The Time-Frequency Interference Terms of the Green’s Function for the Harmonic Oscillator

The Time-Frequency Interference Terms of the Green's Function for the Harmonic Oscillator

Lorenzo Galleani

Abstract. The harmonic oscillator is a fundamental prototype for all types of resonances, and hence plays a key role in the study of physical systems governed by differential equations. The time-frequency representation of its Green's function, obtained through the Wigner distribution, reveals the time-varying frequency structure of resonances. Unfortunately, the Wigner distribution of the Green's function is affected by strong interference terms with a highly oscillatory structure. We characterize these interference terms by evaluating the ambiguity function of the Green's function. The obtained result shows that, in the ambiguity domain, the interference terms are localized and separate from the resonance component, and hence they can be reduced by a proper filtering.

Department of Electronics and Telecommunications
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy

1. Introduction

Differential equations model a wide variety of deterministic and random physical phenomena. A common approach to study them are transformation techniques, such as frequency analysis (the Fourier transform) [1] and the Laplace transform [2]. An effective approach is also time-frequency analysis [3], [4], a body of techniques for the characterization of signals whose frequency content changes with time. Conversely from frequency analysis, where the Fourier transform connects the time and frequency domains, in time-frequency analysis there are infinite time-frequency representations, or distributions, such as the Wigner distribution [3], [5], [6]

$$W_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x^*(t-\tau/2)x(t+\tau/2)e^{-i\tau\omega}d\tau. \qquad (1)$$

In [7] we have obtained the Wigner distribution of the Green's function for the harmonic oscillator, a fundamental model for resonant phenomena defined as

$$\frac{d^2x(t)}{dt^2} + 2\mu\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t), \tag{2}$$

where $f(t)$ is the forcing term, or input, $x(t)$ is the solution, also referred to as output or response, and we consider the case $\mu < \omega_0$, which gives rise to a resonance at the frequency

$$\omega_c = \sqrt{\omega_0^2 - \mu^2}. \tag{3}$$

The Green's function is defined as the solution $h(t)$ when the forcing term is a Dirac delta function [8]. Since the delta function is the ideal impulse, the Green's function is also referred to as the impulse response. The advantage of the Green's function is that, for any forcing term, the solution of (2) can be written through the convolution integral

$$x(t) = \int_{-\infty}^{+\infty} h(t-t')f(t')dt'. \tag{4}$$

The convolution property holds also in the time-frequency domain [3]

$$W_x(t,\omega) = \int_{-\infty}^{+\infty} W_h(t-t',\omega)W_f(t',\omega)dt'. \tag{5}$$

The Green's function is a cornerstone for the analysis and design of physical systems and devices, and it can be used for any ordinary differential equation with constant coefficients [1], as well as for partial differential equations.

The Wigner distribution of the Green's function for the harmonic oscillator is given by [7]

$$W_h(t,\omega) = \frac{1}{4\omega_c^2}W_{h_L}(t,\omega-\omega_c) + \frac{1}{4\omega_c^2}W_{h_L}(t,\omega+\omega_c) - \frac{1}{2\omega_c^2}W_{h_L}(t,\omega)\cos 2\omega_c t, \tag{6}$$

where

$$W_{h_L}(t,\omega) = u(t)e^{-2\mu t}\frac{\sin 2\omega t}{\pi\omega} \tag{7}$$

is the Wigner distribution of the Green's function $h_L(t)$ corresponding to the first-order differential equation

$$\frac{dx(t)}{dt} + \mu x(t) = f(t), \tag{8}$$

and $u(t)$ is the Heaviside step function defined as $u(t) = 1$ for $t \geq 0$, and $u(t) = 0$ for $t < 0$. When $f(t)$ is white Gaussian noise, (8) is the Langevin equation [9], a fundamental model for random phenomena. The quantity $\mu > 0$ is referred to as the damping coefficient.

Unfortunately, due to its quadratic nature, the Wigner distribution is affected by interference terms, highly oscillatory components which make the understanding and interpretation of the time-frequency structure of signals a difficult problem [10]- [12]. A common approach to reduce the interference terms is to Fourier transform the Wigner distribution, thus obtaining the ambiguity function [3]. Because of their oscillatory behavior, in the ambiguity domain the interference terms are mostly located away from the origin, and they can be therefore reduced by a proper lowpass filtering [4].

