Instanton bundles on the Segre threefold with Picard number three

INSTANTON BUNDLES ON THE SEGRE THREEFOLD WITH PICARD NUMBER THREE

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Abstract. We study instanton bundles E on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We construct two different monads which are the analog of the monads for instanton bundles on $\mathbb{P}^3$ and on the flag threefold $F(0,1,2)$. We characterize the Gieseker semistable cases and we prove the existence of μ-stable instanton bundles generically trivial on the lines for any possible $c_2(E)$. We also study the locus of jumping lines.

Instanton bundles on $\mathbb{P}^3$ were first defined in [4] by Atiyah, Drinfel'd, Hitchin and Manin. Its importance arises from quantum physics; in fact these particular bundles correspond (through the Penrose-Ward transform) to self dual solutions of the Yang-Mills equation over the real sphere S^4. We recall that a mathematical instanton bundle E with charge (or quantum number k) on $\mathbb{P}^3$ is a stable rank two vector bundle E with $c_1(E) = 0$, $c_2(E) = k$ and with the property (called instatonic condition) that

$$H^1(E(-2)) = 0$$

Every instanton of charge k on $\mathbb{P}^3$ can be represented as the cohomology of a monad (a three-term self dual complex)

In [19], Hitchen showed that the only twistor spaces of four dimensional (real) differential varieties which are Kähler (and a posteriori, projective) are $\mathbb{P}^3$ and the flag variety $F(0,1,2)$, which is the twistor space of $\mathbb{P}^2$.

On $F(0,1,2)$ instanton bundles has been studied in [8], [12] and more recently in [24]. This is Fano threefold with Picard number two. Let us call h_1 and h_2 the two generators. In [24] has been given the following definition:

a rank two vector bundle E on the Fano threefold $F(0,1,2)$ is an instanton bundle of charge k if the following properties hold:

- $c_1(E) = 0, c_2(E) = kh_1h_2$;
- $h^0(E) = 0$ and E is μ-semistable;

Notice that, when the Picard number is one, the condition $H^0(E) = 0$ implies the μ-stability. When the Picard number is higher than one, however, this is not true and it is natural to consider also μ-semistable bundles (see [12] and [24] Remark 2.2.)

In [15] (see also [20] in the case $i_F = 2$ and [28] for details in the case of the Delpezzo therefold of degree 5), the author generalize the notion of instanton bundle on $\mathbb{P}^3$ to any Fano threefold with Picard number one. In this line may be generalized also the definition on $F(0,1,2)$ to any Fano threefold with Picard number higher than one (see [**?**] for the case of the blow up of the projective 3-space at a point).

In this paper we consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which has the same index and degree of $F(0,1,2)$ but Picard number three. Let us call h_1, h_2 and h_3 the three generators. The only difference with respect to the definition of instanton bundle on $F(0,1,2)$ is that on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the

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second Chern class is $c_2(E) = k_1h_2h_3 + k_2h_1h_3 + k_3h_1h_2$ instead of $c_2(E) = kh_1h_2$. By using a Beilinson type spectral sequence with suitable full exceptional collections we construct two different monads which are the analog of the monads for instanton bundles on $\mathbb{P}^3$ and on $F(0, 1, 2)$. We show that the Gieseker semistable intanton bundles are extensions of line bundles and can be obtained as pulbacks from $\mathbb{P}^1 \times \mathbb{P}^1$. The cases where the degree of $c_2(E)$ is minimal, namely $k = k_1 + k_2 + k_2 = 2$, has been studied in [10]. In fact we get, up to twist, Ulrich bundles. Here we show that all these Ulrich bundles are generically trivial on the lines. So we use this case as a starting step in order to prove by induction the existence of μ-stable instanton bundles generically trivial on the lines for any possible $c_2(E)$. Finally we also study the locus of jumping lines.

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1. First properties of instantons bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let V_1, V_2, V_3 be three 2-dimensional vector spaces with the coordinates $[x_{1i}], [x_{2j}], [x_{3k}]$ respectively with $i, j, k \in \{1, 2\}$. Let $X \cong \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$ and then it is embedded into $\mathbb{P}^7 \cong \mathbb{P}(V)$ by the Segre map where $V = V_1 \otimes V_2 \otimes V_3$.

The intersection ring $A(X)$ is isomorphic to $A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \otimes A(\mathbb{P}^1)$ and so we have

$$A(X) \cong \mathbb{Z}[h_1, h_2, h_3]/(h_1^2, h_2^2, h_3^2).$$

We may identify $A^1(X) \cong \mathbb{Z}^{\oplus 3}$ by $a_1h_1 + a_2h_2 + a_3h_3 \mapsto (a_1, a_2, a_3)$. Similarly we have $A^2(X) \cong \mathbb{Z}^{\oplus 3}$ by $k_1e_1 + k_2e_2 + k_3e_3 \mapsto (k_1, k_2, k_3)$ where $e_1 = h_2h_3, e_2 = h_1h_3, e_3 = h_1h_2$ and $A^3(X) \cong \mathbb{Z}$ by $ch_1h_2h_3 \mapsto c$. Then X is embedded into $\mathbb{P}^7$ by the complete linear system $h = h_1 + h_2 + h_3$ as a subvariety of degree 6 since $h^3 = 6$.

If E is a rank two bundle with the Chern classes $c_1 = (a_1, a_2, a_3)$, $c_2 = (k_1, k_2, k_3)$ we have:

$$c_1(E(s_1, s_2, s_3)) = (a_1 + 2s_1, a_2 + 2s_2, a_3 + 2s_3)$$
$$c_2(E(s_1, s_2, s_3)) = c_2 + c_1 \cdot (s_1, s_2, s_3) + (s_1, s_2, s_3)^2$$

for $(s_1, s_2, s_3)) \in \mathbb{Z}^{\oplus 3}$.

Let's recall the Riemann-Roch formula:

$$\chi(E) = (a_1 + 1)(a_2 + 1)(a_3 + 1) + 1 - \frac{1}{2}((a_1, a_2, a_3) \cdot (k_1, k_2, k_3) + 2(k_1 + k_2 + k_3)) \tag{1.1}$$

Recall that for each sheaf F on X the *slope* of F with respect to h is the rational number $\mu(F) := c_1(F)h^2/rk(F)$ and the reduced Hilbert polynomial $p_E(t)$ of a bundle E over X is $p_E(t) := \chi(E(th))/rk(E)$.

We say that a vector bundle E is *μ-stable* (resp. *μ-semistable*) with respect to h if $\mu(G) < \mu(E)$ (resp. $\mu(G) \leq \mu(E)$) for each subsheaf G with $0 < rk(G) < rk(E)$.

On the other hand, E is said to be Gieseker semistable with respect to h if for all G as above one has

$$p_G(t) \geq p_E(t),$$

and Gieseker stable again if equality cannot hold in the above inequality.

