on $\partial\Omega$. This minimization problem leads to the equation

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0.$$

Figure 3: The elicoid as a soap film.

The difficulty with this equation, which is again $\mathbf{H} = 0$, is hidden in the growth of A: indeed A has linear growth in the gradient, so that the natural Sobolev space where one could considers the minimization problem for A is $W^{1,1}(\Omega)$. But this space is not reflexive, hence the direct methods of the Calculus of Variations cannot be easily applied. In order to treat functionals as A one has to move to the space of *functions of bounded variations*, but in this case one has to consider also discontinuous solutions, which are not physical. To overcome the difficulty an idea could be to modify the area functional in such a way it becomes a functional with superlinear growth. This is the key point of the Douglas-Radó approach, which we are going to describe in the next Section.

3. Disc-type Plateau problem

During the 19th century, the Plateau problem for 2-dimensional surfaces was solved for many special boundary curves Γ. A general treatment for that arrived in 1930 independently by J. Douglas (1897-1965) and T. Radó (1896-1965). A simplification has been given independently by R. Courant (1888-1972) and L. Tonelli (1885-1946). In this section we will sketch the approach of Courant and Tonelli following Dierkes et al. [6]. The idea is to consider surfaces parametrized on a disc in the plane. In other words

$$D = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$$

and $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth parametrization, while, roughly speaking, the trace of $\mathbf{X}$ on ∂D is a smooth parametrization of a prescribed closed curve Γ in $\mathbb{R}^3$.

3.1. The parametric area functional: lack of compactness

If we wish to apply the direct method of the Calculus of Variations in order to solve the disc-type Plateau problem, we have to consider the area functional

$$\mathbf{A}(\mathbf{X}) = \int_D |\partial_1 \mathbf{X} \wedge \partial_2 \mathbf{X}|\, du$$

for which we have to check semicontinuity and compactness with respect to a suitable topology on a suitable domain. The weak lower semicontinuity of $\mathbf{A}$ in some Sobolev space is not a problem: it turns out that $\mathbf{A}$ is weakly lower semicontinuous in $W^{1,p}(D;\mathbb{R}^3)$ for any $p \geqslant 2$. The key point here is that $|\partial_1 \mathbf{X} \wedge \partial_2 \mathbf{X}|$ is a convex function of the determinants of the 2×2 minors of $\nabla \mathbf{X}$ (what is called *polyconvex* function), and the lower semicontinuity follows from standard results (see Dacorogna [5]). Concerning the compactness, unfortunately the set

$$\{\mathbf{X} : \mathbf{A}(\mathbf{X}) \leqslant c\}$$

is not bounded in any reasonable Sobolev norm. The main obstruction is the fact that F is invariant under reparametrization, that is for any diffeomorphism $\phi \colon D \to D$ we have

$$\mathbf{A}(\mathbf{X}) = \mathbf{A}(\mathbf{X} \circ \phi).$$

Hence, taking suitable ϕ we can make any Sobolev norm of $\mathbf{X} \circ \phi$ as large as we want. On the other hand, this invariance may help: indeed, we could use it in order to restrict the search of minimizer to a much smaller and better behaved class of surfaces.

3.2. The Dirichlet functional

Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we have

$$|\mathbf{v} \wedge \mathbf{w}| \leqslant |\mathbf{v}|\,|\mathbf{w}| \leqslant \frac{|\mathbf{v}|^2 + |\mathbf{w}|^2}{2}.$$

If $\mathbf{v}, \mathbf{w}$ are the two columns of the matrix M then the previous estimate reads as

$$|\mathbf{v} \wedge \mathbf{w}| \leqslant \frac{1}{2}|M|^2.$$

Moreover, the equality holds true if and only if $|\mathbf{v}| = |\mathbf{w}|$ and $\langle \mathbf{v}, \mathbf{w}\rangle = 0$. Thanks to the previous considerations we can say that

$$\mathbf{A}(\mathbf{X}) = \int_D |\partial_1 \mathbf{X} \wedge \partial_2 \mathbf{X}|\, du \leqslant \frac{1}{2}\int_D |\nabla \mathbf{X}|^2\, du$$

and the equality holds true if and only if $\mathbf{X}$ is conformal. This suggests that we could deal with the Dirichlet functional instead of the area functional and this should be better since the Dirichlet functional has superlinear growth in the gradient so that we can work in Sobolev spaces where good compactness properties hold true.

3.3. Setting of the disc-type Plateau problem

In this paragraph we state the rigorous formulation of the disc-type Plateau problem. Fix $\mathbf{X} \in W^{1,2}(D;\mathbb{R}^3)$. We denote $C = \partial D$ and

$$\mathbf{X}_{|C} : C \to \mathbb{R}^3$$

denotes the trace of $\mathbf{X}$ on C; it is well known that $\mathbf{X}_{|C} \in L^2(C;\mathbb{R}^3)$. Now it comes the main point: we have to say that $\mathbf{X}_{|C}$ is a prescribed curve in $\mathbb{R}^3$. Fix a Jordan curve Γ in $\mathbb{R}^3$ which is oriented by a fixed homeomorphism $\gamma\colon C \to \Gamma$. Let $\varphi\colon C \to \Gamma$. We say that φ is *weakly monotonic* if φ is continuous, surjective, and there exists a non-decreasing function $\tau\colon [0,2\pi] \to \mathbb{R}$ such that $\tau(2\pi) = \tau(0) + 2\pi$ and

$$\varphi(e^{i\theta}) = \gamma(e^{i\tau(\theta)}), \quad \forall\theta \in [0,2\pi].$$

Roughly speaking, φ is weakly monotonic if the image points $\varphi(w)$ traverse Γ in a constant direction when w moves on C in a constant direction. Denoting by $\mathcal{E}\colon [0,2\pi] \to C$ the map $\mathcal{E}(\theta) = e^{i\theta}$, we can rewrite the weak monotonic condition as $\mathcal{E} \circ \tau = \gamma^{-1} \circ \varphi \circ \mathcal{E}$. As a consequence of this formula, one easily obtains that if $\{\varphi_h\}$ is a sequence of weakly monotonic maps $C \to \Gamma$ which converges uniformly to $\varphi\colon C \to \Gamma$, then φ is weakly monotonic. We are therefore ready to define the right domain. Let

$$\mathcal{C}(\Gamma) = \{\mathbf{X} \in W^{1,2}(D;\mathbb{R}^3) : \mathbf{X}_{|C} : C \to \Gamma \text{ is weakly monotonic}\}.$$

The class $\mathcal{C}(\Gamma)$ turns out to be invariant under conformal transformations: remember that $\sigma\colon \overline{D} \to \overline{D}$ is said to be *conformal* if $|\partial_1\sigma| = |\partial_2\sigma|$ and $\langle\partial_1\sigma, \partial_2\sigma\rangle = 0$. The idea should be minimize

$$\mathbf{D}(\mathbf{X}) = \frac{1}{2}\int_D |\nabla\mathbf{X}|^2\, du$$

on $\mathcal{C}(\Gamma)$. Indeed, assume that we have found $\mathbf{X}_0 \in \mathcal{C}(\Gamma)$ such that $\mathbf{X}_0$ is conformal and

$$\mathbf{D}(\mathbf{X}_0) = \min_{\mathbf{X}\in\mathcal{C}(\Gamma)} \mathbf{D}(\mathbf{X}).$$

If $\mathbf{X} \in \mathcal{C}(\Gamma)$ we can pass to a conformal $\mathbf{X}^c$ by means of a conformal transformations of coordinates, and therefore

$$\mathbf{A}(\mathbf{X}) = \mathbf{A}(\mathbf{X}^c) = \mathbf{D}(\mathbf{X}^c) \geqslant \mathbf{D}(\mathbf{X}_0) = \mathbf{A}(\mathbf{X}_0).$$

3.4. Proof of the existence of $\mathbf{X}_0$

In this paragraph we prove that if Γ has finite length then the problem

$$\min_{\mathbf{X}\in\mathcal{C}(\Gamma)} \mathbf{D}(\mathbf{X})$$

has a conformal solution $\mathbf{X}_0 \in C^0(\overline{D};\mathbb{R}^3)$ which is harmonic on D.

Step 1: Reduction to $\mathcal{C}^*(\Gamma)$. In order to solve the minimization problem we have to find a minimizing sequence $(\mathbf{X}_h)$ whose boundary values $\{\mathbf{X}_{h|_C}\}$ contains a subsequence that converges uniformly on C. The selection of such a minimizing sequence will be achieved by the following artifice: we fix some points on Γ and work only with parametrizations which fix these points; since it is enough to fix just three points, this is the so called *three points condition*. Precisely, fix three different points $w_1, w_2, w_3 \in C$ and three different points $Q_1, Q_2, Q_3 \in \Gamma$ such that $\gamma(w_k) = Q_k$. Let

$$\mathcal{C}^*(\Gamma) := \{\mathbf{X} \in \mathcal{C}(\Gamma) : \mathbf{X}_{|_C}(w_k) = Q_k, k = 1,2,3\}.$$

If we denote

$$e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} \mathbf{D}, \quad e^*(\Gamma) := \inf_{\mathcal{C}^*(\Gamma)} \mathbf{D}$$

of course we have $e^*(\Gamma) \geqslant e(\Gamma)$. On the other hand, if $\mathbf{X} \in \mathcal{C}(\Gamma)$ then there exist three different points $\zeta_1, \zeta_2, \zeta_3 \in C$ such that $\mathbf{X}_{|_C}(\zeta_k) = Q_k$ for $k = 1,2,3$. Let us take a conformal map $\sigma \colon \overline{D} \to \overline{D}$ such that $\sigma(w_k) = \zeta_k$ for $k = 1,2,3$. Then $\mathbf{X} \circ \sigma \in \mathcal{C}^*(\Gamma)$ and since $\mathbf{D}$ is invariant under conformal transformation we also have that $\mathbf{D}(\mathbf{X} \circ \sigma) = \mathbf{D}(\mathbf{X})$. This means that actually $e(\Gamma) = e^*(\Gamma)$.

Step 2: $\mathcal{C}^*(\Gamma) \neq \varnothing$. In order to show that the problem $\inf_{\mathcal{C}^*(\Gamma)} \mathbf{D}$ is well posed we have to ensure that $\mathcal{C}^*(\Gamma)$ is non-empty. It is possible to prove (see for instance [6] pages 254-255) that if Γ has finite length (for instance if φ is Lipschitz continuous) then $\mathcal{C}^*(\Gamma) \neq \varnothing$; we also observe that this condition is only sufficient.