We obtain the ambiguity function of the Green's function for the harmonic oscillator, and we show that, similarly to the Wigner distribution $W_h(t, \omega)$, it can be written with respect to the ambiguity function of the Langevin equation. The time-frequency interference terms of the Green's function have a simple structure in the ambiguity domain, which we discuss in detail. Our results can pave the way for the design of interference mitigation filters which take advantage of the structure of the differential equation defining the signal $x(t)$.

We note that an alternative approach for the time-frequency study of differential equations is to transform the differential equation in the time domain to an equivalent differential equation in the time-frequency domain, whose structure is often more complicated than the original equation, but whose solution is often easier to get and more revealing than in the time-domain [13]- [16].

The article is organized as follows. In Sect. 2 we define the ambiguity function and give some of its properties. In Sect. 3 we obtain the ambiguity function for the Langevin equation and for the harmonic oscillator, and we use it to discuss the structure of the interference terms of these differential equations. Finally, Sect. 4 summarizes the obtained results.

2. The Ambiguity Function

The ambiguity function of a signal $x(t)$, also referred to as the characteristic function, is defined as [3]

$$A_x(\theta, \tau) = \int_{-\infty}^{+\infty} x^*(t - \tau/2) x(t + \tau/2) e^{i\theta t} dt, \tag{9}$$

and it plays a fundamental role in radars, where θ is the Doppler frequency and τ the time delay. This definition is known as the symmetric ambiguity function, and it is connected through a Fourier transformation to the Wigner distribution,

$$A_x(\theta, \tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(t, \omega) e^{i\theta t + i\tau\omega} dt d\omega. \tag{10}$$

Therefore, the magnitude $|A_x(\theta, \tau)|$ describes the oscillatory structure of the time-frequency representation of $x(t)$. Actually, (10) is an inverse Fourier transform, but since the Wigner distribution is real, then $|A_x(\theta, \tau)|$ is even with respect to θ and τ, and therefore adopting a definition for $A_x(\theta, \tau)$ which connects it to the Wigner distribution through a direct Fourier transformation would not change $|A_x(\theta, \tau)|$.

The cross-ambiguity function of two signals $x(t)$ and $y(t)$ is defined as

$$A_{x,y}(\theta,\tau)=\int_{-\infty}^{+\infty}x^*(t-\tau/2)y(t+\tau/2)e^{i\theta t}dt. \tag{11}$$

We now give some properties of the ambiguity function which are useful for our analysis. These properties can be easily proved from the definition (9) and from (10).

Multiplication by a constant. If

$$y(t)=cx(t), \tag{12}$$

then

$$A_y(\theta,\tau)=|c|^2\,A_x(\theta,\tau). \tag{13}$$

Multiplication by constants (cross-ambiguity function). If

$$y_1(t) = c_1x_1(t), \tag{14}$$
$$y_2(t) = c_2x_2(t), \tag{15}$$

then

$$A_{y_1,y_2}(\theta,\tau)=c_1^*c_2A_{x_1,x_2}(\theta,\tau). \tag{16}$$

Complex frequency modulation. When

$$y(t)=x(t)e^{i\omega_0 t}, \tag{17}$$

it is

$$A_y(\theta,\tau)=A_x(\theta,\tau)e^{i\omega_0\tau}. \tag{18}$$

Complex frequency modulation (cross-ambiguity function). When

$$x_1(t) = x(t)e^{i\omega_0 t}, \tag{19}$$
$$x_2(t) = x(t)e^{-i\omega_0 t}, \tag{20}$$

it is

$$A_{x_1,x_2}(\theta,\tau)=A_x(\theta-2\omega_0,\tau). \tag{21}$$

Sum of two signals. If

$$y(t)=x_1(t)+x_2(t), \tag{22}$$

then

$$A_y(\theta,\tau)=A_{x_1}(\theta,\tau)+A_{x_2}(\theta,\tau)+A_{x_1,x_2}(\theta,\tau)+A_{x_2,x_1}(\theta,\tau). \tag{23}$$

