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LINEAR ELASTIC COMPOSITES WITH STATISTICALLY ORIENTED SPHEROIDAL INCLUSIONS

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Summary

The purpose of this chapter is to critically review some results that our groups obtained in previous works, which were devoted to the investigation of the elastic properties of composite materials with a statistical distribution of spheroidal inclusions. These studies were motivated by our interest in the description of mechanical properties of fibre-reinforced biological tissues (such as articular cartilage), starting from the internal structure of these tissues. After an introduction to tensor algebra, which defines the notation and clarifies the mathematical framework adopted in the chapter, we present, in a covariant setting inspired by Differential Geometry, Walpole's representation of isotropic and transversely isotropic second- and fourth-order tensors, along with its properties. Hence, starting from Eshelby's seminal work on the problem of an inclusion in an infinite matrix, we briefly review the theories developed by Hill, Walpole and Weng for the determination of the overall elasticity tensor of materials with one or more inclusion phases. Then, we discuss in detail the cases of composite materials with aligned spheroidal inclusions and with statistically oriented spheroidal inclusions. Emphasis is put on extending Walpole's formula to the case of inclusions aligned according to some probability density of orientation, both in the transversely isotropic and the isotropic case.

Keywords: covariant formalism, tensor algebra, material symmetries, inclusions, ellipsoidal inclusions, spheroidal inclusions, linear elasticity, elasticity tensor, elastic moduli, statistical orientation.

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Dedicated to the memory of Prof. Gaetano Giaquinta (1945-2016)

1 Introduction

From the 1950s to the 1970s, Eshelby published several papers (e.g., Eshelby, 1951, 1957, 1975) that turned out to be of fundamental importance in the development of the theory of materials with defects or inclusions. In this chapter, we are particularly interested in his work on the ellipsoidal inclusion (Eshelby, 1957), which is at the basis of the theory of materials reinforced by one or more phases of ellipsoidal inclusions, whose shape ranges from flat discs (which could represent cracks, if assigned a null elasticity tensor) to spherical inclusions to fibre-like inclusions.

The theory for the determination of the elasticity tensor of a composite material with inclusions has been developed by, among others, Hill (1963, 1965), Hashin (1963) and Walpole (1966a,b, 1969). The case of aligned inclusions has been thoroughly studied by Weng and his group (Weng, 1984, 1990; Qiu and Weng, 1990). A few cases of composites with non-aligned inclusions have been studied in the 1980s. The first work we are aware of is that by Chou and Nomura (1981), who studied a short-fibre composite in which the directions of alignment of the fibres lay on the surface of a cone. Tandon and Weng (1986), Weng (1990) and Qiu and Weng (1990) studied the case of randomly oriented spheroidal inclusions.

In this chapter, we report our method of solution for the general case of statistically oriented inclusions and, in particular, for the case of probability density being transversely isotropic with respect to a given direction. This is the core of one of our first works (Federico et al., 2004), which here we would like to present from a more mature point of view (twelve years are not so few...), and in a more general setting. Furthermore, we take this chance to correct a few imprecisions in our original work and in some subsequent ones.

Originally, we were motivated by our interest in modelling articular cartilage as a composite comprised of a proteoglycan matrix with spheroidal inclusions, representing the chondrocytes (i.e., cartilage cells) and collagen fibres (Federico et al., 2005). This method was able to predict the elastic behaviour of articular cartilage only for a given type of deformation (i.e., either in compression or in tension). Indeed, the method had been conceived to model "pure" linear elasticity, rather than to capture the tension-compression asymmetry caused by the fact that the collagen fibres bear load when extended, but almost no load when contracted. Such non-linear effect was highlighted by Soltz and Ateshian (2000), who modelled cartilage by means of the *conewise linear elastic model* developed by Curnier et al. (1995), which adopts different elasticity tensors in tension and compression. However, considering the sign of deformation explicitly in the original paper (Federico et al., 2004) and its application to articular cartilage (Federico et al., 2005) would have prevented direct averaging integration of the elasticity tensors over all possible directions. In fact, this difficulty emerged also in our subsequent non-linear works (see Federico, 2015, and references therein). Despite the limitations of our early paper (Federico et al., 2004), its methods have served as the basis for several other projects in our research groups, both in the linear and non-linear settings (see, again, Federico, 2015, for an account of all works in this "family").

2 Theoretical Background

We shall exclusively deal with the theory of small deformations, and therefore we shall make no distinction between reference and current configuration of a deformable body, which we shall simply regard as an open subset $\mathcal{B}$ of the physical space $\mathcal{S}$. Consequently, we shall not make distinction between uppercase and lowercase symbols as typically done in modern Continuum Mechanics (see, e.g., Marsden and Hughes, 1983), and shall exclusively use lowercase indices. However, we

decided to keep the distinction between vector and covector quantities, which is reflected in the distinction between contravariant and covariant indices, respectively. The rationale for this choice is twofold. First, we believe that, even when Cartesian coordinates are used and the difference between vectors and covectors fades under orthogonal transformations (and *exclusively* under orthogonal transformations), it is good practice to keep the distinction, from a didactical point of view. Second, the theory of composite materials has traditionally been developed under the tacit assumption of Cartesian coordinates, and our own past work is no exception; thus, we found appealing the idea of attempting to systematically employ a covariant formalism instead.

We start by presenting the description of the physical space $\mathcal{S}$ as an affine space. Then, we introduce our general notation for tensors and tensor spaces, the metric tensor and contractions of tensors. We continue by introducing the symmetries of second- and fourth-order tensors, and the material symmetries of isotropy and transverse isotropy, along with the corresponding representations of second- and fourth-order tensors satisfying these symmetries. In particular, we present Walpole's formalism for the representation of transversely isotropic fourth-order tensors. Finally, we briefly recall some key relations from the Theory of Linear Elasticity.

2.1 Affine Spaces, Open Subsets and Tangent Spaces

We cannot but agree with, e.g., Marsden and Hughes (1983) or Epstein (2010), when they say that *differentiable manifolds* are the most general and most appropriate setting for the description of Mechanics. However, in many cases, the much simpler structure of *affine space* is sufficient for a reasonably rigorous presentation. An affine space is in fact a *trivial* differentiable manifold (i.e., a differentiable manifold that can be covered by a single chart) and is the minimal structure that allows to develop Differential Calculus and to attach vectors and tensors at any point in space.

An affine space consists of a set $\mathcal{S}$, called the *point space*, a vector space $\mathcal{V}$, called the *modelling* space, and a map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to \mathcal{V}$ that, for every pair of points x, y in $\mathcal{S}$, yields a vector in $\mathcal{V}$ denoted $\mathcal{F}(x, y) = y - x = \boldsymbol{w}$, called the *oriented segment* from x to y. The map $\mathcal{F}$ must be anti-commutative, i.e., $[x - y] = -[y - x]$, and must satisfy the triangle rule, i.e., $y - x = [y - z] + [z - x]$, and the axiom of arbitrary origin, i.e., for every $x \in \mathcal{S}$ and $\boldsymbol{w} \in \mathcal{V}$ there exists one, and only one, $y \in \mathcal{S}$, such that $y - x = \boldsymbol{w}$. At every point $x \in \mathcal{S}$, the set of all vectors emanating from x is defined by

$$T_x\mathcal{S} = \{\boldsymbol{w}_x = y - x : y \in \mathcal{S}\}, \tag{1}$$

where the notation $\boldsymbol{w}_x$ means that "$\boldsymbol{w}$ is attached at x". The space $T_x\mathcal{S}$ is a vector space called *tangent space* of $\mathcal{S}$ at point x. The *tangent bundle* of $\mathcal{S}$, denoted by $T\mathcal{S}$, is defined as the disjoint union of all tangent spaces $T_x\mathcal{S}$ for all $x \in \mathcal{S}$.

