POLITECNICO DI TORINO
Repository ISTITUZIONALE

Quantum resources for hybrid communication via qubit-oscillator states

Original
Quantum resources for hybrid communication via qubit-oscillator states / Tufarelli, T.; Girolami, D.; Vasile, R.; Bose, S.; Adesso, G.. - In: PHYSICAL REVIEW A. - ISSN 1050-2947. - 86:5(2012), p. 052326. [10.1103/PhysRevA.86.052326]

Availability:
This version is available at: 11583/2849529 since: 2020-10-22T15:47:29Z

Publisher:
AMER PHYSICAL SOC

Published
DOI:10.1103/PhysRevA.86.052326

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

12 October 2024

Quantum resources for hybrid communication via qubit-oscillator states

Tommaso Tufarelli,[1] Davide Girolami,[2] Ruggero Vasile,[2] Sougato Bose,[3] and Gerardo Adesso[2]
[1]*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, United Kingdom*
[2]*School of Mathematical Sciences, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*
[3]*Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom*
(Received 1 May 2012; revised manuscript received 11 October 2012; published 26 November 2012)

We investigate a family of qubit-oscillator states as resources for hybrid quantum communication. They result from a mechanism of qubit-controlled displacement on the oscillator. For large displacements, we obtain analytical formulas for entanglement and other nonclassical correlations, such as entropic and geometric discord, in those states. We design two protocols for quantum communication using the considered resource states: a hybrid teleportation and a hybrid remote-state preparation. The latter, in its standard formulation, is shown to have a performance limited by the initial mixedness of the oscillator, echoing the behavior of the geometric discord. If one includes a further optimization over nonunitary correcting operations performed by the receiver, the performance is improved to match that of teleportation, which is directly linked to the amount of entanglement. Both protocols can then approach perfect efficiency even if the oscillator is originally highly thermal. We discuss the critical implications of these findings for the interpretation of general quantum correlations.

DOI: 10.1103/PhysRevA.86.052326 PACS number(s): 03.67.Mn, 03.65.Ta, 03.67.Hk, 42.50.Dv

I. INTRODUCTION

Understanding, identifying, and exploiting "quantumness" in composite systems represent essential steps to grasp the fundamental implications of quantum theory [1] and are of particular relevance in the race for efficient information and communication technology applications defying the classical boundaries [2]. Nonlocality and entanglement are clearcut signatures of nonclassicality [3,4]. However, it has recently been acknowledged that, focusing on aspects of *correlations* among quantum systems, manifestations of quantumness can exist even in the absence of entanglement [5]. Quantum discord and related measures of general quantum correlations are receiving widespread attention [6] as they promise to enable a supraclassical speedup in computational frameworks where entanglement is not robust enough to endure [7]. Some protocols have been identified that appear to take advantage of discord-like correlations, rather than entanglement, for their functionality [8]; still, basic questions about the interpretation of discord remain unanswered.

Some steps have been undertaken to understand the interplay between entanglement and general quantum correlations [8,9]. A hierarchic relation is proven for two qubits [10], involving the negativity $\mathcal{N}$ [11] as an entanglement measure and the so-called geometric discord $\mathcal{D}_G$ [12] as a nonclassicality indicator: $\sqrt{\mathcal{D}_G} \geqslant \mathcal{N}$. Less is known about the structure of general quantum correlations in high-dimensional systems, with the exception of continuous variable Gaussian states [13–16].

Here we study hybrid bipartite systems composed of a two-level system A (qubit) and a harmonic oscillator B (qumode). We consider a class of states ρ_{AB} in which correlations are induced by the action of qubit-controlled displacements on the oscillator [17]. These states can be implemented in several setups [18] and the required interaction can be exploited for universal quantum computation [19].

The main aim of this paper is to construct protocols for quantum communication using these resource states and to analyze their performance in connection with the contents of different types of correlations in the states ρ_{AB}.

We define two state-transfer protocols based on the shared states ρ_{AB}; see Fig. 1. One is a hybrid teleportation scheme [20] where Bob can teleport an unknown qubit state $|\psi_Q^{\rm in}\rangle$ to Alice, and whose fidelity is proven to approach unity for a particular subclass of resource states. The second is a hybrid remote-state preparation protocol [21] where Alice can measure the qubit to remotely prepare Bob's oscillator in some (known to Alice) state $|\psi_B^{\rm goal}\rangle$. In this case, if the receiver Bob can only perform unitary corrections, the fidelity is bounded in general by the initial purity of the oscillator [22–24]; however, if we allow Bob to perform nonunitary corrections, the figure of merit can increase and match that associated to teleportation.

We thus analyze in detail the nature of correlations in the states ρ_{AB} and their role for the performance of the different protocols. We obtain analytical formulas for negativity $\mathcal{N}$ [11], entropic quantum discord $\mathcal{D}_Z$ [5], and geometric discord $\mathcal{D}_G$ [12] of the states ρ_{AB}. We find that such states can be maximally entangled and maximally discordant in the limit of large displacements, while their geometric discord is limited by the initial purity of the oscillator and can be thus arbitrarily small. We argue that this is a direct consequence of the particular geometry of the state-space induced by the Hilbert-Schmidt norm, which enters the definition of $\mathcal{D}_G$ [12], leading us to conclude that $\mathcal{D}_G$ cannot be regarded, in general, as a proper "measure" of nonclassical correlations, in agreement with the conclusions of Ref. [25].

This work provides three main advances: First, from a practical viewpoint, it presents workable protocols for hybrid quantum communication, which can be useful as building blocks in any light-matter-interfaced implementation of quantum information processing [19,26,27]. Second, from a technical viewpoint, it provides useful methods for the analytical evaluation of correlation quantifiers in bipartite systems with subsystems of different dimensionality [28]. Third, from a physical perspective, it exposes the need for a mathematically sound and physically meaningful approach

 ©2012 American Physical Society

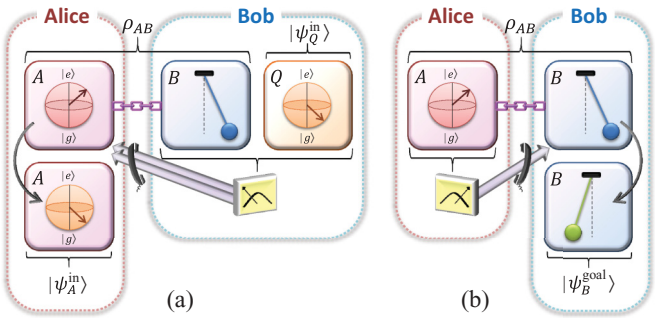

FIG. 1. (Color online) (a) Schemes for hybrid teleportation and (b) hybrid remote-state preparation via the shared resource states ρ_{AB} of qubit A and oscillator B. Details are provided in the text.

to delve into the nature and structure of general nonclassical correlations in quantum states.

The organization of the paper is as follows. In Sec. II we define the studied model and calculate analytically various measures of nonclassical correlations for the states ρ_{AB}. In Sec. III we define protocols for teleportation and remote-state preparation using the shared resource states ρ_{AB} and calculate their fidelity, relating it to the correlations present in the states. In Sec. IV we draw our conclusions. Some technical derivations are deferred to appendixes.

II. THE STATES AND THEIR CORRELATIONS

We consider a family of qubit-oscillator states $\rho_{AB} = U_{AB}(\beta)(\rho_A^0 \otimes \rho_B^0)U_{AB}^\dagger(\beta)$ obtained by applying the unitary $U_{AB}(\beta) = D(\sigma_3\beta)$ to an initially uncorrelated generic state $\rho_A^0 \otimes \rho_B^0$. This interaction induces a qubit-controlled displacement on the oscillator; that is, a displacement where the sign of the parameter $\pm\beta$ is determined by the eigenvalue of the qubit Pauli operator σ_3 [17,18]. This corresponds, for example, to the evolution of the joint system via a coupling Hamiltonian $H \propto \sigma_3(b + b^\dagger)$, which can be realized experimentally in a number of setups [18,29]. This type of interaction has relevant applications for the state reconstruction of oscillator networks probed by a single qubit [30] and for quantum computation based on light-matter interfaces [19]. We describe the state of qubit A before the interaction as

$$\rho_A^0 = \begin{pmatrix} p & r \\ r^* & 1-p \end{pmatrix}$$

in the standard basis $\{|e\rangle, |g\rangle\}$, with $0 \leqslant p \leqslant 1$ and $|r|^2 \leqslant p(1-p)$. The states of the hybrid system after the interaction take the form

$$\begin{aligned}\rho_{AB} = {} & p|e\rangle\langle e| \otimes D(\beta)\rho_B^0 D^\dagger(\beta) \\ & + (1-p)|g\rangle\langle g| \otimes D^\dagger(\beta)\rho_B^0 D(\beta) \\ & + r|e\rangle\langle g| \otimes D(\beta)\rho_B^0 D(\beta) \\ & + r^*|g\rangle\langle e| \otimes D^\dagger(\beta)\rho_B^0 D^\dagger(\beta). \end{aligned} \tag{1}$$

We focus on the regime of large displacements $|\beta| \to \infty$, which in practice means $|\beta|$ large enough such that the overlap between the two phase-space domains, associated to ρ_B^0 displaced by β and to ρ_B^0 displaced by $-\beta$, becomes negligible.