Step 3: The Courant-Lebesgue Lemma. Let $\mathbf{X} \in C^0(\overline{D}; \mathbb{R}^3) \cap C^1(D; \mathbb{R}^3)$ and assume that $\mathbf{D}(\mathbf{X}) \leqslant M$ for some $M \geqslant 0$. Let $z_0 \in C$ and $r > 0$ small. Denote

$$S_r(z_0) = D \cap B_r(z_0), \quad C_r(z_0) = \overline{D} \cap \partial B_r(z_0).$$

Since $z_0 \in C$ we can write

$$C_r(z_0) = \{z_0 + re^{i\theta} : \theta_1(r) \leqslant \theta \leqslant \theta_2(r)\}$$

for some $\theta_i(r)$ with $0 < \theta_2(r) - \theta_1(r) < \pi$. Let

$$Z(r,\theta) = \mathbf{X}(z_0 + re^{i\theta})$$

defined in its natural domain. We have

$$\int_0^r \frac{1}{\rho} \int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta|^2 \, d\theta \, d\rho \leqslant \int_0^r \int_{\theta_1(\rho)}^{\theta_2(\rho)} \left(|Z_\rho|^2 + \frac{|Z_\theta|^2}{\rho^2} \right) \rho \, d\theta \, d\rho = \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du$$

Fix $\delta \in (0,1)$ small. The previous estimate gives

$$\int_\delta^{\sqrt{\delta}} \frac{1}{\rho} \int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta|^2 \, d\theta \, d\rho \leqslant \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du. \tag{4}$$

Observe now that the set

$$J = \left\{ \rho \in (\delta, \sqrt{\delta}) : \int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta|^2 \, d\theta \int_\delta^{\sqrt{\delta}} \frac{1}{r} \, dr \leqslant \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du \right\}$$

has positive 1-dimensional Lebesgue measure. Indeed, if $\mathcal{L}^1(J) = 0$ then we would obtain

$$\int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta|^2 \, d\theta > \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du \left(\int_\delta^{\sqrt{\delta}} \frac{1}{r} \, dr \right)^{-1/2}$$

for $\mathcal{L}^1$-almost all $\rho \in J$. Multiplying the previous inequality by $\frac{1}{\rho}$ and integrating on $(\delta, \sqrt{\delta})$ we would arrive to

$$\int_\delta^{\sqrt{\delta}} \frac{1}{\rho} \int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta|^2 \, d\theta \, d\rho > \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du,$$

which contradicts (4). Now, for any $\rho \in J$ and for any θ, θ' with $\theta_1(\rho) \leqslant \theta \leqslant \theta' \leqslant \theta_2(\rho)$ we obtain, by Hölder inequality,

$$\begin{aligned}
\int_\theta^{\theta'} |Z_\theta| \, d\theta &\leqslant \left(\int_\theta^{\theta'} |Z_\theta|^2 \, d\theta \right)^{1/2} |\theta - \theta'|^{1/2} \\
&\leqslant \left(\int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du \right)^{1/2} \left(\int_\delta^{\sqrt{\delta}} \frac{1}{r} \, dr \right)^{-1/2} |\theta - \theta'|^{1/2} \\
&= \left(\frac{2}{\log(1/\delta)} \int_{S_r(z_0)} |\nabla \mathbf{X}|^2 \, du \right)^{1/2} |\theta - \theta'|^{1/2} \\
&\leqslant \left(\frac{4M\pi}{\log(1/\delta)} \right)^{1/2}
\end{aligned}$$

from which

$$|Z(\rho, \theta') - Z(\rho, \theta)| \leqslant \int_\theta^{\theta'} |Z_\theta| \, d\theta \leqslant \left(\frac{4M\pi}{\log(1/\delta)} \right)^{1/2}.$$

In other words, we have proved the *Courant-Lebesgue Lemma*: for any $z_0 \in C$ and for any $\delta \in (0,1)$ there exists $\rho \in (\delta, \sqrt{\delta})$ such that

$$|\mathbf{X}(z) - \mathbf{X}(z')| \leqslant \left(\frac{4M\pi}{\log(1/\delta)} \right)^{1/2}$$

where $\{z, z'\} = C \cap \partial B_\rho(z_0)$.

Step 4: A topological remark. Since Γ is the topological image of C, it is possible to prove that for any $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ such that any $P, Q \in \Gamma$ with

$$0 < |P - Q| < \ell(\varepsilon)$$

decompose Γ into two arcs $\Gamma_1(P,Q)$ and $\Gamma_2(P,Q)$ in such a way $\operatorname{diam}\Gamma_1(P,Q) < \varepsilon$.

Step 5: The key estimate on $\mathbf{X}_{|C}$. Let $\mathbf{X} \in \mathcal{C}^*(\Gamma) \cap C^0(\overline{D};\mathbb{R}^3) \cap C^1(D;\mathbb{R}^3)$ and assume that $\mathbf{D}(\mathbf{X}) \leqslant M$ for some $M \geqslant 0$. Let $\delta_0 \in (0,1)$ be such that

$$2\sqrt{\delta_0} < \min_{j\neq k} |w_j - w_k|.$$

If $0 < \varepsilon < \min_{j\neq k} |Q_j - Q_k|$ we choose $\delta > 0$ such that

$$\left(\frac{4M\pi}{\log 1/\delta}\right)^{1/2} < \ell(\varepsilon) \quad \text{and} \quad \delta < \delta_0.$$

We use now the Courant-Lebesgue Lemma: take an arbitrary point $z_0 \in C$ and let $\rho \in (\delta, \sqrt{\delta})$ be such that

$$|\mathbf{X}(z) - \mathbf{X}(z')| \leqslant \left(\frac{4M\pi}{\log 1/\delta}\right)^{1/2}$$

where $\{z, z'\} = C \cap \partial B_\rho(z_0)$. Then $|\mathbf{X}(z) - \mathbf{X}(z')| < \ell(\varepsilon)$ hence

$$\operatorname{diam}\Gamma_1(\mathbf{X}(z), \mathbf{X}(z')) < \varepsilon.$$

Since $\varepsilon < \min_{j\neq k} |Q_j - Q_k|$ the arc $\Gamma_1(\mathbf{X}(z), \mathbf{X}(z'))$ contains at most one of the points Q_j. On the other hand $\mathbf{X}(C \cap \overline{B_\rho(z_0)})$ contains at most one of the points Q_j because of our choice of δ. Therefore it must be

$$\mathbf{X}(C \cap \overline{B_\rho(z_0)}) = \Gamma_1(\mathbf{X}(z), \mathbf{X}(z')).$$

As a consequence, we get

$$|\mathbf{X}(w) - \mathbf{X}(w')| < \varepsilon \quad \forall w, w' \in C \cap B_\rho(z_0)$$

which implies the key estimate

$$|\mathbf{X}(w) - \mathbf{X}(w')| < \varepsilon \quad \forall w, w' \in C \text{ such that } |w - w'| < \delta. \tag{5}$$

Step 6: Minimization by direct method of the Calculus of Variations. We are going to solve $\min_{\mathcal{C}^*(\Gamma)} \mathbf{D}$. Let us take a minimizing sequence $\{\mathbf{X}_h\}$, that is $\mathbf{D}(\mathbf{X}_h) \to e^*(\Gamma)$. Let

$$\mathbf{Z}_h \in C^0(\overline{D};\mathbb{R}^3) \cap C^2(D;\mathbb{R}^3) \cap W^{1,2}(D;\mathbb{R}^3)$$

be the unique solution to the problem

$$\begin{cases} \Delta\mathbf{Z}_h = 0 & \text{on } D \\ \mathbf{Z}_h = \mathbf{X}_h & \text{on } C. \end{cases}$$

This solution minimizes $\mathbf{D}$ among all functions $\mathbf{X} \in W^{1,2}(D;\mathbb{R}^3)$ such that

$$(\mathbf{X} - \mathbf{X}_h)_{|C} = 0.$$

As a consequence, we deduce that $\mathbf{D}(\mathbf{Z}_h) \leqslant \mathbf{D}(\mathbf{X}_h)$ and since by construction $\mathbf{Z}_h \in \mathcal{C}^*(\Gamma)$, we can say that $\{\mathbf{Z}_h\}$ is still a minimizing sequence. The advantage is that $\mathbf{Z}_h$ is harmonic in D for any $h \in \mathbb{N}$. Now, since $\mathbf{Z}_h$ is minimizing for sure it holds $\mathbf{D}(\mathbf{Z}_h) \leqslant M$ for some $M > 0$. Applying (5) we can conclude that $\{\mathbf{Z}_{h|_C}\}$ is equicontinuous. Moreover, since $\mathbf{Z}_h(C) = \Gamma$ the family $\{\mathbf{Z}_{h|_C}\}$ is also uniformly bounded. We may therefore apply the Ascoli-Arzelà Theorem and then, up to a subsequence (not relabeled), $\mathbf{Z}_{h|_C} \to \varphi$ uniformly on C, where $\varphi\colon C \to \Gamma$ is weakly monotonic. Now the conclusion follows using standard properties of harmonic functions. Since $\mathbf{Z}_{h|_C} \to \varphi$ uniformly on ∂D, we have that $\mathbf{Z}_h \to \mathbf{Z}$ uniformly on $\overline{D}$, where $\mathbf{Z}$ is continuous on $\overline{D}$ and harmonic on D. Moreover, $\nabla \mathbf{Z}_h \to \nabla \mathbf{Z}$ uniformly on every $D' \subset\subset D$, from which

$$\int_{D'} |\nabla \mathbf{Z}_h|^2 \, du \to \int_{D'} |\nabla \mathbf{Z}|^2 \, du.$$

Thus, for every $D' \subset\subset D$ we have

$$\liminf_h \int_D |\nabla \mathbf{Z}_h|^2 \, du \geqslant \liminf_h \int_{D'} |\nabla \mathbf{Z}_h|^2 \, du = \int_{D'} |\nabla \mathbf{Z}|^2 \, du$$

which means that, when $D' \nearrow D$,

$$e^*(\Gamma) = \lim_h \mathbf{D}(\mathbf{Z}_h) \geqslant \mathbf{D}(\mathbf{Z}).$$

This concludes the proof of the fact that the problem

$$\min_{\mathcal{C}^*(\Gamma)} \mathbf{D}$$

has a solution $\mathbf{X}_0$ which is continuous on $\overline{D}$ and harmonic on D.

Step 7: Conformality of minimizers. Consider a vector field $\lambda = (\mu, \nu) \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. For ε small take the family of maps $\tau_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\tau_\varepsilon(u) = u - \varepsilon\lambda(u).$$

Choose some open set D_0 with $D \subset\subset D_0$. Then it is easy to see that $\tau_\varepsilon \colon D_0 \to \tau_\varepsilon(D_0)$ is an orientation-preserving C^1-diffeomorphism of D_0 onto its image provided that $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$. Take the inverse mapping σ_ε, which is well defined on $D^*_\varepsilon = \tau_\varepsilon(D)$. Then

$$\sigma_\varepsilon(w) = w + \varepsilon\lambda(w) + o(\varepsilon), \quad \varepsilon \to 0.$$

Consider now $\mathbf{X} \in W^{1,2}(D; \mathbb{R}^3)$ and construct the family of functions

$$\mathbf{X}_\varepsilon \colon \overline{D^*_\varepsilon} \to \mathbb{R}^3, \quad \mathbf{X}_\varepsilon = \mathbf{X} \circ \sigma_\varepsilon.$$

The idea is to exploit the *first inner variation of* $\mathbf{D}$ *in the direction of* λ that is the quantity defined by

$$\delta\mathbf{D}(\mathbf{X}, \lambda) = \frac{d}{d\varepsilon}_{|\varepsilon=0} \frac{1}{2} \int_{D^*_\varepsilon} |\nabla \mathbf{X}_\varepsilon|^2 \, dw.$$

In order to compute the right-hand side first of all observe that

$$\int_{D_\varepsilon^*} |\nabla \mathbf{X}_\varepsilon|^2\, dw = \int_D |\nabla \mathbf{X}_\varepsilon \circ \tau_\varepsilon|^2 |\det \nabla \tau_\varepsilon|\, du.$$

We have $\nabla \mathbf{X}_\varepsilon(w) = \nabla \mathbf{X}(\sigma_\varepsilon(w))\nabla\sigma_\varepsilon(w)$, hence

$$\nabla \mathbf{X}_\varepsilon(\tau_\varepsilon(u)) = \nabla \mathbf{X}(u)\nabla\sigma_\varepsilon(\tau_\varepsilon(u)).$$