The definition of tangent space given in Equation (1), however, applies exclusively to affine spaces and *not* to subsets of an affine space. This is crucial, because deformable bodies are often seen as open subsets $\mathcal{B} \subset \mathcal{S}$. The problem for the case of a subset is that, if $x \in \mathcal{B}$, there exist tangent vectors $\boldsymbol{w}_x \in T_x\,\mathcal{B}$ such that $y = x + \boldsymbol{w}_x$ does *not* belong to $\mathcal{B}$, i.e., the "tip of the arrow" lies outside $\mathcal{B}$ (see the example in Figure 1, right). Therefore, in order to properly define the tangent space $T_x\mathcal{B}$ at a point $x \in \mathcal{B}$, we need to use the definition inherited from Differential Geometry. In this definition, the tangent space $T_x\mathcal{S}$ is the set of all vectors that are tangent at x to all possible regular curves $\Gamma : [a, b] \to \mathcal{S} : s \mapsto \Gamma(s)$ such that $\Gamma(s_0) = x$, with $s_0 \in]a, b[$, i.e., the vectors

$$\boldsymbol{w}_x = \lim_{h \to 0} \frac{\Gamma(s_0 + h) - \Gamma(s_0)}{h} = \Gamma'(s_0) \in T_x\mathcal{S}, \tag{2}$$

in which the numerator of the limit is the difference between point $\Gamma(s_0 + h)$ and $\Gamma(s_0)$, which is a vector *secant* to Γ, and the limit is precisely the tangent at x (Figure 1, left). For the case of an affine space $\mathcal{S}$, this definition of tangent space $T_x\mathcal{S}$ coincides with that in Equation (1). However, although the definition in Equation (1) does not make sense for an open subset $\mathcal{B}$, that in Equation (2) can be automatically inherited by $T_x\mathcal{B}$ just by saying that $x \in \mathcal{B}$ and the curves Γ are such that $\Gamma : [a, b] \to \mathcal{B} : s \mapsto \Gamma(s)$, with $\Gamma(s_0) = x$, and $s_0 \in]a, b[$.

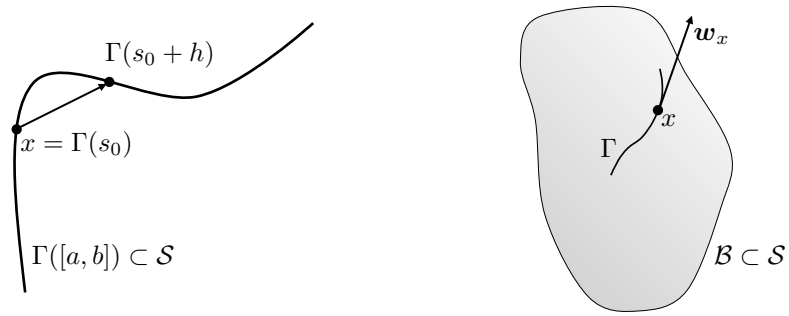

Figure 1: Left: The geometrical definition of tangent vector at a point x in the affine space $\mathcal{S}$ as the tangent at x to a curve passing by x, obtained as the limit of the secant passing by x. Right: A body $\mathcal{B}$ is an open subset of the physical space $\mathcal{S}$, which is considered as an affine space. Considering all regular curves Γ passing by $x \in \mathcal{B}$, the tangent space $T_x\mathcal{B}$ is the set of the tangent vectors $\boldsymbol{w}_x$ that are each tangent at x to one of the curves Γ.

Usually, the affine space of Classical Mechanics is constructed by assuming that both the point space and the modelling space are $\mathbb{R}^3$, and is often denoted by $\mathbb{E}^3$. Here, we shall assume that $\mathcal{S} \equiv \mathbb{E}^3$.

2.2 Tensors

For our purposes, we shall refer to tensors on the tangent bundle $T\mathcal{B}$ of a material body $\mathcal{B}$, but these definitions are completely general and could be used in the tangent bundle $T\mathcal{S}$ of the physical space $\mathcal{S}$ or even in a generic vector space $\mathcal{V}$ of dimension n. Whenever we give examples, we use second- or fourth-order tensors, which are the types of tensors that are relevant in the subsequent sections of this chapter. This section is largely based on a previous work on non-linear elasticity (Federico, 2015), and is adapted to the setting of the small-displacement theory.

A *covector*, or *linear form*, or *one-form* is a linear map

$$\boldsymbol{\varphi} : T\mathcal{B} \to \mathbb{R} : \boldsymbol{u} \mapsto \boldsymbol{\varphi}\,\boldsymbol{u} \equiv \boldsymbol{\varphi}(\boldsymbol{u}), \tag{3}$$

where we use simple juxtaposition to indicate the action of the covector $\boldsymbol{\varphi}$ on the vector $\boldsymbol{u}$. The space of all covectors on the tangent bundle $T\mathcal{B}$ is the *dual* of $T\mathcal{B}$, and is denoted $T^\star\mathcal{B}$ and called *cotangent bundle*. If one looks at a point $x \in \mathcal{B}$, the dual of the tangent space $T_x\mathcal{B}$ is the *cotangent space* $T_x^\star\mathcal{B}$. It is possible to prove that, given a basis $\{\boldsymbol{e}_i\}_{i=1}^3$ in $T\mathcal{B}$, the covectors $\{\boldsymbol{e}^i\}_{i=1}^3$, defined by

$$\boldsymbol{e}^i\,\boldsymbol{u} \equiv \boldsymbol{e}^i(\boldsymbol{u}) = u^i, \tag{4}$$

constitute a basis for the cotangent bundle $T^\star\mathcal{B}$. The basis $\{\boldsymbol{e}^i\}_{i=1}^3$ is called the *dual basis* of $\{\boldsymbol{e}_i\}_{i=1}^3$. Each of the basis covectors $\boldsymbol{e}^i$ has a very precise geometrical meaning, as it associates, with every

vector $\boldsymbol{u}$, the i-th component u^i of $\boldsymbol{u}$ with respect to the vector basis $\{\boldsymbol{e}_i\}_{i=1}^3$ and, for this reason, the basis covectors $\boldsymbol{e}^i$ are often called *projections*. The definition in Equation (4) implies

$$\boldsymbol{e}^i\,\boldsymbol{e}_j \equiv \boldsymbol{e}^i(\boldsymbol{e}_j) = \delta^i{}_j, \tag{5}$$

where $\delta^i{}_j$ is the Kronecker symbol. If, in the definition (3) of covector, we express vector $\boldsymbol{u}$ in the basis $\{\boldsymbol{e}_i\}_{i=1}^3$, we have

$$\boldsymbol{\varphi}\,\boldsymbol{u} = \boldsymbol{\varphi}(u^i\,\boldsymbol{e}_i) = u^i\,\boldsymbol{\varphi}\,\boldsymbol{e}_i = u^i\,\varphi_i \tag{6}$$

where

$$\varphi_i = \boldsymbol{\varphi}\,\boldsymbol{e}_i \equiv \boldsymbol{\varphi}(\boldsymbol{e}_i). \tag{7}$$

Using the definition of basis covector, it follows that the covector $\boldsymbol{\varphi}$ can be decomposed as a linear combination of the basis covectors $\{\boldsymbol{e}^i\}_{i=1}^3$, i.e., as

$$\boldsymbol{\varphi} = \varphi_i\,\boldsymbol{e}^i, \tag{8}$$

where the coefficients φ_i take the meaning of components of $\boldsymbol{\varphi}$ with respect to $\{\boldsymbol{e}^i\}_{i=1}^3$.

In finite dimension, which is the case we are interested in, the relation between a vector space and its dual is symmetric in the sense that $T\mathcal{B}$ can be identified with the bi-dual space $T^{\star\star}\mathcal{B}$ (the set of all linear maps from $T^\star\mathcal{B}$ into $\mathbb{R}$). Therefore, the vectors of $T\mathcal{B}$ can be made to act on the covectors of $T^\star\mathcal{B}$ as linear forms, and the action of a vector $\boldsymbol{u}$ on a covector $\boldsymbol{\varphi}$ is identical of that of $\boldsymbol{\varphi}$ on $\boldsymbol{u}$

$$\boldsymbol{\varphi}\,\boldsymbol{u} = \varphi_i\,u^i = u^i\varphi_i = \boldsymbol{u}\boldsymbol{\varphi}. \tag{9}$$

Consequently, the basis vectors $\{\boldsymbol{e}_i\}_{i=1}^3$ are the projections in $T^\star\mathcal{B}$, i.e.,

$$\boldsymbol{e}_i\,\boldsymbol{\varphi} = \boldsymbol{\varphi}\,\boldsymbol{e}_i \;= \varphi_i. \tag{10}$$

We also say that the expression $\boldsymbol{\varphi}\,\boldsymbol{u} = \boldsymbol{u}\,\boldsymbol{\varphi} = u^i\,\varphi_i$ is the *contraction* of $\boldsymbol{\varphi}$ and $\boldsymbol{u}$.