Definition 1.1. A μ-semistable vector bundle E on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is called an instanton bundle of charge k if and only if $c_1(E) = 0$,

$$H^0(E) = H^1(E(-h)) = 0$$

and $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$ with $k_1 + k_2 + k_3 = k$.

Remark 1.2. It is worthwhile to point out that, exactly as in the case of $F(0,1,2)$ (see [24] Remark 2.2), the condition $H^0(E)=0$ does not follow from the rest of the conditions defining an instanton bundle. Indeed we may consider the rank two aCM bundles with $c_1(E)=0$ and $H^0(E)\neq 0$ given in [10] Theorem B.

Proposition 1.3 (Generalized Hoppe's criterion)**.** *Let E be a rank two holomorphic vector bundle over a polycyclic variety X and let L be a polarization on X. E is μ-(semi)stable if and only if*

$$H^0(X, E\otimes\mathcal{O}_X(B))=0$$

for all $B\in Pic(X)$ such that $\delta_L(B)\underset{(<)}{\leq}-\mu_L(E)$, where $\delta_L(B)=\deg_L(\mathcal{O}_X(B))$.

Now we characterize strictly Gieseker semistable instanton bundles on X

Proposition 1.4. *Let E be an instanton bundle of charge k. If E is not μ-stable then $k=2l^2$ for some $l\in\mathbb{Z}$, $l\neq 0$. Moreover $c_2(E)=2l^2e_i$, $i=1,2,3$ and E can be constructed as an extension*

$$0\to\mathcal{O}_X(-lh_i+lh_j)\to E\to\mathcal{O}_X(lh_i-lh_j)\to 0 \tag{1.2}$$

with $i\neq j$. In particular $H^0(E)=0$.

Proof. Suppose $H^0(X, E(ah_1+bh_2-(a+b)h_3))\neq 0$ for some $a,b\in\mathbb{Z}$. So E fits into an exact sequence

$$0\to\mathcal{O}_X\to E(ah_1+bh_2-(a+b)h_3)\to\mathcal{I}_Z(2ah_1+2bh_2-2(a+b)h_3)\to 0$$

where $Z\subset X$ is a subscheme of X. Since $H^0(E(ah_1+bh_2-(a+b)h_3)\otimes\mathcal{O}_X(-h_j))=0$ for all $j=1,2,3$ by proposition 1.3, we have that $Z\subset X$ is either empty or purely 2-codimensional. Suppose we are dealing with the latter case, since E in Gieseker semistable we have that

$$P_{\mathcal{O}_X}(t)\leq P_{E(ah_1+bh_2-(a+b)h_3)}(t)\leq P_{\mathcal{I}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)$$

and

$$P_{\mathcal{I}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)=P_{\mathcal{O}_X(2ah_1+2bh_2-2(a+b)h_3)}(t)-P_{\mathcal{O}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)$$

where $P(t)$ is the Hilbert polynomial. So we have

$$\begin{aligned}P_{\mathcal{O}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)&\leq P_{\mathcal{O}_X(2ah_1+2bh_2-2(a+b)h_3)}(t)-P_{\mathcal{O}_X}(t)\\&=(2a+t+1)(2b+t+1)(t+1-2a-2b)-(t+1)^3\\&=-4(t+1)(a^2+b^2+ab)<0\text{ for }t>>0.\end{aligned}$$

contradicting Serre's vanishing theorem. Se we can conclude that Y is empty and E fits into

$$0\to\mathcal{O}_X(-ah_1-bh_2+(a+b)h_3)\to E\to\mathcal{O}_X(ah_1+bh_2-(a+b)h_3)\to 0.$$

Now computing $c_2(E)$ we obtain

$$\begin{aligned}c_2(E)&=(-ah_1-bh_2+(a+b)h_3)\cdot(ah_1+bh_2-(a+b)h_3)\\&=2b(a+b)e_1+2a(a+b)e_2-2abe_3.\end{aligned}$$

Since E is an istanton bundle on X, all the summands of $c_2(E)$ must be nonnegative. In fact by Proposition 2.3 they represent the dimension of a cohomology group. So either one between a and b is 0 (but not both since the charge k must be greater than two) or $a=-b$. In all three cases we obtain the desired result. $\square$

2. Monads

We will use the following version (explained in [3]) of the Beilinson spectral sequence (see also [27, Corollary 3.3.2], and [16, Section 2.7.3] and [6, Theorem 2.1.14].)

Theorem 2.1. *Let X be a smooth projective variety with a full exceptional collection $\langle E_0,\dots,E_n\rangle$ where $E_i=\mathcal{E}_i^*[-k_i]$ with each $\mathcal{E}_i$ a vector bundle and $(k_0,\dots,k_n)\in\mathbb{Z}^{\oplus n+1}$ such that there exists a sequence $\langle F_n=\mathcal{F}_n,\dots,F_0=\mathcal{F}_0\rangle$ of vector bundles satisfying*

$$\operatorname{Ext}^k(E_i,F_j)=H^{k+k_i}(\mathcal{E}_i\otimes\mathcal{F}_j)=\begin{cases}\mathbb{C} & \text{if } i=j=k\\ 0 & \text{otherwise}\end{cases}\tag{2.1}$$

Then for any coherent sheaf A on X there is a spectral sequence in the square $-n\le p\le 0$, $0\le q\le n$ with the E_1-term

$$E_1^{p,q}=\operatorname{Ext}^q(E_{-p},A)\otimes F_{-p}=H^{q+k_{-p}}(\mathcal{E}_{-p}\otimes A)\otimes\mathcal{F}_{-p}$$

which is functorial in A and converges to

$$E_\infty^{p,q}=\begin{cases}A & \text{if } p+q=0\\ 0 & \text{otherwise.}\end{cases}\tag{2.2}$$

Let $D^b(X)$ be the the bounded derived category of coherent sheaves on a smooth projective variety X. An object $E\in D^b(X)$ is called exceptional if $\operatorname{Ext}^\bullet(E,E)=\mathbb{C}$. We recall that a set of exceptional objects $E_1,\dots,E_n$ on X is called an exceptional collection if $\operatorname{Ext}^\bullet(E_i,E_j)=0$ for $i>j$. An exceptional collection is full when $\operatorname{Ext}^\bullet(E_i,A)=0$ for all i implies $A=0$, or equivalently when $\operatorname{Ext}^\bullet(A,E_i)=0$ for all i also implies $A=0$.

