On non-linear dependence of multivariate subordinated Lévy processes

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Abstract

Multivariate subordinated Lévy processes are widely employed in finance for modeling multivariate asset returns. We propose to exploit non-linear dependence among financial assets through multivariate cumulants of these processes, for which we provide a closed form formula by using the multi-index generalized Bell polynomials. Using multivariate cumulants, we perform a sensitivity analysis, to investigate non-linear dependence as a function of the model parameters driving the dependence structure.

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1 Introduction

Lévy processes are widely used in finance to model asset returns being more versatile than Gaussian driven processes as they can model skewness and excess kurtosis. Their characteristic function describes the distribution of each independent increment through the Lévy-Khintchine representation. In the following we focus our attention on the moment generating function (mgf) and its relation with the cumulant generating function (cgf). In particular, if $\{\boldsymbol{L}(t), t \geq 0\}$ is a $\mathbb{R}^n$-valued Lévy process with mgf $M_{\boldsymbol{L}(t)}(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{R}^n$ at each t, the Lévy-Khintchine representation allows us to work with $\exp\left[tK_{\boldsymbol{Y}}(\boldsymbol{z})\right]$, where $K_{\boldsymbol{Y}}(\boldsymbol{z})$ is the cgf of $\boldsymbol{Y} = \boldsymbol{Y}(1)$, the time one distribution of the Lévy process.

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Deviation of asset prices from Normality is connected with trade activity, since the seminal work of Clark (1973). Trade activity is usually modeled using a stochastic change of time: time runs fast when there are a lot of orders, while it slows down when trade is stale. The subordination of a Lévy process $\boldsymbol{L}(t)$ by a univariate subordinator $T(t)$, i.e. a Lévy process on $\mathbb{R}_+ = [0, \infty)$ with increasing trajectories, independent of $\boldsymbol{L}(t)$, defines a new process $\boldsymbol{Y}(t)$ by the composition $\boldsymbol{Y}(t) := \boldsymbol{L}(T(t))$. Unfortunately, the resulting models exhibit several shortcomings including the lack of independence between asset returns and a limited span of linear correlations. Furthermore, there is empirical evidence that trading activity is different across assets (Harris (1986)). From the theoretical perspective, multivariate subordination allowing different assets to have different time-changes was introduced in the work of Barndorff-Nielsen et al. (2001). Given a $\mathbb{R}^n$-valued multiparameter Lévy process $\{\boldsymbol{L}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$ as defined in Barndorff-Nielsen et al. (2001), the $\mathbb{R}^n$-valued subordinated Lévy process $\boldsymbol{Y}(t)$ is the composition $\boldsymbol{Y}(t) = \boldsymbol{L}(\boldsymbol{T}(t))$, where $\boldsymbol{T}(t)$ is a multivariate subordinator, i.e. a Lévy process on $\mathbb{R}^d_+$ whose trajectories are increasing in each coordinate, independent of $\boldsymbol{L}(t)$.

In Section 2, we give a closed form formula of joint (or cross) cumulants of $\boldsymbol{Y}(t)$ through the multi-index generalized (complete exponential) Bell polynomials introduced in Di Nardo et al. (2011). We use an *umbral* evaluation operator (Di Nardo (2015)) to recover the contribution of joint cumulants of the d-dimensional subordinator. For multivariate subordinated Brownian motions, this closed form formula further simplifies by taking advantage of the well-known property that cumulants of Brownian motion are zero when their order is greater than two. The case of subordinated Brownian motions is of particular interest in finance for two reasons: first they link deviation of normality of asset returns to trade activity and second they often have analytical characteristic functions. A subclass of multivariate subordinated Brownian motions widely used in finance because of their economic interpretation is the so called $\rho\alpha$-models, see Luciano and Semeraro (2010). These models exhibit a flexible dependence structure and allow to model also high correlations. They also incorporate non-linear dependence as discussed in Luciano and Semeraro (2010): however this feature has not be investigated so far.

In Section 3, we propose to use joint cumulants to study higher order dependence and its behaviour in time. Indeed, as well known, the covariance matrix completely describes the dependence structure among components of a multivariate process only for Gaussian ones. Co-skewness and co-kurtosis measure extreme deviations or dispersions undergone by the components, as in the univariate case they measure asymmetry and fat-tailedness. Moreover, higher order cumulants play an important role in the analysis of non-Gaussian data and allow to detect higher order cross-correlations, a critical feature that exacerbates during financial turmoils, see Domino et al. (2018). If joint cumulants are asymptotically zero, central limit theorems can be investigated to forecast the market behavior or viceversa to choose among different models the one which better incorporates non-linear dependence. As case study, we investigate the non-linear dependence structure of $\rho\alpha$-models describing asset returns as a superposition of an idiosyncratic component, due to the asset specific trades, and a systematic one, due to the overall trade. We focus our attention on the Normal Inverse Gaussian (NIG) specification, whose one dimensional marginals are NIG processes and discuss the role played by the model parameters in driving non-linear dependence.