A *tensor* of order $r+s=m$ on the tangent bundle $T\mathcal{B}$ is a *multilinear form*, i.e., a map of the type

$$\mathbb{T} : \underbrace{T^\star\mathcal{B}\times\ldots\times T^\star\mathcal{B}}_{r\text{ times}}\times\underbrace{T\mathcal{B}\times\ldots\times T\mathcal{B}}_{s\text{ times}}\to\mathbb{R}, \tag{11a}$$

$$\mathbb{T} : (\boldsymbol{\varphi}^1,...\boldsymbol{\varphi}^r,\boldsymbol{u}_1,...\boldsymbol{u}_s)\mapsto\mathbb{T}(\boldsymbol{\varphi}^1,...\boldsymbol{\varphi}^r,\boldsymbol{u}_1,...\boldsymbol{u}_s), \tag{11b}$$

that is linear in each of the $r+s$ arguments separately. The space of all tensors of the type in Equation (11) is denoted $[T\mathcal{B}]^r{}_s$, a notation that will be justified later on (Equation (17)).

The *tensor product* of the r vectors $\boldsymbol{v}_1,...,\boldsymbol{v}_r$ in $T\mathcal{B}$ and the s covectors $\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^s$ in $T^\star\mathcal{B}$ is the tensor $\boldsymbol{v}_1\otimes...\otimes\boldsymbol{v}_r\otimes\boldsymbol{\psi}^1\otimes...\otimes\boldsymbol{\psi}^s$ in $[T\mathcal{B}]^r{}_s$ such that, for every $\boldsymbol{\varphi}^1,...,\boldsymbol{\varphi}^r$ in $T^\star\mathcal{B}$ and for every $\boldsymbol{u}_1,...,\boldsymbol{u}_s$ in $T\mathcal{B}$,

$$[\boldsymbol{v}_1\otimes...\otimes\boldsymbol{v}_r\otimes\boldsymbol{\psi}^1\otimes...\otimes\boldsymbol{\psi}^s](\boldsymbol{\varphi}^1,...,\boldsymbol{\varphi}^r,\boldsymbol{u}_1,...,\boldsymbol{u}_s) = \boldsymbol{v}_1(\boldsymbol{\varphi}^1)\;...\;\boldsymbol{v}_r(\boldsymbol{\varphi}^r)\;\boldsymbol{\psi}^1(\boldsymbol{u}_1)\;...\;\boldsymbol{\psi}^s(\boldsymbol{u}_s). \tag{12}$$

The tensor $\boldsymbol{v}_1\otimes...\otimes\boldsymbol{v}_r\otimes\boldsymbol{\psi}^1\otimes...\otimes\boldsymbol{\psi}^s$ is said to have r *vector legs* $\boldsymbol{v}_1,...,\boldsymbol{v}_r$ and s *covector legs* $\boldsymbol{\psi}^1,...,\boldsymbol{\psi}^s$.

With the definition of tensor product of vectors and covectors, and using multilinearity, we can derive the component expression of any tensor $\mathbb{T}$ in $[T\mathcal{B}]^r{}_s$ with respect to a given basis $\{\boldsymbol{e}_i\}_{i=1}^n$ and dual basis $\{\boldsymbol{e}^i\}_{i=1}^n$. Indeed, we have

$$\begin{aligned}\mathbb{T}(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^r, \boldsymbol{u}_1, ..., \boldsymbol{u}_s) &= \mathbb{T}(\varphi^1_{i_1}\, \boldsymbol{e}^{i_1}, ..., \varphi^r_{i_r}\, \boldsymbol{e}^{i_r}, u_1^{j_1}\, \boldsymbol{e}_{j_1}, ..., u_s^{j_s}\, \boldsymbol{e}_{j_s}) \\ &= \varphi^1_{i_1} ... \varphi^r_{i_r}\, u_1^{j_1} ... u_s^{j_s}\, \mathbb{T}(\boldsymbol{e}^{i_1}, ..., \boldsymbol{e}^{i_r}, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_s}) \\ &= \varphi^1_{i_1} ... \varphi^r_{i_r}\, u_1^{j_1} ... u_s^{j_s}\, \mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s}, \end{aligned} \tag{13}$$

where

$$\mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s} = \mathbb{T}(\boldsymbol{e}^{i_1}, ..., \boldsymbol{e}^{i_r}, \boldsymbol{e}_{j_1}, ..., \boldsymbol{e}_{j_s}) \tag{14}$$

are the components of $\mathbb{T}$. By analogy with the indices of vectors and covectors, the indices $i_1 \dots i_r$ are called *contravariant* and the indices $j_1 \dots j_s$ are called *covariant*. Using Equations (4), (10) and (12), we obtain

$$\begin{aligned}&\mathbb{T}(\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^r, \boldsymbol{u}_1, ..., \boldsymbol{u}_s) \\ &= \varphi^1_{i_1} ... \varphi^r_{i_r}\, u_1^{j_1} ... u_s^{j_s}\, \mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s} \\ &= \boldsymbol{e}_{i_1}(\boldsymbol{\varphi}^1) ... \boldsymbol{e}_{i_r}(\boldsymbol{\varphi}^r)\, \boldsymbol{e}^{j_1}(\boldsymbol{u}_1) ... \boldsymbol{e}^{j_s}(\boldsymbol{u}_s)\, \mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s} \\ &= \mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s}\, [\boldsymbol{e}_{i_1} \otimes ... \otimes \boldsymbol{e}_{i_r} \otimes \boldsymbol{e}^{j_1} \otimes ... \otimes \boldsymbol{e}^{j_s}](\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^r, \boldsymbol{u}_1, ..., \boldsymbol{u}_s). \end{aligned} \tag{15}$$

By dropping the arguments $\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^r, \boldsymbol{u}_1, ..., \boldsymbol{u}_s$ on both sides of Equation (15), we obtain the component representation of $\mathbb{T}$ in the *tensor basis* $\{\boldsymbol{e}_{i_1} \otimes ... \otimes \boldsymbol{e}_{i_r} \otimes \boldsymbol{e}^{j_1} \otimes ... \otimes \boldsymbol{e}^{j_s}\}_{i_1,\dots,i_r,j_1,\dots j_s=1}^n$ of the tensor space $[T\mathcal{B}]^r{}_s$ as

$$\mathbb{T} = \mathrm{T}^{i_1 \dots i_r}{}_{j_1 \dots j_s}\, \boldsymbol{e}_{i_1} \otimes ... \otimes \boldsymbol{e}_{i_r} \otimes \boldsymbol{e}^{j_1} \otimes ... \otimes \boldsymbol{e}^{j_s}. \tag{16}$$

A tensor $\mathbb{T}$ in $[T\mathcal{B}]^r{}_s$ is said to have r vector legs and s covector legs, and $[T\mathcal{B}]^r{}_s$ can be represented as the *tensor product of spaces* (Bishop and Goldberg, 1968)

$$[T\mathcal{B}]^r{}_s = \underbrace{T\mathcal{B} \otimes \ldots \otimes T\mathcal{B}}_{r \text{ times}} \otimes \underbrace{T^\star\mathcal{B} \otimes \ldots \otimes T^\star\mathcal{B}}_{s \text{ times}}. \tag{17}$$

With a widely accepted abuse of terminology, we shall often refer to a tensor in $[T\mathcal{B}]^r{}_s$ as being "r times contravariant and s times covariant" although, rigorously speaking, the adjectives contravariant and covariant refer to tensor *indices* and tensor *components*.