Definition 2.2. Let E be an exceptional object in $D^b(X)$. Then there are functors $\mathbb{L}_E$ and $\mathbb{R}_E$ fitting in distinguished triangles

$$\mathbb{L}_E(T)\to\operatorname{Ext}^\bullet(E,T)\otimes E\to T\to\mathbb{L}_E(T)[1]$$

$$\mathbb{R}_E(T)[-1]\to T\to\operatorname{Ext}^\bullet(T,E)^*\otimes E\to\mathbb{R}_E(T)$$

The functors $\mathbb{L}_E$ and $\mathbb{R}_E$ are called respectively the *left* and *right mutation functor.*

Now we construct the full exceptional collections that we will use in the next theorems: Let us consider on the three copies of $\mathbb{P}^1$ the full exceptional collection $\{\mathcal{O}_{\mathbb{P}^1}(-1),\mathcal{O}_{\mathbb{P}^1}\}$. We may obtain the full exceptional collection $\langle E_7,\dots,E_0\rangle$ (see [25]):

$$\begin{gathered}\{\mathcal{O}_X(-h)[-4],\mathcal{O}_X(-h_2-h_3)[-4],\mathcal{O}_X(-h_1-h_3)[-3],\\ \mathcal{O}_X(-h_1-h_2)[-2],\mathcal{O}_X(-h_3)[-2],\mathcal{O}_X(-h_2)[-1],\mathcal{O}_X(-h_1),\mathcal{O}_X\}\end{gathered}\tag{2.3}$$

The associated full exceptional collection $\langle F_7=\mathcal{F}_7,\dots,F_0=\mathcal{F}_0\rangle$ of Theorem 2.1 is

$$\{\mathcal{O}_X(-h),\mathcal{O}_X(-h_2-h_3),\mathcal{O}_X(-h_1-h_3),\mathcal{O}_X(-h_1-h_2),\mathcal{O}_X(-h_3),\mathcal{O}_X(-h_2),\mathcal{O}_X(-h_1),\mathcal{O}_X\}\tag{2.4}$$

From (2.3) with a left mutations of the pair $\{\mathcal{O}_X(-h_1),\mathcal{O}_X\}$ we obtain:

$$\begin{gathered}\{\mathcal{O}_X(-h)[-4],\mathcal{O}_X(-h_2-h_3)[-4],\mathcal{O}_X(-h_1-h_3)[-3],\\ \mathcal{O}_X(-h_1-h_2)[-2],\mathcal{O}_X(-h_3)[-2],\mathcal{O}_X(-h_2)[-1],\mathcal{O}_X(-2h_1),\mathcal{O}_X(-h_1)\}\end{gathered}\tag{2.5}$$

From the above collection with a left mutations of the pair $\{\mathcal{O}_X(-h_2)[-1],\mathcal{O}_X(-2h_1)\}$ we obtain:

$$\begin{aligned}(2.6)\qquad &\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ &\mathcal{O}_X(-h_1-h_2)[-2], \mathcal{O}_X(-h_3)[-2], A[-1], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1)\}\end{aligned}$$

where A is given by the extension

$$0 \to \mathcal{O}_X(-2h_1) \to A \to \mathcal{O}_X(-h_2)^{\oplus 2} \to 0 \tag{2.7}$$

From the above collection with a left mutations of the pair $\{\mathcal{O}_X(-h_3), A\}$ we obtain:

$$\begin{aligned}(2.8)\qquad &\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ &\mathcal{O}_X(-h_1-h_2)[-2], B[-2], \mathcal{O}_X(-h_3)[-2], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1)\}\end{aligned}$$

where B is given by the extension

$$0 \to A \to B \to \mathcal{O}_X(-h_3)^{\oplus 2} \to 0 \tag{2.9}$$

Making the respective right mutation of (2.4) we obtain the full exceptional collection $\langle F_7 = \mathcal{F}_n, \dots, F_0 = \mathcal{F}_0\rangle$ of Theorem 2.1:

(2.10)
$$\{\mathcal{O}_X(-h), \mathcal{O}_X(-h_2-h_3), \mathcal{O}_X(-h_1-h_3), \mathcal{O}_X(-h_1-h_2), \mathcal{O}_X, \mathcal{O}_X(h_3), \mathcal{O}_X(h_2), \mathcal{O}_X(h_1)\}$$

It is easy to check that the conditions (2.1) are satisfied.

Theorem 2.3. *Let E be a charge k instanton bundle on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$, then E is the cohomology of a monad of the form*

$$0 \to \begin{array}{c}\mathcal{O}_X^{k_3}(-h_1-h_2)\\ \oplus\\ \mathcal{O}_X^{k_2}(-h_1-h_3)\\ \oplus\\ \mathcal{O}_X^{k_1}(-h_2-h_3)\end{array} \to \begin{array}{c}\mathcal{O}_X^{k_2+k_3}(-h_1)\\ \oplus\\ \mathcal{O}_X^{k_1+k_3}(-h_2)\\ \oplus\\ \mathcal{O}_X^{k_1+k_2}(-h_3)\end{array} \to \mathcal{O}_X^{k-2} \to 0 \tag{2.11}$$

Conversely any μ-semistable bundle defined as the cohomology of such a monad is a charge k instanton bundle.

Proof. We consider the Beilinson type spectral sequence associated to an instanton bundle E and identify the members of the graded sheaf associated to the induced filtration as the sheaves mentioned in the statement of Theorem 2.1. We consider the full exceptional collection $\langle E_7, \dots, E_0\rangle$ given in (2.3) and the full exceptional collection $\langle F_7, \dots, F_0\rangle$ given in (2.4).

First of all, let's observe that since $H^0(E) = 0$ we have $H^0(E(-D)) = 0$ for every effective divisor D. Furthermore by Serre's duality we have also $H^2(E(K + D)) = 0$ for all effective divisors D. Since $c_1(E) = 0$ using Serre's duality and $H^1(E(-h)) = 0$ we obtain

$$H^i(E(-h)) = H^{3-i}(E(-h)) = 0 \text{ for all } i.$$

We want to show that for each twist in the table, there's only one non vanishing cohomology group, so that we can use the Riemann Roch formula to compute the dimension of the remaining cohomology group. Let's consider the pull-back of the Euler sequence from one of the $\mathbb{P}^1$ factors

$$0 \to \mathcal{O}_X(-h_a) \to \mathcal{O}_X^2 \to \mathcal{O}_X(h_a) \to 0 \tag{2.12}$$

and tensor it by $E(-h)$. We have

$$0 \to E(-2h_a - h_b - h_c) \to E^2(-h) \to E(-h_b - h_c) \to 0$$

with $ia, b, c \in \{1, 2, 3\}$ and they are all different from each other. Since $H^i(E(-h)) = 0$ for all i and $H^0(E(-2h_a - h_b - h_c)) = H^3(E(-2h_a - h_b - h_c)) = 0$, considering the long exact sequence induced in cohomology we have $H^2(E(-h_b - h_c)) = 0$. Now we want to show that $H^2(E(-h_a)) = 0$ for all $a \in \{1, 2, 3\}$. Tensor (2.12) by $E(-h_b)$ with $b \neq a$ and we have:

$$0 \to E(-2h_a - h_b) \to E^2(-h_a - h_b) \to E(-h_b) \to 0.$$

Consider the long exact sequence induced in cohomology we have that $H^2(E(-h_b)) = 0$ since $H^2(E(-h_a - h_b)) = H^3(E(-2h_a - h_b)) = 0$. Finally if we tensor (2.12) by $E(-h_a)$ and we consider the long exact sequence in cohomology we obtain $H^2(E) = 0$. Now let's compute the Euler characteristic of E tensored by a line bundle $\mathcal{O}_X(D)$ so that we are able to compute all the numbers in the Beilinson's table. Combining (1.1) and (**??**) we have

$$\chi(E(D)) = \frac{1}{6}(14D^3 + 6c_2(E)D) + h(D^2 - c_2(E)) + \frac{1}{3}(2Dh^2 + 2D(e_1 + e_2 + e_3)) + 2. \tag{2.13}$$

So we have

- $h^1(E) = -\chi(E) = 2 - k_1 - k_2 - k_3 = 2 - k$.
- $h^1(E(-h_i)) = -\chi(E(-h_i)) = k_i - k$.
- $h^1(E(-h_i - h_j)) = -\chi(E(-H - i - h_j)) = k_i + k_j - k$.