2 Cumulants of multivariate subordinated Lévy processes

Let us consider the multiparameter Lévy process $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{A}\boldsymbol{Z}(\boldsymbol{s})$ where $\boldsymbol{Z}(\boldsymbol{s}) = (Z_1(s_1), \ldots, Z_d(s_d))$, $\boldsymbol{s} \in \mathbb{R}^d_+$, is a multiparameter Lévy process with independent components and $\boldsymbol{A} = (a_{ij})_{n\times d} \in \mathbb{R}^{n\times d}$. According to Barndorff-Nielsen et al. (2001), the subordinated process $\boldsymbol{Y}(t)$

$$\boldsymbol{Y}(t) := \boldsymbol{A}\boldsymbol{Z}(\boldsymbol{T}(t)) = \boldsymbol{A}(Z_1(T_1(t)), \cdots, Z_d(T_d(t)))^T \tag{2.1}$$

is a Lévy process. It's a straightforward consequence of Theorem 4.7 in Barndorff-Nielsen et al. (2001) to prove that

$$K_{\boldsymbol{Y}}(\boldsymbol{z}) = K_{\boldsymbol{T}}\left(K_{Z_1}\left(\sum_{m=1}^{n} a_{m1} z_m\right), \ldots, K_{Z_d}\left(\sum_{m=1}^{n} a_{md} z_m\right)\right), \; \boldsymbol{z} \in \mathbb{R}^n. \tag{2.2}$$

Indeed if $\boldsymbol{\delta}_j = (\delta_{j1}, \ldots, \delta_{jd})$ with Kronecker's δ_{jk}, then $\boldsymbol{L}(\boldsymbol{\delta}_j) = (a_{1j}Z_j(1), \ldots, a_{nj}Z_j(1))^T$ has mgf

$$M_j(\boldsymbol{z}) = E\left[\exp\left(Z_j(1)\sum_{m=1}^{n} a_{mj} z_m\right)\right] = M_{Z_j}\left(\sum_{m=1}^{n} a_{mj} z_m\right). \tag{2.3}$$

Since $\log M_{Z_j}$ is the cgf of $Z_j(1)$ and $K_{\boldsymbol{Y}}(\boldsymbol{z}) = \log M_{\boldsymbol{Y}}(\boldsymbol{z})$, then (2.2) follows by plugging (2.3) in $M_{\boldsymbol{Y}}(\boldsymbol{z}) = \exp[K_{\boldsymbol{T}}(\log M_1(\boldsymbol{z}), \ldots, \log M_d(\boldsymbol{z}))]$. In order to recover the $\boldsymbol{i}$-th cumulant $c_{\boldsymbol{i}}(\boldsymbol{Y})$ of $\boldsymbol{Y}(1)$, we need to expand in formal power series the cgf $K_{\boldsymbol{Y}}(\boldsymbol{z})$ given in (2.2). To this aim, let us recall the notion of multi-index partition introduced in Di Nardo et al. (2011).

Definition 2.1. *A partition of a multi-index $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ is a matrix Λ of non-negative integers with n rows and no zero columns in lexicographic order, such that $\lambda_{s1} + \lambda_{s2} + \cdots = i_s$ for $s = 1, 2, \ldots, n$.*

As for integer partitions, let us fix some notation:

- $|\Lambda|$ is the sum of all components of Λ
- $\Lambda = (\boldsymbol{\lambda}_1^{r_1}, \boldsymbol{\lambda}_2^{r_2}, \ldots) \vdash \boldsymbol{i}$ denotes the multi-index partition of $\boldsymbol{i}$ with r_1 columns equal to $\boldsymbol{\lambda}_1$, r_2 columns equal to $\boldsymbol{\lambda}_2$ and so on, with $\boldsymbol{\lambda}_1 < \boldsymbol{\lambda}_2 < \ldots$
- $\mathfrak{m}(\Lambda) = (r_1, r_2, \ldots)$ is the vector of multiplicities of $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots$
- $l(\Lambda) = |\mathfrak{m}(\Lambda)| = r_1 + r_2 + \cdots$ is the number of columns of Λ and $\Lambda! = \prod_{j=1}^{l(\Lambda)} (\boldsymbol{\lambda}_j!)^{r_j}$ where $\boldsymbol{\lambda}_j! = \lambda_{1j}!\lambda_{2j}! \cdots$
- given the multi-indexed sequence $\{g_{\boldsymbol{j}}\}$, the product $g_\Lambda = \prod_{j=1}^{l(\Lambda)} g_{\boldsymbol{\lambda}_j}^{r_j}$ is said associated to the sequence through $\Lambda \vdash \boldsymbol{i}$, in particular $g_\Lambda = 1$ if $\Lambda \vdash \boldsymbol{i} = \boldsymbol{0}$.