It is easy to see that

$$\nabla\sigma_\varepsilon(\tau_\varepsilon(u))_{|\varepsilon=0} = \begin{pmatrix} \partial_1\mu(u) & \partial_2\nu(u) \\ \partial_1\nu(u) & \partial_2\nu_\nu(u) \end{pmatrix}.$$

After a straightforward computation, we obtain

$$2\delta\mathbf{D}(\mathbf{X},\lambda) = \int_D [(|\partial_1\mathbf{X}|^2 - |\partial_2\mathbf{X}|^2)(\partial_1\mu - \partial_2\nu) + 2\langle \partial_1\mathbf{X}, \partial_2\mathbf{X}\rangle(\partial_2\mu + \partial_1\nu)]\, du.$$

Fix now arbitrary functions $\rho, \sigma \in C_c^\infty(D)$ and find $h, k \in C^\infty(D)$ in such a way

$$\begin{cases} \Delta h = \rho & \text{on } D \\ h = 0 & \text{on } C \end{cases}, \quad \begin{cases} \Delta k = \sigma & \text{on } D \\ k = 0 & \text{on } C. \end{cases}$$

Therefore, taking $\mu = \partial_1 h + \partial_2 k$ and $\nu = -\partial_2 h + \partial_1 k$ we get

$$2\delta\mathbf{D}(\mathbf{X},\lambda) = \int_D [(|\partial_1\mathbf{X}|^2 - |\partial_2\mathbf{X}|^2)\rho + 2\langle \partial_1\mathbf{X}, \partial_2\mathbf{X}\rangle\sigma]\, du. \tag{6}$$

Now we finally apply this formula. To do this, we choose $\mathbf{X} = \mathbf{X}_0$. Since $\overline{D}$ and $\overline{D_\varepsilon^*}$ are diffeomorphic there is a conformal map $k_\varepsilon\colon D \to D_\varepsilon^*$ of D onto D_ε^*, by virtue of the Riemann Mapping Theorem[†]. Moreover, since ∂D_ε^* is a Jordan curve, a classical result grants that k_ε can be extended to a homeomorphism $\overline{D} \to \overline{D_\varepsilon^*}$. It follows that $\mathbf{Y}_\varepsilon = \mathbf{X}_\varepsilon \circ k_\varepsilon \in \mathcal{C}(\Gamma)$, so that

$$\mathbf{D}(\mathbf{X}_0) \leqslant \mathbf{D}(\mathbf{Y}_\varepsilon), \quad |\varepsilon| < \varepsilon_0.$$

But $\mathbf{D}$ is invariant under conformal transformation, therefore

$$\mathbf{D}(\mathbf{Y}_\varepsilon) = \frac{1}{2}\int_{D_\varepsilon^*} |\nabla \mathbf{X}_\varepsilon|^2\, du,$$

[†]The Riemann Mapping Theorem states that if U is a non-empty simply connected open subset of $\mathbb{C}$ which is not $\mathbb{C}$, then there exists a biholomorphic mapping $f\colon U \to D$. The idea of the proof can be explained easily: given $z_0 \in U$, we ask for f which maps U to D with $f(z_0) = 0$. Assume U bounded with smooth boundary is smooth. Write $f(z) = (z - z_0)e^{u(z)+iv(z)}$, where u, v are to be determined. Since we require $|f| = 1$ on ∂U, we need $u(z) = -\log|z - z_0|$ on ∂U. But u is the real part of a holomorphic function, hence u is harmonic function. We then solve the Laplace equation with $-\log|z - z_0|$ on ∂U, and therefore we find v by means of Cauchy-Riemann conditions.

which gives

$$\mathbf{D}(\mathbf{X}_0) \leqslant \frac{1}{2}\int_{D_\varepsilon^*} |\nabla \mathbf{X}_\varepsilon|^2\,du, \quad |\varepsilon| < \varepsilon_0.$$

As a consequence, it must be $\delta\mathbf{D}(\mathbf{X}_0,\lambda) = 0$ for any $\lambda \in C^1(\overline{D};\mathbb{R}^3)$. Using formula (6) we conclude that

$$|\partial_1 \mathbf{X}_0|^2 - |\partial_2 \mathbf{X}_0|^2 = 0, \quad \langle \partial_1 \mathbf{X}_0, \partial_2 \mathbf{X}_0 \rangle = 0$$

which means that $\mathbf{X}_0$ is conformal.

3.5. Some remarks

We conclude the section relative to the disc-type Plateau problem with some remarks. First of all about uniqueness: for a given boundary curve Γ there may exist a lot of solutions of different genus, orientable and non-orientable and so on. Concerning regularity, it has been proven (see for instance Gulliver [10]) that disc-type minimal surfaces cannot have singularities in the interior, so they are smooth surfaces. This does not means that disc-type solutions are embedded, and they also may have self-intersection, which is physically inconsistent. Hence, solutions of disc-type with self-intersection for sure are not a good model for soap films. All of these phenomena are related in

Figure 4: A soap film which bounds a Jordan wire but it is not of disc-type: the disc-type solution indeed should have a self-intersection.

some sense with the topology of the surface and the topology of the boundary curve.

If we wish to obtain more general existence results for soap films, we have to move to a more general frameworks which do not care about topology in some sense. We are going to describe these more general frameworks in the coming sections.

4. A review on measure theory

In this section we review the fundamental notions of measure theory that we need in the rest of the paper. We refer to the book by Ambrosio-Fusco-Pallara [3] for details.

4.1. Measures

We recall that a *measure space* is a pair $(X,\mathcal{E})$ where $X \neq \varnothing$ and $\mathcal{E}$ is a σ-algebra on X, that is:

(a) $\varnothing, X \in \mathcal{E}$,

(b) $X \backslash E \in \mathcal{E}$ whenever $E \in \mathcal{E}$,

(c) for any sequence (E_h) in $\mathcal{E}$ we have

$$\bigcup_{h=0}^{\infty} E_h \in \mathcal{E}.$$

A function $\mu\colon \mathcal{E} \to [0,+\infty]$ is said to be a *positive measure* on $(X,\mathcal{E})$ if

(a) $\mu(\varnothing) = 0$,

(b) for any sequence (E_h) of pairwise disjoint elements of $\mathcal{E}$ it holds

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h) \quad (\sigma\text{-additivity}).$$

We denote by $\mathcal{B}(\mathbb{R}^n)$ the *Borel* σ*-algebra* on $\mathbb{R}^n$, namely the smallest σ-algebra on $\mathbb{R}^n$ containing all the open subsets of $\mathbb{R}^n$; the elements of $\mathcal{B}(\mathbb{R}^n)$ are called *Borel subsets* of $\mathbb{R}^n$. A positive measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is also called a *Borel measure* on $\mathbb{R}^n$. A positive measure μ is called *finite* if $\mu(X) < +\infty$. A function $\mu\colon \mathcal{E} \to \mathbb{R}^m$, $m \in \mathbb{N}$ with $m \geqslant 1$, is said to be a *vector-valued measure* on $(X,\mathcal{E})$ if the previous conditions (a)-(b) hold true; in the case $m = 1$ we also say that μ is a *real-valued measure* on $(X,\mathcal{E})$. We denote by $|\mu|$ the *total variation* of μ, defined by

$$|\mu|(E) = \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)| : E = \bigcup_{h=0}^{\infty} E_h, E_h \in \mathcal{E} \text{ are pairwise disjoint}\right\}.$$

It is possibile to prove that $|\mu|$ is a positive finite measure on $(X,\mathcal{E})$. Let us mention the *polar decomposition* of μ: there exists a unique function $\eta \in L^1(X,|\mu|)^m$ such that

$|\eta| = 1$ and $\mu = \eta|\mu|$. Finally, if μ_h, μ are vector-valued measures on X we say that μ_h converges to μ *weakly**, and we write $\mu_h \rightharpoonup^* \mu$, if

$$\lim_{h\to\infty} \int_X u\, d\mu_h = \int_X u\, d\mu, \quad \forall u \in C_0(X)$$

where $C_0(X)$ is the space of all continuous functions $X \to \mathbb{R}$ vanishing at infinity. An important property is the lower semicontinuity of the total variation: if (μ_h) is a sequence of vector-valued measures on X and $\mu_h \rightharpoonup^* \mu$ then

$$|\mu|(X) \leqslant \liminf_h |\mu_h|(X).$$

4.2. Hausdorff measures

We recall the notion of Hausdorff measure which plays the role of length or area for subsets of higher dimensional spaces avoiding parametrization. Let $E \subseteq \mathbb{R}^n$, let $d \in [0, +\infty)$ and let $\delta > 0$. We define

$$\mathcal{H}^d_\delta(E) = \frac{\alpha_d}{2^d} \inf \left\{ \sum_{h=0}^{\infty} (\operatorname{diam} E_h)^d : E \subset \bigcup_{h=0}^{\infty} E_h \text{ and } \operatorname{diam} E_h \leqslant \delta \right\}$$

where α_d is a suitable renormalization constant (we will more precise about that in a moment). We also let

$$\mathcal{H}^d(E) = \lim_{\delta\to 0} \mathcal{H}^d_\delta(E) = \sup_{\delta>0} \mathcal{H}^d_\delta(E)$$

and we say that $\mathcal{H}^d(E)$ is the *d-dimensional Hausdorff measure of E*. It turns out that both $\mathcal{H}^d_\delta$ and $\mathcal{H}^d$ are σ-*subadditive* on $\mathbb{R}^n$, namely for any sequence (E_h)

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) \leqslant \sum_{h=0}^{\infty} \mu(E_h), \quad \mu = \mathcal{H}^d_\delta, \mathcal{H}^d.$$

Nevertheless, it is not true that $\mathcal{H}^d_\delta$ is σ-additive on disjoint Borel subsets of $\mathbb{R}^n$: actually, it is true that

$$\mathcal{H}^d_\delta(E \cup F) = \mathcal{H}^d_\delta(E) + \mathcal{H}^d_\delta(F)$$

whenever $\operatorname{dist}(E,F) > \delta$. This is one of the reasons why we need to send $\delta \to 0$. Indeed, $\mathcal{H}^d$ becomes σ-additive on disjoint Borel subsets of $\mathbb{R}^n$, hence a Borel measure on $\mathbb{R}^n$. It is sufficient to bserve that

$$\mathcal{H}^d(E \cup F) = \mathcal{H}^d(E) + \mathcal{H}^d(F)$$

holds true whenever $\operatorname{dist}(E,F) > 0$, and the σ-additivity follows from the well known Carathéodory's Theorem. We also have that $\mathcal{H}^d$ is invariant under isometries and scales as a d-dimensional volume:

$$\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E), \quad \forall \lambda \geqslant 0.$$

Moroever, if we choose α_d as the volume of the unit ball in $\mathbb{R}^d$, we have

$$\mathcal{L}^d = \mathcal{H}^d = \mathcal{H}^d_\delta$$

where $\mathcal{L}^d$ is the Lebesgue measure in $\mathbb{R}^d$.

5. Minimization of the Hausdorff measure

In this section we briefly discuss the possibility to minimize directly the Hausdorff measure on a suitable class of sets.