Since a tensor leg can be a vector or a covector, there are 2^m possible spaces of tensors of order m. For instance, there is only one type of space of zero-order tensors (scalars in $[T\mathcal{B}]^0_0 \equiv \mathbb{R}$), two types of spaces of first-order tensors (vectors in $[T\mathcal{B}]^1_0 \equiv T\mathcal{B}$ and covectors in $[T\mathcal{B}]^0_1 \equiv T^\star\mathcal{B}$), 4 types of spaces of second-order tensors, 16 types of spaces of fourth-order tensors. The table below summarises the situation and reports some examples of the 16 types of fourth-order tensors. Here, we shall exclusively deal with the first four types of fourth-order tensors.

Order	Types	Spaces	Components	Notes
0	2^0	$[T\mathcal{B}]^0_0 \equiv \mathbb{R}$	a	scalars
1	2^1	$[T\mathcal{B}]^1_0 \equiv T\mathcal{B}$	a^i	vectors
		$[T\mathcal{B}]^0_1 \equiv T^\star\mathcal{B}$	a_i	covectors
2	2^2	$[T\mathcal{B}]^2_0$	a^{ij}	"contravariant"
		$[T\mathcal{B}]^0_2$	a_{ij}	"covariant"
		$[T\mathcal{B}]^1{}_1$	$a^i{}_j$	"mixed"
		$[T\mathcal{B}]_1{}^1$	$a_i{}^j$	"mixed"
4	2^4	$[T\mathcal{B}]^4_0$	A^{ijkl}	"contravariant"
		$[T\mathcal{B}]^0_4$	A_{ijkl}	"covariant"
		$[T\mathcal{B}]^2{}_2$	$\mathrm{A}^{ij}{}_{kl}$	"mixed"
		$[T\mathcal{B}]_2{}^2$	$\mathrm{A}_{ij}{}^{kl}$	"mixed"
		...	...	
		$[T\mathcal{B}]^1{}_1{}^1{}_1$	$\mathrm{A}^i{}_j{}^k{}_l$	
		$[T\mathcal{B}]_1{}^1{}_1{}^1$	$\mathrm{A}_i{}^j{}_k{}^l$	
		...	...	

2.3 Tensor Contractions and Tensor as Linear Maps

So far, we have seen tensors as multilinear maps, whose legs are all contracted at the same time with vectors or covector arguments, as appropriate. However, one could contract part of the legs of a tensor with all or part of the legs of another tensor. In this work, we are going to see single and double contractions.

Given a tensor whose last leg is a vector and another tensor whose first leg is a covector, or vice versa, we call *single contraction* the contraction of the last leg of the first tensor with the first leg of the second tensor, and denote it by simple juxtaposition. For instance, for a "contravariant" second-order tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$ and a "covariant" second-order tensor $\boldsymbol{c}$ in $[T\mathcal{B}]^0_2$, the contraction $\boldsymbol{a}\,\boldsymbol{c}$ has components $a^{ij}c_{jk}$. The same type of contraction occurs between, e.g., a "mixed" tensor $\boldsymbol{l}$ in $[T\mathcal{B}]^1{}_1$ and a vector $\boldsymbol{u}$ in $T\mathcal{B}$, and the single contraction is the usual $\boldsymbol{l}\,\boldsymbol{u}$ with components $l^i{}_j\,u^j$.

The double contraction of two tensors works similarly to the simple contraction, except that one contracts the last two legs of the first tensor and the first two legs of the second tensor. As with the single contraction, the contracting legs must be of opposite type. Double contraction is denoted by a colon. For example, for a fourth-order tensor $\mathbb{T}$ in $[T\mathcal{B}]^2{}_2$ and a second-order tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$, the double contraction $\mathbb{T} : \boldsymbol{a}$ has components $\mathrm{T}^{ij}{}_{kl}a^{kl}$.

Tensors can also be regarded as linear maps between tensor spaces. For instance, a "mixed" second-order tensor, i.e., a tensor $\boldsymbol{l}$ in $[T\mathcal{B}]^1{}_1$ could be seen as the linear map $\boldsymbol{l} : T\mathcal{B} \to T\mathcal{B} : \boldsymbol{u} \mapsto \boldsymbol{l}\,\boldsymbol{u}$ (in components, $l^i{}_j\,u^j$). Similarly, a "contravariant" fourth-order tensor $\mathbb{T}$ in $[T\mathcal{B}]^4_0$ could be regarded as the linear map $\mathbb{T} : [T\mathcal{B}]^0_2 \to [T\mathcal{B}]^2_0 : \boldsymbol{c} \mapsto \mathbb{T} : \boldsymbol{c}$ (in components, $\mathrm{T}^{ijkl}c_{kl}$). Rigorously speaking, a tensor seen as a linear map between two tensor spaces should be somehow notationally distinguished from its multilinear form counterpart. However, since context and, above all, index notation prevent any possible confusion, the customary practice is to use the same symbol $\mathbb{T}$ for the tensor employed in both manners.

2.4 Metric Tensor and Scalar Products

The physical space $\mathcal{S}$ is assumed to be equipped with a metric tensor $\boldsymbol{g}$, which is inherited by the body $\mathcal{B}$. A metric tensor is a symmetric and positive definite tensor in $[T\mathcal{B}]^0_2$, such that, for every pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $T\mathcal{B}$,

$$\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) \equiv \boldsymbol{u}\,\boldsymbol{g}\,\boldsymbol{v} \equiv \langle \boldsymbol{u},\boldsymbol{v}\rangle \equiv \boldsymbol{u}.\boldsymbol{v} = u^i\, g_{ij}\, v^j. \tag{18}$$

Symmetry means that $\langle \boldsymbol{u},\boldsymbol{v}\rangle = \langle \boldsymbol{v},\boldsymbol{u}\rangle$, and positive definiteness means that, for every $\boldsymbol{u} \neq \boldsymbol{0}$, $\langle \boldsymbol{u},\boldsymbol{u}\rangle > 0$. The equivalent notations $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) \equiv \boldsymbol{u}\,\boldsymbol{g}\,\boldsymbol{v} \equiv \langle \boldsymbol{u},\boldsymbol{v}\rangle \equiv \boldsymbol{u}.\boldsymbol{v}$ denote the *scalar product* of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The metric tensor induces the *Euclidean norm* $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{g}(\boldsymbol{u},\boldsymbol{u})} \equiv \sqrt{\langle \boldsymbol{u},\boldsymbol{u}\rangle}$. A basis $\{\boldsymbol{e}_i\}_{i=1}^3$ is called *orthonormal* with respect to $\boldsymbol{g}$ if $\boldsymbol{g}(\boldsymbol{e}_i,\boldsymbol{e}_j) \equiv \langle \boldsymbol{e}_i,\boldsymbol{e}_j\rangle = \delta_{ij}$, i.e., if the matrix representation of the metric tensor is the identity. Positive definiteness of $\boldsymbol{g}$ also implies invertibility, and the inverse is the tensor $\boldsymbol{g}^{-1}$ valued in $[T\mathcal{B}]^2_0$ such that, for every pair of covectors $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ in $T^\star\mathcal{B}$,

$$\boldsymbol{g}^{-1}(\boldsymbol{\varphi},\boldsymbol{\psi}) \equiv \boldsymbol{\varphi}\,\boldsymbol{g}^{-1}\,\boldsymbol{\psi} \equiv \langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle \equiv \boldsymbol{\varphi}.\boldsymbol{\psi} = \varphi_i\,(\boldsymbol{g}^{-1})^{ij}\,\psi_j. \tag{19}$$