Real frequency modulation. From the previous properties, if

$$y(t)=x(t)\sin\omega_0 t, \tag{24}$$

then

$$A_y(\theta,\tau) = \frac{1}{2}A_x(\theta,\tau)\cos\omega_0\tau - \frac{1}{4}A_x(\theta-2\omega_0,\tau) - \frac{1}{4}A_x(\theta+2\omega_0,\tau). \tag{25}$$

Sum of two Wigner distributions. If

$$W_y(t,\omega) = c_1 W_{x_1}(t,\omega) + c_2 W_{x_2}(t,\omega), \tag{26}$$

then

$$A_y(\theta,\tau) = c_1 A_{x_1}(\theta,\tau) + c_2 A_{x_2}(\theta,\tau). \tag{27}$$

Frequency translation of the Wigner distribution. If

$$W_y(t,\omega) = W_x(t,\omega-\omega_0), \tag{28}$$

then

$$A_y(\theta,\tau) = A_x(\theta,\tau)e^{i\omega_0\tau}. \tag{29}$$

Complex frequency modulation of the Wigner distribution. If

$$W_y(t,\omega) = W_x(t,\omega)e^{i\omega_0 t}, \tag{30}$$

then

$$A_y(\theta,\tau) = A_x(\theta+\omega_0,\tau). \tag{31}$$

Real frequency modulation of the Wigner distribution. By using the previous properties, if

$$W_y(t,\omega) = W_x(t,\omega)\cos\omega_0 t, \tag{32}$$

then

$$A_y(\theta,\tau) = \frac{1}{2}A_x(\theta+\omega_0,\tau) + \frac{1}{2}A_x(\theta-\omega_0,\tau). \tag{33}$$

3. The Interference Terms of the Harmonic Oscillator

We first obtain the ambiguity function $A_{h_L}(\theta,\tau)$ of the Green's function for the Langevin equation (8), and then we use it to obtain the ambiguity function $A_h(\theta,\tau)$ of the Green's function for the harmonic oscillator (2).

3.1. The Ambiguity Function for the Langevin Equation

The Green's function of the Langevin equation (8) is given by [1]

$$h_L(t) = u(t)e^{-\mu t}. \tag{34}$$

The corresponding ambiguity function is given by

$$A_{h_L}(\theta,\tau) = \int_{-\infty}^{+\infty} h_L^*(t-\tau/2)h_L(t+\tau/2)e^{i\theta t}dt, \tag{35}$$

$$= \int_{-\infty}^{+\infty} u(t-\tau/2)u(t+\tau/2)e^{(-2\mu+i\theta)t}dt. \tag{36}$$

We note that

$$u(t-\tau/2)u(t+\tau/2) = u(t-\tau/2), \quad \text{for } \tau \geq 0, \tag{37}$$

$$u(t-\tau/2)u(t+\tau/2) = u(t+\tau/2), \quad \text{for } \tau < 0. \tag{38}$$

Therefore

$$u(t-\tau/2)u(t+\tau/2) = u(t-|\tau|/2). \tag{39}$$

Substituting,

$$A_{h_L}(\theta,\tau) = \int_{-\infty}^{+\infty} u(t-|\tau|/2)e^{(-2\mu+i\theta)t}dt, \tag{40}$$

$$= \int_{|\tau|/2}^{+\infty} e^{(-2\mu+i\theta)t}dt. \tag{41}$$

Finally,

$$A_{h_L}(\theta,\tau) = \frac{e^{(-\mu+i\theta/2)|\tau|}}{2\mu - i\theta}. \tag{42}$$

In the Appendix we confirm this result by (inverse) Fourier transforming the Wigner distribution $W_{h_L}(t,\omega)$ in (7), whose oscillatory structure is described by the magnitude

$$|A_{h_L}(\theta,\tau)| = \frac{e^{-\mu|\tau|}}{\sqrt{4\mu^2+\theta^2}}. \tag{43}$$