5.1. Minimization of $\mathcal{H}^d$

Since the notion of Hausdorff measure, a possible direct strategy could be to look at the surface simply as a set and try to minimize $\mathcal{H}^d$ on a suitable class. Let us see very briefly how this approach could be investigated. Consider the class $\mathcal{F}$ of all non-empty closed and connected subsets of a given compact domain D in $\mathbb{R}^n$. On $\mathcal{F}$ take the *Hausdorff distance*:

$$d_H(E,F) = \inf\{r \in [0,+\infty] : E \subset F_r,\, F \subset E_r\}, \quad E_r = \bigcup_{x \in E} B_r(x).$$

It turns out that $(\mathcal{F}, d_H)$ is a compact metric space. Moreover, if Γ is any subset of D the subclass

$$\mathcal{F}_\Gamma = \{E \in \mathcal{F} : \Gamma \subset E\}$$

is closed in $\mathcal{F}$, hence compact too. If we are thinking to direct methods in the Calculus of Variations compactness is fine, but what about lower semicontinuity of the Hausdorff measure with respect to the Hausdorff distance? By a well known theorem due to Gołab it is possibile to prove that $\mathcal{H}^1$ is lower semicontinuous on $\mathcal{F}$. Putting together the compactness result and the lower semicontinuity property it is possible to have the existence of sets with minimal length: for every $\Gamma \subset D$ there exists some connected and closed set E which minimizes $\mathcal{H}^1$ among all closed and connected sets which contain Γ. Can we apply a similar argument for the Plateau problem? The first main difficulty in stating the Plateau problem in this framework is represented by the boundary condition: what does it mean that a set spans some curve Γ? A possibility could be the following one: given a closed curve Γ in $\mathbb{R}^3$ find a compact set E_0 which minimizes $\mathcal{H}^2$ among all sets E such that Γ is homotopic to a constant in E. Another difficulty is that the semicontinuity of $\mathcal{H}^1$ depends heavily on the connectedness of the sets: if we drop this assumption, lower semicontinuity fails. The real problem is that no topological assumptions can ensure the semicontinuity of $\mathcal{H}^2$ (or $\mathcal{H}^d$ when $d > 1$). However, a direct approach to the Plateau problem in this direction has been investigated in the ’60s mainly by Reifenberg [14]: his proof of the existence result is rather complicated and it involves algebraic topology so we will not enter in details. Nevertheless, we will come back later on the set approach to Plateau problem.

6. The approach via sets of finite perimeter

In this section we briefly describe the theory of sets of finite perimeter which dates back to Caccioppoli but here we will define them by means of the distributional approach which is due to De Giorgi; for details we refer to the book by Ambrosio-Fusco-Pallara [3] or to the monograph by Maggi [13].

6.1. Sets of finite perimeter

In order to apply successfully the direct method of the Calculus of Variations we would like to define a class $\mathcal{F}$ of sets in $\mathbb{R}^n$ such that:

(a) $\mathcal{F}$ is endowed with a topology with good compactness properties so that sets with smooth boundaries belong to $\mathcal{F}$ and are dense;

(b) a notion of perimeter $\mathcal{P}(E)$ for any $E \in \mathcal{F}$ such that the map $E \mapsto \mathcal{P}(E)$ is lower semicontinuous on $\mathcal{F}$ and $\mathcal{P}$ extends $\mathcal{H}^{n-1}$: precisely, $\mathcal{P}(E) = \mathcal{H}^{n-1}(\partial E)$ whenever ∂E is a smooth hypersurface in $\mathbb{R}^n$;

(c) for every $E \in \mathcal{F}$ there is a sequence $E_h \to E$ such that ∂E_h are smooth and $\mathcal{H}^{n-1}(\partial E_h) \to \mathcal{P}(E)$.

It is evident that this program might work only for hypersurfaces, and this is one of the main drawbacks of the approach via sets of finite perimeter. However, let us see how it works. The key observation is that the boundary of a set is related with the distributional derivative of its the characteristic function. For instance, if $E = [a,b]$ then

$$\mathbf{1}_E' = \delta_a - \delta_b$$

which is a finite measure concentrated on $\{a,b\} = \partial E$. Moreover,

$$|\mathbf{1}_E'|(\mathbb{R}) = 2 = \mathcal{H}^0(\partial E).$$

Let us try to see how to extract the correct information from that, in a general situation. Let $E \subset \mathbb{R}^n$ be a Borel set with finite Lebesgue measure. We say that E is a *set of finite perimeter* if the distributional derivative of $\mathbf{1}_E$ is a vector-valued measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $\mathcal{B}(\mathbb{R}^n))$ is the σ-algebra of all Borel subsets of $\mathbb{R}^n$ (namely the smallest σ-algebra which contains all the open subsets of $\mathbb{R}^n$). In other words, there exist $\mu_1, \dots, \mu_n$ real-valued measures such that

$$\int_E \frac{\partial \varphi}{\partial x_i}\, dx = -\int \varphi\, d\mu_i, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), i = 1, \dots, n.$$

Equivalently, we can require

$$\int_E \operatorname{div} \phi\, dx = -\int \langle \phi, \eta \rangle\, d|\mu|, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),$$

where $\mu = \eta|\mu|$ is the polar decomposition of μ. The measure μ is therefore uniquely determined and is denoted by $D\mathbf{1}_E$. We thus define

$$\mathcal{P}(E) = |D\mathbf{1}_E|(\mathbb{R}^n)$$

which is called *perimeter of* E. We endow the class $\mathcal{F}$ of sets of finite perimeter with the L^1 distance, namely

$$d(E,F) = \|\mathbf{1}_E - \mathbf{1}_F\|_{L^1}.$$

Notice that $d(E,F) = \mathcal{L}^n(E\Delta F)$, where $E\Delta F$ denotes the symmetric difference between E and F. With this choice it is immediate to see that $E \mapsto \mathcal{P}(E)$ is lower semicontinuous: indeed, observe that

$$\mathcal{P}(E) = \sup_{\phi\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\,|\phi|\leqslant 1} \int_E \operatorname{div}\phi\, dx$$

namely $E \mapsto \mathcal{P}(E)$ is the supremum of a family of continuous functions. Let us see why sets with smooth boundary enter in this definition. If E is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary then for any $\phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ we have, applying the Divergence Theorem,

$$\int_E \operatorname{div}\phi\, dx = -\int_{\partial E} \langle \phi, \nu_E \rangle\, d\mathcal{H}^{n-1}$$

where ν_E is the inner unit normal of ∂E. This formula implies that E has finite perimeter and

$$D\mathbf{1}_E = \nu_E \cdot \mathbf{1}_{\partial E} \cdot \mathcal{H}^{n-1}$$

that is $\mathcal{P}(E) = \mathcal{H}^{n-1}(\partial E)$. It is also possible to show that for any set of finite perimeter E there exists a sequence of smooth sets E_h such that $E_h \xrightarrow{L^1} E$ and $\mathcal{P}(E_h) \to \mathcal{P}(E)$. We finally notice that we have also the compactness property: if $\{E_h\}$ is a sequence of sets of finite perimeter contained in a fixed ball and with uniformly bounded perimeter, then up to a subsequence (not relabeled) $E_h \xrightarrow{L^1} E$ where E has finite perimeter. The proof of the compactness property is not hard but is based on something we did not mention, that is the theory of *functions of bounded variation*. We briefly sketch the argument. If Ω is an open set in $\mathbb{R}^n$ a function $u \in L^1(\Omega)$ is said to be a function of *bounded variation* (the space of all these functions is denoted by $BV(\Omega)$) if the distributional derivative Du is a vector-valued measure on $\mathbb{R}^n$, namely there exist $\mu_1,\dots,\mu_n$ real-valued measures such that

$$\int u \frac{\partial \varphi}{\partial x_i}\, dx = -\int \varphi\, d\mu_i, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),\, i = 1,\dots,n.$$

Of course, E has finite perimeter if and only if $\mathbf{1}_E \in BV(\mathbb{R}^n)$. It turns out that on $BV(\Omega)$ we can put the weak* convergence

$$u_h \rightharpoonup^* u \Longleftrightarrow \begin{cases} u_h \xrightarrow{L^1} u \\ Du_h \rightharpoonup^* Du. \end{cases}$$

Moreover, and this is the key remark, this convergence is really a weak* convergence in the sense of functional analysis: it can be proved that $BV(\Omega)$ is the dual of a separable Banach space and the weak* convergence induced by such a duality is exactly what we have defined as weak* convergence. Compactness therefore follows from Banach-Alaoglu Theorem if we have $||u_h||_{L^1} \leqslant c$ and $|Du_h|(\Omega) \leqslant c$. If we translate these two conditions in terms of sets of finite perimeter we get the required compactness for sets of finite perimeter. We conclude the section concerning general properties of sets of finite perimeter with the structure theorem, which is due to De Giorgi and Federer. Before stating it, we recall the notion of *rectifiability*. By means of the Hausdorff measure it is possible to define a first very weak notion of surface. Let E be a Borel subset of $\mathbb{R}^n$. We say that E is *d-rectifiable* if

$$E = E_0 \cup \bigcup_{h=1}^{\infty} E_h$$

where $\mathcal{H}^d(E_0) = 0$ and E_h is contained in the image of a Lipschitz function $f_h \colon \mathbb{R}^d \to \mathbb{R}^n$. It is possibile to prove that the smoothness of the regular part of E can be strengthened: indeed, it turns out that E is d-rectifiable if and only if

$$E = E_0 \cup \bigcup_{h=1}^{\infty} E_h$$

where $\mathcal{H}^d(E_0) = 0$ and E_h is contained in a d-dimensional surface of class C^1. The class of d-rectifiable sets is the largest class for which it is still possible to give a notion of tangent space: it turns out that there is a Borel map τ that associate at any E a subspace of $\mathbb{R}^n$ of dimension d such that for every surface S of class C^1 and dimension d contained in E there holds

$$\tau(p) = \operatorname{Tan}(S,p), \quad \mathcal{H}^d\text{-a.e. } p \in S \cap E.$$

Moreover, such τ is unique up to a $\mathcal{H}^d$-null subset of E. We let

$$\operatorname{Tan}(E,p) = \tau(p)$$

and we call it the *approximate tangent space to E at p*. This construction looks strange, but we have to understand that the notion of tangent space to a rectifiable set is not defined in any pointwise way, in particular it does not make sense to specify $\operatorname{Tan}(E,p)$ at some given point p (like to specify the value at x of some $f \in L^p$). The key observation is that if S_1, S_2 are two C^1 surfaces with dimension d in $\mathbb{R}^n$ then $\operatorname{Tan}(S_1,p) = \operatorname{Tan}(S_2,p)$ for $\mathcal{H}^d$-a.e. $x \in S_1 \cap S_2$. In order to clarify this, think that S_i are locally the graph of maps $\mathbb{R}^d \to \mathbb{R}^{n-d}$ of class C^1. Let $f_1, f_2 \colon \mathbb{R}^d \to \mathbb{R}$ of class C^1 and let

$$I = \{x \in \mathbb{R}^d : f_1(x) = f_2(x) \text{ and } df_1(x) \neq df_2(x)\}.$$

Then I is a C^1 surface with dimension $d-1$ (or it is empty).