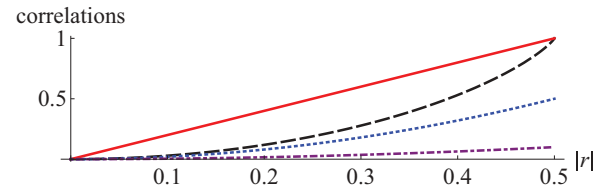

FIG. 2. (Color online) Plot of negativity [Eq. (2)] (solid line), lower bound on entropic discord [Eq. (4)] (dashed line), and geometric discord [Eq. (3)] for $\mu_B^0 = 0.5$, 0.1 (dotted and dot-dashed lines, respectively) calculated for the states ρ_{AB} with $p = 1/2$, as a function of $|r|$. The plotted quantities are dimensionless and correspond to $|\beta| \to \infty$.

Let us calculate the correlations in the states of Eq. (1). Nonlocality properties have been studied in Ref. [31].

Entanglement can be quantified by the (normalized) negativity [11] $\mathcal{N}(\rho_{AB}) = (\|\rho_{AB}^{\mathsf{T}_A}\|_1 - 1)$, where $\|M\|_1 = \mathrm{Tr}|M|$ denotes the trace norm and $\rho_{AB}^{\mathsf{T}_A}$ is obtained by partial transposition with respect to the qubit only [32]. We find the following rigorous result, whose proof is provided in Appendix A:

$$\lim_{|\beta|\to\infty} \mathcal{N}(\rho_{AB}) = 2|r|. \tag{2}$$

Independently of the initial state of the oscillator (in particular, no matter how thermal it is), one can always find a $|\beta|$ large enough such that the qubit-oscillator states are asymptotically maximally entangled, when the qubit is initially in a pure equatorial state, $p = |r| = 1/2$. Then the state ρ_{AB} reproduces a proper Schrödinger cat state, if one interprets the qubit as the "microscopic" degree of freedom and the qumode as the "macroscopic" one [33].

Let us now consider more general types of nonclassical correlations. The geometric discord $\mathcal{D}_G(\rho_{AB})$ quantifies how far (in the Hilbert-Schmidt norm) a bipartite quantum state is from the set of classical-quantum states [12]. It can be measured experimentally with direct, nontomographic methods [34–36]. Given a quantum state ρ_{AB} of a $\mathbb{C}^2 \otimes \mathbb{C}^d$ system, with A being a qubit and B being an arbitrary (finite or infinite) d-dimensional system, the normalized geometric discord $\mathcal{D}_G(\rho_{AB})$ is defined as [12,37] $\mathcal{D}_G(\rho_{AB}) = 2\inf_{\Pi_A} \|\rho_{AB} - \Pi_A(\rho_{AB})\|_2^2$, where the infimum is over all complete (nondegenerate) von Neumann measurements $\Pi_A \equiv \{\Pi_A^k\}$ on the qubit A, with $\Pi_A(\rho_{AB}) = \sum_k (\Pi_A^k \otimes \mathbb{1}_B)\rho_{AB}(\Pi_A^k \otimes \mathbb{1}_B)$, and $\|M\|_2 = [\mathrm{Tr}(MM^\dagger)]^{1/2}$ is the Hilbert-Schmidt norm. In Appendix B, we provide a useful and compact analytical framework for the calculation of $\mathcal{D}_G$ in $2 \times d$ systems including the case $d = \infty$, which we need here. We obtain

$$\lim_{|\beta|\to\infty} \mathcal{D}_G(\rho_{AB}) = 4\mu_B^0|r|^2. \tag{3}$$

The geometric discord can be made arbitrarily small by decreasing the initial purity $\mu_B^0 \equiv \mathrm{tr}_B[(\rho_B^0)^2]$ of the oscillator (see Fig. 2). This result shows that the conjectured ordering relation $\sqrt{\mathcal{D}_G} \geqslant \mathcal{N}$, which holds for all two-qubit states [10], is violated when at least one subsystem of a bipartite system has large dimension, as in our case. For the states ρ_{AB}, the ordering is actually *reversed*, revealing the quirky situation of states with possibly maximum entanglement yet asymptotically vanishing (for $\mu_B^0 \to 0$) geometric discord.

We then ask: are those maximally entangled states infinitesimally close to the classical-quantum border in other metrics? To address the question, we evaluate the conventional entropic quantum discord $\mathcal{D}_Z$ [5] on the states ρ_{AB}. We recall the definition, $\mathcal{D}_Z(\rho_{AB}) = \inf_{\Pi_A}\{\mathcal{I}(\rho_{AB}) - \mathcal{I}[\Pi_A(\rho_{AB})]\}$, where $\mathcal{I}$ denotes the quantum mutual information. We observe that, by means of a local operation on Bob's side, one can map ρ_{AB} onto an effective two-qubit state $\tilde{\rho}_{AB}$. Entanglement measures [3] as well as the discord $\mathcal{D}_Z$ with measurements on A [38] (but crucially not $\mathcal{D}_G$ [39]) are monotonic under such operations on B, so that by evaluating those correlation measures on $\tilde{\rho}_{AB}$ one obtains lower bounds to the corresponding measures for ρ_{AB} [28,40,41]. In particular, as proven in Appendix C, Bob can choose two orthonormal vectors $|\tilde{e}\rangle$, $|\tilde{g}\rangle$ and design a local operation which, in the limit of large displacements, converts the state ρ_{AB} to the "digitalized" $\tilde{\rho}_{AB}^{\rm dig} = p|e\tilde{e}\rangle\langle e\tilde{e}| + r|e\tilde{e}\rangle\langle g\tilde{g}| + r^*|g\tilde{g}\rangle\langle e\tilde{e}| + (1-p)|g\tilde{g}\rangle\langle g\tilde{g}|$, without resorting to postselection. This effective two-qubit state can be achieved independently of the initial purity of B. The entropic discord in the digitalized state reads

$$\mathcal{D}_Z\left(\tilde{\rho}_{AB}^{\rm dig}\right) = -p\log_2(p) - (1-p)\log_2(1-p) + \frac{1}{2}\left\{\log_2[p(1-p) - |r|^2] + \zeta\log_2\left(\frac{1+\zeta}{1-\zeta}\right)\right\}, \tag{4}$$

with $\zeta = [(1-2p)^2 + 4|r|^2]^{1/2}$. In the case $p = |r| = 1/2$, the entropic discord $\mathcal{D}_Z(\tilde{\rho}_{AB}^{\rm dig})$ converges to 1 like the entanglement, implying $\mathcal{D}_Z(\rho_{AB}) \to 1$ in the original states of Eq. (1) as well.

This demonstrates that the Hilbert-Schmidt norm is generally unsuitable for defining quantitative distance-based measures of correlations—as recognized for entanglement in Ref. [42] and very recently for geometric discord in Ref. [25]—with its deficiency being even more critically exposed in systems with large dimension. We can then reassess some findings in the recent literature, such as $\mathcal{D}_G$ failing to capture the resource power of the discrete quantum computation with one bit [12,36], as evidence of this deficiency. Nevertheless, when the purity of the states is fixed, and/or when the dimension of the Hilbert space is small enough (e.g., for two qubits), the geometric discord returns a reliable quantification of nonclassical correlations [10,24,43]. In general, given its computability and experimental accessibility [34,35], it can still play a useful role if one correctly regards it not as a measure by itself, but as a valid *lower bound* to regular measures of nonclassical correlations such as the relative entropy of discord [44,45] (alias one-way deficit [46]). Such a bound becomes looser with increasing dimension, as evidenced by our analysis, and as emerges from the study of continuous variable Gaussian states [16].

III. HYBRID QUANTUM COMMUNICATION PROTOCOLS

We now question whether the different aspects of correlations identified in the states of Eq. (1) (see Fig. 2 for a comparison) can be endowed with operational meanings. We design two hybrid quantum communication protocols which employ the family of states ρ_{AB}, with Alice operating the qubit A and Bob operating the oscillator B, as shared resources.