So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2 - h_3)$	$\mathcal{O}_X(-h_1 - h_3)$	$\mathcal{O}_X(-h_1 - h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	0	0	0	0	h^4
0	0	0	k_3	$k_1 + k_2$	0	0	0	h^3
0	0	0	0	0	$k_1 + k_3$	0	0	h^2
0	0	0	0	0	0	$k_2 + k_3$	$k - 2$	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2 - h_3)[-4]$	$E(-h_1 - h_3)[-3]$	$E(-h_1 - h_2)[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	E	

Using Beilinson's theorem we retrieve the monad (2.11).

Conversely let E be a μ-semistable bundle defined as the cohomology of a monad (2.11). We may consider the two short exact sequences:

$$0 \to G \to \begin{array}{c} \mathcal{O}_X^{k_2+k_3}(-h_1) \\ \oplus \\ \mathcal{O}_X^{k_1+k_3}(-h_2) \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(-h_3) \end{array} \to \mathcal{O}_X^{k-2} \to 0 \tag{2.14}$$

and

$$0 \to \begin{matrix} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ \mathcal{O}_X^{k_2}(-h_1-h_3) \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{matrix} \to G \to E \to 0. \tag{2.15}$$

We deduce that $H^0(G) = H^0(E) = 0$. By (2.14) and (2.15) tensored by $\mathcal{O}_X(-h)$ we obtain $H^1(G(-h)) = H^1(E(-h)) = 0$ so E is an instanton. □

Proposition 2.4. *Let E be an instanton bundle on X, then $h^1(E(-h-D)) = 0$ for every effective divisor D.*

Proof. Let us consider the two short exact sequences (2.14) and (2.15) tensored by $\mathcal{O}_X(-h+D)$. By Kunneth formula we have that $h^i(\mathcal{O}_X(-h-D)) = 0$ for all i, and thus taking the cohomology of (2.14) we get $h^i(K(-h-D)) = 0$ for $i \neq 3$. Combining this with the induced sequence in cohomology of (2.15) we obtain $h^0(E(-h-D)) = h^1(E(-h-D)) = 0$. □

Theorem 2.5. *Let E be a charge k instanton bundle on X with $c_2(E) = k_1e_1+k_2e_2+k_3e_3$, then E is the cohomology of a monad of the form*

$$0 \to \begin{matrix} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ \mathcal{O}_X^{k_2}(-h_1-h_3) \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{matrix} \to \mathcal{O}_X^{3k+2} \to \begin{matrix} \mathcal{O}_X^{k_2+k_3}(h_1) \\ \oplus \\ \mathcal{O}_X^{k_1+k_3}(h_2) \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(h_3) \end{matrix} \to 0 \tag{2.16}$$

Conversely any μ-semistable bundle with $H^0(E) = 0$ defined as the cohomology of such a monad is a charge k instanton bundle.

Proof. We consider the Beilinson type spectral sequence associated to an instanton bundle E and identify the members of the graded sheaf associated to the induced filtration as the sheaves mentioned in the statement of Theorem 2.1. We consider the full exceptional collection $\langle E_7, \ldots, E_0\rangle$ given in (2.8) and the full exceptional collection $\langle F_7, \ldots, F_0\rangle$ given in (2.10).

First of all, let's observe that since since E is μ-semistable, by Hoppe's criterion we have $H^0(E(-D)) = 0$ for every effective divisor D. Furthermore we have all the vanishing computed in Theorem 2.3 Moreover by (2.7) and (2.9 tensored by E we get $\chi(E \otimes B) = \chi(E \otimes A) + 2\chi(E(-h_3)) = \chi(E(-2h_1)) + 2\chi(E(-h_3)) + 2\chi(E(-h_2)) = -2 + k_1 - k_2 - k_3 - 2(k_1+k_2) - 2(k_1+k_3) = -2-3k$. So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2-h_3)$	$\mathcal{O}_X(-h_1-h_3)$	$\mathcal{O}_X(-h_1-h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	a	0	0	0	h^4
0	0	0	k_3	b	k_1+k_2	0	0	h^3
0	0	0	0	0	0	k_1+k_3	0	h^2
0	0	0	0	0	0	0	k_2+k_3	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2-h_3)[-4]$	$E(-h_1-h_3)[-3]$	$E(-h_1-h_2)[-2]$	$E\otimes B[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	

where $a-b=-2-3k$. Since the spectral sequence converges to an object in degree 0 and there no maps involving a we deduce that $a=0$ and $b=3k+2$. So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2-h_3)$	$\mathcal{O}_X(-h_1-h_3)$	$\mathcal{O}_X(-h_1-h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	0	0	0	0	h^4
0	0	0	k_3	3k+2	k_1+k_2	0	0	h^3
0	0	0	0	0	0	k_1+k_3	0	h^2
0	0	0	0	0	0	0	k_2+k_3	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2-h_3)[-4]$	$E(-h_1-h_3)[-3]$	$E(-h_1-h_2)[-2]$	$E\otimes B[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	

Using Beilinson's theorem we retrieve the monad (2.16).

Conversely let E be a μ-semistable bundle defined as the cohomology of a monad (2.11). We may consider the two short exact sequences:

$$0\to G\to \mathcal{O}_X^{3k+2}\to \begin{array}{c}\mathcal{O}_X^{k_2+k_3}(h_1)\\ \oplus\\ \mathcal{O}_X^{k_1+k_3}(h_2)\\ \oplus\\ \mathcal{O}_X^{k_1+k_2}(h_3)\end{array}\to 0 \tag{2.17}$$

and

$$0\to \begin{array}{c}\mathcal{O}_X^{k_3}(-h_1-h_2)\\ \oplus\\ \mathcal{O}_X^{k_2}(-h_1-h_3)\\ \oplus\\ \mathcal{O}_X^{k_1}(-h_2-h_3)\end{array}\to G\to E\to 0. \tag{2.18}$$

By (2.17) and (2.18) tensored by $\mathcal{O}_X(-h)$ we obtain $H^1(G(-h))=H^1(E(-h))=0$ so E is an instanton. □

Remark 2.6. Let us remark that the monad (2.16) is the analog of the monad for instanton bundles on $\mathbb{P}^3$

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k}\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^3}^{\oplus 2k+2}\xrightarrow{\beta}\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k}\to 0,$$

and the monad (2.11) is the analog of the second monad for instanton bundles on $\mathbb{P}^3$ (see for instance [1] display (1.1))

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k}\xrightarrow{\alpha}\Omega_{\mathbb{P}^3}(1)^{\oplus k}\xrightarrow{\beta}\mathcal{O}_{\mathbb{P}^3}^{\oplus 2k-2}\to 0.$$

A very similar behavior was shown for the two monads for instanton bundles on the flag threefold in [24].