To illustrate our result, we show $W_{h_L}(t,\omega)$ in Fig. 1, and $|A_{h_L}(\theta,\tau)|$ in Fig. 2, for the case $\mu = 5$. From Fig. 1 we see that, at $t = 0$, the delta function at the input generates an initial spread over all frequencies, which then concentrates about the zero frequency. Therefore, this first-order equation can be interpreted as a resonant system whose resonance frequency is zero. The arc-shaped waves propagating from the origin of the ambiguity plane are interference terms. As Fig. 2 shows, the frequency spectrum of the Wigner distribution $W_{h_L}(t,\omega)$ is mainly concentrated about the origin, an expected result since $|A_{h_L}(\theta,\tau)|$ is made by the product of the Cauchy-like distribution $1/\sqrt{4\mu^2+\theta^2}$ and the symmetric exponential function $e^{-\mu|\tau|}$. The tails of the ambiguity function are mainly due to the interference terms of $W_{h_L}(t,\omega)$, which oscillates more than the resonant component at $\omega = 0$. The component at $t = 0$ contributes also to the tails of the ambiguity function.

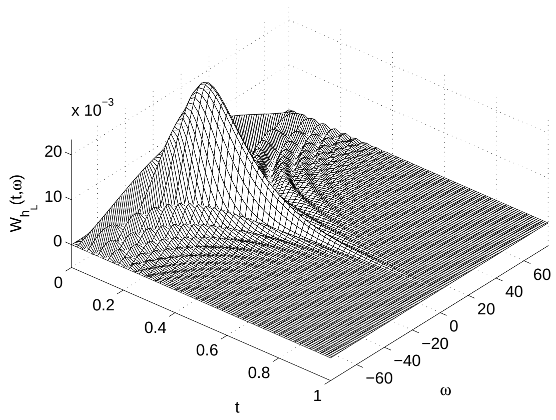

FIGURE 1. Wigner distribution of the Green's function for the Langevin equation. The delta function at the input generates a time-frequency response made by an initial spread over all frequencies, which then concentrates about the zero frequency. This first-order equation can be interpreted as a system with a resonance at the zero frequency. The arc-shaped waves propagating from the origin are interference terms.

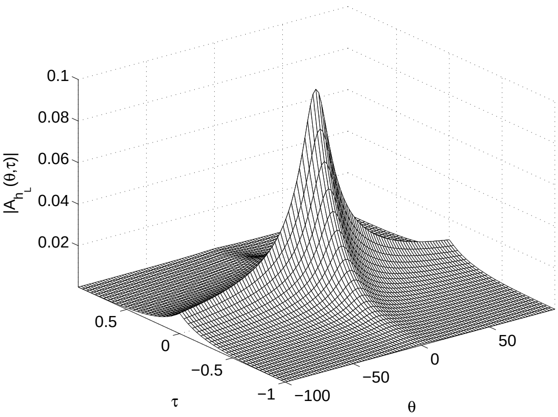

FIGURE 2. Magnitude of the ambiguity function of the Green's function for the Langevin equation. This function has a peak at the origin of the ambiguity plane, and has tails on the θ and τ axes. These tails are mainly due to the interference terms of the Wigner distribution $W_{h_L}(t, \omega)$ in Fig. 1.

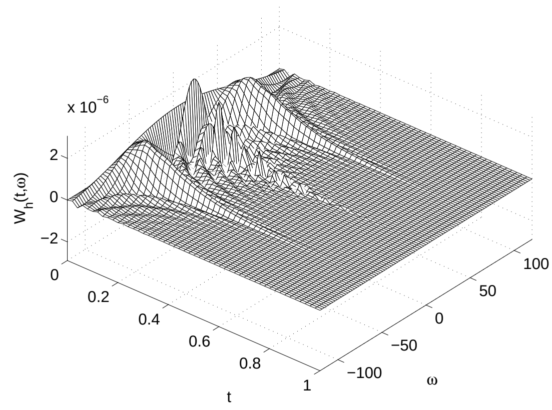

FIGURE 3. Wigner distribution of the Green's function for the harmonic oscillator. The delta function at the input generates a time-frequency response made by an initial spread over all frequencies, which then concentrates on the resonant frequency ω_c, as well as on its symmetric counterpart at $-\omega_c$. The oscillating components centered about the time axis are interference terms.