A. Hybrid teleportation

Alice and Bob share a state ρ_{AB} of the form (1) [see Fig. 1(a) for reference]. Bob (the sender) wishes to teleport [20] an unknown state $|\psi_Q^{\rm in}\rangle = \eta|e\rangle + \gamma|g\rangle$ of an input qubit Q to Alice (the receiver). Bob then makes a joint measurement on the input qubit Q and the oscillator B, communicates the outcome to Alice, who can then implement a correction on her qubit A. The final state $\rho_A^{\rm out}$ of A can be shown (see below) to have a fidelity $\mathcal{F}_{\rm TEL} = \langle\psi_Q^{\rm in}|\rho_A^{\rm out}|\psi_Q^{\rm in}\rangle$, averaged over the uniform distribution of the input state, given by

$$\lim_{|\beta|\to\infty}\bar{\mathcal{F}}_{\rm TEL} = \tfrac{2}{3}(1+|r|). \tag{5}$$

Notice that for any $|r| > 0$ the average fidelity $\bar{\mathcal{F}}_{\rm TEL}$ exceeds the classical benchmark achievable by measure-and-prepare schemes, $\mathcal{F}_{\rm TEL}^{\rm cl} = 2/3$ [47]. We can then define a *payoff*

$$\mathcal{P}_{\rm TEL} = \left(1 - \mathcal{F}_{\rm TEL}^{\rm cl}\right)^{-1}\max\left\{0, \bar{\mathcal{F}}_{\rm TEL} - \mathcal{F}_{\rm TEL}^{\rm cl}\right\} \tag{6}$$

quantifying the better-than-classical performance of the teleportation protocol. It is immediate to see that in the limit of large displacements,

$$\mathcal{P}_{\rm TEL} = \mathcal{N}(\rho_{AB}) = 2|r|. \tag{7}$$

This shows that entanglement, in the form of negativity, is clearly the resource for this protocol [see Fig. 3(a)].

We now provide the explicit steps of the protocol and prove the result announced in Eq. (5), which can be formalized as follows: For any $\epsilon > 0$, the state (1) can be used for teleportation of a generic qubit state $|\psi_Q^{\rm in}\rangle$ from Bob to Alice, with average fidelity $\bar{\mathcal{F}}_{\rm TEL} \geqslant (2/3)(1+|r|) - \epsilon$ and success probability $P \geqslant 1 - \epsilon$.

We first choose a cutoff integer N and define $\epsilon_N = \sum_{N+1}^{\infty} s_n$, where the initial state of the oscillator B in Eq. (1) has been expressed in its eigenbasis as $\rho_B^0 = \sum_n s_n|\psi_n\rangle\langle\psi_n|$. We can find a lower bound to the fidelity by just assuming that, with probability ϵ_N, the protocol fails. That is, we may use $\rho'_{\rm AB} = (1-\epsilon_N)\rho_{AB}^{(N)} + \epsilon_N\rho_{AB}^{\perp}$ where $\rho_{AB}^{(N)}$ is the truncated version of $\rho_{\rm AB}$ and $\rho_{AB}^{\perp}$ is a state that yields zero teleportation fidelity. From now on we assume that Alice and Bob share the truncated resource $\rho_{AB}^{(N)}$, and the derived results will hold with probability $P \geqslant 1 - \epsilon_N$. One can always choose N large enough so that $\epsilon_N < \epsilon$.

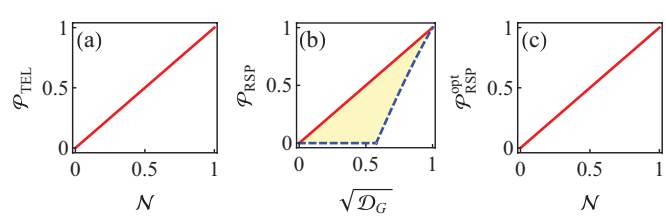

FIG. 3. (Color online) (a) Payoffs for hybrid teleportation and (b) for hybrid remote-state preparation with unitary corrections and (c) optimized with nonunitary corrections plotted versus the correlations in the shared states, measured by negativity in panels (a) and (c) and by the square root of geometric discord in panel (b); see text for details on the boundary curves in panel (b). The plotted quantities are dimensionless and correspond to $|\beta| \to \infty$.

The initial state of the complete ABQ system is $\rho_{ABQ}^{\rm in} = |\psi_Q^{\rm in}\rangle\langle\psi_Q^{\rm in}| \otimes \rho_{AB}^{(N)}$. The state $\rho_{AB}^{(N)}$ can be expanded in terms of the qubit basis and the $2(N+1)$ states $\{D(\beta)|\psi_n\rangle, D^\dagger(\beta)|\psi_n\rangle\}_{n=0,\ldots,N}$. If we choose $|\beta|$ large enough, these states will be effectively orthonormal. Then, Bob performs a hybrid Bell-state measurement corresponding to the following $4(N+1)$ orthonormal vectors: $|\phi_m^\pm\rangle = \frac{1}{\sqrt{2}}[|e\rangle D(\beta)|\psi_m\rangle \pm |g\rangle D^\dagger(\beta)|\psi_m\rangle]$, and $|\xi_m^\pm\rangle = \frac{1}{\sqrt{2}}[|e\rangle D^\dagger(\beta)|\psi_m\rangle \pm |g\rangle D(\beta)|\psi_m\rangle]$. These measurements can be carried out as follows: Bob performs the disentangling operation $U^\dagger = D^\dagger(\sigma_3\beta)$ on the input qubit Q and his oscillator B then measures the qubit Q in the basis $\{|e\rangle \pm |g\rangle\}$, and the oscillator in the effectively orthogonal basis $\{|\psi_m\rangle, D(2\beta) \pm D^\dagger(2\beta)|\psi_m\rangle\}_{m=0}^N$ [48]. Then, upon receiving two classical bits from Bob (as all values of m give the same results), Alice first performs a local phase rotation $|e\rangle \to \frac{|r|}{r}|e\rangle$ on qubit A; furthermore, she may or may not perform the corrections $|e\rangle \leftrightarrow |g\rangle$ and $|e\rangle \to -|e\rangle$, depending on Bob's outcomes. After tedious calculations, we find that there are only two possible (unnormalized) states that Alice obtains:

$$\begin{aligned}\rho_A^{{\rm out}\phi} &= p|\eta|^2|e\rangle\langle e| + (1-p)|\gamma|^2|g\rangle \\ &\quad\times \langle g| + |r|\eta\gamma^*|e\rangle\langle g| + |r|\eta^*\gamma|g\rangle\langle e|, \\ \rho_A^{{\rm out}\xi} &= (1-p)|\eta|^2|e\rangle\langle e| + p|\gamma|^2|g\rangle \\ &\quad\times \langle g| + |r|\eta\gamma^*|e\rangle\langle g| + |r|\eta^*\gamma|g\rangle\langle e|.\end{aligned}$$

The input-output fidelity is then given by

$$\mathcal{F}_{\rm TEL} = \sum_{j=\phi,\xi} \langle\psi_Q^{\rm in}|\rho_A^{{\rm out}j}|\psi_Q^{\rm in}\rangle = |\eta|^4 + |\gamma|^4 + 4|r||\eta|^2|\gamma|^2. \tag{8}$$

Averaging η and γ over the Bloch sphere, we obtain Eq. (5).

B. Hybrid remote-state preparation

We now describe a different protocol; see Fig. 1(b) for reference. Alice and Bob share again a state ρ_{AB} of the form (1). Without loss of generality, we can assume $r = |r|$ (this is true up to a local unitary on A). Alice (the preparer) wishes to remotely prepare [21] Bob's oscillator in the target superposition state, known to Alice,

$$|\psi_B^{\rm goal}\rangle = (|\beta\rangle + {\rm e}^{-i\varphi}|-\beta\rangle)\Big/\sqrt{2(1+\cos\varphi {\rm e}^{-2|\beta|^2})}, \tag{9}$$

for some phase φ. For this purpose, Alice measures qubit A in the basis $|\pm_A\rangle = (|e\rangle \pm {\rm e}^{i\varphi}|g\rangle)/\sqrt{2}$ and classically communicates the one-bit outcome "$\pm$," obtained with probability $P_\pm$, to Bob. If Alice obtains "$+$," Bob does nothing, otherwise he applies a π phase shift Φ_B to B, which ideally (for large $|\beta|$) maps $|\beta\rangle \to |\beta\rangle$ and $-|\beta\rangle \to -|-\beta\rangle$ and can be implemented in phase space via a combination of displacements and photon subtraction [49]. The qumode B after the correction can be in one of two possible unnormalized states: $\rho_B^{{\rm out}+} = \langle +_A|\rho_{AB}|+_A\rangle$ and $\rho_B^{{\rm out}-} = \Phi_B\langle -_A|\rho_{AB}|-_A\rangle\Phi_B^\dagger$. The fidelity of the protocol, averaged over the distribution of the target state, is then

$$\bar{\mathcal{F}}_{\rm RSP} = (2\pi)^{-1}\int d\varphi \langle\psi_B^{\rm goal}|\big(\rho_B^{{\rm out}+} + \rho_B^{{\rm out}-}\big)|\psi_B^{\rm goal}\rangle. \tag{10}$$