As in the case of instanton bundles on the projective space and flag varieties, the two monads (2.16) and (2.11) are closely related. Indeed, sequence (2.17) fits in the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & \mathcal{O}^{\oplus 3k+2} \arrow[r, "\beta"] \arrow[d] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(h_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(-h_i) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus 4k} \arrow[r, "\beta'"] \arrow[d] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(h_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}^{\oplus k-2} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus k-2} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

So we get sequence (2.14) as the first column. Morever sequence (2.18) fits in the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_i}(-h_{i+1}-h_{i+2}) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_i}(-h_{i+1}-h_{i+2}) \arrow[r] \arrow[d] & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(-h_i) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{O}^{\oplus k-2} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus k-2} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

which is the display of monad (2.11).

3. Existence of instanton for every charge

Now we will construct, through an induction process, stable k-instanton bundles on the flag variety for each charge k.

Step1: Base case of induction. Splitting type of Ulrich bundles.

Observe that for $k = 2$ the bundle constructed above are actually Ulrich bundles (up to twisting by $\mathcal{O}_X(-h)$). For the study of Ulrich bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ see [**?**]. We have two possible alternatives for the second Chern class of an Ulrich bundle:

a) $c_2(E) = 2e_i$ for some $i \in \{1, 2, 3\}$.
b) $c_2(E) = e_i + e_j$ with $i \neq j$.

We show that in both in case $a)$ there exists an Ulrich bundle which has trivial restriction to a generic line of each family.

By 2.3 we see that every rank two Ulrich bundle with this second Chern class is the pullback from a quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$. In this case there exist strictly semistable Ulrich bundle, realized as extensions

$$0 \to \mathcal{O}_X(h_j - h_k) \to E \to \mathcal{O}_X(h_k - h_j) \to 0 \tag{3.1}$$

with $j \neq k \neq i \neq j$. For these vector bundles, by restricting (3.1) to a line in each family, we observe that in the family $h_j h_k$ there are not jumping lines, i.e. $E_l = \mathcal{O}_l^2$ for each $l \in |h_j h_k|$. On the other hand, E_l is never trivial when $l \in |h_i h_k|$ or $l \in |h_j h_i|$. For a characterization of strictly semistable instantons see REF. Let us focus on stable Ulrich bundles. They are pull back via the projection on the quadric, of stable bundles on Q. By [?] every such bundle can be deformed to a stable bundle which is trivial when restricted to the generic line of each family.

Now let us consider case b). The details of what follows can be found in [?]. Up to a permutation of the indices we can assume $c_2(E) = e_2 + e_3$ Let us denote by H a general hyperplane section in $\mathbb{P}^7$ and let S be $S = X \cap H$. S is a del Pezzo surface of degree 6, given as the blowup of $\mathbb{P}^2$ in 3 points. Let us denote by F the pullback to S of the class of a line in $\mathbb{P}^2$ and by E_i the exceptional divisors. Take a general curve C of class $3F - E_1$, so that C is a smooth irreducible elliptic curve of degree 8. Moreover we have $h^0(C, \mathcal{N}_{C|X}) = 16$ and $h^1(C, \mathcal{N}_{C|X}) = 0$, so the Hilbert scheme $\mathcal{H} = \mathcal{H}_1^8$ is smooth of dimension 16 [?, Proposition 6.3] and the general deformation of C in $\mathcal{H}_1^8$ is non-degenerate [?, Proposition 6.6]. Let $\mathcal{C} \subset X \times B \to B$ a flat family of curves in $\mathcal{H}$ with special fibre $\mathcal{C}_{b_0} \cong C$ over b_0. To each curve in the family $\mathcal{C}$ we can associate a rank two vector bundle via the Serre's correspondence:

$$0 \to \mathcal{O}_X(-h) \to E_b \to I_{C_b|X}(h) \to 0 \tag{3.2}$$

where C_b is the curve in $\mathcal{C}$ over $b \in B$. The general fiber C_b correspond via (3.2) to rank two Ulrich bundle of the desired c_2.

Now choose a line L in S, such that $L \cap C$ is a single point x. In order to do so, we deal with the classes of F and E_i in $A^2(X)$. One obtain that the classes of F, E_1, E_2 and E_3 are $e_1 + e_2 + e_3$, e_1, e_2 and e_3 respectively. In particular, there exists a line L in the system $|E_1|$ (corresponding to $|e_1|$ in $A^2(X)$) which it intersect the curve C in the class $3F - E_1$ in one point. It follows that $I_{C|X}(1) \otimes \mathcal{O}_L \cong \mathcal{O}_x \oplus \mathcal{O}_L$. Tensoring (3.2) by $\mathcal{O}_L$ we obtain a surjection

$$E_{b_{0|L}} \to \mathcal{O}_x \oplus \mathcal{O}_L \to 0.$$

In particular $E_{b_{0|L}}$ cannot be $\mathcal{O}_L(-t) \oplus \mathcal{O}_L(t)$ for any $t > 0$, thus $E_{b_{0|L}}$ is trivial, which is equivalent to $h^0(L, E_{b_{0|L}}(-1)) = 0$. By semicontinuity we have that $h^0(L, E_{b_{|L}}(-1)) = 0$ for all b in an open neighborhood of $b_0 \in B$, thus the vector bundle corresponding to the general fiber C_b is trivial over the line L. Since this is an open condition on the variety of lines contained in X, it takes place for the general line in $|e_1|$.

To deal with the other families of lines let us consider a general quadric Q in $|h_1|$. C is a smooth, irreducible, non-degenerate elliptic curve in the class $2e_1 + 3e_2 + 3e_3$. $\mathrm{Pic}(Q) = \mathbb{Z}^2$ generated by two lines $< l, m >$ which correspond respectively to e_3 and e_2. Since Q_1 is general then $Z = C \cap Q$ consist of two points. Following the previous strategy, we say that E restricted to a generic line of the family e_2 (resp. e_3) is trivial if Z is not contained in a line of the ruling m (resp. l). Since Z cannot be contained in a line in both rulings of the quadric, E_L is trivial on the generic line of one of two families. Suppose is trivial on e_3. We have two possible cases for the position of the point in Z:

- Z is contained in a line of the ruling m.
- Z is not contained in a line of the ruling m.