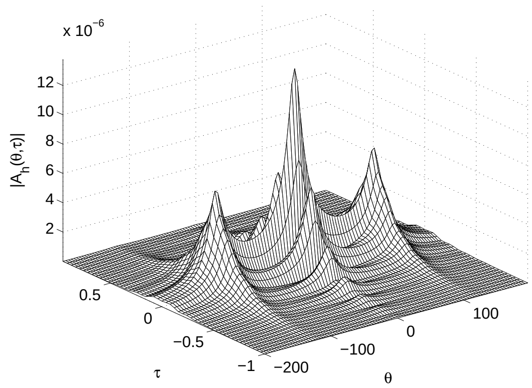

FIGURE 4. Magnitude of the ambiguity function of the Green's function for the harmonic oscillator. This function is made by three components. The first has a peak at the origin, and it represents the resonances at ω_c and $-\omega_c$, merged together in the ambiguity domain. The other two components are located on the $\tau = 0$ axis, at $\theta = 2\omega_c$ and $\theta = -2\omega_c$, and they represent the interference terms of the Wigner distribution $W_h(t, \omega)$ in Fig. 3. These interference terms can be filtered out by a proper masking of the ambiguity function.

3.2. The Ambiguity Function for the Harmonic Oscillator

The Green's function for the harmonic oscillator can be written as [7]

$$h(t) = \frac{1}{\omega_c} h_L(t) \sin \omega_c t. \tag{44}$$

By using the properties (13) and (25) we immediately obtain

$$A_h(\theta,\tau) = \frac{1}{2\omega_c^2} A_{h_L}(\theta,\tau)\cos\omega_c\tau - \frac{1}{4\omega_c^2} A_{h_L}(\theta-2\omega_c,\tau) - \frac{1}{4\omega_c^2} A_{h_L}(\theta+2\omega_c,\tau). \tag{45}$$

An alternative way to derive this result is to apply the properties (27), (29), and (33) to $W_h(t,\omega)$ in (6), obtaining

$$A_h(\theta,\tau) = \frac{1}{4\omega_c^2} A_{h_L}(\theta,\tau) e^{i\omega_c\tau} + \frac{1}{4\omega_c^2} A_{h_L}(\theta,\tau) e^{-i\omega_c\tau} \tag{46}$$

$$- \frac{1}{4\omega_c^2} A_{h_L}(\theta+2\omega_c,\tau) - \frac{1}{4\omega_c^2} A_{h_L}(\theta-2\omega_c,\tau). \tag{47}$$

Combining the first two terms returns (45).

To illustrate our result, we show $W_h(t,\omega)$ in Fig. 3 and $|A_h(\theta,\tau)|$ in Fig. 4, for $\mu = 5$ and $\omega_c = 60$. From Fig. (3) we see that the input delta function at $t = 0$ generates an initial spread over all frequencies, which eventually concentrates on the resonance frequency ω_c, and on its symmetric counterpart at $-\omega_c$. The oscillating components between these two resonances are interference terms. From (6), aside from the constants, the resonance at frequency ω_c is described by the term $W_{h_L}(t,\omega-\omega_c)$ (its negative counterpart by $W_{h_L}(t,\omega+\omega_c)$), whereas the interference terms between the two resonances are described by the oscillating term $W_{h_L}(t,\omega)\cos 2\omega_c t$. Figure (4) shows that the ambiguity function is made by three components. The component centered about the origin represents the resonant components at ω_c and $-\omega_c$, which are merged in the single term $A_{h_L}(\theta,\tau)\cos\omega_c\tau$ in (45). In the ambiguity domain, the interference terms are instead split up in the two terms $A_{h_L}(\theta-2\omega_c,\tau)$ and $A_{h_L}(\theta+2\omega_c,\tau)$ in (45), which, in Fig. 4, correspond to the two components centered about $\theta = 2\omega_c$, $\tau = 0$, and $\theta = -2\omega_c$, $\tau = 0$.