Now, we review the notion of density. For any $t \in [0,1]$ and for any E Borel set in $\mathbb{R}^n$,

$$E^t = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\alpha_n r^n} = t \right\}.$$

We say that x has *density* t if $x \in E^t$. We let

$$\partial^* E = \mathbb{R}^n \backslash (E^0 \cup E^1).$$

The sets E^0 and E^1 could be considered as the measure theoretic exterior and interior of E respectively. Thus $\partial^* E$, called *essential boundary of* E, in the sense of measure theory should be the "nice" part of ∂E. Indeed, if now E has finite perimeter set in $\mathbb{R}^n$ then the De Giorgi structure Theorem says that:

(a) $\partial^* E$ is $(n-1)$-rectifiable and $\mathcal{P}(E) = \mathcal{H}^{n-1}(\partial^* E) < +\infty$;

(b) $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ has density $1/2$;

(c) there exists a Borel map $\nu \colon \partial^* E \to \mathbb{R}^n$ such that if

$$E_{x,r} = \frac{1}{r}(E - x)$$

then for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ we have $\mathbf{1}_{E_{x,r}} \to \mathbf{1}_{H_{\nu(x)}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where

$$H_v = \{x \in \mathbb{R}^n : \langle x, v \rangle \geqslant 0\};$$

(d) $D\mathbf{1}_E = \nu \cdot \mathbf{1}_{\partial^* E} \cdot \mathcal{H}^{n-1}$

The map ν is also called *approximate inner normal to* E. Statement (c) says that if we blow up centering in a point on $\partial^* E$ we obtain an half plane orthogonal to $\nu(x)$.

6.2. Plateau problem in the context of sets of finite perimeter

We are ready to apply the theory of sets of finite perimeter to the Plateau problem. Fix Ω a bounded open and convex subset of $\mathbb{R}^3$ (we do in $\mathbb{R}^3$ but more generally it can be done in $\mathbb{R}^n$). Let Γ be a curve on $\partial\Omega$ which is the boundary, relative to $\partial\Omega$, of some $\Sigma_0 \subset \partial\Omega$. We then construct a smooth, bounded and open set $E_0 \subset \mathbb{R}^3 \backslash \Omega$ such that $\partial E_0 \cap \partial\Omega = \Sigma_0$. Of course we are assuming that $\partial\Omega, \Gamma, \Sigma_0$ are regular enough. The idea is to look at minimizers of $\mathcal{P}(E)$ among all sets of finite perimeter such that $E \backslash \Omega = E_0$. In order to find a closed relation, we relax this last condition in $\mathcal{L}^3((E \backslash \Omega) \Delta E_0) = 0$. Then, applying the direct method of the Calculus of Variations we can say that if X denotes the class of all sets of finite perimeter E such that $\mathcal{L}^3((E \backslash \Omega) \Delta E_0) = 0$ then the problem

$$\min_{E \in X} \mathcal{P}(E)$$

has a solution. One might wonder why we did not follow a simpler argument, that is minimize the perimeter among all sets E contained in $\overline{\Omega}$ such that $\partial^* E \cap \partial\Omega = \Sigma_0$. The reason is that the measure theoretic version $\mathcal{H}^2((\partial^* E \cap \partial\Omega) \Delta \Sigma_0) = 0$ is not closed: behind this fact there is the lack of weak*-continuity of the trace operator of *BV* functions.

6.3. Concluding remarks on the approach via sets of finite perimeter

The approach presented in this section imposes strong constraints on the geometry of the boundary curve Γ. Actually, the theory of sets of finite perimeter is not really suited for the Plateau problem. A typical problem for which sets of finite perimeter are a good framework is the following one: find the domain E in $\mathbb{R}^3$ which minimizes

$$\mathcal{H}^2(\partial E) + \int_E f(x)\,dx + \text{additional constraints (e.g. volume prescribed).}$$

Concerning regularity, one would like to show that the minimizer found in the class of sets of finite perimeter is smooth enough in order to say that is a soap film. However, regularity results are hard to prove and the theory of sets of finite perimeter hides some of the deep technical difficulties inherent to the Plateau problem. In order to say something, consider the simplest issue: the regularity of a set E in $\mathbb{R}^n$ which minimizes the perimeter with respect to all possible compact supported perturbations. Then it is possibile to show that $\partial E \backslash S$ is smooth, where S is a closed set of singularities, and:

(a) if $2 \leqslant n \leqslant 7$ then S is empty (in fact, ∂E is analytical);

(b) if $n = 8$ then S has no accumulation points in E;

(c) if $n \geqslant 9$ then $\mathcal{H}^d(S) = 0$ for every $d > n-8$.

This regularity statement can be obtained combining a lot of results proved by Almgren, Allard, Bombieri, De Giorgi, Federer, Schoen, Simon, and others, in the study of area minimizing currents and stationary varifolds. An explicit example showing this result is given by the cone

$$E = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$$

which turns out to be a minimizer with respect to compactly supported perturbations: this is the famous result conjectured by Simons and its minimality was finally proved by Bombieri, De Giorgi and Giusti in 1969 (see [4]). In any case, if we want to remain in $\mathbb{R}^3$, using sets of finite perimeter we cannot hope to model soap films which develop singularities.

7. The approach via currents

We want to describe another distributional approach to the Plateau problem, namely the approach via currents. In some sense such an approach is the real distributional approach since the space of currents is defined as the dual of a suitable space exactly as in the classical theory of distributions. The notion of current goes back to De Rham and related works on differential geometry, but soon this tool entered analysis in order to have a suitable weak notion of surface, and this is due mainly to Federer and Fleming. Here we only sketch the theory of currents; for details see Federer [9] or Simon [15].

7.1. Covectors and simple vectors

First of all we need some elements of multilinear algebra. Let V be a real vector space of dimension n; we denote by V^* the dual of V. For any $d \in \{0,\ldots,n\}$ a *d-covector on V* is simply a linear map $\alpha\colon V^d \to \mathbb{R}$ which is *alternating*, that is

$$\alpha(v_{\sigma(1)},\ldots,v_{\sigma(d)}) = \operatorname{sgn}(\sigma)\alpha(v_1,\ldots,v_d)$$

whenever $\sigma \in S_d$, the set of all permutations on $\{1,\ldots,d\}$, and $\operatorname{sgn}(\sigma)$ stands for the sign of σ. The vector space of all d-covectors on V is denoted by $\Lambda^d(V)$. By convention, we let $\Lambda^0(V) = V$. Moreover, $\Lambda^1(V) = V^*$. We introduce the so called *exterior product* in $\Lambda^d(V)$: for any $\alpha \in \Lambda^d(V)$ and $\beta \in \Lambda^{d'}(V)$ let $\alpha \wedge \beta \in \Lambda^{d+d'}(V)$ defined by means of

$$\alpha \wedge \beta(v_1,\ldots,v_{d+d'}) = \frac{1}{d!d'!}\sum_{\sigma\in S_{d+d'}} \alpha(v_{\sigma(1)},\ldots,v_{\sigma(d)})\beta(v_{\sigma(d+1)},\ldots,v_{\sigma(d+d')}).$$

The meaning of the normalization constant will be explained later. By construction, we have

$$\beta \wedge \alpha = (-1)^{dd'}\alpha\wedge\beta, \quad \alpha\wedge\alpha = 0, \quad \alpha\wedge(\beta\wedge\eta) = (\alpha\wedge\beta)\wedge\eta.$$

Now, we want to construct a basis in $\Lambda^d(V)$. Let us fix a basis $\{e_1,\ldots,e_n\}$ in V. Consider the *dual basis of* $\{e_1,\ldots,e_n\}$ denoted by $\{dx^1,\ldots,dx^n\}$ where

$$dx^i \in V^*, \quad dx^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The dual basis $\{dx^1,\ldots,dx^n\}$ is a basis in $V^* = \Lambda^1(V)$. For any $i_1,\ldots,i_d \in \{1,\ldots,n\}$ we have $dx^{i_1}\wedge\cdots\wedge dx^{i_d} \in \Lambda^d(V)$. It is possible to show that

$$\{dx^{i_1}\wedge\cdots\wedge dx^{i_d} : i_j \in \{1,\ldots,n\}\}$$

is a basis in $\Lambda^d(V)$. As a consequence of that and remembering the properties of $\wedge$, any $\alpha \in \Lambda^d(V)$ can be written in a unique way as

$$\alpha = \sum_{i_1<\cdots<i_d} \alpha_{i_1,\ldots,i_d}dx^{i_1}\wedge\cdots\wedge dx^{i_d}, \quad \alpha_{i_1,\ldots,i_d} \in \mathbb{R}.$$

As a consequence,

$$\dim\Lambda^d(V) = \binom{n}{d}.$$

We remark that

$$dx^{i_1}\wedge\cdots\wedge dx^{i_d}(v_1,\ldots,v_d) = \det A \tag{7}$$

where A is the $d\times d$ matrix defined by $A_{j\ell} = dx^{i_j}(v_\ell)$, that is the matrix whose ℓ-th column is given by the coordinates of v_ℓ with respect to the basis $\{e_1,\ldots,e_n\}$ (in order to have (7) the normalization constant in the definition of $\wedge$ plays a role). Using d-covectors we can define the simple d-vectors, which are the main objects we need. Define, on V^d, the equivalence relation $\sim$ given by

$$(v_1,\dots,v_d) \sim (v'_1,\dots,v'_d) \Longleftrightarrow \alpha(v_1,\dots,v_d) = \alpha(v'_1,\dots,v'_d) \quad \forall\alpha \in \Lambda^d(V).$$

We call *simple d-vector* any element $[v_1,\dots,v_d] \in V/\sim$; we also write 0 for $[0,\dots,0]$. It is possible to prove that

(a) $(v_1,\dots,v_d) \sim (0,\dots,0)$ if and only if $v_1,\dots,v_d$ are linearly dependent;

(b) $(v_1,\dots,v_d) \sim (v'_1,\dots,v'_d) \nsim (0,\dots,0)$ then

$$\operatorname{span}\{v_1,\dots,v_d\} = \operatorname{span}\{v'_1,\dots,v'_d\}.$$

Moreover, the matrix of change of basis has determinant 1.

Assume now that V is endowed with a scalar product. For any $v_1,\dots,v_d \in V$, let $R(v_1,\dots,v_d)$ be the rectangle spanned by $v_1,\dots,v_d$. Notice that if

$$(v_1,\dots,v_d) \sim (v'_1,\dots,v'_d) \nsim (0,\dots,0)$$

then $R(v_1,\dots,v_d)$ and $R(v'_1,\dots,v'_d)$ have the same d-dimensional volume: such a volume is denoted by $|[v_1,\dots,v_d]|$ and is called *norm*[‡] of the simple d-vector $[v_1,\dots,v_d]$. Recall that if W is a vector space then an *orientation of W* is an equivalence class of bases where two bases are equivalent if the change of basis matrix has positive determinant. Therefore, if again

$$(v_1,\dots,v_d) \sim (v'_1,\dots,v'_d) \nsim (0,\dots,0)$$

and $W = \operatorname{span}\{v_1,\dots,v_d\}$ then $(v_1,\dots,v_d)$ and $(v'_1,\dots,v'_d)$ induce on W the same orientation. This means that the map

$$0 \neq [v_1,\dots,v_d] \mapsto (W, \text{orientation of } W, |[v_1,\dots,v_d]|)$$

is well defined. It is also possible to show that such a map is one-to-one. This is the main point: we have that unitary simple d-vectors are in one-to-one correspondence with oriented d-planes in V, and this permits to have an algebra on the set of oriented d-planes in V.

7.2. Orientation of d-dimensional surfaces

We want to define what an orientation on a surface is, since forms, as we will se later, can be integrated only on oriented surfaces. Let S be a smooth d-dimensional surface in $\mathbb{R}^n$. An *orientation on S* is a continuous[§] map that assigns to each $x \in S$ a unit simple

[‡]Be careful, this cannot be a norm in the sense of normed spaces since the space of all simple d-vectors is not linear.