In order to study which case actually can occur we compute $h^0(I_{Z|Q}(m))$. Using Serre's correspondence we have the following short exact sequence

$$0 \to \mathcal{O}_Q \to E_Q(l+m) \to I_{Z|Q}(2l+2m) \to 0 \tag{3.3}$$

Setting $E_Q(h) = F$ and twisting (3.3) by $\mathcal{O}_Q(-2l-m)$ we obtain

$$H^i(Q, F(-2l-m)) = H^i(Q, I_{Z|Q}(m))$$

for all i. Furthermore we also have the short exact sequence

$$0 \to E(-h_1) \to E \to F(-l-m) \to 0 \tag{3.4}$$

which tensorized by $\mathcal{O}_Q(-l)$ and using the cohomology computations in 2.3 gives us

$$h^0(Q, F(-2l-m)) = h^1(Q, F(-2l-m)) \leq 1.$$

In particular, by 1.3 we have that E has trivial splitting type on the generic line of each family if and only if the restriction of E to a generic quadric of the family h_1 is semistable.

Because of this we will only use the case $c_2(E) = 2e_i$ as the base case of induction.

Step 2: Defining a sheaf G with increased c_2.

Let's consider a charge k instanton bundle E on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$, $k_1 + k_2 + k_3 = k$ and suppose $E_{|_l} = \mathcal{O}_l^2$, with l is a generic line of the first family e_1

So let's consider the short exact sequence

$$0 \to G \to E \to \mathcal{O}_l \to 0. \tag{3.5}$$

G is a torsion free sheaf which is not locally free. Using the resolution of $\mathcal{O}_l$:

$$0 \to \mathcal{O}_X(-h_2-h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_l \to 0 \tag{3.6}$$

we obtain $c_1(\mathcal{O}_l) = 0$ and $c_2(\mathcal{O}_l) = -e_1$ so using the sequence (3.5) we have that $c_1(G) = 0$, $c_2(G) = (k_1+1)e_1 + k_2e_2 + k_3e_3$ and $c_3(G) = 0$.

Now applying the functor $\mathrm{Hom}(E, -)$ to (3.5) we obtain $\mathrm{Ext}^2(E, G) = 0$, in fact we have $Ext^2(E, E) = 0$ by hypothesis and $\mathrm{Ext}^1(E, \mathcal{O}_l) = 0$ by Serre's duality since $E_{|_l} = \mathcal{O}_l^2$. Now apply the contravariant functor $\mathrm{Hom}(-, G)$ to (3.5). We have the following sequence

$$\mathrm{Ext}^2(E, G) \to \mathrm{Ext}^2(G, G) \to \mathrm{Ext}^3(\mathcal{O}_l, G)$$

Now we show that $\mathrm{Ext}^3(\mathcal{O}_l, G) = 0$ in order to obtain $\mathrm{Ext}^2(G, G) = 0$. By Serre's duality we have $\mathrm{Ext}^3(\mathcal{O}_l, G) = \mathrm{Hom}(G, \mathcal{O}_l(-2h))$. Consider the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(A, B)) \Rightarrow \mathrm{Ext}^{p+q}(A, B)$$

with $A, B \in \mathrm{Coh}(X)$. Setting $A = G$ and $B = \mathcal{O}_l(-2h)$ we obtain

$$\mathrm{Hom}(G, \mathcal{O}_l(-2h)) = H^0(\mathcal{H}om(G, \mathcal{O}_l(-2h))).$$

Now apply the functor $\mathcal{H}om(-, \mathcal{O}_l(-2h))$ to the sequence (3.5), we obtain

$$0 \to \mathcal{H}om(\mathcal{O}_l, \mathcal{O}_l(-2h)) \to \mathcal{H}om(E, \mathcal{O}_l(-2h)) \to \mathcal{H}om(G, \mathcal{O}_l(-2h)) \to \mathcal{E}xt^1(\mathcal{O}_l, \mathcal{O}_l(-2h)) \to 0. \tag{3.7}$$

Now $\mathcal{H}om(\mathcal{O}_l, \mathcal{O}_l(-2h)) \cong \mathcal{O}_l(-2h)$, $\mathcal{H}om(E, \mathcal{O}_l(-2h)) \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_l) \otimes E_{|_l}^\vee(-2h) \cong \mathcal{O}_l^2(-2h)$ and $\mathcal{E}xt^1(\mathcal{O}_l, \mathcal{O}_l(-2h)) \cong N_l(-2h) = \mathcal{O}_l^2(-2h)$. If we split (3.7) in two short exact sequences we obtain

$$0 \to \mathcal{O}_l(-2h) \to \mathcal{H}om(G, \mathcal{O}_l(-2h)) \to \mathcal{O}_l^2(-2h) \to 0.$$

We deduce $\mathcal{H}om(G, \mathcal{O}_l(-2h)) \cong \mathcal{O}_l^3(-2h)$, thus $H^0(\mathcal{H}om(G, \mathcal{O}_l(-2h))) \cong H^0(\mathcal{O}_l^3(-2h)) = 0$. Finally we obtain $\mathrm{Ext}^3(\mathcal{O}_l, G) \cong \mathrm{Hom}(G, \mathcal{O}_l(-2h)) = 0$ from which it follows $\mathrm{Ext}^2(G, G) = 0$. This implies that $M_X(2, 0, c_2(G))$ is smooth in the point correspondent to G and we can compute its dimension using the Hirzebruch-Riemann-Roch formula. We have $c_1(G\otimes G^\vee)=c_3(G\otimes G^\vee)=0$ and $c_2(G\otimes G^\vee)=4c_2(G)$, so

$$h^1(G\otimes G^\vee)=4c_2(G)h-3=4k+1.$$

Furthermore tensoring (3.5) by $\mathcal{O}_l(-h)$ and using the fact that $E_{|l}=\mathcal{O}_l^{\oplus 2}$, we obtain $H^0(G\otimes\mathcal{O}_l(-h))=0$.

Step 3: Deforming G to a locally free sheaf F.

Now we take a deformation of G in $M_X(2,0,c_2(G))$ and let's call it F. For semicontinuity F satisfies

$$H^0(X,F\otimes\mathcal{O}_l(-h))=0 \text{ and } H^1(X,F(-h))=0$$

Our goal is to show that F is locally free. Let's take E' and l' two deformation in a neighborhood of E and l respectively. The strategy is to show that if F is not locally free, then he would fit into a sequence

$$0\to F\to E'\to\mathcal{O}_{l'}\to 0.$$

But such F are parametrized by a family of dimension $4k$: indeed we have a $(4k-3)$-dimensional family for the choice of E', 2 for the choice of a line in the first family and we have 1 for $\mathbb{P}^1=\mathbb{P}(H^0(l',E_{|l'}))$, since $E_{l'}\cong\mathcal{O}_{l'}^2$. But we showed that G, and hence F, moves over a $(4k+1)$-dimensional component in $M_X(2,0,c_2(G))$, so F must be locally free.