When considered as a linear map $\boldsymbol{g} : T\mathcal{B} \to T^\star\mathcal{B}$, the metric tensor $\boldsymbol{g}$ is said to be used to "lower contravariant indices", by mapping the vector $\boldsymbol{u}$ into the associated covector $\boldsymbol{u}^\flat = \boldsymbol{g}\,\boldsymbol{u}$, with components $u_i = g_{ij}u^j$. Analogously, the inverse metric tensor $\boldsymbol{g}^{-1}$, seen as the linear map $\boldsymbol{g}^{-1} : T^\star\mathcal{B} \to T\mathcal{B}$, is said to "raise covariant indices", by mapping the covector $\boldsymbol{\varphi}$ into the associated vector $\boldsymbol{\varphi}^\sharp = \boldsymbol{g}^{-1}\boldsymbol{\varphi}$, with components $\varphi^i = (\boldsymbol{g}^{-1})^{ij}\varphi_j$. The metric tensor and its inverse can be used to lower and rise, respectively, the indices of tensors of any order. For instance, given the "contravariant" fourth-order tensor $\mathbb{T}$ in $[T\mathcal{B}]^4_0$, its "covariant" associated tensor is denoted $\mathbb{T}^\flat$ and has components $\mathrm{T}_{ijkl} = g_{ip}\,g_{jq}\,g_{kr}\,g_{ls}\,\mathrm{T}^{pqrs}$. In particular, if we raise the indices of the metric tensor $\boldsymbol{g}$ itself by means of the inverse metric tensor $\boldsymbol{g}^{-1}$, we have the important identity

$$\boldsymbol{g}^\sharp = \boldsymbol{g}^{-1}\boldsymbol{g}\,\boldsymbol{g}^{-1} = \boldsymbol{g}^{-1}, \qquad g^{il} = (\boldsymbol{g}^{-1})^{ij}\,g_{jk}\,(\boldsymbol{g}^{-1})^{kl} = (\boldsymbol{g}^{-1})^{il}. \tag{20}$$

The scalar product induced by the metric tensor $\boldsymbol{g}$ can be extended to pairs of tensors of the same type of any order, by contracting each pair of homologous indices by means of the metric tensor or its inverse, as appropriate. For instance, given two tensors $\mathbb{T}$ and $\mathbb{Z}$ in $[T\mathcal{B}]^1{}_1{}^1{}_1$, their scalar product is $\langle \mathbb{T},\mathbb{Z}\rangle = \mathrm{T}^i{}_j{}^k{}_l\,g_{ip}\,g^{jq}\,g_{kr}\,g^{ls}\,\mathrm{Z}^p{}_q{}^r{}_s$.

Finally, we use a single low dot to indicate that the metric tensor (or its inverse) is involved in the contraction of two tensors such that the last leg of the first tensor and the first leg of the second tensor are of the same type. For instance, given two "contravariant" tensors $\boldsymbol{a},\boldsymbol{b}$ in $[T\mathcal{B}]^2_0$, the expression $\boldsymbol{a}.\boldsymbol{b}$ stands for $\boldsymbol{a}\,\boldsymbol{g}\,\boldsymbol{b}$, which has components $(\boldsymbol{a}.\boldsymbol{b})^{il} = a^{ij}\,g_{jk}\,b^{kl}$.

2.5 Symmetries of Second- and Fourth-Order Tensors

For a "covariant" second-order tensor $\boldsymbol{c}$ in $[T\mathcal{B}]^0_2$, the *transpose* is defined as the tensor $\boldsymbol{c}^T$ in $[T\mathcal{B}]^0_2$ such that, for every pair of vectors $\boldsymbol{u},\boldsymbol{v}$ in $T\mathcal{B}$, $\boldsymbol{u}\,\boldsymbol{c}\,\boldsymbol{v} = \boldsymbol{v}\,\boldsymbol{c}^T\boldsymbol{u}$, which in components reads $u^i\,c_{ij}\,v^j = v^j\,(\boldsymbol{c}^T)_{ji}\,u^i$, implying $(\boldsymbol{c}^T)_{ji} = c_{ij}$. The transpose of a "contravariant" second-order tensor is defined analogously. For the case of a "mixed" tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^1{}_1$, the transpose is the tensor $\boldsymbol{a}^T$ in $[T\mathcal{B}]_1{}^1$ such that, for every vector $\boldsymbol{v}$ in $T\mathcal{B}$ and every covector $\boldsymbol{\varphi}$ in $T^\star\mathcal{B}$, $\boldsymbol{\varphi}\,\boldsymbol{a}\,\boldsymbol{v} = \boldsymbol{v}\,\boldsymbol{a}^T\boldsymbol{\varphi}$, which in components reads $\varphi_i\,a^i{}_j\,v^j = v_j\,(\boldsymbol{a}^T)_j{}^i\,u^i$, implying $(\boldsymbol{a}^T)_j{}^i = a^i{}_j$. Note that, while a "covariant" and its transpose, or a "contravariant" tensor and its transpose, belong to the *same* space, a "mixed" tensor and its transpose belong to *different* spaces.

A "covariant" second-order tensor $\boldsymbol{c}$ in $[T\mathcal{B}]^0_2$ is called symmetric if $\boldsymbol{c} = \boldsymbol{c}^T$, which, in components, means $c_{ij} = c_{ji}$. The symmetry of a "contravariant" tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$ is defined analogously. For the case of a "mixed" tensor, speaking about equality of the tensor and its transpose has no meaning, as they belong to *different* spaces. Thus, symmetry of a "mixed" tensor $\boldsymbol{l}$ in $[T\mathcal{B}]^1{}_1$ is defined in terms of the symmetry of its "covariant" counterpart $\boldsymbol{l}^\flat = \boldsymbol{g}\,\boldsymbol{l}$ or, equivalently, in terms of the symmetry of its "contravariant" counterpart $\boldsymbol{l}^\sharp = \boldsymbol{l}\,\boldsymbol{g}^{-1}$; indeed, we have that $\boldsymbol{l}^\sharp$ is symmetric if, and only if $\boldsymbol{l}^\flat$ is such.

When a fourth-order tensor is viewed as a linear map between spaces of second-order tensors, its transpose can be defined in a way similar to that of a second-order tensor. For the purposes of our presentation, let us restrict our attention to "covariant" tensors in $[T\mathcal{B}]^0_4$, "contravariant" tensors in $[T\mathcal{B}]^4_0$, and "mixed" tensors in $[T\mathcal{B}]^2{}_2$ and $[T\mathcal{B}]_2{}^2$ (actually, the transpose of fourth-order tensors of any other type is defined exactly in the same way, but we do not need these tensors here). For instance, the transpose of $\mathbb{A}$ in $[T\mathcal{B}]^4_0$ is defined as the tensor $\mathbb{A}^T$ in $[T\mathcal{B}]^4_0$ such that, for every $\boldsymbol{c}, \boldsymbol{d}$ in $[T\mathcal{B}]^0_2$, the identity $\boldsymbol{c} : \mathbb{A} : \boldsymbol{d} = \boldsymbol{d} : \mathbb{A}^T : \boldsymbol{c}$ holds. In components, this reads $c_{ij}\,\mathrm{A}^{ijkl}\,d_{kl} = d_{kl}\,[\mathbb{A}^T]^{klij}\,c_{ij}$, i.e., $[\mathbb{A}^T]^{klij} = \mathrm{A}^{ijkl}$.

Fourth-order tensors admit a variety of symmetries. Here we are interested in those called *major* and *minor* symmetry. A "contravariant" fourth-order tensor $\mathbb{A}$ in $[T\mathcal{B}]^4_0$ is said to have *major symmetry* (or *diagonal symmetry*) if $\mathbb{A}^T = \mathbb{A}$. The same definition holds for a "covariant" tensor $\mathbb{B}$ in $[T\mathcal{B}]^0_4$. The case of "mixed" tensors is of course a little more complicated. We are interested in the case of a "mixed" tensor $\mathbb{T}$ in $[T\mathcal{B}]^2{}_2$ or in $[T\mathcal{B}]_2{}^2$. The major symmetry of such tensor is checked by looking at the major symmetry of either its "contravariant" counterpart $\mathbb{T}^\sharp$ or of its "covariant" counterpart $\mathbb{T}^\flat$, as $\mathbb{T}^\sharp$ is major-symmetric if, and only if, $\mathbb{T}^\flat$ is major-symmetric. Another important symmetry of fourth-order tensors is called *minor symmetry* (or *pair symmetry*), and it is straightforward to define for tensors in which the two legs (indices) within the first pair and within the second pair are of the same type, i.e., for tensors in $[T\mathcal{B}]^4_0$, $[T\mathcal{B}]^0_4$, $[T\mathcal{B}]^2{}_2$ and $[T\mathcal{B}]_2{}^2$. For instance, a tensor $\mathbb{A}$ in $[T\mathcal{B}]^4_0$ is said to possess minor symmetry on the first pair of legs (indices) if, for every $\boldsymbol{c}$ in $[T\mathcal{B}]^0_2$, one has $\boldsymbol{c} : \mathbb{A} = \boldsymbol{c}^T : \mathbb{A}$, and on the second pair of legs (indices) if $\mathbb{A} : \boldsymbol{c} = \mathbb{A} : \boldsymbol{c}^T$. In components, these symmetries read $\mathrm{A}^{ijkl} = \mathrm{A}^{jikl}$ and $\mathrm{A}^{ijkl} = \mathrm{A}^{ijlk}$, respectively. If a tensor enjoys minor symmetry on both the first and the second pair of legs, we simply say that it "enjoys minor symmetry". When there is no danger of confusion, we say that a fourth-order tensor is *symmetric* if it enjoys both major and minor symmetry.[1]