Example 1. The partitions of the multi-index $(2, 1)$ are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \vdash \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Let us consider the partition $\Lambda = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2^2)$ with $\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{\lambda}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus $l(\Lambda) = 3, |\Lambda| = 3, \mathfrak{m}(\Lambda) = (1, 2)$, and $\Lambda! = \boldsymbol{\lambda}_1!(\boldsymbol{\lambda}_2!)^2$ where $\boldsymbol{\lambda}_1! = \boldsymbol{\lambda}_2! = 1!0!$. The product $g_\Lambda = g_{\boldsymbol{\lambda}_1} g_{\boldsymbol{\lambda}_2}^2 = g_{0,1} g_{1,0}^2$ is said associated to the sequence $\{g_{\boldsymbol{j}}\}$.

The $\boldsymbol{i}$-th coefficient of $\exp\left(\sum_{k=1}^d x_k g_k(\boldsymbol{z})\right)$ with $g_k(\boldsymbol{z}) = \sum_{\boldsymbol{j}:|\boldsymbol{j}|>0} g_{k,\boldsymbol{j}} \frac{\boldsymbol{z}^{\boldsymbol{j}}}{\boldsymbol{j}!}$ is the $\boldsymbol{i}$-th generalized (complete exponential) Bell polynomial (Di Nardo et al. (2011))

$$\mathfrak{B}_{\boldsymbol{i}}(x_1, \ldots, x_d) = \boldsymbol{i}! \sum_{\substack{\Lambda \vdash \boldsymbol{s}_1, \ldots, \tilde{\Lambda} \vdash \boldsymbol{s}_d \\ \boldsymbol{s}_1 + \cdots + \boldsymbol{s}_d = \boldsymbol{i}}} x_1^{l(\Lambda)} \cdots x_d^{l(\tilde{\Lambda})} \frac{g_{1,\Lambda} \cdots g_{d,\tilde{\Lambda}}}{\Lambda! \cdots \tilde{\Lambda}!\, \mathfrak{m}(\Lambda)! \cdots \mathfrak{m}(\tilde{\Lambda})!}, \tag{2.4}$$

where $g_{k,\Lambda}$ is associated to the sequence $\{g_{k,\boldsymbol{j}}\}$ through $\Lambda \vdash \boldsymbol{s}_k$ for $k = 1, \ldots, d$.

Theorem 2.1. *The $\boldsymbol{i}$-th cumulant of $\boldsymbol{Y}$ is $c_{\boldsymbol{i}}(\boldsymbol{Y}) = E\left(\mathfrak{B}_{\boldsymbol{i}}(T_1, \ldots, T_d)\right)$ where*

i) E is an umbral evaluation linear operator (Di Nardo et al. (2011)) such that $E(T_1^{i_1} \cdots T_d^{i_d}) = c_{\boldsymbol{i}}(\boldsymbol{T}), \boldsymbol{i} \in \mathbb{N}^n$, the $\boldsymbol{i}$-th joint cumulant of $\boldsymbol{T}$;

ii) $g_{k,\Lambda}$ is associated to $\{g_{k,\boldsymbol{\lambda}_j}\}$ with $g_{k,\boldsymbol{\lambda}_j} = c_{|\boldsymbol{\lambda}_j|}(Z_k)(\boldsymbol{a}_{.k})^{\boldsymbol{\lambda}_j}$ where $\{c_{|\boldsymbol{\lambda}_j|}(Z_k)\}$ are cumulants of Z_k and $\boldsymbol{a}_{.k}$ is the k-th column of $\boldsymbol{A}$.