Given such F let's consider the natural sequence

$$0\to F\to F^{\vee\vee}\to T\to 0. \tag{3.8}$$

Let us denote by Y the support of T. Since we supposed F not locally free, we have that $Y\neq\emptyset$. Furthermore T is supported in codimension at least two. We say that Y has pure dimension one.

In fact twisting (3.8) by $\mathcal{O}_X(-h)$ we observe that if $H^0(X,F^{\vee\vee}(-h))\neq 0$ then a nonzero global section of $F^{\vee\vee}$ will induce via pull-back a subsheaf K of F with $c_1(K)=h$, which is not possible since F is stable. So we have $H^0(X,F^{\vee\vee}(-h))\cong H^1(X,F(-h))=0$ which implies $H^0(X,T(-h))=0$. In particular Y has no embedded points, i.e. is pure of dimension one. We want to show that Y is actually a line.

Let H be a general hyperplane section which does not intersect the points where $F^{\vee\vee}$ is not locally free. Tensor (3.5) by $\mathcal{O}_H$. Since H is general the sequence remains exact and $\mathcal{O}_{l\cap H}$ is supported at one point, which represent the point where G_H fails to be reflexive (in this case also locally free). F is a deformation of G and because of the choice of H, restricting (3.8) to H does not affect the exactness of the short exact sequence. Moreover T_H is supported on points where F_H is not reflexive. Since being reflexive is an open condition, by semicontinuity T_H is supported at most at one point. But Y cannot be empty and is purely one dimensional, $Y\cap H$ consists of one point and Y must be a line L. Furthermore by semicontinuity T is of generic rank one and we have $c_2(T)h=-1$ (see [**?**, Example 15.3.1]).

Now we prove that $F^{\vee\vee}$ is locally free. Twist (3.8) by $\mathcal{O}_X(th)$ with $t<<0$. Considering the long exact sequence induced in cohomology we have $h^1(X,T(t))\leq h^2(X,F(t))$ because $h^1(X,F^{\vee\vee})=0$ by Serre's vanishing. Observe that $c=c_3(F^{\vee\vee})$ and $c_2(T)$ are invariant for twists.

By computing the Chern classes using (3.8) we have $c_3(T)=c$ and $c_3(T(th))=c-2thc_2(T)$. For $t<<0$ we have

$$h^1(T(th))=-\chi(T(th))=(t+1)hc_2(T)-\frac{c}{2}.$$

By semicontinuity we have $h^2(F(th))\leq h^2(G(th))$, but using (3.5) for $t<<0$ we have $h^2(G(th))=h^1(\mathcal{O}_{l_1}(t))=-(t+1)$ using Hirzebruch-Riemann-Roch formula or the sequence

(3.6). Now we have

$$(t+1)hc_2(T) - \frac{c}{2} = h^1(T(th)) \leq h^2(X, F(t)) \leq -(t+1)$$

so that

$$hc_2(T) \geq -1 + \frac{c}{2(t+1)}, \tag{3.9}$$

which holds for all $t << 0$. Now using (3.9) and substituting $hc_2(T) = -1$ we get $c \leq 0$. Since $F^{\vee\vee}$ is reflexive, $c \geq 0$ so we obtain $c_3(T) = c = 0$.

Now it remains to show that $F^{\vee\vee}$ is a deformation of E. The first step is to show that L is a deformation of the line l. In order to do so we compute the class of L in $A^2(X)$, which is represented by $c_2(T) = a_1e_1 + a_2e_2 + a_3e_3$. Consider a divisor $D = \beta_1h_1 + \beta_2h_2 + \beta_3h_3$, by (1.1) and $c = 0$ we have

$$h^1(L, T(D)) = (D+2)c_2(T).$$

Suppose $\beta_i << 0$ for all i. Then

$$a_1(\beta_1+1) + a_2(\beta_2+1) + a_3(\beta_3+1) = h^1(L, T(D)) = h^2(X, F(D)) \leq h^2(X, G(D)) \tag{3.10}$$

where the last inequality is by semicontinuity. Furthermore $\beta_i << 0$ implies that $h^1(X, E(D)) = h^2(X, E(D)) = 0$ and thus

$$h^2(X, G(D)) = h^1(l, \mathcal{O}_l(D)) = -1 - \beta_1. \tag{3.11}$$

We showed that $a_1 + a_2 + a_3 = c_2(T)h = -1$ and combining this with (3.10) and (3.11) we obtain

$$a_2(\beta_2 - \beta_1) + a_3(\beta_3 - \beta_1) \leq 0$$

for all $\beta_i << 0$, thus we must have $a_2 = a_3 = 0$ and $a_1 = -1$, i.e. L lives in a neighborhood of l. It remains to show that $F^{\vee\vee}$ is a deformation of E.

Since $c = 0$ we have that $F^{\vee\vee}$ is locally free and we computed $c_2(T) = -e_1$, so we get $c_2(F^{\vee\vee}) = k_1e_1 + k_2e_2 + k_3e_3$, which implies that $F^{\vee\vee}$ has the same Chern classes as E. Therefore, $F^{\vee\vee}$ is a flat deformation of E and also semistable, so $F^{\vee\vee}$ lies in a neighborhood of E in $M_X(2, 0, c_2(E))$. To summarize, we showed that if F is not locally free it fits into a sequence

$$0 \to F \to E' \to \mathcal{O}_{l'} \to 0$$

with E' and l' flat deformation of E and l. But we observed that this is not possible, thus F must be locally free.

Remark 3.1. Using this argument, we are able to construct a good component of the moduli space instanton bundle, where each element behaves well when restricted to the generic line of each family. We observed that

4. Jumping lines

Let's recall the definition of a jumping line:

Definition 4.1. Let E be a rank two vector bundle on X with $c_1(E) = 0$. A jumping line for E is a line L such that $H^0(E_L(-r)) = 0$ for some $r > 0$. The largest such integer is called the order of the jumping line L.

Let's consider a line in the first family $e_1 = h_2h_3$. Then we have the following resolution

$$0 \to \mathcal{O}_X(-h_2 - h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_L \to 0 \tag{4.1}$$

Let $\mathcal{H}$ be the Hilbert scheme of lines of the family h_2h_3. In particular we have $\mathcal{H} = \mathbb{P}^1 \times \mathbb{P}^1$, and we will denote by l and m the generators of $\mathrm{Pic}(\mathcal{H})$. Writing the sequence (4.1) with respect to global section of $\mathcal{O}_X(-h_2)\oplus\mathcal{O}_X(-h_3)$ we get the description of the universal line $\mathcal{L}\subset X\times\mathcal{H}$

(4.2)

$$0\to\mathcal{O}_X(-h_2-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}(-1,-1)\to\mathcal{O}_X(-h_2)\boxtimes\mathcal{O}_{\mathcal{H}}(-1,0)\oplus\mathcal{O}_X(-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}(0,-1)\to\mathcal{O}_{X\times\mathcal{H}}\to\mathcal{O}_{\mathcal{L}}\to 0.$$

Let's denote by $\mathcal{D}^1_E$ the locus of jumping lines (from the first family) of an istanton bundle E, and by i its embedding in $\mathcal{H}$. Let's consider the following diagram

$$\begin{tikzcd} & \mathcal{L}\subset X\times\mathcal{H} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & \mathcal{H} \end{tikzcd} \tag{4.3}$$

where q and p are the projection to the first and second factor respectively.