2.6 Isotropic Second- and Fourth-Order Tensors

Isotropy is the invariance of a material property under any arbitrary rotation. A "mixed" second-order tensor $\boldsymbol{l}$ in $[T\mathcal{B}]^1{}_1$ is isotropic if, and only if, it is proportional to the identity tensor $\boldsymbol{i}$, i.e., if $\boldsymbol{l} = l\,\boldsymbol{i}$ (in components, $l^i{}_j = l\,\delta^i{}_j$). A "contravariant" tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$ is said to be isotropic if the associated "mixed" tensor $\boldsymbol{a}\,\boldsymbol{g}$ is isotropic, which implies that $\boldsymbol{a} = a\,\boldsymbol{g}^{-1}$ (i.e., $a^{ij} = a\,g^{ij}$). Similarly, a "covariant" tensor $\boldsymbol{c}$ in $[T\mathcal{B}]^0_2$ is said to be isotropic if such is the associated "mixed" tensor $\boldsymbol{g}^{-1}\boldsymbol{c}$, from which $\boldsymbol{c} = c\,\boldsymbol{g}$ (i.e., $c_{ij} = c\,g_{ij}$). We remark that, as a consequence of the definition of isotropic second-order tensor, it follows that any isotropic second-order tensor is symmetric.

Let us consider the subspace $([T\mathcal{B}]^2{}_2, \mathrm{Sym})$ of $[T\mathcal{B}]^2{}_2$ of all tensors with major and minor symmetry. Since tensors in $[T\mathcal{B}]^2{}_2$ are "mixed", major symmetry of a tensor $\mathbb{T}$ is understood in the

[1] In our past works, we have called a fourth-order tensor with both major and minor (diagonal- and pair-) symmetry "fully symmetric", but we are not going to use this nomenclature here, as it can be confusing. Indeed, what is normally called fully or completely symmetric is a tensor that is invariant under *any* permutation of the indices.

sense of Section 2.5, i.e., in relation to the symmetry of the "contravariant" counterpart $\mathbb{T}^\sharp$ or of the symmetry of the "covariant" counterpart $\mathbb{T}^\flat$. The *symmetric identity* in $([T\mathcal{B}]^2{}_2, \mathrm{Sym})$ is defined with the help of the special tensor products $\underline{\otimes}$ and $\overline{\otimes}$ introduced by Curnier et al. (1995) as

$$\mathbb{I} = \tfrac{1}{2}(\boldsymbol{i} \underline{\otimes} \boldsymbol{i} + \boldsymbol{i} \overline{\otimes} \boldsymbol{i}), \qquad \mathrm{I}^{ij}{}_{kl} = \tfrac{1}{2}(\delta^i{}_k \delta^j{}_l + \delta^i{}_l \delta^j{}_k), \tag{21}$$

where $\boldsymbol{i}$, with components $\delta^i{}_j$, is the identity second-order tensor in $[T\mathcal{B}]^1{}_1$. Since $\mathbb{I}$ is the identity, it is invariant under rotations and is therefore clearly isotropic. The symmetric identity is such that, for every *symmetric* second-order tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$, $\mathbb{I} : \boldsymbol{a} = \boldsymbol{a}$.

The tensor basis of the subspace $([T\mathcal{B}]^2{}_2, \mathrm{Sym}, \mathrm{Iso})$ of the symmetric *and* isotropic tensors is found by decomposing the symmetric identity into (Walpole, 1981, 1984; Federico, 2012)

$$\mathbb{I} = \mathbb{K} + \mathbb{M}, \tag{22}$$

where

$$\mathbb{K} = \tfrac{1}{3}\, \boldsymbol{g}^{-1} \otimes \boldsymbol{g}, \qquad \mathrm{K}^{ij}{}_{kl} = \tfrac{1}{3}\, g^{ij} g_{kl}, \tag{23a}$$

$$\mathbb{M} = \mathbb{I} - \mathbb{K}, \qquad \mathrm{M}^{ij}{}_{kl} = \tfrac{1}{2}(\delta^i{}_k \delta^j{}_l + \delta^i{}_l \delta^j{}_k) - \tfrac{1}{3}\, g^{ij} g_{kl}, \tag{23b}$$

are the *spherical operator* and the *deviatoric operator*, such that, for every symmetric tensor $\boldsymbol{a}$ in $[T\mathcal{B}]^2_0$, $\mathbb{K} : \boldsymbol{a} = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{a})\, \boldsymbol{g}^{-1}$ is the spherical part of $\boldsymbol{a}$, and $\mathbb{M} : \boldsymbol{a} = \boldsymbol{a} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{a})\, \boldsymbol{g}^{-1}$ is the deviatoric part of $\boldsymbol{a}$, where $\mathrm{tr}(\,\cdot\,)$ is the natural trace operator, such that $\mathrm{tr}(\boldsymbol{a}) = \boldsymbol{g} : \boldsymbol{a} = g_{ij}\, a^{ij}$. The tensors $\{\mathbb{K}, \mathbb{M}\}$ constitute the basis of the space $([T\mathcal{B}]^2{}_2, \mathrm{Sym}, \mathrm{Iso})$ of the symmetric and isotropic tensors. We remark that *all* isotropic fourth-order tensors enjoy minor symmetry (Jog, 2006), and that there exist isotropic fourth-order tensors which do *not* enjoy major symmetry (the additional basis tensor is the *skew-symmetriser* $\mathbb{W} = \frac{1}{2}(\boldsymbol{i} \underline{\otimes} \boldsymbol{i} - \boldsymbol{i} \overline{\otimes} \boldsymbol{i})$; see Jog, 2006).

The bases of the spaces $([T\mathcal{B}]^4_0, \mathrm{Sym}, \mathrm{Iso})$ and $([T\mathcal{B}]^0_4, \mathrm{Sym}, \mathrm{Iso})$ are obtained by raising and lowering, respectively, the indices of $\{\mathbb{K}, \mathbb{M}\}$, or by decomposing the "contravariant" symmetric identity $\mathbb{I}^\sharp$ and the "covariant" symmetric identity $\mathbb{I}^\flat$, respectively. The resulting tensors are (Federico, 2012)

$$\mathbb{I}^\sharp = \tfrac{1}{2}(\boldsymbol{g}^{-1} \underline{\otimes} \boldsymbol{g}^{-1} + \boldsymbol{g}^{-1} \overline{\otimes} \boldsymbol{g}^{-1}), \quad \mathrm{I}^{ijkl} = \tfrac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}), \tag{24a}$$

$$\mathbb{K}^\sharp = \tfrac{1}{3}\, \boldsymbol{g}^{-1} \otimes \boldsymbol{g}^{-1}, \qquad \mathrm{K}^{ijkl} = \tfrac{1}{3}\, g^{ij} g^{kl}, \tag{24b}$$

$$\mathbb{M}^\sharp = \mathbb{I}^\sharp - \mathbb{K}^\sharp, \qquad \mathrm{M}^{ijkl} = \tfrac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \tfrac{1}{3}\, g^{ij} g^{kl}, \tag{24c}$$

and

$$\mathbb{I}^\flat = \tfrac{1}{2}(\boldsymbol{g} \underline{\otimes} \boldsymbol{g} + \boldsymbol{g} \overline{\otimes} \boldsymbol{g}), \qquad \mathrm{I}_{ijkl} = \tfrac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}), \tag{25a}$$

$$\mathbb{K}^\flat = \tfrac{1}{3}\, \boldsymbol{g} \otimes \boldsymbol{g}, \qquad \mathrm{K}_{ijkl} = \tfrac{1}{3}\, g_{ij} g_{kl}, \tag{25b}$$

$$\mathbb{M}^\flat = \mathbb{I}^\flat - \mathbb{K}^\flat, \qquad \mathrm{M}_{ijkl} = \tfrac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}) - \tfrac{1}{3}\, g_{ij} g_{kl}. \tag{25c}$$