Proof. From (2.2) the cgf of $\boldsymbol{Y}$ is a composition $f[g_1(\boldsymbol{z}), \ldots, g_d(\boldsymbol{z})]$, with $g_k(\boldsymbol{z}) = K_{Z_k}(\sum_{m=1}^{n} a_{mk} z_m)$ and $f(\tilde{\boldsymbol{z}}) = K_{\boldsymbol{T}}(\tilde{\boldsymbol{z}}), \tilde{\boldsymbol{z}} = (\tilde{z}_1, \ldots, \tilde{z}_d)$. The $\boldsymbol{i}$-th coefficient of $f[g_1(\boldsymbol{z}), \ldots, g_d(\boldsymbol{z})]$ is computed by expanding $\exp\left(\sum_{k=1}^{d} x_k g_k(\boldsymbol{z})\right)$ in formal power series (Di Nardo et al. (2011)) and by replacing the product $x_1^{l(\Lambda)} \cdots x_d^{l(\tilde{\Lambda})}$ with the $(l(\Lambda), \ldots, l(\tilde{\Lambda}))$-th coefficient of the formal power series $f(\tilde{\boldsymbol{z}})$, that is the joint cumulant $c_{l(\Lambda),\ldots,l(\tilde{\Lambda})}(\boldsymbol{T})$. By further expanding K_{Z_k} in formal power series, we have $g_{k,\boldsymbol{\lambda}} = c_{|\boldsymbol{\lambda}|}(Z_k)(\boldsymbol{a}_{.k})^{\boldsymbol{\lambda}}$ where $c_{|\boldsymbol{\lambda}|}(Z_k)$ is the $|\boldsymbol{\lambda}|$-th cumulant of Z_k and $\boldsymbol{a}_{.k}$ is the k-th column of $\boldsymbol{A}$. Thus the result follows from (2.4). □

2.1 Multivariate subordinated Brownian motion

As multiparameter Lévy process $\boldsymbol{Z}(\boldsymbol{s})$ in (2.1) let us consider $\boldsymbol{B}(\boldsymbol{s}) = (B_1(s_1), \ldots, B_d(s_d))^\top, \boldsymbol{s} \in \mathbb{R}_+^d$, obtained from a $\mathbb{R}^d$-valued Brownian motion $\{\boldsymbol{B}(t), t \geq 0\}$ with independent components, drift $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2)$, and the multiparameter Lévy process

$$\boldsymbol{B}_A(\boldsymbol{s}) := \boldsymbol{A}\boldsymbol{B}(\boldsymbol{s}) = \boldsymbol{A}(B_1(s_1), \ldots, B_d(s_d))^\top, \;\; \boldsymbol{s} \in \mathbb{R}_+^d. \tag{2.5}$$

Note that $\boldsymbol{B}_A(t) = \boldsymbol{A}\boldsymbol{B}(t) = \boldsymbol{A}(B_1(t), \ldots, B_n(t))^\top$ has drift $\boldsymbol{\mu}_A = \boldsymbol{A}\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_A = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top$ and that, if $n = d$ and $\boldsymbol{A}$ is the identity matrix, we recover the subcase of independent Brownian motions. Suppose $\boldsymbol{T}(t)$ independent of $\boldsymbol{B}_A(\boldsymbol{s})$ and consider $\boldsymbol{Y}(t) = \boldsymbol{B}_A(\boldsymbol{T}(t))$. Theorem 2.1 allows us to compute the $\boldsymbol{i}$-th cumulant $c_{\boldsymbol{i}}(\boldsymbol{Y})$ by taking advantage of the well-known property that cumulants of a Brownian motion are zero when their order is greater than 2. Indeed in (2.4), the products $g_{k,\Lambda} = \prod_{j=1}^{l(\Lambda)} g_{k,\boldsymbol{\lambda}_j}^{r_j}$ are not zero if and only if $|\boldsymbol{\lambda}_j| \leq 2$ that is $\Lambda \in \mathcal{P}_2(\boldsymbol{s}_k) = \{(\boldsymbol{\lambda}_1^{r_1}, \boldsymbol{\lambda}_2^{r_2}, \ldots) \vdash \boldsymbol{s} : |\boldsymbol{\lambda}| \leq 2\}$, the set of all multi-index partitions whose columns have sum of components not greater than 2. Thus the summation in (2.4) reduces to $\{\Lambda \in \mathcal{P}_2(\boldsymbol{s}_1), \ldots, \tilde{\Lambda} \in \mathcal{P}_2(\boldsymbol{s}_d) \text{ with } \boldsymbol{s}_1 + \cdots + \boldsymbol{s}_d = \boldsymbol{i}\}$. Moreover the sequence $\{g_{k,\boldsymbol{\lambda}_j}\}$ is given by

$$g_{k,\boldsymbol{\lambda}_j} = \begin{cases} a_{mj}\mu_j & \text{if } |\boldsymbol{\lambda}_j| = 1 \text{ and } (\boldsymbol{\lambda}_j)_m = 1 \text{ for } m \in [n], \\ a_{mj}^2\sigma_j^2 & \text{if } |\boldsymbol{\lambda}_j| = 2 \text{ and } (\boldsymbol{\lambda}_j)_m = 2 \text{ for } m \in [n], \\ a_{m_1 j} a_{m_2 j}\sigma_j^2 & \text{if } |\boldsymbol{\lambda}_j| = 2 \text{ and } (\boldsymbol{\lambda}_j)_{m_1} = (\boldsymbol{\lambda}_j)_{m_2} = 1 \text{ for } m_1 \neq m_2 \in [n], \\ 0 & \text{if } |\boldsymbol{\lambda}_j| > 2, \end{cases} \tag{2.6}$$