Clearly the ideal resource for this protocol is obtained for $p = |r| = 1/2$ and $\rho_B^0 = |0\rangle\langle 0|$, where $|0\rangle$ is the ground state of the oscillator. A typical realistic deviation is given by the oscillator being initially in a mixed state. For simplicity, let us focus on ρ_B^0 being in general a Gaussian thermal state [50] $\rho_B^0 = \sum_n s_n|n\rangle\langle n|$, where $s_n = \bar{n}^n/(1+\bar{n})^{n+1}$, with $\bar{n}$ the mean number of thermal excitations. This choice allows us to compute the fidelity analytically (although different choices for ρ_B^0 give rise to qualitatively similar results as verifiable numerically), yielding

$$\lim_{|\beta|\to\infty} \bar{\mathcal{F}}_{\rm RSP} = \frac{\mu_B^0}{1+\mu_B^0}(1+2|r|). \tag{11}$$

In this case, the fidelity is limited by the initial purity of the oscillator, akin to the geometric discord in Eq. (3). For remote-state preparation, the classical threshold corresponds to Bob preparing a completely random guess (unnormalized) state [21,24] $\rho_B^{\rm cl} = (|\beta\rangle\langle\beta| + |-\beta\rangle\langle -\beta|)/2$, yielding

$$\mathcal{F}_{\rm RSP}^{\rm cl} = (2\pi)^{-1}\int d\varphi \langle\psi_B^{\rm goal}|\rho_B^{\rm cl}|\psi_B^{\rm goal}\rangle \underset{|\beta|\to\infty}{\longrightarrow} \frac{1}{2}. \tag{12}$$

Defining again the *payoff* as

$$\mathcal{P}_{\rm RSP} = \big(1-\mathcal{F}_{\rm RSP}^{\rm cl}\big)^{-1}\max\big\{0, \bar{\mathcal{F}}_{\rm RSP} - \mathcal{F}_{\rm RSP}^{\rm cl}\big\}, \tag{13}$$

we then find in the limit of large displacements

$$\begin{aligned}L[\mathcal{D}_G(\rho_{AB})] \leqslant \mathcal{P}_{\rm RSP} &= \max\left\{0, \frac{\mu_B^0(1+4|r|)-1}{1+\mu_B^0}\right\} \\ &\leqslant \sqrt{\mathcal{D}_G(\rho_{AB})}.\end{aligned} \tag{14}$$

For a given $\mathcal{D}_G(\rho_{AB})$, the payoff in the performance of remote-state preparation can never exceed $\sqrt{\mathcal{D}_G}$, which is reached for $\mu_B^0 = 1$ [see solid line in Fig. 3(b)] and admits a lower bound as well, $L[\mathcal{D}_G] = \max\{0, (3\mathcal{D}_G - 1)/(1+\mathcal{D}_G)\}$ [dashed curve in Fig. 3(b)]. The latter is tight for $p = |r| = 1/2$, when the shared states are *maximally entangled*, yet the remote-state preparation succeeds with null or limited payoff. A nonzero payoff implies necessarily a nonzero $\mathcal{D}_G$. Thus, despite its clear shortcomings [25], the geometric discord might still seem to capture the operative performance of qubit-to-oscillator remote-state preparation, in analogy with the case of two-qubit resources [24].

A remark is in order. In the previously described hybrid teleportation (Sec. III A), Bob can optimize his measurement strategy to compensate for the initial mixedness of the oscillator, which therefore does not affect the achievable fidelity. In contrast, in remote-state preparation, the measurement of Alice's qubit necessarily leaves Bob's oscillator in a mixed state, if its initial state ρ_B^0 was mixed. As the protocol aims at preparing a pure state, it is no surprise that the resource mixedness enters in and degrades the figure of merit. We can address this limitation by extending the conventional remote-state preparation primitive [21], to allow Bob to perform general completely positive maps rather than just unitary corrections, aiming to improve the fidelity with the goal state [51]. In our hybrid case, we observe that if Bob performs the operation which "digitalizes" the resource state as discussed above, then the mixedness of the oscillator is effectively bypassed and we obtain a payoff $\mathcal{P}_{\rm RSP}^{\rm opt} = 2|r|$, equal to the hybrid teleportation payoff and coinciding with the

negativity of ρ_{AB} [see Fig. 3(c)]. We now present details of the procedure.

We start with the usual resource state ρ_{AB} given by Eq. (1). Alice wants to prepare remotely a state of the form $|\psi_B^{\rm goal}\rangle = \frac{1}{\sqrt{2}}(|\tilde{e}\rangle + {\rm e}^{-i\varphi}|\tilde{g}\rangle)$, where $|\tilde{e}\rangle$ and $|\tilde{g}\rangle$ are two orthogonal states in Bob's Hilbert space. Alice first rotates her basis such that $r \to |r|$ then performs a projective measurement on her part of the system using the basis $|\pm\varphi\rangle = \frac{1}{\sqrt{2}}(|e\rangle \pm {\rm e}^{-i\varphi}|g\rangle)$. The reduced density matrix of Bob after the measurement (up to a normalization constant) is given by

$$\begin{aligned}&\langle\pm\varphi|\rho_{AB}|\pm\varphi\rangle\\&\quad= pD(\beta)\rho_B^0 D^\dagger(\beta) + (1-p)D^\dagger(\beta)\rho_B^0 D(\beta)\\&\qquad\pm|r|{\rm e}^{i\varphi}D(\beta)\rho_B^0 D(\beta) \pm |r|{\rm e}^{-i\varphi}D^\dagger(\beta)\rho_B^0 D^\dagger(\beta).\end{aligned} \tag{15}$$

In the limit of large displacements, Bob can perform the following local (nonunitary) operation, with probability approaching unity (Note: it is the same "digitalizing" operation detailed in Appendix C):

$$\begin{aligned}D(\beta)\rho_B^0 D^\dagger(\beta) &\to |\tilde{e}\rangle\langle\tilde{e}|,\\ D^\dagger(\beta)\rho_B^0 D(\beta) &\to |\tilde{g}\rangle\langle\tilde{g}|,\\ D(\beta)\rho_B^0 D(\beta) &\to |\tilde{e}\rangle\langle\tilde{g}|.\end{aligned} \tag{16}$$

The state of Bob's mode then becomes

$$\begin{aligned}\rho_B' &= p|\tilde{e}\rangle\langle\tilde{e}| + (1-p)|\tilde{g}\rangle\langle\tilde{g}| \pm |r|{\rm e}^{-i\varphi}|\tilde{e}\rangle\\&\quad\times\langle\tilde{g}| \pm |r|{\rm e}^{i\varphi}|\tilde{g}\rangle\langle\tilde{e}|,\end{aligned} \tag{17}$$

which is now properly normalized. Finally, Bob can remove the $\pm$ signs with a further local unitary correction, obtaining eventually the state

$$\begin{aligned}\rho_B'' &= p|\tilde{e}\rangle\langle\tilde{e}| + (1-p)|\tilde{g}\rangle\langle\tilde{g}| + |r|{\rm e}^{-i\varphi}|\tilde{e}\rangle\\&\quad\times\langle\tilde{g}| + |r|{\rm e}^{i\varphi}|\tilde{g}\rangle\langle\tilde{e}|.\end{aligned} \tag{18}$$

The fidelity between this output state and the target state is

$$\mathcal{F}_{\rm RSP}^{\rm opt} = \langle\psi_B^{\rm goal}|\rho_B''|\psi_B^{\rm goal}\rangle = \frac{1}{2} + |r|, \tag{19}$$

which does not depend on the phase φ and is hence equal to the average fidelity $\bar{\mathcal{F}}_{\rm RSP}^{\rm opt}$. Recalling that for remote-state preparation $\mathcal{F}_{\rm RSP}^{\rm cl} \to \frac{1}{2}$, we finally recover that the payoff, for the optimized scheme incorporating a nonunitary local correction on Bob's side, becomes $\mathcal{P}_{\rm RSP}^{\rm opt} = 2|r|$ in the limit of large displacements, as anticipated above.