[§]Recall that the space of all simple d-vectors is a quotient of a topological space, hence topological too.

d-vector $\tau(x) = [v_1(x), \dots, v_d(x)]$ which spans $\mathrm{Tan}(S,x)$. If S has an orientation and has a boundary $\partial S \neq \emptyset$, there is a canonical way to orient also ∂S if we have fixed an orientation on S. Precisely, for any $x \in \partial S$ we can define the exterior normal $\eta(x)$. Then, if S is oriented by $\tau = [v_1, \dots, v_d]$ we endow ∂S with the orientation $[v'_1, \dots, v'_{d-1}]$ such that

$$[v_1(x), \dots, v_d(x)] = [\eta(x), v'_1(x), \dots, v'_{d-1}(x)], \quad \forall x \in \partial S.$$

7.3. Differential forms

Using d-covectors we are able to introduce differential forms on surfaces. Let Ω be an open set in $\mathbb{R}^n$. A *d-form on* Ω is a "smooth map" ω that assigns to each $x \in \Omega$ an element $\omega(x) \in \Lambda^d(\mathbb{R}^n)$. In order to clarify what smooth means, let us write ω in coordinates. If we fix a basis $\{e_1, \dots, e_n\}$ in $\mathbb{R}^n$ then we can write

$$\omega(x) = \sum_{i_1 < \cdots < i_d} \omega_{i_1,\dots,i_d}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_d}, \quad \omega_{i_1,\dots,i_d} : \Omega \to \mathbb{R}.$$

It is now easy to set what smooth means: simply, $\omega_{i_1,\dots,i_d} \in C^\infty(\Omega)$. It is easy to see that this regularity does not depend on the choice of the basis $\{e_1, \dots, e_n\}$. The most important operation on forms is the *exterior derivative*:

$$d\omega = \sum_{i_1 < \cdots < i_d} d\omega_{i_1,\dots,i_d} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_d}.$$

Here we are assuming that

$$d\omega_{i_1,\dots,i_d} = \sum_{i=1}^{n} \frac{\partial \omega_{i_1,\dots,i_d}}{\partial x_i} dx^i.$$

It turns out that $d\omega$ is a $(d+1)$-form.

7.4. Integration of forms on surfaces

We now move to the integration of forms on surfaces. The main application of the theory of forms is the integration on (oriented) surfaces, and the Stokes formula, which is the key point in order to understand why we need forms for the notion of current. If S is a smooth and oriented d-dimensional surface in $\mathbb{R}^n$, τ is an orientation on S, and ω is a d-form on some open set containing S, we let

$$\int_S \omega = \int_S \langle \omega(x), \tau(x) \rangle \, d\mathcal{H}^d(x)$$

whenever the integral on the right hand-side exists. If now S is also compact and ω is a $(d-1)$-form on some open set containing S we have the *Stokes formula*:

$$\int_{\partial S} \omega = \int_S d\omega.$$

Of course, here we are assuming that ∂S has the canonical orientation induced by τ.

7.5. Vectors

We construct the space of d-vectors on V exploiting the fact that V can be canonically identified with its dual V^*. Precisely, we let $\Lambda_d(V) := \Lambda^d(V^*)$. The duality between V and V^* extends to a duality between $\Lambda^d(V)$ and $\Lambda_d(V)$. We in fact use the natural reflexivity of V, that is V^{**} is canonically isomorphic to V, which means that if $\{e_1,\dots,e_n\}$ is a basis on V then $\{e_1,\dots,e_n\}$ is still a basis on V^{**} simply setting

$$e_i(dx^j) = \delta_i^j.$$

In particular, notice that the quantity $v_1 \wedge \cdots \wedge v_d$ turns out to be well defined whenever $v_1,\dots,v_d \in V$. Moreover, it is possibile to show that for any $\alpha \in \Lambda^d(V)$ there holds

$$\alpha(v_1 \wedge \cdots \wedge v_d) = \alpha(v_1,\dots,v_d).$$

Remember now that $(v_1,\dots,v_d) \sim (v_1',\dots,v_d')$ if and only if $\alpha(v_1,\dots,v_d) = \alpha(v_1',\dots,v_d')$ for any $\alpha \in \Lambda^d(V)$, which the means that

$$\alpha(v_1 \wedge \cdots \wedge v_d) = \alpha(v_1' \wedge \cdots \wedge v_d'), \quad \forall \alpha \in \Lambda^d(V).$$

Thus, $(v_1,\dots,v_d) \sim (v_1',\dots,v_d')$ if and only if $v_1 \wedge \cdots \wedge v_d = v_1' \wedge \cdots \wedge v_d'$. In particular, we can identify the simple d-vector $[v_1,\dots,v_d]$ with $v_1 \wedge \cdots \wedge v_d$. If V is endowed with a scalar product, we can define the *mass norm* on $\Lambda_d(V)$ as the convex envelope of the restriction of the Euclidean norm to simple d-vectors, that is

$$\|v\| := \inf\left\{ \sum_{i=1}^N t_i|v_i| : v_i \text{ is simple and } \sum_{i=1}^N t_i = 1 \right\}.$$

Accordingly, we can define the *comass norm* of $\alpha \in \Lambda^d(V)$ as the dual norm of the mass norm, that is

$$\|\alpha\| := \sup\{|\alpha(v)| : ||v|| \leqslant 1\}.$$

7.6. Currents and Plateau problem in terms of currents

We are ready to define currents. In order to explain the idea, take a d-dimensional surface in $\mathbb{R}^n$ with boundary ∂S. If we mimic the definition of distribution we could consider a linear and “continuous" functional of the type

$$\varphi \mapsto \int_S \varphi \, d\mathcal{H}^d,$$

where, as in the classical theory of distributions, φ belongs to a suitable set of test functions. This seems to be the natural way to have a weak notion of surface. But there is a huge drawback: what is the weak version of ∂S? It is essential to have that, since we are dealing with the Plateau problem. There is no chance: if we decide to follow the natural idea to integrate functions on S we are in trouble with the boundary. The right idea comes from the integration of forms: Stokes formula provides the distributional notion of ∂S. More precisely, if $\mathcal{D}^d(\mathbb{R}^n)$ denotes the set of all d-forms on $\mathbb{R}^n$ with compact support, then the dual space of $\mathcal{D}^d(\mathbb{R}^n)$ is the space of all *d-currents on* $\mathbb{R}^n$, denoted by $\mathcal{D}_d(\mathbb{R}^n)$. Here, dual means topological dual, so in principle one has to construct a topology on $\mathcal{D}^d(\mathbb{R}^n)$: this can be done as in the standard theory of distributions. Accordingly with the dual nature of $\mathcal{D}_d(\mathbb{R}^n)$, if $T_h, T \in \mathcal{D}_d(\mathbb{R}^n)$ we say that $T_h \to T$ if

$$\langle T_h, \omega \rangle \to \langle T, \omega \rangle, \quad \forall \omega \in \mathcal{D}^d(\mathbb{R}^n).$$

Of course, as for distributions, the main example of a current is given by a smooth surface: if S is a smooth d-dimensional oriented surface in $\mathbb{R}^n$ we define $T_S \in \mathcal{D}_d(\mathbb{R}^n)$ by means of

$$\langle T_S, \omega \rangle = \int_S \omega, \quad \forall \omega \in \mathcal{D}^d(\mathbb{R}^n).$$

The next crucial notion is the definition of boundary of a current, and this can be well defined via the Stokes formula: if we look again at the smooth case, we notice that

$$\mathcal{D}^{d-1}(\mathbb{R}^n) \ni \omega \mapsto \int_{\partial S} \omega = \int_S d\omega = \langle \partial T_S, d\omega \rangle$$

defines a $(d-1)$ current which is the canonical current associated to ∂S. Then, in general if $T \in \mathcal{D}_d(\mathbb{R}^n)$ we define the *boundary of T* as $\partial T \in \mathcal{D}_{d-1}(\mathbb{R}^n)$ given by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle.$$

By construction, for oriented surfaces we have

$$\partial T_S = T_{\partial S}.$$

Thus, we have found a weak notion of surface and a corresponding weak notion of its boundary. In order to at least state the Plateau problem it remains to understand what is the "area" of a current. We introduce the *mass* of a current in this way: if $T \in \mathcal{D}_d(\mathbb{R}^n)$ then we let

$$\mathbb{M}(T) := \sup_{||\omega(x)|| \leqslant 1} \langle T, \omega \rangle.$$

For oriented surfaces we have

$$\mathbb{M}(T_S) = \mathcal{H}^d(S).$$

Before going on, let us see an illustrative example. In $\mathbb{R}^2$ take the segment $I = [0,1] \times \{0\}$. We orient I using $\tau(x) = e_1 = (1,0)$, for any $x \in I$. We want to see what the boundary of T_I is. For any 0-form ω with compact support in $\mathbb{R}^2$, that is a smooth function with compact support in $\mathbb{R}^2$, we have

$$\langle \partial T_I, \omega \rangle = \langle T_I, d\omega \rangle = \int_I \langle \frac{\partial \omega}{\partial x_1} dx^1 + \frac{\partial \omega}{\partial x_2} dx^2, e_1 \rangle d\mathcal{H}^1 = \int_0^1 \frac{\partial \omega}{\partial x_1} dx_1 = \omega(1,0) - \omega(0,0)$$

that is

$$\partial T_I = \delta_{(1,0)} - \delta_{(0,0)}.$$

Generalizing this example, if $\gamma\colon [0,1] \to \mathbb{R}^n$ is a smooth function and $C = \gamma(I)$ is oriented by the tangent vector γ' then

$$\partial T_C = \delta_{\gamma(1)} - \delta_{\gamma(0)}.$$

We will come back to this example, now we go on with the theory. Of course, currents of interests have finite mass. These currents can be characterized. Let μ be a real measure on $\mathbb{R}^n$ and let $\tau \in L^1_\mu(\mathbb{R}^n; \Lambda_d(\mathbb{R}^n))$. Define the current $T = \tau\mu$ as

$$\langle T, \omega \rangle = \int \langle \omega(x), \tau(x) \rangle \, d\mu.$$

Then, one easily has $\mathbb{M}(T) \leqslant ||\tau||_1 = |\tau\mu|(\mathbb{R}^n)$, hence T has finite mass. Actually, this is the general case. Indeed, if $\mathbb{M}(T) < +\infty$ then T is a linear functional on $\mathcal{D}^d(\mathbb{R}^n)$ which is bounded with respect to the supremum norm on forms. Hence T can be extended by density to a linear functional on the closure of $\mathcal{D}^d(\mathbb{R}^n)$ with respect to the supremum norm, which is the space of all continuous d-form vanishing at infinity. Therefore, T is represented by a vector-valued measure with values in the dual of $\Lambda^d(\mathbb{R}^n)$, which is $\Lambda_d(\mathbb{R}^n)$, and all such measures can be written as $\tau\mu$ as in the previous example. In view of this equivalence, we can state a first compactness/lower semicontinuity theorem: if (T_h) is a sequence of d-currents with finite mass such that $\mathbb{M}(T_h) \leqslant c$ for some $c > 0$ then, up to a subsequence (not relabeled), $T_h \to T$ in $\mathcal{D}_d(\mathbb{R}^n)$ and

$$\mathbb{M}(T) \leqslant \liminf_h \mathbb{M}(T_h).$$

In particular, T has finite mass. For the Plateau problem we wish also to consider currents such that also the boundary has finite mass. We say that a d-current is a *normal current* if both $\mathbb{M}(T)$ and $\mathbb{M}(\partial T)$ are finite. The advantage is that for normal currents we have compactness and lower semicontinuity of the masses: if (T_h) is a sequence of normal d-currents such that $\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leqslant c$ for some $c > 0$ then, up to a subsequence (not relabeled), $T_h \to T$ in $\mathcal{D}_d(\mathbb{R}^n)$, $\partial T_h \to \partial T$ in $\mathcal{D}_{d-1}(\mathbb{R}^n)$ and

$$\mathbb{M}(T) \leqslant \liminf_h \mathbb{M}(T_h), \quad \mathbb{M}(\partial T) \leqslant \liminf_h \mathbb{M}(\partial T_h).$$