with $[n] = \{1, \ldots, n\}$. Indeed in (2.2) the cgf $K_{Z_k}(\sum_{m=1}^n a_{mk} z_m)$ reduces to $\mu_k \sum_{m=1}^n a_{mk} z_m + \frac{1}{2}\sigma_k^2 \big(\sum_{m=1}^n a_{mk} z_m \big)^2$ as $\{Z_k\}_{k=1}^d$ are Gaussian distributed with mean $\{\mu_k\}_{k=1}^d$ and variance $\{\sigma_k^2\}_{k=1}^d$. Note that when $d = 1, E\left(\mathfrak{B}_{\boldsymbol{i}}(T_1, \ldots, T_d)\right)$ in Theorem 2.1 reduces to

$$c_{\boldsymbol{i}}(\boldsymbol{Y}) = \boldsymbol{i}! \sum_{\Lambda \in \mathcal{P}_2(\boldsymbol{i})} c_{l(\Lambda)}(T) \prod_{s=1}^{l(\lambda)} \frac{g_{\boldsymbol{\lambda}_s}^{r_s}}{(\boldsymbol{\lambda}_s!)^{r_s} r_s!}. \tag{2.7}$$

2.2 The $\rho\alpha$-model

In this section, we consider a class of processes used in finance to model asset returns: the $\rho\alpha$-models. Jevtić et al. (2018) proved that they belong to the class of multivariate subordinated Brownian motions (2.5), by properly choosing the matrix A and the subordinator $\boldsymbol{T}(t)$.

A $\rho\alpha$-model (Luciano and Semeraro (2010)) is constructed by subordinating n independent Brownian motions $B_j(t)$ with n independent subordinators $\{X_j(t)\}$ and by subordinating a multidimensional Brownian motion $\boldsymbol{B}^\rho(t)$ with a unique subordinator $Z(t)$. More in details, let $\boldsymbol{B}^\rho(t) = (B_1^\rho(t), ..., B_n^\rho(t))$ be a multivariate Brownian motion, with correlations $\boldsymbol{\rho}$ and Lévy triplet $(\boldsymbol{\mu}^\rho, \boldsymbol{\Sigma}^\rho, \boldsymbol{0})$, where $\boldsymbol{\mu}^\rho = (\mu_1\alpha_1, \ldots, \mu_n\alpha_n)$, with $\boldsymbol{\mu} \in \mathbb{R}^n$, and $\boldsymbol{\Sigma}^\rho := (\rho_{ij}\sigma_i\sigma_j\sqrt{\alpha_i}\sqrt{\alpha_j})_{ij}$, with $\sigma_j > 0$ and $\rho_{ij} \in [-1, 1]$ for $i, j = 1, \ldots, n$. Assume $\boldsymbol{B}^\rho(t)$ independent of $\boldsymbol{B}(\boldsymbol{s}) = (B_1(s_1), \ldots, B_n(s_n))^\top$ obtained from a $\mathbb{R}^n$-valued Brownian motion $\{\boldsymbol{B}(t), t \geq 0\}$ with independent components, drift $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$. The $\rho\alpha$-model $\boldsymbol{Y}(t)$ is the $\mathbb{R}^n$-valued subordinated process

$$\boldsymbol{Y}(t) = \begin{pmatrix} Y_1^I(t) + Y_1^\rho(t) \\ \vdots \\ Y_n^I(t) + Y_n^\rho(t) \end{pmatrix} = \begin{pmatrix} B_1(X_1(t)) + B_1^\rho(Z(t)) \\ \vdots \\ B_n(X_n(t)) + B_n^\rho(Z(t)) \end{pmatrix}, \tag{2.8}$$

where $X_j(t)$ and $Z(t)$ are independent subordinators, independent of $\boldsymbol{B}(t)$ and $\boldsymbol{B}^\rho(t)$. If $\rho = 0$, the process $\boldsymbol{Y}(t)$ is said an α-model. As $\boldsymbol{Y}^I$ has independent margins, from the additivity property of cumulants, we have $K_{\boldsymbol{Y}}(\boldsymbol{z}) = K_{\boldsymbol{Y}^I}(\boldsymbol{z}) + K_{\boldsymbol{Y}^\rho}(\boldsymbol{z}) = \sum_{j=1}^n g_j(\boldsymbol{z}) + f(\boldsymbol{z})$, where

$$g_j(\boldsymbol{z}) = K_{X_j}\left(\mu_j z_j + \frac{1}{2}\sigma_j^2 z_j^2\right) \quad \text{and} \quad f(\boldsymbol{z}) = K_Z\left(\boldsymbol{z}^T\boldsymbol{\mu}^\rho + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Sigma}^\rho\boldsymbol{z}\right), \tag{2.9}$$