This suggests that the link between remote-state preparation and measures of discord, highlighted for two-qubit systems [23,24], might be due to a nonoptimized version of the protocol used. In the case of the qubit-oscillator resources described here, such a limitation may be relevant where the digitalizing operation of Eq. (16) is experimentally challenging to realize, so that one is constrained to unitary corrections only. However, the resulting connection between geometric discord and protocol performance would be solely due to technological limitations, and not to fundamental quantum-mechanical principles. In fact, we showed how properly accounting for an extra freedom to correct for the resource mixedness allows the protocol to reach a performance only dependent on the amount of shared entanglement.

It will be interesting to test how this result is modified once additional decoherence sources in the implementation of the hybrid protocols are considered, to see whether the link with entanglement will persist also for remote-state preparation, or some form of discord would emerge as an essential operational ingredient.

IV. CONCLUDING REMARKS

We conclude that, broadly speaking, various indicators of "quantumness" in composite systems [3,8] can be justified by operational interpretations related to their role in different tasks. To wit, entanglement is a resource for teleportation [20,47] and superdense coding [52] (among all), geometric discord is such for a non-globally-optimized remote-state preparation scheme [24] (as shown here), entropic discord is interpreted through quantum-state merging [53], and relative entropy of discord [45,46] is the cost of entanglement distribution via separable states [54]. The list is likely to grow in the future, although care is needed to ensure it does not derail away from physical grounds.

On the practical side, we demonstrated that robust quantum correlations in the form of entropic discord and entanglement can be engineered in coupled qubit-oscillator systems regardless of the temperature of the oscillator, provided the interaction generates enough displacement. This indicates that, for example, in systems of nano- or optomechanical oscillators [55] coupled to a two-level probe, there is no need to cool the oscillator down to its ground state in order for quantum communication to be achieved efficiently. Although this may require generalized (nonunitary) operations that can be challenging to realize with current technology, the question of implementing those operations in a realistic experimental setup can be a source of further interesting research outside the scope of this paper. This may in the future relax the need for ground-state cooling in favor of generalized operations. In fact, while ground-state cooling has attracted experimental efforts for a long time and is hence well developed, our findings support the view that the experimental realization of generalized quantum operations, a comparatively young subject, may also deserve consideration and could have a substantial practical impact for the realistic implementation of quantum technologies.

ACKNOWLEDGMENTS

We thank C. Brukner, E. Chitambar, B. Dakić, A. Ferraro, B. Garraway, S. Gharibian, V. Giovannetti, T. Nakano, M. G. A. Paris, M. Piani, S. Ragy, V. Vedral, and especially M. S. Kim for fruitful discussions. We thank the anonymous referee for having suggested how to improve the remote-state preparation protocol. We acknowledge financial support from the Royal Society, the Wolfson Foundation, the University of Nottingham (ECRKTA/2011), the UK EPSRC (Grant RDF/BtG/0612b/31), and the Qatar National Research Fund (NPRP 4-554-1-084).

APPENDIX A: NEGATIVITY OF QUBIT-OSCILLATOR STATES ρ_{AB}

Here we derive Eq. (2), which can be formalized as follows:

Theorem 1. For any $\epsilon > 0$, it is possible to find a $\beta \in \mathbb{C}$ such that

$$\mathcal{N}(\rho_{AB}) \geqslant 2|r| - \epsilon, \tag{A1}$$

where $\mathcal{N}$ is the negativity [11] (twice the modulus of the sum of the negative eigenvalues of $\rho_{AB}^{\mathsf{T}_A}$).

Proof. The partial transpose of the state (1) with respect to the qubit is

$$\begin{aligned}\rho_{AB}^{\mathsf{T}_A} = {} & p|e\rangle\langle e| \otimes D(\beta)\rho_B^0 D^\dagger(\beta) \\ & + (1-p)|g\rangle\langle g| \otimes D^\dagger(\beta)\rho_B^0 D(\beta) \\ & + r|g\rangle\langle e| \otimes D(\beta)\rho_B^0 D(\beta) \\ & + r^*|e\rangle\langle g| \otimes D^\dagger(\beta)\rho_B^0 D^\dagger(\beta).\end{aligned} \tag{A2}$$

Let us consider the test states

$$|\phi_m\rangle = \frac{1}{\sqrt{2}}[|e\rangle D^\dagger(\beta)|\psi_m\rangle - \mathrm{e}^{i\phi}|g\rangle D(\beta)|\psi_m\rangle], \tag{A3}$$

where ψ_m are the eigenvectors of the initial oscillator state $\rho_B^0 = \sum_m s_m|\psi_m\rangle\langle\psi_m|$, and ϕ is the complex argument of r; that is, $\mathrm{e}^{i\phi} = r/|r|$. The expectation value of the partial transpose on the test states is

$$\begin{aligned}\langle\phi_m|\rho_{AB}^{\mathsf{T}_A}|\phi_m\rangle = {} & -|r|s_m + \frac{p}{2}\langle\psi_m|D(2\beta)\rho_B^0 D^\dagger(2\beta)|\psi_m\rangle \\ & + \frac{1-p}{2}\langle\psi_m|D^\dagger(2\beta)\rho_B^0 D(2\beta)|\psi_m\rangle.\end{aligned} \tag{A4}$$

Now, let us make use of the following Lemma (proven below):

Lemma 2. Given any oscillator density matrix ρ and any oscillator pure state ψ, one has

$$\lim_{|\alpha|\to\infty} \langle\psi|D(\alpha)\rho D^\dagger(\alpha)|\psi\rangle = 0. \tag{A5}$$

Let us then pick a cutoff N such that $\sum_m^N p_m \geqslant 1 - \epsilon'$, where $\epsilon' > 0$ and, by using the property (A5), let us choose β such that for any $m \leqslant N$ one has $\langle\psi_m|D(2\beta)\rho_B^0 D^\dagger(2\beta)|\psi_m\rangle < \epsilon'$ and also $\langle\psi_m|D^\dagger(2\beta)\rho_B^0 D(2\beta)|\psi_m\rangle < \epsilon'$. Then, from Eq. (A4) it follows that

$$\sum_m^N \langle\phi_m|\rho_{AB}^{\mathsf{T}_A}|\phi_m\rangle \leqslant -|r| + \epsilon'(|r| + N/2).$$

By choosing $\epsilon' = (\epsilon/2)(|r| + N/2)^{-1}$, and using the fact that the test states are orthonormal, we obtain Eq. (A1). ■

Proof of Lemma 2. Given any density operator ρ, one has

$$\lim_{|\alpha|\to\infty} \chi_\rho(\alpha) = 0,$$

$\chi_\rho(\alpha) = \mathrm{tr}\,[\rho D(\alpha)]$ being its characteristic function. This is a consequence of the fact that

$$\mathrm{tr}[\rho^2] = \pi^{-1}\int d^2\alpha|\chi_\rho(\alpha)|^2 \leqslant 1.$$

In particular, by taking a projector $\rho = |\varphi\rangle\langle\varphi|$, we have

$$\lim_{|\alpha|\to\infty} \langle\varphi|D(\alpha)|\varphi\rangle = 0.$$

Then, by substituting $|\varphi\rangle$ with $|\varphi_1\rangle \pm |\varphi_2\rangle$, $|\varphi_1\rangle \pm i|\varphi_2\rangle$ in the above result, it is easy to show that, for any pair of vectors $|\varphi_1\rangle$, $|\varphi_2\rangle$, one has

$$\lim_{|\alpha|\to\infty} \langle\varphi_1|D(\alpha)|\varphi_2\rangle = 0. \tag{A6}$$

To prove (A5), we decompose $\rho = \sum_n q_n|\varphi_n\rangle\langle\varphi_n|$ and write

$$\langle\psi|D(\alpha)\rho D^\dagger(\alpha)|\psi\rangle = \sum_n q_n|\langle\psi|D(\alpha)|\varphi_n\rangle|^2. \tag{A7}$$

We now choose N such that $\sum_{N+1}^\infty q_n \leqslant \epsilon/2$ and, thanks to Eq. (A6), we can choose α such that $|\langle\psi|D(\alpha)|\varphi_n\rangle|^2 < \epsilon/2$ for any $n \leqslant N$. It follows that

$$\begin{aligned}&\sum_n q_n|\langle\psi|D(\alpha)|\varphi_n\rangle|^2 \\ &\quad = \sum_n^N q_n|\langle\psi|D(\alpha)|\varphi_n\rangle|^2 \\ &\qquad + \sum_{n=N+1}^\infty q_n|\langle\psi|D(\alpha)|\varphi_n\rangle|^2 \\ &\quad \leqslant \frac{\epsilon}{2}\sum_n^N q_n + \sum_{N+1}^\infty q_n \leqslant \frac{\epsilon}{2}\sum_n^\infty q_n + \frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{A8}$$