In particular, T is normal. Indeed, let us apply the compactness and lower semicontinuity theorem for currents with finite mass both to T_h and ∂T_h. Up to subsequences, $T_h \to T$, $\partial T_h \to U$, and

$$\mathbb{M}(T) \leqslant \liminf_h \mathbb{M}(T_h), \quad \mathbb{M}(U) \leqslant \liminf_h \mathbb{M}(\partial T_h).$$

It is sufficient to prove that $U = \partial T$. Let $\omega \in \mathcal{D}^{d-1}(\mathbb{R}^n)$. Then

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle = \lim_h \langle T_h, d\omega \rangle = \lim_h \langle \partial T_h, \omega \rangle = \langle U, \omega \rangle$$

and this yields the conclusion. Thanks to the previous theorem, we can solve the Plateau problem in terms of normal currents. Let T_0 be a given normal d-current on $\mathbb{R}^n$. Then the problem

$$\min\{\mathbb{M}(T) : T \text{ is a normal } d\text{-current and } \partial T = \partial T_0\}$$

has a solution. Notice that in this formulation of the Plateau problem we fix T_0 normal d-current and we ask for minimizers in the class

$$\{T \text{ is a normal } d\text{-current and } \partial T = \partial T_0\}.$$

The natural way to set the problem would be fix T_0 normal $(d-1)$-current and ask for minimizers in the class

$$\{T \text{ is a normal } d\text{-current and } \partial T = T_0\}.$$

But in this last case one has to prove that this class is notempty, and this could be not trivial. In other words, we are saying that the admissible boundary data are all the objects which are really obtained as the boundary of something. Nevertheless, this solution to the Plateau problem is not satisfactory because the class of normal currents is too large. Let us see an example. Let T be the 1-current on $\mathbb{R}^2$ given by $T = \tau\mu$ where μ is the Lebesgue measure on the square $Q = [-1,1]^2$ and $\tau(x) = e_1 = (1,0)$ for any $x \in Q$. Notice that $\mathbb{M}(T) = 4$. We want to find ∂T. Let ω be a 0-form (that is, a function) with compact support on $\mathbb{R}^2$. Then we have

$$\langle \partial T, \omega\rangle = \langle T, d\omega\rangle = \int_Q \langle \frac{\partial\omega}{\partial x_1} dx^1 + \frac{\partial\omega}{\partial x_2} dx^2, e_1\rangle\, dx = \int_Q \frac{\partial\omega}{\partial x_1}\, dx$$

$$= \int_{-1}^{1} \omega(1,x_2) - \omega(-1,x_2)\, dx_2 = \int \omega\tau'\, d\mu'$$

where μ' is $\mathcal{H}^1$ restricted to $I_\pm = \{\pm 1\} \times [-1,1]$ and $\tau' = +1$ on I_+ and $\tau' = -1$ on I_-. In particular, $\mathbb{M}(\partial T) = 4$ hence T is a normal current. This example suggests that working with normal currents we might obtain, in general, very mild solutions. The idea is to consider currents which, in some sense, are “supported" on at least a rectifiable set. We say that $T \in \mathcal{D}_d(\mathbb{R}^n)$ is a *d-rectifiable current* if there exist:

(a) a d-rectifiable set E in $\mathbb{R}^n$,

(b) an *orientation* τ *on* E, that is a Borel map that to $\mathcal{H}^d$-a.e. $x \in E$ assigns a unit simple d-vector $\tau(x)$ which spans $\mathrm{Tan}(E,x)$,

(c) a *multiplicity function*, that is a summable (with respect to the $\mathcal{H}^d$ measure) function $m\colon E \to \mathbb{R}$,

such that

$$\langle T, \omega\rangle = \int_E \langle \omega(x), \tau(x)\rangle m(x)\, d\mathcal{H}^d(x), \quad \forall \omega \in \mathcal{D}^d(\mathbb{R}^n).$$

In this case, we denote T by $[E,\tau,m]$. We notice that it holds

$$\mathbb{M}([E,\tau,m]) = \int_E |m|\, d\mathcal{H}^d.$$

If S is a smooth d-dimensional surface oriented by τ then

$$T_S = [S,\tau,1].$$

Notice that at a first sight it seems that we can treat also non-orientable surfaces in the framework of (integral) currents, since we are not assuming any continuity of the orientation. Indeed, by definition an orientation of a rectifiable set is simply a Borel choice of a unit simple d-vector which spans the approximate tangent space $\mathcal{H}^d$-a.e. on E. Actually something wrong happens if we wish to preserve the "physical" boundary: indeed, a discontinuity of the orientation affects the boundary of T. As an example take

$$T = [[0,1],-e_1,1] + [[1,2],e_1,1].$$

Then,

$$\partial T = \delta_2 + \delta_0 - 2\delta_1.$$

As a consequence, if we use the framework of integral currents we find only good models for orientable soap films, since any discontinuity in the orientation produces some boundary which is not physical. As for normal currents, we wish to consider d-rectifiable currents such that also the boundary is rectifiable. We then try to look at compactness and lower semicontinuity for such a currents. The bad thing is that there is no compactness: let us sketch an example. Let

$$E_h = \bigcup_{k=0}^{h-1} [0,1] \times \left\{ \frac{1}{2h} \right\}, \quad T_h = \left[E_h, e_1, \frac{1}{h} \right].$$

First of all notice that both T_h and ∂T_h are rectifiable and $\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) = 3$. Nevertheless, it is possible to prove that

$$T_h = \left[E_h, e_1, \frac{1}{h} \right] \to T = e_1 \mathcal{L}^2 \llcorner [0,1]^2$$

and T is not rectifiable. This is due to the fact that the multiplicity is arbitrarily close to 0. Assuming integer multiplicity it is possible to prove what we need, that is the celebrated *Federer-Fleming Compactness Theorem*, which is, probably, the most important result in the theory of currents. Precisely, we say that $[E,\tau,m]$ is a *d-rectifiable current with integer multiplicity* if m takes values in $\mathbb{Z}$, and finally we say that $T \in \mathcal{D}_d(\mathbb{R}^n)$ is a *k-integral current* if both T and ∂T are rectifiable currents with integer multiplicity. The Federer-Fleming Compactness Theorem states that if (T_h) is a sequence of integral d-currents such that $\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leqslant c$ for some $c > 0$ then, up to a subsequence (not relabeled), $T_h \to T$ in $\mathcal{D}_d(\mathbb{R}^n)$, $\partial T_h \to \partial T$ in $\mathcal{D}_{d-1}(\mathbb{R}^n)$ and

$$\mathbb{M}(T) \leqslant \liminf_h \mathbb{M}(T_h), \quad \mathbb{M}(\partial T) \leqslant \liminf_h \mathbb{M}(\partial T_h).$$

Moreover, T is an integral current. As for normal currents, thanks to the Federer-Fleming Theorem we can solve the Plateau problem in terms of integral currents, which is, in some sense, the "right" formulation of the Plateau problem in terms of currents. Let T_0 be a given integral d-current on $\mathbb{R}^n$. Then the problem

$$\min\{\mathbb{M}(T) : T \text{ is a integral } d\text{-current and } \partial T = \partial T_0\}$$

has a solution.

7.7. Concluding remarks on the theory of currents

First of all notice that, in principle, one could obtain, as a solution of the Plateau problem in the context of integral currents, a current with some multiplicity different from 1. This can happen, think to minimizing sequences of currents which attach in the limit in some region with positive mass. Actually, the right object to minimize should be the *size* of a current and not the mass, where the size is defined as

$$\mathbb{S}([E,\tau,m]) = \mathcal{H}^d(\{x \in E : m(x) \neq 0\}).$$

But the situation for size minimizers is far from clear. Even if $d = 2$ and S is the current of integration on a smooth curve, there is no general existence result for an integral current T such that $\partial T = S$ and $\mathbb{S}(T)$ is minimal. The main problem is compactness: from a bound on the size we are not able, in general, to deduce a bound on the mass, hence we are not in position to apply the Federer-Fleming compactness result. Another drawback of the use of integral currents is that we cannot treat non-orientable boundaries since, as we have already observed, the discontinuity of the orientation produces, in general, new boundaries. However, there is a possibility to obtain also non-orientable soap films using currents: it is sufficient to work with rectifiable currents *modulo* ν, where $\nu \geqslant 2$ is an integer. More precisely, two rectifiable currents T and S are *congruent modulo* ν if $T - S = \nu Q$ for some current Q. In particular, we can say that T and $-T$ are congruent modulo 2, and this permits, in principle, to solve the Plateau problem in a more general context using the equivalence classes of rectifiable currents modulo ν. We point out that non-orientable surfaces occur as soap films. We also remark that there is an alternative perspective to currents for working with non-oriented objects, which is the theory of *varifolds*, introduced by Almgren in '70 and developed mainly by Allard and Hutchinson in view of the applications to variational problems which involve curvatures of surfaces. We do not want to enter in details, the interested reader can consult for instance the original paper by Almgren [1]. We just point out that for varifolds the main difficulty is to produce a good definition of boundary. We conclude with some remarks on the regularity. How much regular are the soap films produced by integral currents? In $\mathbb{R}^2$ the situation is almost perfect: indeed, it is possible to prove that if T is a mass-minimizing 1-integral current in $\mathbb{R}^2$ then the "interior part" of T (the part of T which is not in the boundary of T) consists of disjoint line segments. In the case $2 \leqslant n \leqslant 7$, if T is a mass-minimizing $(n-1)$-integral current in $\mathbb{R}^n$ then the interior part of T is a smooth embedded hypersurface: if we go back to Figure 4 the soap film solution corresponds to an embedded solution, which could be

Figure 5: A Möbius strip-like soap film

the mass-minimizing integral current. When $n > 7$ we already know that the regularity is lost, since the result of Bombieri-De Giorgi-Giusti [4]. In any case, as for sets of finite perimeter, we cannot hope to have the singularities developed by some soap films. However, notice that such singularities appear when we have, as boundary, curved objects which are not well covered by the theory of currents: for instance, in the Figure 1 we can see a singular soap film, surely not covered by currents, but here the problem is that this soap film cannot be reduced to a rectifiable current whose boundary, in the sense of currents, is the set of the edges of a cube.

8. Minimal sets approach

Perhaps the best model for the soap films is represented by the Almgren minimal sets, introduced by Almgren in [2]. The idea is to come back, in some sense, to the set approach, as the already cited one by Reifenberg [14]. The surface is a (d-rectifiable) set and we minimize the d-dimensional Hausdorff measure among a suitable class of sets.