Lemma 4.2. *$\mathcal{D}^1_E$ is the support of the sheaf $R^1p_*(q^*(E(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})$.*

Proof. See [**?**, p. 108] for a proof for $\mathbb{P}^n$. Since the argument is local, it can be generalized to our case. □

Proposition 4.3. *Let E be a generic instanton on X with $c_2=k_1e_1+k_2e_2+k_3e_3$. Then $\mathcal{D}^1_E$ is a divisor given by $\mathcal{D}^1_E=k_2l+k_3m$ equipped with a sheaf G fitting into*

$$0\to\mathcal{O}_{\mathcal{H}}^{k_3}(-1,0)\oplus\mathcal{O}_{\mathcal{H}}^{k_2}(0,-1)\to\mathcal{O}_{\mathcal{H}}^{k_2+k_3}\to i_*G\to 0 \tag{4.4}$$

Proof. By Lemma 4.2 a line L is jumping for E if and only if the point of $\mathcal{H}$ corresponding to L lies in the support of $R^1p_*(q^*(E(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})$.

Let us consider the Fourier-Mukai functor

$$\Phi_{\mathcal{L}}:D^b(X)\to D^b(\mathcal{H})$$

with kernel the structure sheaf of $\mathcal{L}$. We need to compute the transform of the bundles appearing in the monad (2.11) tensorixed by $\mathcal{O}_X(-h_1)$.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1-h_2))$.

  By (4.2) tensored by $\mathcal{O}_X(-2h_1-h_2)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1-2h_2))=1$ we get $R^ip_*(q^*(\mathcal{O}_X(-2h_1-h_2))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for $i\neq 1$. Using the projection formula we obtain

  $$R^1p_*(q^*(\mathcal{O}_X(-2h_1-h_2))\boxtimes\mathcal{O}_{\mathcal{L}})\cong R^2p_*(q^*(\mathcal{O}_X(-2h_1-2h_2)))\boxtimes\mathcal{O}_{\mathcal{H}}(-1,0).$$

  Observe that by Grauert's theorem we have that $R^2p_*(q^*(\mathcal{O}_X(-2h_1-2h_2)))$ is a rank one vector bundle on $\mathcal{H}$. Using Grothendieck-Riemann-Roch theorem it follows trivially that

  $$c_1(R^2p_*q^*(\mathcal{O}_X(-2h_1-2h_2)))=0.$$

  In fact consider the diagram (4.3). Since X is a threefold and $\mathcal{H}$ is a surface, we have that after being pulled-back on $X\times\mathcal{H}$ and push-forwarded to $\mathcal{H}$ all the cycles on X became either zero or points. So we obtain

  $$R^1p_*(q^*(\mathcal{O}_X(-2h_1-h_2))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}(-1,0).$$

  We continue with the other terms of the monad (2.11). The computations are completely analogous.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1-h_3))$.

  By (4.2) tensored by $\mathcal{O}_X(-2h_1-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1-2h_3))=1$ we get $R^ip_*(q^*(\mathcal{O}_X(-2h_1-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for $i\neq 1$ and

  $$R^1p_*(q^*(\mathcal{O}_X(-2h_1-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}(0,-1).$$

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_2-h_3))$.
  By (4.2) tensored by $\mathcal{O}_X(-h_1-h_2-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^ip_*(q^*(\mathcal{O}_X(-h_1-h_2-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.
- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_2))$.
  By (4.2) tensored by $\mathcal{O}_X(-h_1-h_2)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^ip_*(q^*(\mathcal{O}_X(-h_1-h_2))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.
- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_3))$.
  By (4.2) tensored by $\mathcal{O}_X(-h_1-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^ip_*(q^*(\mathcal{O}_X(-h_1-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.
- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1-h_3))$.
  By (4.2) tensored by $\mathcal{O}_X(-2h_1)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1))=1$ we get $R^ip_*(q^*(\mathcal{O}_X(-2h_1))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for $i\neq 1$ and
  $$R^1p_*(q^*(\mathcal{O}_X(-2h_1))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}.$$
- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1))$.
  By (4.2) tensored by $\mathcal{O}_X(-h_1)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^ip_*(q^*(\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.

Now we apply the $\Phi_{\mathcal{L}}$ to the moand (2.11). First we apply $\Phi_{\mathcal{L}}$ to the sequence

$$0\to K\to \begin{matrix}\mathcal{O}_X^{k_3}(-h_1-h_2)\\ \oplus\\ \mathcal{O}_X^{k_2}(-h_1-h_3)\\ \oplus\\ \mathcal{O}_X^{k_1}(-h_2-h_3)\end{matrix}\to\mathcal{O}_X^{k-2}\to 0$$

we get $R^ip_*q^*(K\otimes\mathcal{O}_X(-h_1))=0$ for $i\neq 1$ and

$$R^1p_*(q^*(K\otimes\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}.$$

From

$$0\to \begin{matrix}\mathcal{O}_X^{k_3}(-h_1-h_2)\\ \oplus\\ \mathcal{O}_X^{k_2}(-h_1-h_3)\\ \oplus\\ \mathcal{O}_X^{k_1}(-h_2-h_3)\end{matrix}\to K\to E\to 0$$

we get

$$0\to R^0p_*(q^*(E\otimes\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})\to\mathcal{O}_{\mathcal{H}}^{k_3}(-1,0)\oplus\mathcal{O}_{\mathcal{H}}^{k_2}(0,-1)\xrightarrow{\gamma}\mathcal{O}_{\mathcal{H}}^{k_2+k_3}.$$

so γ is a $(k_2+k_3)\times(k_2+k_3)$ made by a $(k_2+k_3)\times(k_3)$ linear matrix in the first variables of $\mathcal{H}$ and a $(k_2+k_3)\times(k_2)$ linear matrix in the second variables of $\mathcal{H}$. We observe that $\mathrm{Ker}(\gamma)$ is zero since is a torsion free sheaf which is zero outside $\mathcal{D}^1_E$, and $\mathrm{Coker}(\gamma)\cong R^1p_*(q^*(E\otimes\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})$ is an extension to $\mathcal{H}$ of a rank 1 sheaf on $\mathcal{D}^1_E$ denoted by G. That is a divisor k_3l+k_2m given by the vanishing of the determinant of γ

□

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