Then, given any $\epsilon > 0$, we have found α such that

$$\langle\psi|D(\alpha)\rho D^\dagger(\alpha)|\psi\rangle < \epsilon, \tag{A9}$$

which proves Eq. (A5). ■

APPENDIX B: GEOMETRIC DISCORD OF THE QUBIT-OSCILLATOR STATES ρ_{AB}

1. Geometric Discord for $2 \otimes \infty$ systems

Given a quantum state ρ_{AB} of a $\mathbb{C}^2 \otimes \mathbb{C}^d$ system, with A being a qubit and B being an arbitrary (finite or infinite) d-dimensional system, the normalized geometric discord $\mathcal{D}_G(\rho_{AB})$ is defined as [12,37]

$$\mathcal{D}_G(\rho_{AB}) = 2\inf_{\Pi_A} \|\rho_{AB} - \Pi_A(\rho_{AB})\|_2^2,$$

where the infimum is over all complete (nondegenerate) von Neumann measurements $\Pi_A \equiv \{\Pi_A^k\}$ on the qubit A, with $\Pi_A(\rho_{AB}) = \sum_k(\Pi_A^k \otimes \mathbb{1}_B)\rho_{AB}(\Pi_A^k \otimes \mathbb{1}_B)$, and $\|M\|_2 = [\mathrm{Tr}(MM^\dagger)]^{1/2}$ is the Hilbert-Schmidt norm.

Noting that any projective measurement induces a dephasing of the qubit on some orthonormal basis $\{|c\rangle, |c^\perp\rangle\}$ [i.e., $\Pi_A(\rho_{AB}) = (\rho_{AB} + U_A\rho_{AB}U_A^\dagger)/2$, with $U_A = |c\rangle\langle c| - |c^\perp\rangle\langle c^\perp|$ being a "root-of-unity" operation on A [56–58]], we can recast the problem into an optimization over such local unitaries, $\mathcal{D}_G(\rho_{AB}) = 2\inf_{U_A}\|[\rho_{AB} - (U_A \otimes \mathbb{1}_B)\rho_{AB}(U_A^\dagger \otimes \mathbb{1}_B)]/2\|_2^2$. Introducing a generic projector $P_A^{\mathbf{e}} = |c\rangle\langle c| = (\mathbf{e}\cdot\boldsymbol{\sigma})/2$ on the qubit, where $\boldsymbol{\sigma} = (\mathbb{1}, \vec{\sigma})$ is a four-vector of Pauli matrices, and $\mathbf{e} = (1, \hat{e})$ with $\hat{e} \in \mathbb{R}^3$ a unit vector, the geometric discord can be expressed as follows [57]:

$$\begin{aligned}\mathcal{D}_G(\rho_{AB}) = \inf_{\mathbf{e}} \big\{4\mathrm{tr}\big[&\rho_{AB}^2\big(P_A^{\mathbf{e}} \otimes \mathbb{1}_B\big) \\ &- \rho_{AB}\big(P_A^{\mathbf{e}} \otimes \mathbb{1}_B\big)\rho_{AB}\big(P_A^{\mathbf{e}} \otimes \mathbb{1}_B\big)\big]\big\}.\end{aligned} \tag{B1}$$

This minimization can be solved in closed form. We define a "partial Fano representation" [59] of ρ_{AB} by expanding only the qubit A in the Bloch basis:

$$\rho_{AB} = \tfrac{1}{2}(\mathbf{v}\cdot\boldsymbol{\sigma}),$$

with the four-vector of operators $\mathbf{v} = \mathrm{tr}_A(\rho_{AB}\boldsymbol{\sigma}) \equiv (v_0, \vec{v})$, where $v_0 \equiv \rho_B$ is the reduced density matrix of B. Like in relativity theory we use Greek indices to indicate the components 0,1,2,3 and Roman indices to indicate the "spatial" components 1,2,3 only.

Let us evaluate each term appearing inside the minimization in Eq. (B1). For the first, we have

$$\begin{aligned} 4\mathrm{tr}\big[\rho_{AB}^2 P_A^{\mathbf{e}}\big] &= \tfrac{1}{2}\mathrm{tr}[(\mathbf{v}\cdot\boldsymbol{\sigma})^2(\mathbf{e}\cdot\boldsymbol{\sigma})] \\ &= \tfrac{1}{2}\mathrm{tr}_B[v_\mu v_\nu]e_\eta \mathrm{tr}_A[\sigma_\mu\sigma_\nu\sigma_\eta], \end{aligned}$$

where sum over the repeated indices is understood as in Einstein's convention. To evaluate the second term, we begin by noting that, if $P_A^{\mathbf{e}} = (\mathbf{e}\cdot\boldsymbol{\sigma}) = |c\rangle\langle c|$, then

$$\begin{aligned} \mathrm{tr}\big[\rho_{AB} P_A^{\mathbf{e}} \rho_{AB} P_A^{\mathbf{e}}\big] &= \mathrm{tr}_B\{\mathrm{tr}_A[\rho_{AB}|c\rangle\langle c|\rho_{AB}|c\rangle\langle c|]\} \\ &= \mathrm{tr}_B[\langle c|\rho_{AB}|c\rangle\langle c|\rho_{AB}|c\rangle] \\ &= \mathrm{tr}_B\{\mathrm{tr}_A[\rho P_{AB}]\mathrm{tr}_A[\rho P_{AB}]\}. \end{aligned}$$

Then we have

$$\begin{aligned} &4\mathrm{tr}\big[\rho_{AB} P_A^{\mathbf{e}} \rho_{AB} P_A^{\mathbf{e}}\big] \\ &= \tfrac{1}{4}\mathrm{tr}_B\left\{\mathrm{tr}_A\left[(\mathbf{v}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma})\right]\mathrm{tr}_A\left[(\mathbf{v}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma})\right]\right\} \\ &= \tfrac{1}{4}\mathrm{tr}_B\left[v_\mu v_\nu\right]e_\eta e_\tau \mathrm{tr}_A\left[\sigma_\mu\sigma_\eta\right]\mathrm{tr}_A\left[\sigma_\nu\sigma_\tau\right] \\ &= \mathrm{tr}_B\left[v_\mu v_\nu\right]e_\mu e_\nu. \end{aligned}$$

For the last equality we have used the fact that $\mathrm{tr}_A[\sigma_\mu\sigma_\eta] = 2\delta_{\mu\eta}$. We now define the 4×4 matrix

$$\mathcal{S}_{\mu\nu} = \mathrm{tr}_B[v_\mu v_\nu].$$

Note that the matrix $\mathcal{S}$ is symmetric, due to the cyclic invariance of the trace. Then Eq. (B1) can be rewritten as $\mathcal{D}_G(\rho_{AB}) = \inf_{\mathbf{e}} T$, with

$$\begin{aligned} T &= \tfrac{1}{2}\mathcal{S}_{\mu\nu}e_\eta \mathrm{tr}_A[\sigma_\mu\sigma_\nu\sigma_\eta] - \mathcal{S}_{\mu\nu}e_\mu e_\nu \\ &= \tfrac{1}{2}\mathcal{S}_{ij}e_\eta \mathrm{tr}_A[\sigma_i\sigma_j\sigma_\eta] - \mathcal{S}_{ij}e_i e_j, \end{aligned} \tag{B2}$$

where we have used the fact that the contributions with $\mu = 0$ or $\nu = 0$ vanish (this is easy to check in the above formula noting that $e_0 = 1$). To simplify further the above expression, we evaluate explicitly the term

$$e_\eta \mathrm{tr}_A[\sigma_i\sigma_j\sigma_\eta] = 2\delta_{ij} + 2i\epsilon_{ijk}e_k.$$

Since $\mathcal{S}$ is symmetric, we have $\mathcal{S}_{ij}\epsilon_{ijk} = 0$, and (B2) simplifies to

$$T = \mathcal{S}_{ii} - \mathcal{S}_{ij}e_i e_j.$$

Defining $S = (\mathcal{S})_{ij}$ as the 3×3 "spatial" sub-block of $\mathcal{S}$, we can write

$$T = \mathrm{tr}\,[S] - \vec{e}^{\,\mathsf{T}} S\vec{e}.$$

Since $|\vec{e}\,| = 1$, the minimization of such expression yields Eq. (B3):

$$\mathcal{D}_G(\rho_{AB}) = \mathrm{tr}(S) - \lambda_{\max}(S). \tag{B3}$$

Recall that $S = \mathrm{tr}_B[\vec{v}\,\vec{v}^{\,\mathsf{T}}]$, with $\vec{v} = \mathrm{tr}_A[\rho_{AB}\vec{\sigma}]$.