The interference terms can be reduced by filtering the ambiguity function through the product

$$M_h(\theta,\tau) = G(\theta,\tau)A_h(\theta,\tau), \tag{48}$$

where $G(\theta,\tau)$ is the filter and $M_h(\theta,\tau)$ is the filtered ambiguity function. Since, as previously discussed, the interference terms are located on the $\tau = 0$ axis and centered about the frequencies $\pm 2\omega_c$, an effective choice for the cut-off frequency θ_c of the filter can be $\theta_c < \omega_c$. Therefore, the specifications for the lowpass filter are $|G(\theta,\tau)| = 1$ for $\theta \leq \theta_c$, and $|G(\theta,\tau)| = 0$ for $\theta > \theta_c$, whereas no filtering is needed on the τ axis. Because of (3), the parameters of the interference mitigation filter are linked to the coefficients μ and ω_0 of the differential equation governing the harmonic oscillator. We also note that, in the time-frequency domain, the filtering

(48) corresponds to the smoothing [3]

$$C_h(t,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(t-t',\omega-\omega')W_h(t',\omega')dt'd\omega', \tag{49}$$

where

$$g(t,\omega) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\theta,\tau)e^{-i\theta t - i\tau\omega}d\theta d\tau. \tag{50}$$

Note that, in general, the filtering (48) does not produce a proper Wigner distribution, because not every real function of time and frequency is a Wigner distribution. This fact is known as the representability problem [3]. Anyway, filtering is advantageous because the resulting smoothed Wigner distribution clearly highlights the time-frequency spectrum of systems modeled by differential equations, as shown in [14].

Furthermore, for an arbitrary input $f(t)$, the Wigner distribution $W_x(t,\omega)$ of the output of the harmonic oscillator is given by the convolution (5) between the Wigner distribution $W_h(t,\omega)$ of the impulse response and the Wigner distribution $W_f(t,\omega)$ of the input. Clearly, $W_f(t,\omega)$ is, in general, affected by interference terms, which can be strong, and, consequently, the resulting output $W_x(t,\omega)$ can also have strong interference terms. In general, the structure of such interference terms depend on the type of input signal. Nevertheless, they will have a highly oscillatory nature, therefore the common countermeasure of smoothing them can still be applied.

4. Summary of Results

The Langevin equation defined as

$$\frac{dx(t)}{dt} + \mu x(t) = f(t), \tag{51}$$

with damping coefficient $\mu > 0$ has a Green's function given by

$$h_L(t) = u(t)e^{-\mu t}, \tag{52}$$

whose corresponding ambiguity function is

$$A_{h_L}(\theta,\tau) = \frac{e^{(-\mu+i\theta/2)|\tau|}}{2\mu - i\theta}. \tag{53}$$

The harmonic oscillator defined as

$$\frac{d^2x(t)}{dt^2} + 2\mu\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t), \tag{54}$$

where $\mu < \omega_0$, has a Green's function given by

$$h(t) = \frac{1}{\omega_c}h_L(t)\sin\omega_c t, \tag{55}$$

where

$$\omega_c = \sqrt{\omega_0^2 - \mu^2}. \tag{56}$$

The corresponding ambiguity function is given by

$$A_h(\theta,\tau) = \frac{1}{2\omega_c^2} A_{h_L}(\theta,\tau)\cos\omega_c\tau - \frac{1}{4\omega_c^2} A_{h_L}(\theta-2\omega_c,\tau) - \frac{1}{4\omega_c^2} A_{h_L}(\theta+2\omega_c,\tau). \quad (57)$$

5. Conclusions

We have obtained the ambiguity function of the Green's function for the harmonic oscillator. The obtained result has a simple connection to the ambiguity function of the Green's function for the Langevin equation. The ambiguity function for the harmonic oscillator is made by three terms. The first, centered about the origin of the ambiguity domain, describes the resonant behavior of the harmonic oscillator. The second and third terms, located away from the origin of the ambiguity domain, represent the interference terms of the Wigner distribution of the Green's function. These interference terms can be filtered out by masking the ambiguity function, an operation corresponding to smoothing the Wigner distribution in the time-frequency domain.