8.1. Almgren minimal sets and Taylor regularity

Let $S \subset \mathbb{R}^n$ be a closed set and $A \subset \mathbb{R}^n$ be an open set. We say that S is a *d-dimensional minimal set in A* (briefly *minimal set* if we do not need further details) if for any closed ball $C \subset A$ and every Lipschitz map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi_{|\mathbb{R}^n \setminus C} = id$ and $\varphi(C) \subset C$ we have

$$\mathcal{H}^d(S) \leqslant \mathcal{H}^d(\varphi(S)).$$

Roughly speaking, a minimal set is such that if we apply any local deformation of the set the d-dimensional Hausdorff measure increases. The reason for which minimal sets are the best model for soap films stems in the regularity theorem for such objects.

Indeed, J. Taylor in 1976 [16] proved that the singularities of 2-dimensional minimal sets in $\mathbb{R}^3$ are precisely those produced by soap films. The analysis of Taylor is very deep. First of all, she proved that if S is a minimal set then S is d-rectifiable, but there is a more detailed analysis of the blow up around points of S. From rectifiability, we already know that at $\mathcal{H}^d$-a.e. $x \in S$ there exists the approximate tangent space $\mathrm{Tan}(S,x)$ defined by the *blow up procedure*, that is looking at the limit

$$\lim_{r\to 0^+} \frac{1}{r}(S \cap B_r(x) - x) \tag{8}$$

The key point of the Taylor's approach is to understand what happens if we do the blow up (8) centering at any point of S. Taylor proved that

$$C = \lim_{r\to 0^+} \frac{1}{r}(S \cap B_r(x) - x)$$

always exists and, by construction, is a *cone*, that is $rC = C$ for any $r > 0$. Moreover, the fact that S was a minimal set reflects on C: since the definition of minimal sets requires only local perturbations it is not difficult to believe that C turns out to be a minimal set too, a so called *minimal cone*. Having proved that, it comes the last part: finding all the possible minimal cones and this should correspond to all possible singularities for minimal sets. Actually, only 1-dimensional minimal cones in $\mathbb{R}^2$, 1-dimensional minimal cones in $\mathbb{R}^3$ and 2-dimensional minimal cones in $\mathbb{R}^3$ are completely classified; higher dimensions and codimensions are far from clear still today. The three 2-dimensional minimal cones in $\mathbb{R}^3$ are:

(a) the plane configuration (this happens when the blow up procedure gives the tangent space);

(b) the $\mathbb{Y}$-*configuration*: three plane sheets crossing on a line and forming a 120° angle;

(c) the $\mathbb{T}$-*configuration*: four lines crossing in a point (called *tetrahedrical* point) and forming a $109,47°$ angle.

All these three configurations are realized by some minimal sets represented by soap films. The flat surface of course produces an example of plane configuration. Concerning plane sheets that meet at 120° and lines meeting at $109,47°$, see Figure 6. The two possible singularities of minimal sets in $\mathbb{R}^3$ ($\mathbb{Y}$ and $\mathbb{T}$) are precisely the only singularities conjectured by Plateau. For this reason, the fact that a soap film can only have $\mathbb{Y}$ and $\mathbb{T}$ singularities are known still today as *Plateau's laws*.

8.2. Plateau problem in the context of minimal sets

As we have already mentioned in the Reifenberg's approach, the main difficulty of the set point of view is to have a good notion of boundary. Very recently a framework for that has been investigated and some existence results have been proved. In this section

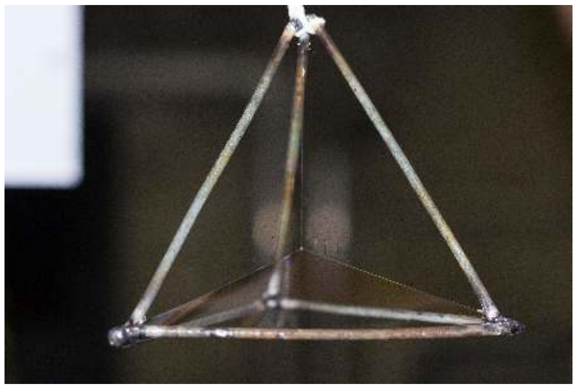

Figure 6: The soap films created by a tetrahedral boundary.

we will state what is proved in [8] by De Lellis, Ghiraldin, and Maggi. This paper is motivated by a very elegant idea introduced by Harrison (see related papers [11] and [12]) in order to give a definition of boundary. Let us give the following definition [8, Def. 3] (see [12] for the original approach). Let $n \geqslant 3$ and let H be a closed subset of $\mathbb{R}^n$. Let

$$\mathcal{C}_H = \{\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n \backslash H \text{ smooth embedding of } \mathbb{S}^1 \text{ into } \mathbb{R}^n\}.$$

Let $\mathcal{C} \subset \mathcal{C}_H$. We say that $\mathcal{C}$ is *closed by homotopy* if together with any $\gamma \in \mathcal{C}$ the set $\mathcal{C}$ contains all elements belonging to the homotopy class $[\gamma] \in \pi_1(\mathbb{R}^n \backslash H)$, where $\pi_1(X)$ denotes the fundamental group of X. Let $\mathcal{C} \subset \mathcal{C}_H$ and let K be a relatively closed set in $\mathbb{R}^n \backslash H$. We say that K is a *$\mathcal{C}$-spanning set of H* if

$$K \cap \gamma(\mathbb{S}^1) \neq \varnothing \quad \forall \gamma \in \mathcal{C}.$$

We denote by $\mathcal{F}(H, \mathcal{C})$ the class of all relatively closed sets in $\mathbb{R}^n \backslash H$ which are $\mathcal{C}$-spanning set of H. Roughly speaking, $K \in \mathcal{F}(H, \mathcal{C})$ means that the set K has a boundary which lies on H. When H is a closed curve in $\mathbb{R}^3$ this corresponds to the fact that the soap film K wets all the curve H, which is precisely what we want. In order to understand better this let us discuss the typical choice when H is a $(n-2)$-dimensional closed submanifold of $\mathbb{R}^n$, which is the idea of Harrison [11]. Let K be relatively closed in $\mathbb{R}^n \backslash H$ and let K_i be the connected components of K. We say that K spans H if for any i and for any $\gamma \in \mathcal{C}_H$ the linking number between γ and K_i has modulus 1 while the linking number between γ and K_j is 0 for any $j \neq i$. The class of all of these γ's is closed by homotopy. We now continue to follow [8] where the following existence theorem has been proved. Let $n \geqslant 3$, let H be a closed subset of $\mathbb{R}^n$, and let $\mathcal{C} \subset \mathcal{C}_H$ be closed by homotopy. Assume that there exists $K \in \mathcal{F}(H, \mathcal{C})$ such that $\mathcal{H}^{n-1}(K) < +\infty$. Then, the problem

$$\min_{K \in \mathcal{F}(H, \mathcal{C})} \mathcal{H}^{n-1}(K)$$

has a solution which is a $(n-1)$-dimensional minimal set in $\mathbb{R}^n \backslash H$. We only sketch the idea of the proof. A difficult part, and we do not enter in details on that, is the proof of the existence of a minimizing sequence which consists of $(n-1)$-rectifiable sets. If we take a minimizing sequence (K_h) of $(n-1)$-rectifiable sets, we can consider the corresponding associated measures $\mu_h = \mathcal{H}^{n-1} \llcorner K_h$. Then, up to a subsequence, $\mu_h \rightharpoonup^* \mu$ in $\mathbb{R}^n \backslash H$. Now, it is possibile to prove, using arguments of Geometric Measure Theory, that

$$\mu \geqslant \theta \mathcal{H}^{n-1} \llcorner K, \quad \text{on subsets of } \mathbb{R}^n \backslash H,$$

where $\theta \geqslant 1$ and $K = \operatorname{spt}\mu \backslash H$ is $(n-1)$-rectifiable. In particular, we get

$$\liminf_h \mathcal{H}^{n-1}(K_h) \geqslant \mathcal{H}^{n-1}(K).$$

Hence, the direct method of the Calculus of Variations should apply. The only thing we have to be careful about is the closedness of the spanning condition. Suppose by contradiction that some loop $\gamma \in \mathcal{C}$ does not intersect K. Since both γ and K are compact, we can find some $\varepsilon > 0$ such that $U_{2\varepsilon}(\gamma)$ does not intersect K and is contained in $\mathbb{R}^n \backslash H$: here $U_r(\gamma)$ denotes the tubular neighborhood of $\gamma(\mathbb{S}^1)$. Hence $\mu(U_{2\varepsilon}(\gamma)) = 0$ and thus

$$\lim_h \mathcal{H}^{n-1}(K_h \cap U_\varepsilon(\gamma)) = 0. \tag{9}$$

Notice now that if ε is small there is a diffeomorphism $\Phi\colon \mathbb{S}^1 \times B^{n-1}_\varepsilon(0) \to U_\varepsilon(\gamma)$ such that $\Phi_{|_{\mathbb{S}^1 \times \{0\}}} = \gamma$. Let $y \in B^{n-1}_\varepsilon(0)$ and set $\gamma_y = \Phi_{|_{\mathbb{S}^1 \times \{y\}}}$. Then $\gamma_y \in [\gamma]$ represents an element of $\pi_1(\mathbb{R}^n \backslash H)$. As a consequence, it must be $K_h \cap \gamma_y(\mathbb{S}^1) \neq \varnothing$. It is now not difficult to conclude that

$$\mathcal{H}^{n-1}(K_h \cap U_\varepsilon(\gamma)) \geqslant c$$

for some $c > 0$ independent on h, which contradicts (9). Finally, it is possibile to show that the solution is a $(n-1)$-dimensional minimal set thanks again to the fact that K is a limit of a minimizing sequence.

This approach furnishes a good answer to the Plateau problem: when H is a Jordan curve in $\mathbb{R}^3$ we obtain the existence of a minimal set K in $\mathbb{R}^3 \backslash H$ that spans H. Therefore, “the boundary of K is H” and, by Taylor’s regularity, K can have singularities but only of Plateau’s type.

References

[1] ALMGREN F.J., *The theory of varifolds*, Mimeographed notes, Princeton, 1965.

[2] ALMGREN F.J., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. AMS **165** (1976), 199 pages.

[3] AMBROSIO L, FUSCO N. AND PALLARA D., *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, 2000.

[4] BOMBIERI E, DE GIORGI E AND GIUSTI E., *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[5] DACOROGNA B., *Direct methods in the Calculus of Variations*, Springer-Verlag New York, 2008.

[6] DIERKES U., HILDEBRANDT S. AND SAUVIGNY F., *Minimal surfaces*, Springer-Verlag Berlin Heidelberg, 2010.

[7] DO CARMO M.P., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[8] DE LELLIS C., GHIRALDIN F AND MAGGI F., *A direct approach to Plateau's problem*, J. Eur. Math. Soc. **19** (2017), 2219–2240.

[9] FEDERER F., *Geometric Measure Theory*, Springer-Verlag Berlin Heidelberg, 1996.

[10] GULLIVER R., *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. **97** (1973), 275–305.

[11] HARRISON J., *Soap film solutions to Plateau's problem*, J. Geom. Anal. **24** (2014), 271–297.

[12] HARRISON J. AND PUG H., *Existence and soap film regularity of solutions to Plateau's problem*, Adv. Calc. Var. **9** (2016), 357–394.

[13] MAGGI F., *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge University Press, 2012.

[14] REIFENBERG E.H., *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. **104** (1960), 1–92.

[15] SIMON L., *Lectures on Geometric Measure Theory*, Proceedings of the Centre for Mathematical Analysis, Australian Mathematical Society, vol. 3, 1983.

[16] TAYLOR J., *The structure of singularities in soap-bubble-like and soap-films-like minimal surfaces*, Ann. of Math. **103** (1976), no. 3, 489–539.

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