Equation (B3) encompasses the known formulas for two-qubit [12,37] and qubit-qudit states [34,57,60] but is valid as well for states of a qubit and a qumode, for which $d = \infty$. In the latter case, it can be convenient to adopt a hybrid Hilbert-space–phase-space picture to describe qubit-oscillator states ρ_{AB} [18,61]. Let b ($b^\dagger$) denote the annihilation (creation) operator for the qumode, with $[b,b^\dagger] = 1$, and let $D(\beta) = \exp(\beta b^\dagger - \beta^* b)$ denote the corresponding Weyl displacement operator, with $\beta \in \mathbb{C}$. We can define the *characteristic vector* associated to the state ρ_{AB} as $\boldsymbol{\chi}(\beta) \equiv (\chi_0(\beta), \vec{\chi}(\beta)) = \mathrm{tr}[\rho_{AB}\boldsymbol{\sigma}D(\beta)] = \mathrm{tr}_B[\mathbf{v}D(\beta)]$, where the zeroth component $\chi_0(\beta) = \mathrm{tr}_B[\rho_B D(\beta)]$ is the conventional characteristic function of the oscillator [50], describing its marginal state in phase space. The matrix S appearing in Eq. (B3) is then $S = \pi^{-1}\int d^2\beta\,\vec{\chi}(\beta)\vec{\chi}^{\,\dagger}(\beta)$. Similar representations can be provided by employing, for example, the Wigner distribution to describe the oscillator [18,41,61].

2. Explicit calculation for states ρ_{AB}

To calculate $\mathcal{D}_G(\rho_{AB})$, we need the spatial components of the vector $\mathbf{v}$ (see previous subsection). These are

$$v_1 = rD(\beta)\rho_B^0 D(\beta) + r^* D^\dagger(\beta)\rho_B^0 D^\dagger(\beta), \tag{B4}$$

$$v_2 = irD(\beta)\rho_B^0 D(\beta) - ir^* D^\dagger(\beta)\rho_B^0 D^\dagger(\beta), \tag{B5}$$

$$v_3 = pD(\beta)\rho_B^0 D^\dagger(\beta) - (1-p)D^\dagger(\beta)\rho_B^0 D(\beta). \tag{B6}$$

The matrix S can now be calculated according to $S_{ij} = \mathrm{tr}_B[v_i v_j]$. We are interested in the regime of large displacements. Suppose that $|\beta|$ is large compared to the phase-space extension of the initial oscillator state ρ_B^0. Then the matrix S converges to

$$S \underset{|\beta|\to\infty}{\longrightarrow} \begin{pmatrix} 2|r|^2 & 0 & 0 \\ 0 & 2|r|^2 & 0 \\ 0 & 0 & 2p(1-p)+1 \end{pmatrix} \mathrm{tr}_B\big[\big(\rho_B^0\big)^2\big]. \tag{B7}$$

From the above, as well as the condition $|r|^2 \leqslant p(1-p)$, we can directly see that $\lambda_{\max}(S) = [2p(1-p)+1]\mathrm{tr}_B[(\rho_B^0)^2]$. Hence, the geometric discord of the state ρ_{AB}, for the case of large displacements, is given by Eq. (3),

$$\mathcal{D}_G(\rho_{AB}) \underset{|\beta|\to\infty}{\longrightarrow} 4|r|^2\mathrm{tr}_B\big[\big(\rho_B^0\big)^2\big].$$

Note that $\mathcal{D}_G(\rho_{AB}) \leqslant \mathrm{tr}_B[(\rho_B^0)^2]$; that is, the geometric discord is smaller than the purity of the initial state of the oscillator.

To see that Eq. (B7) is correct in the limit of large displacements, we can proceed as follows. When we calculate $\mathrm{tr}_B[v_i v_j]$, we get terms of the form $\mathrm{tr}_B\,[\rho_B D_1 D_2 \rho_B D_3 D_4]$, where each one of the D_j's can be either $D(\beta)$ or $D^\dagger(\beta)$. In the limit of large $|\beta|$, however, only those that evaluate to $\mathrm{tr}_B[(\rho_B^0)^2]$ survive. All the others can be shown to be negligible by using the fact that $|\mathrm{tr}_B\,[\rho_B D(\alpha)]| \to 0$ for $|\alpha| \gg |\beta|$ (see also Lemma 2 in the previous appendix). For example, using

Glauber's P representation we have $\rho_B = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, and we can see that [denoting $D(\beta) \equiv D$], for instance,

$$\begin{aligned} &|\mathrm{tr}_B[\rho_B(D^\dagger)^2\rho_B D^2]| \\ &\leqslant \int d^2\alpha d^2\alpha' |P(\alpha)P(\alpha')| \left|\langle\alpha'|D^2|\alpha\rangle\right|^2 \\ &= \int d^2\alpha d^2\alpha' |P(\alpha)P(\alpha')| \mathrm{e}^{-|2\beta+\alpha-\alpha'|^2} \\ &\xrightarrow[|\beta|\to\infty]{} 0, \end{aligned} \tag{B8}$$

and so on. The above and all the other terms can be evaluated explicitly (e.g., in case ρ_B^0 is a Gaussian state), but this restriction is not crucial for the validity of Eq. (3).

APPENDIX C: DIGITALIZATION OF QUBIT-OSCILLATOR STATES ρ_{AB}

Theorem 3. For any $\epsilon > 0$, it is possible to find a $\beta \in \mathbb{C}$ and a local operation on Bob, $\sum_j O_j \rho_{\mathrm{AB}} O_j^\dagger$, with $\sum_j O_j^\dagger O_j = \mathbb{1}$, such that the state ρ_{AB} [Eq. (1)] is converted into the digitalized two-qubit state

$$\begin{aligned} \tilde{\rho}_{AB}^{\mathrm{dig}} = \; & p|e\rangle\langle e| \otimes |\tilde{e}\rangle\langle\tilde{e}| + (1-p)|g\rangle\langle g| \otimes |\tilde{g}\rangle\langle\tilde{g}| \\ & + r|e\rangle\langle g| \otimes |\tilde{e}\rangle\langle\tilde{g}| + r^*|g\rangle\langle e| \otimes |\tilde{g}\rangle\langle\tilde{e}|, \end{aligned} \tag{C1}$$

with probability $P \geqslant 1-\epsilon$ and fidelity $\mathcal{F} > 1-\epsilon$.

Proof. Recall $\rho_B^0 = \sum_n s_n|\psi_n\rangle\langle\psi_n|$. Let us fix $\epsilon_N = \sum_{n=N+1}^\infty s_n$; clearly $\lim_{N\to\infty}\epsilon_N = 0$. Let us now consider the two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ spanned by the bases $\mathcal{B}_1 = \{D(\beta)|\psi_1\rangle, \ldots, D(\beta)|\psi_N\rangle\}$ and $\mathcal{B}_2 = \{D(\beta)^\dagger|\psi_1\rangle, \ldots, D(\beta)^\dagger|\psi_N\rangle\}$. The bases $\mathcal{B}_1$ and $\mathcal{B}_2$ may overlap with each other; nevertheless, in the limit $|\beta| \to \infty$, $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$ becomes an orthonormal set. Let us then choose $|\beta|$ large enough such that this orthonormality is verified for all practical purposes. We denote by $\mathcal{H}_{12} = \mathcal{H}_1 \oplus \mathcal{H}_2$ the subspace spanned by the basis $\mathcal{B}$. Let us consider the local operation on Bob, corresponding to

$$O_j = |\tilde{e}\rangle\langle\psi_j|D^\dagger(\beta) + |\tilde{g}\rangle\langle\psi_j|D(\beta),$$

where $j = 0, \ldots, N$. To see that those operators are quasicomplete, we evaluate

$$\begin{aligned} \sum_j O_j^\dagger O_j = D(\beta)\sum_j^N |\psi_j\rangle\langle\psi_j|D^\dagger(\beta) \\ + D^\dagger(\beta)\sum_j^N |\psi_j\rangle\langle\psi_j|D(\beta) = \mathcal{P}_{12}, \end{aligned} \tag{C2}$$

where $\mathcal{P}_{12}$ is the projection on the subspace $\mathcal{H}_{12}$. Let us in general denote the complete operation with $\{O_{j=1}^N, O_\perp\}$. It is irrelevant how specifically we complete the operation on the remainder of Bob's Hilbert space; we can just assume that the fidelity of the output state is zero if we go outside $\mathcal{H}_{12}$ (this happens with probability $< \epsilon_N$). Now we can easily see that

$$\lim_{\beta\to\infty}\sum_{j=0}^N O_j\rho_{\mathrm{AB}}O_j^\dagger = \sum_{j=0}^N s_j\tilde{\rho}_{AB}^{\mathrm{dig}},$$

so that the output of the complete operation is some state

$$\tilde{\rho}_{AB}^{\mathrm{out}} = (1-\epsilon_N)\rho_{AB}^{\mathrm{dig}} + \epsilon_N\tilde{\rho}_{AB}^{\perp},$$

where $\tilde{\rho}_{AB}^{\perp}$ is some state orthogonal to $\tilde{\rho}_{AB}^{\mathrm{dig}}$. It is clear that, if we choose ϵ_N small enough, this state $\tilde{\rho}_{AB}^{\mathrm{out}}$ has fidelity $\mathcal{F} > 1-\epsilon$ with the target state $\tilde{\rho}_{AB}^{\mathrm{dig}}$ of Eq. (C1). ■