6. Appendix

By using the property (10), the ambiguity function of the Green's function for the Langevin equation can be obtained from

$$A_{h_L}(\theta,\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_{h_L}(t,\omega) e^{i\theta t + i\tau\omega} dt d\omega. \quad (58)$$

Substituting $W_{h_L}(t,\omega)$ from (1), gives

$$\begin{aligned} A_{h_L}(\theta,\tau) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(t) e^{-2\mu t}\frac{\sin 2\omega t}{\pi\omega} e^{i\theta t + i\tau\omega} dt d\omega, & (59)\\ &= \int_{-\infty}^{+\infty} \frac{1}{\pi\omega}\left[\frac{1}{2i}\int_0^{+\infty} e^{(-2\mu+i(\theta+2\omega))t} dt \right.\\ &\qquad \left. -\frac{1}{2i}\int_0^{+\infty} e^{(-2\mu+i(\theta-2\omega))t} dt\right] e^{i\tau\omega} d\omega, & (60)\\ &= \int_{-\infty}^{+\infty} \frac{1}{\pi\omega}\left[\frac{1}{2i}\frac{1}{2\mu - i(\theta+2\omega)} - \frac{1}{2i}\frac{1}{2\mu - i(\theta-2\omega)}\right] e^{i\tau\omega} d\omega, & (61)\\ &= \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{1}{(2\mu - i\theta)^2 + 4\omega^2} e^{i\tau\omega} d\omega, & (62)\\ &= \frac{e^{(-\mu+i\theta/2)|\tau|}}{2\mu - i\theta}. & (63) \end{aligned}$$

which is (42).

References

[1] G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, John Wiley & Sons, 1989.

[2] C. J. Savant, *Fundamentals of the Laplace transformation*, Mc-Graw Hill, 1962.

[3] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.

[4] B. Boashash (Ed.), *Time-Frequency Signal Analysis and Processing: A Comprehensive Reference*, Academic Press, 2015.

[5] T. A. C. M. Claasen and W. F. G. Mecklenbrauker, "The Wigner distribution - A tool for time-frequency signal analysis. Part I: Continuous time signals," *Philips Journal of Research*, vol. 35, no. 3, pp. 217-250, 1980.

[6] E. P. Wigner, On the quantum correction for thermodynamic equilibrium, *Physical Review*, vol. 40, pp. 749-759, 1932.

[7] L. Galleani, "A time-frequency relationship between the Langevin equation and the harmonic oscillator," in M. W. Wong and H. Zhu (Eds.), *Pseudo-Differential Operators: Groups, Geometry and Applications*, Birkhäuser-Springer, pp. 119-131, 2017.

[8] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Dover, 1987.

[9] L. Cohen, "The history of noise," *IEEE Signal Processing Magazine*, vol. 22, no. 6, pp. 20-45, Nov. 2005.

[10] F. Hlawatsch, "Interference terms in the Wigner distribution," *Digital signal processing*, vol. 84, pp. 363-367, 1984.

[11] H. I. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE Trans. on Acoust., Speech, and Signal Processing.*, vol. 37, no. 6, pp. 862-871, June 1989.

[12] J. Jeong and W. J. Williams, "Kernel design for reduced interference distributions," *IEEE Trans. Sig. Process.*, vol. 40, no. 2, pp. 402-412, Feb. 1992.

[13] L. Galleani, "Time-Frequency Representation of MIMO Dynamical Systems," *IEEE Trans. Sig. Process.*, vol. 61 n. 17, pp. 4309-4317, 2013.

[14] L. Galleani, "Response of Dynamical Systems to Nonstationary Inputs," *IEEE Trans. Sig. Process.*, vol. 60, no. 11, pp. 5775-5786, 2012.

[15] L. Galleani, "The transient spectrum of a random system," *IEEE Trans. Sig. Process.*, vol. 58, no. 10, pp. 5106-5117, 2010.

[16] L. Galleani and L. Cohen, "Direct Time-Frequency Characterization of Linear Systems Governed by Differential Equations," *IEEE Signal Processing Letters*, vol. 11, no. 9, pp. 721-724, 2004.

Lorenzo Galleani
e-mail: galleani@polito.it