and joint cumulants

$$c_{\boldsymbol{i}}(\boldsymbol{Y}) = \begin{cases} \sum_{j=1}^n g_{j,\boldsymbol{i}} + f_{\boldsymbol{i}} & \text{if } \boldsymbol{i} = (0, \ldots, i_m, \ldots, 0), \\ f_{\boldsymbol{i}} & \text{otherwise,} \end{cases} \tag{2.10}$$

with $g_{j,\boldsymbol{i}}$ and $f_{\boldsymbol{i}}$ the $\boldsymbol{i}$-th coefficient of $g_j(\boldsymbol{z})$ and $f(\boldsymbol{z})$ in (2.9) respectively. By using (2.7) we have

$$g_{j,\boldsymbol{i}} = \boldsymbol{i}! \sum_{\Lambda \vdash \boldsymbol{i}} c_{l(\Lambda)}(X_j) \prod_{s=1}^{l(\Lambda)} \frac{(g_{j,\boldsymbol{\lambda}_s})^{r_s}}{r_s!(\boldsymbol{\lambda}_s!)^{r_s}} \qquad \text{and} \qquad f_{\boldsymbol{i}} = \boldsymbol{i}! \sum_{\Lambda \vdash \boldsymbol{i}} c_{l(\Lambda)}(Z) \prod_{s=1}^{l(\Lambda)} \frac{(\tilde{g}_{\boldsymbol{\lambda}_s})^{r_s}}{r_s!(\boldsymbol{\lambda}_s!)^{r_s}}, \tag{2.11}$$

where $c_{l(\Lambda)}(X_j)$ and $c_{l(\Lambda)}(Z)$ are the $l(\Lambda)$-th cumulants of X_j and Z respectively, $\{g_{j,\boldsymbol{\lambda}_s}\}$ and $\{\tilde{g}_{\boldsymbol{\lambda}_s}\}$ are the coefficients of the inner power series of K_{X_j} and K_Z respectively, as given in (2.9). To discuss non-linear dependence for the bivariate case in Section 3, we consider the normalized cumulants

$$\bar{c}_{i,j}(t) = \frac{c_{i,j}[\boldsymbol{Y}(t)]}{(c_2[Y_1(t)])^{i/2}(c_2[Y_2(t)])^{j/2}}, \quad i,j \in \mathbb{N}, \tag{2.12}$$

as a function of the time scale. For $i + j \leq 4$ it's straightforward to get

$$\begin{array}{rclrcl} \bar{c}_{1,1}(t) & = & a(b_{1,1} + d_{1,1}\rho_{12}) & \bar{c}_{1,2}(t) & = & \frac{a}{\sqrt{t}}(b_{1,2} + d_{1,2}\rho_{12}) \\ \bar{c}_{1,3}(t) & = & \frac{a}{t}(b_{1,3} + d_{1,3}\rho_{12}) & \bar{c}_{2,2}(t) & = & \frac{a}{t}(b_{2,2} + d_{2,2}\rho_{12} + e_{2,2}\rho_{12}^2) \end{array} \tag{2.13}$$

where $b_{i,j}, d_{i,j}$ and $e_{i,j}$ are functions of the marginal parameters $(\mu_i, \sigma_i, \alpha_i)$. Cumulants of $\boldsymbol{Y}(t)$ increase linearly in t so that co-skewness measures are proportional to $1/\sqrt{t}$, while co-kurtosis measures are proportional to $1/t$, converging to Gaussian values asymptotically. Notice that a is a scale parameter for all cross-cumulants, driving the general level of dependence, both linear and non-linear. Furthermore, the Brownian motion correlation ρ_{12}, providing an extra-term in (2.13), affects non only asset correlation measured by $\bar{c}_{1,1}(t)$, but also non-linear dependence measured by the other cross-cumulants. Thus the $\rho\alpha$-models not only span a wider range of linear dependence compared to the α-models, but they can also incorporate higher non-linear dependencies.

3 A case study

To show the role played by cumulants in analyzing non-linear dependence over time, let us consider a bivariate price process $\{\boldsymbol{S}(t),\, t \geq 0\}$ such that $\boldsymbol{S}(t) = \boldsymbol{S}(0)\exp(\boldsymbol{d}t + \boldsymbol{Y}(t))$, where $\boldsymbol{d} \in \mathbb{R}^2$ is the drift term, not affecting the dependence structure, and $\boldsymbol{Y}(t)$ is a bivariate Lévy process. Since we are in the class of multivariate Lévy models, the centered asymptotic distribution of daily logreturns is a bivariate Normal distribution $N(\boldsymbol{0}, \Sigma)$ where Σ is the constant covariance matrix of the process (see Jammalamadaka et al. (2004)). This analysis is performed considering the NIG specification of the $\rho\alpha$-model.