[1] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[3] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[4] N. Brunner, N. Gisin, and V. Scarani, New J. Phys. **7**, 88 (2005).
[5] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001); L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).
[6] Z. Merali, Nature (London) **474**, 24 (2011).
[7] E. Knill and R. Laflamme, Phys. Rev. Lett. **81**, 5672 (1998); A. Datta, A. Shaji, and C. M. Caves, *ibid.* **100**, 050502 (2008); B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, *ibid.* **101**, 200501 (2008).
[8] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, arXiv:1112.6238 [Rev. Mod. Phys. (to be published)].
[9] M. Piani and G. Adesso, Phys. Rev. A **85**, 040301(R) (2012).
[10] D. Girolami and G. Adesso, Phys. Rev. A **84**, 052110 (2011).
[11] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998); J. Lee, M. S. Kim, Y. J. Park, and S. Lee, J. Mod. Opt. **47**, 2151 (2000); G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[12] B. Dakić, V. Vedral, and C. Brukner, Phys. Rev. Lett. **105**, 190502 (2010).
[13] G. Adesso and A. Datta, Phys. Rev. Lett. **105**, 030501 (2010).
[14] P. Giorda and M. G. A. Paris, Phys. Rev. Lett. **105**, 020503 (2010).
[15] L. Mišta Jr., R. Tatham, D. Girolami, N. Korolkova, and G. Adesso, Phys. Rev. A **83**, 042325 (2011).
[16] G. Adesso and D. Girolami, Int. J. Quantum Inform. **9**, 1773 (2011).
[17] T. Tufarelli, M. S. Kim, and S. Bose, Phys. Rev. A **80**, 062317 (2009); S. Bose and G. S. Agarwal, New J. Phys. **8**, 34 (2006).
[18] T. Tufarelli, Ph.D. thesis, University College London, 2012, (unpublished); arXiv:1209.1837.
[19] T. P. Spiller, K. Nemoto, S. L. Braunstein, W. J. Munro, P. van Loock, and G. J. Milburn, New J. Phys. **8**, 30 (2006).
[20] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[21] A. K. Pati, Phys. Rev. A **63**, 014302 (2000); C. H. Bennett, D. P. Di Vincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).

[22] A similar dependence on the resource state mixedness has been reported for the two-qubit version of remote-state preparation [23], whose figure of merit has been operationally linked to the geometric discord [24].
[23] R. Chaves and F. de Melo, Phys. Rev. A **84**, 022324 (2011).
[24] B. Dakić *et al.*, Nat. Phys. **8**, 666 (2012).
[25] M. Piani, Phys. Rev. A **86**, 034101 (2012).
[26] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Nature (London) **414**, 413 (2001); H. J. Kimble, *ibid.* **453**, 1023 (2008).
[27] A. Furusawa and P. van Loock, *Quantum Teleportation and Entanglement: A Hybrid Approach to Optical Quantum Information Processing* (Wiley-VCH Verlag, Berlin, 2011).
[28] K. Kreis and P. van Loock, Phys. Rev. A **85**, 032307 (2012).
[29] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001); H. Walther, B. T. H. Varcoe, B.-G. Englert, and T. Becker, Rep. Prog. Phys. **69**, 1325 (2006); F. Xue, Y. D. Wang, C. P. Sun, H. Okamoto, H. Yamaguchi, and K. Semba, New J. Phys. **9**, 35 (2007); O. Gamel and D. F. V. James, Phys. Rev. A **82**, 052106 (2010); J. M. Gambetta, A. A. Houck, and A. Blais, Phys. Rev. Lett. **106**, 030502 (2011).
[30] T. Tufarelli, M. S. Kim, and S. Bose, Phys. Rev. A **83**, 062120 (2011); T. Tufarelli, A. Ferraro, M. S. Kim, and S. Bose, *ibid.* **85**, 032334 (2012).
[31] J. Li, G. McKeown, F. L. Semião, and M. Paternostro, Phys. Rev. A **85**, 022116 (2012).
[32] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[33] E. Schrödinger, Naturwissenschaften **23**, 807 (1935); For an English translation see J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, New Jersey, 1983), Sec. I.11.
[34] D. Girolami and G. Adesso, Phys. Rev. Lett. **108**, 150403 (2012).
[35] J.-S. Jin, F.-Y. Zhang, C.-S. Yu, and H.-S. Song, J. Phys. A **45**, 115308 (2012).
[36] G. Passante, O. Moussa, and R. Laflamme, Phys. Rev. A **85**, 032325 (2012).
[37] S. Luo and S. Fu, Phys. Rev. A **82**, 034302 (2010).
[38] A. Streltsov, H. Kampermann, and D. Bruss, Phys. Rev. Lett. **106**, 160401 (2011).
[39] X. Hu, H. Fan, D. L. Zhou, and W.-M. Liu, arXiv:1203.6149.
[40] S. Bose, I. Fuentes-Guridi, P. L. Knight, and V. Vedral, Phys. Rev. Lett. **87**, 050401 (2001).
[41] J. Rigas, O. Gühne, and N. Lütkenhaus, Phys. Rev. A **73**, 012341 (2006).
[42] M. Ozawa, Phys. Lett. A **268**, 158 (2000).
[43] D. Girolami and G. Adesso, Phys. Rev. A **83**, 052108 (2011).
[44] J. Watrous, *Theory of Quantum Information* (2008); Lecture notes available at http://www.cs.uwaterloo.ca/∼watrous/quant-info/lecture-notes/all-lectures.pdf.
[45] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010); M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, *ibid.* **106**, 220403 (2011).
[46] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).
[47] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999); D. Bruss and C. Macchiavello, Phys. Lett. A **253**, 249 (1999); F. Verstraete and H. Verschelde, Phys. Rev. Lett. **90**, 097901 (2003).
[48] For example, if ρ_B^0 is a Gaussian thermal state, then $|\psi_m\rangle = |m\rangle$ are the Fock states, and Bob needs to be able to discriminate photon numbers going from $|0\rangle$ to $|N\rangle$, as well as the more complicated states given by $[D(2\beta) \pm D^\dagger(2\beta)]|m\rangle$.
[49] P. Marek and J. Fiurášek, Phys. Rev. A **82**, 014304 (2010); R. Blandino, F. Ferreyrol, M. Barbieri, P. Grangier, and R. Tualle-Brouri, New J. Phys. **14**, 013017 (2012).
[50] A. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).
[51] A similar generalization has been considered for continuous-variable teleportation in J. Fiurasek, Phys. Rev. A **66**, 012304 (2002).
[52] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[53] V. Madhok and A. Datta, Phys. Rev. A **83**, 032323 (2011); D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter, *ibid.* **83**, 032324 (2011).
[54] A. Streltsov, H. Kampermann, and D. Bruss, Phys. Rev. Lett. **108**, 250501 (2012); T. K. Chuan, J. Maillard, K. Modi, T. Paterek, M. Paternostro, and M. Piani, *ibid.* **109**, 070501 (2012).
[55] D. Vitali, S. Gigan, A. Ferreira, H. R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, Phys. Rev. Lett. **98**, 030405 (2007).
[56] A. Monras, G. Adesso, S. M. Giampaolo, G. Gualdi, G. B. Davies, and F. Illuminati, Phys. Rev. A **84**, 012301 (2011).
[57] S. Gharibian, Phys. Rev. A **86**, 042106 (2012); A. Streltsov, S. M. Giampaolo, W. Roga, D. Bruss, and F. Illuminati, arXiv:1206.4075.
[58] T. Nakano, M. Piani, and G. Adesso, arXiv:1211.4022.
[59] U. Fano, Rev. Mod. Phys. **55**, 855 (1983).
[60] S. Vinjanampathy and A. R. P. Rau, J. Phys. A **45**, 095303 (2012).
[61] N. Klenner, M. Doucha, and J. Weis, J. Phys. A **19**, 3831 (1986).