The tensors $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ and $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$ constitute the bases of the spaces $([T\mathcal{B}]^4_0, \mathrm{Sym}, \mathrm{Iso})$ and $([T\mathcal{B}]^0_4, \mathrm{Sym}, \mathrm{Iso})$, respectively. It is important to recall how to obtain the representation of a symmetric isotropic tensor, and we show this in the case that is most important for our purposes, i.e., that of a "contravariant" tensor. A symmetric isotropic tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \mathrm{Sym}, \mathrm{Iso})$ can be shown to admit the representation (Walpole, 1981, 1984)

$$\mathbb{T} = \langle \mathbb{T}, \mathbb{K}^\sharp \rangle\, \mathbb{K}^\sharp + \tfrac{1}{5} \langle \mathbb{T}, \mathbb{M}^\sharp \rangle\, \mathbb{M}^\sharp, \tag{26}$$

where

$$\langle \mathbb{T}, \mathbb{K}^\sharp \rangle = \mathrm{T}^{ijkl}\, g_{ip}\, g_{jq}\, g_{kr}\, g_{ls} \left(\tfrac{1}{3}\, g^{pq}\, g^{rs} \right) = \tfrac{1}{3}\, \mathrm{T}^{i}{}_{i}{}^{k}{}_{k}, \tag{27a}$$

$$\langle \mathbb{T}, \mathbb{M}^\sharp \rangle = \mathrm{T}^{ijkl}\, g_{ip}\, g_{jq}\, g_{kr}\, g_{ls} \left(\tfrac{1}{2}(g^{pr}\, g^{qs} + g^{ps}\, g^{qr}) - \tfrac{1}{3}\, g^{pq}\, g^{rs} \right) = \mathrm{T}^{ij}{}_{ij} - \tfrac{1}{3}\, \mathrm{T}^{i}{}_{i}{}^{k}{}_{k}. \tag{27b}$$

Note that, in the second of Equations (27), we obtain a term $\frac{1}{2}(\mathrm{T}^{ij}{}_{ij} + \mathrm{T}^{ij}{}_{ji})$, which reduces to $\mathrm{T}^{ij}{}_{ij}$ because of the minor symmetry of $\mathbb{T}$. We remark that, if $\mathbb{T}$ is a generic, not necessarily symmetric and isotropic tensor in $[T\mathcal{B}]^4_0$, the right-hand side of Equation (26) yields the *projection* of $\mathbb{T}$ onto the isotropic subspace $([T\mathcal{B}]^4_0, \mathrm{Sym}, \mathrm{Iso})$ of $[T\mathcal{B}]^4_0$, i.e.,

$$\mathbb{T}_{\mathrm{iso}} = \langle \mathbb{T}, \mathbb{K}^\sharp \rangle\, \mathbb{K}^\sharp + \tfrac{1}{5} \langle \mathbb{T}, \mathbb{M}^\sharp \rangle\, \mathbb{M}^\sharp \neq \mathbb{T}, \tag{28}$$

where the coefficients on the right-hand side are found in precisely the same way as in Equation (27).

A thorough analysis of the properties of *idempotence* and *orthogonality* (Walpole, 1981, 1984) enjoyed by the tensors of each of the bases $\{\mathbb{K}, \mathbb{M}\}$, $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ and $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$ is discussed in a previous work (Federico, 2012), in the same covariant formalism used here. The idempotence and orthogonality of isotropic basis tensors implies that multiplication and inversion of isotropic tensors are performed by multiplying and inverting the individual scalars of the decomposition (26).

2.7 Transversely Isotropic Second- and Fourth-Order Tensors

The set

$$\mathbb{S}^2\mathcal{B} = \{\boldsymbol{m} \in T\mathcal{B} : \|\boldsymbol{m}\| = 1\}, \tag{29}$$

where $\|\boldsymbol{m}\| = \sqrt{\boldsymbol{m}.\boldsymbol{m}}$ is the Euclidean norm of vector $\boldsymbol{m}$, is the subset of all *unit* vectors in the tangent bundle $T\mathcal{B}$, and is called the (bundle) *unit sphere* in the body $\mathcal{B}$. When the point x is fixed, one speaks about the *unit sphere* $\mathbb{S}^2_x\mathcal{B} = \{\boldsymbol{m} \in T_x\mathcal{B} : \|\boldsymbol{m}\| = 1\}$ at x. Transverse isotropy with respect to $\boldsymbol{m}$ is defined as the symmetry (i.e., the invariance) with respect to rotations about $\boldsymbol{m}$. The direction identified by $\boldsymbol{m}$ is called *symmetry axis* and the class of equivalence of the planes orthogonal to $\boldsymbol{m}$ is called *transverse plane*.

The subspace of $[T\mathcal{B}]^2_0$ of all second-order "contravariant" symmetric tensors with transverse isotropy with respect to a direction $\boldsymbol{m}$ is denoted $([T\mathcal{B}]^2_0, \boldsymbol{m})$. The basis of $([T\mathcal{B}]^2_0, \boldsymbol{m})$ is given by (Walpole, 1981, 1984; Federico, 2012)

$$\boldsymbol{a} = \boldsymbol{m} \otimes \boldsymbol{m}, \tag{30a}$$

$$\boldsymbol{t} = \boldsymbol{g}^{-1} - \boldsymbol{a}, \tag{30b}$$

where $\boldsymbol{t}$ is the complement of tensor $\boldsymbol{a}$ to $\boldsymbol{g}^{-1}$, which serves as the "contravariant identity" in $[T\mathcal{B}]^2_0$. Evidently, both $\boldsymbol{a}$ and $\boldsymbol{t}$ are invariant under reflections $\boldsymbol{m} \mapsto -\boldsymbol{m}$, i.e., the sense of $\boldsymbol{m}$ is irrelevant. Tensors $\boldsymbol{a}$ and $\boldsymbol{t}$ take the geometrical meaning of *axial projection operator* and *transverse projection operator*, respectively. Indeed, contraction of $\boldsymbol{a}$ and $\boldsymbol{t}$, by means of the metric tensor $\boldsymbol{g}$, with a vector $\boldsymbol{v}$ in $T\mathcal{B}$ yields the axial and transverse *vectorial* components of $\boldsymbol{v}$, respectively, as[2]

$$\boldsymbol{v}_{\parallel} = \boldsymbol{a}.\boldsymbol{v} = (\boldsymbol{m}.\boldsymbol{v})\boldsymbol{m}, \tag{31a}$$

$$\boldsymbol{v}_{\perp} = \boldsymbol{t}.\boldsymbol{v} = \boldsymbol{v} - (\boldsymbol{m}.\boldsymbol{v})\boldsymbol{m}. \tag{31b}$$

[2] In two of our past works (Equations (2.8) in Federico and Grillo (2012), and Equations (96) in Federico (2015)), we regrettably forgot to set the font in bold for the parallel and transverse components of a vector $\boldsymbol{v}$ with respect to a direction $\boldsymbol{m}$, and we may have therefore given the misleading impression that we were referring to scalar components when, in fact, we meant to speak about *vectorial* components.

In the jargon of composite and fibre-reinforced materials, tensor $\boldsymbol{a}$ is often called the *structure tensor* or *fabric tensor* of direction $\boldsymbol{m}$. Tensor $\boldsymbol{t}$ is often simply called *projector* (Bonet and Wood, 2008; Gurtin et al., 2010). It is sometimes convenient to explicitly indicate the dependence of $\boldsymbol{a}$ and $\boldsymbol{t}$ on the direction $\boldsymbol{m}$, in which case we say that $\{\boldsymbol{a}(\boldsymbol{m}), \boldsymbol{t}(\boldsymbol{m})\}$ is the basis of the space $([T\mathcal{B}]^2_0, \boldsymbol{m})$. As seen in the case of isotropy, transversely isotropic second-order tensors are necessarily symmetric.