3.1 NIG specification

Recall that a NIG process $Y(t)$ has no Gaussian component, is of infinite variation and can be constructed by subordination as $Y(t) = \beta\delta T(t) + \delta^2 B(T(t))$, where $T \sim \mathrm{IG}(1, \delta\sqrt{\gamma^2 - \beta^2})$ is independent of the standard Brownian motion $B(t)$. In particular the time one distribution $Y(1)$ has a NIG distribution $\mathrm{NIG}(\gamma, \delta, \beta)$ with parameters $\gamma > 0, |\beta| < \gamma$ and $\delta > 0$.

Now, let us consider the $\rho\alpha$-model (2.8), with $X_j \sim \mathrm{IG}(1 - a\sqrt{\alpha_j}, \alpha_j^{-1/2})\, j = 1, ..., n$ and $Z \sim \mathrm{IG}(a, 1)$ the time one distributions of the subordinators $X_j(t)$ and $Z(t)$ respectively. If we choose the parameters of $\boldsymbol{B}^\rho(t)$ and of the subordinators so that:

$$\alpha_j^{-1/2} = \delta_j\sqrt{\gamma_j^2 - \beta_j^2} \qquad \text{and} \qquad \mu_j = \beta_j\delta_j^2,\ \sigma_j = \delta_j \tag{3.1}$$

with $\gamma_j, \delta_j \in \mathbb{R}_+, \beta_j \in \mathbb{R}$ and $|\beta_j| < \gamma_j$ for $j = 1, \ldots, n$, the subordinated process $\boldsymbol{Y}(t)$ in (2.8) has the remarkable property that its one dimensional marginal processes have NIG distributions $\mathrm{NIG}(\gamma_j, \delta_j, \beta_j)$ for $t = 1$. The process $\boldsymbol{Y}(t)$ is named $\rho\alpha$-NIG process. Thus $c_{\boldsymbol{i}}(\boldsymbol{Y})$ is given in (2.10) for suitable replacements of the parameters in (2.11), taking into account that if $X \sim \mathrm{IG}(a, b)$, its cumulants are $c_1(X) = a/b$ and $c_k(X) = a(2k-3)!!/b^{2k-1}$ for $k = 2, 3, \ldots$.

3.2 Numerical results

In this section, we use the cross-cumulants (2.13) to study the evolution in time of non-linear dependence in a bivariate $\rho\alpha$-NIG model as a function of the common parameters ρ and a. In fact, dependence is driven by the common subordinator Z representing the systematic component in two ways: first, through the common parameter a which defines the distribution of the common time change; second, through the action of the Brownian motions' correlation ρ on Z. The reference parameter set is given in Jevtić et al. (2018) to which we refer for further details[1]. We plot the normalized cumulants $\bar{c}_{i,j}(t)$ in (2.12) up to the fourth order as a function of the Brownian correlation ρ. We consider two scenarios corresponding to different values of the scaling parameter a. Since $a \in (0, 2.1)$ and the limit value $a = 0$ corresponds to independence, we choose the intermediate value $a = 1.05$ and the higher boundary value $a = 2.1$. For each scenario, we plot the evolution of cross-cumulants for three levels of time to maturity. Comparing the two

[1] The $\rho\alpha$-NIG specification has been calibrated by the generalized method of moments (GMM) to a bivariate basket composed by Goldman Sachs and Morgan Stanley US daily logreturns from January 3, 2011 to December 31st, 2015. Marginal parameters are: $\gamma_1 = 85.4175, \gamma_2 = 64.2544, \delta_1 = 0.0248, \delta_2 = 0.0335, \beta_1 = -8.8886, \delta_2 = -13.5988$. The marginal drifts not involved in the dependence structure are $d_1 = 0.0027$ and $d_2 = 0.0074$

scenarios, it is evident that a drives both linear and non-linear dependence and it allows to reach maximal correlation. Nevertheless, also in the first scenario, where the maximal attainable asset correlation is 0.5, moving ρ, we are able to incorporate non-linear dependence. The evolution in time confirms that higher order cumulants go to zero according to the rates in equation (2.13).

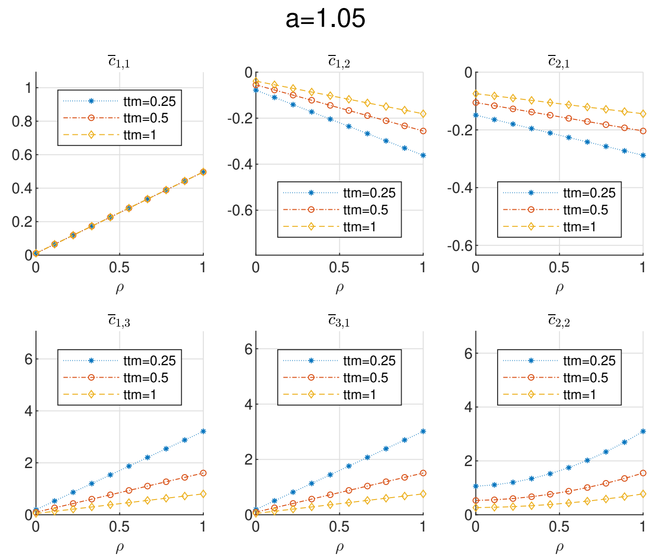

Figure 1: Scenario 1. Normalized cross-cumulants.

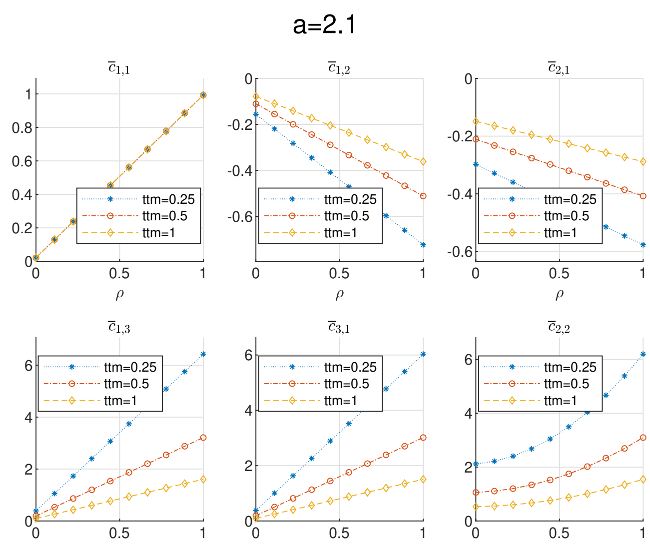

Figure 2: Scenario 2. Normalized cross-cumulants.

4 Conclusions

Cumulants are a powerful tool to measure non-linear dependence. We take advantage of the Lévy-Kinthchin representation of multiparameter subordinated Lévy processes to find their cumulants in closed form. We use them to study non-linear dependence captured by some class of processes widely used in finance to model asset returns. Indeed higher order statistics and suitable tests of hypotheses have been employed aiming to identify nonlinear processes Masson (2001). Nevertheless, the closed formula we found has other possible uses, as for example the estimate of the process parameters since the maximum likelihood estimation is computationally cumbersome within a multivariate framework. For large samples or high frequency data, the closed formulae of multivariate cumulants can be matched to multivariate k-statistics, unbiased estimators with minimum variance Di Nardo (2015), and thus estimates of parameters can be recovered by an analogous of GMM. This is in the agenda of our future research.

References

Barndorff-Nielsen, O. E., Pedersen, J., Sato, K., 2001. Multivariate subordination, self-decomposability and stability. *Advances in Applied Probability.* 33(1), 160–187

Clark, P. K. 1973. A subordinated stochastic process model with finite variance for speculative prices., *Econometrica: journal of the Econometric Society.*, 41 (1), 135–155

Di Nardo, E., 2015. Symbolic calculus in mathematical statistics: a review. *Séminaire Lotharingien de Combinatoire*, 67:B67a.

Di Nardo, E., Guarino, G., Senato, D., 2011. A new algorithm for computing the multivariate Faa di Bruno's formula. *Applied Mathematics and Computation*, 217(13), 6286–6295.

Domino, K., Gawron, P., Pawela, Ł., 2018. Efficient computation of higher-order cumulant tensors. *SIAM Journal on Scientific Computing*, 40(3), A1590–A1610.

Harris, L., 1986. Cross-security tests of the mixture of distributions hypothesis. *Journal of financial and Quantitative Analysis*, 21(1), 39–46.

Jammalamadaka, S. R., Rao, T. S., Terdik, G., 2004. Higher order cumulants of random vectors and applications to statistical inference and time series. *Sankhya. The Indian Journal of Statistics*, 68(2), 326–356.

Jevtić, P., Marena, M., Semeraro, P., 2019. Multivariate marked poisson processes and market related multidimensional information flows. *International Journal of Theoretical and Applied Finance*, 22(2), 1850058.

Luciano, E., Semeraro, P., 2010. Multivariate time changes for Lévy asset models: Characterization and calibration. *Journal of Computational and Applied Mathematics*, 233(8), 1937–1953.

Masson, A., Shishkov, B. B., Lefeuvre, F., 2001. On the use of higher-order statistical tests in the analysis of time series associated with space data. *Traitement du Signal*, 18, 59–78.