The basis of the subspace of $[T\mathcal{B}]^4_0$ of all tensors with transverse isotropy with respect to direction $\boldsymbol{m}$, denoted $([T\mathcal{B}]^4_0, \boldsymbol{m})$, has been obtained in two different versions by Walpole (1981, 1984). Initially, Walpole (1981) used a tensor basis allowing for a representation in the form of a 6×1 array, which has been extensively used by other authors (Weng, 1990; Qiu and Weng, 1990; Bhattacharyya and Weng, 1994; Wu and Herzog, 2002; Federico et al., 2004, 2005). Later, Walpole (1984) perfected the representation, with new normalisation constants for the basis tensors, which allows for an extremely convenient representation in an array constituted by a 2×2 matrix and 2 scalars (Walpole (1984) has also provided similar representations for all other symmetry groups). This later representation (Walpole, 1984) has been used in more recent works (Federico, 2015; Federico et al., 2015), developed within a covariant framework, and we do so in this chapter too.

The basis of $([T\mathcal{B}]^4_0, \boldsymbol{m})$ proposed by Walpole (1984) is obtained (similarly to that proposed in the older work; Walpole, 1981) by means of suitable tensor products, as

$$\mathbb{U}_{11} = \boldsymbol{a} \otimes \boldsymbol{a}, \qquad (\mathbb{U}_{11})^{ijkl} = a^{ij}\, a^{kl}, \tag{32a}$$

$$\mathbb{U}_{12} = \tfrac{\sqrt{2}}{2}\, \boldsymbol{a} \otimes \boldsymbol{t}, \qquad (\mathbb{U}_{12})^{ijkl} = \tfrac{\sqrt{2}}{2}\, a^{ij}\, t^{kl}, \tag{32b}$$

$$\mathbb{U}_{21} = \tfrac{\sqrt{2}}{2}\, \boldsymbol{t} \otimes \boldsymbol{a}, \qquad (\mathbb{U}_{21})^{ijkl} = \tfrac{\sqrt{2}}{2}\, t^{ij}\, a^{kl}, \tag{32c}$$

$$\mathbb{U}_{22} = \tfrac{1}{2}\, \boldsymbol{t} \otimes \boldsymbol{t}, \qquad (\mathbb{U}_{22})^{ijkl} = \tfrac{1}{2}\, t^{ij}\, t^{kl}, \tag{32d}$$

$$\mathbb{V}_1 = \tfrac{1}{2}\, (\boldsymbol{t} \underline{\otimes} \boldsymbol{t} + \boldsymbol{t} \overline{\otimes} \boldsymbol{t} - \boldsymbol{t} \otimes \boldsymbol{t}), \qquad (\mathbb{V}_1)^{ijkl} = \tfrac{1}{2}\, (t^{ik}\, t^{jl} + t^{il}\, t^{jk} - t^{ij}\, t^{kl}), \tag{32e}$$

$$\mathbb{V}_2 = \tfrac{1}{2}\, (\boldsymbol{a} \underline{\otimes} \boldsymbol{t} + \boldsymbol{a} \overline{\otimes} \boldsymbol{t} + \boldsymbol{t} \underline{\otimes} \boldsymbol{a} + \boldsymbol{t} \overline{\otimes} \boldsymbol{a}), \qquad (\mathbb{V}_2)^{ijkl} = \tfrac{1}{2}\, (a^{ik}\, t^{jl} + a^{il}\, t^{jk} + t^{ik}\, a^{jl} + t^{il}\, a^{jk}). \tag{32f}$$

The transversely isotropic basis in Equation (32) is denoted $\{\mathbb{U}_{\alpha\beta}, \mathbb{V}_\gamma\}^2_{\alpha,\beta,\gamma=1}$ and, when it is convenient to explicitly indicate the dependence of the basis tensors on the direction $\boldsymbol{m}$, one says that $\{\mathbb{U}_{\alpha\beta}(\boldsymbol{m}), \mathbb{V}_\gamma(\boldsymbol{m})\}^2_{\alpha,\beta,\gamma=1}$ is the basis of the space $([T\mathcal{B}]^4_0, \boldsymbol{m})$. A tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ is expressed as

$$\mathbb{T} = \overline{\mathbb{T}}^{\alpha\beta} \mathbb{U}_{\alpha\beta} + \overline{\mathbb{T}}^{\gamma} \mathbb{V}_\gamma, \tag{33}$$

where Einstein's summation convention is understood for $\alpha, \beta, \gamma \in \{1, 2\}$ and the components $\overline{\mathbb{T}}^{\alpha\beta}$ and $\overline{\mathbb{T}}^{\gamma}$ are obtained by the scalar product of $\mathbb{T}$ with each of the basis tensors, with some normalisation constants:[3]

$$\overline{\mathbb{T}}^{\alpha\beta} = \langle \mathbb{T}, \mathbb{U}_{\alpha\beta} \rangle, \qquad \overline{\mathbb{T}}^{\gamma} = \tfrac{1}{2}\, \langle \mathbb{T}, \mathbb{V}_\gamma \rangle. \tag{34}$$

In the basis of Equation (32) the tensors $\mathbb{U}_{\alpha\beta}$ constitute an algebra isomorphic to that of 2×2 matrices (Walpole, 1984), which allows for grouping the Walpole components $\overline{\mathbb{T}}^{\alpha\beta}$ and $\overline{\mathbb{T}}^{\gamma}$ of Equation (34) into the array

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} \overline{\mathbb{T}}^{11} & \overline{\mathbb{T}}^{12} \\ \overline{\mathbb{T}}^{21} & \overline{\mathbb{T}}^{22} \end{bmatrix}, \overline{\mathbb{T}}^1, \overline{\mathbb{T}}^2 \right\} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^{\gamma}\}, \tag{35}$$

[3] In some previous works (Federico, 2010a,b), we used the normalisation constants of the later work by Walpole (1984), but kept the formalism with the 6×1 array formalism of the earlier work by Walpole (1981). We candidly admit that this was an infelicitous choice on our part. Also, because of an incautious copy-and-paste operation from the definitions of the basis tensors, we reported (see Appendices in Federico, 2010a,b) the wrong coefficients for the scalar products in Equation (34).

which we call *Walpole array representation* of tensor $\mathbb{T}$. Note the compact notation $\overline{\mathbb{T}} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^{\gamma}\}$.

It is precisely for the form of the array in Equation (35) that we find Walpole's formalism (Walpole, 1984) to be very convenient. Indeed, *all* operations on transversely isotropic tensors in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ can be performed by working on the 2×2 matrix and the 2 scalars of the Walpole array of each tensor. Linear combination of tensors in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ can be obtained via the linear combination of the matrices and the individual scalars. Given a tensor $\mathbb{T}$ with Walpole array $\overline{\mathbb{T}} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^{\gamma}\}$, the Walpole array of the transpose $\mathbb{T}$ is obtained by simply transposing the 2×2 matrix, i.e., $\overline{\mathbb{T}^T} = \overline{\mathbb{T}}^T = \{[\overline{\mathbb{T}}^{\alpha\beta}]^T, \overline{\mathbb{T}}^{\gamma}\}$. Moreover, since $\mathbb{U}_{12}^T = \mathbb{U}_{21}$, major (diagonal) symmetry of a tensor $\mathbb{T}$ is attained if $\overline{\mathbb{T}}^{12} = \overline{\mathbb{T}}^{21}$, in which case $\mathbb{T}$ has only 5 independent components, rather than the 6 independent components of the general case.[4] We also remark that positive definiteness of a tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ can be checked extremely simply: $\mathbb{T}$ is positive definite if, and only if, the 2×2 matrix $[\overline{\mathbb{T}}^{\alpha\beta}]$ is positive definite and the 2 scalars $\overline{\mathbb{T}}^{\gamma}$ are strictly positive. We remark that *all* transversely isotropic fourth-order tensors (i.e., tensors of the space $([T\mathcal{B}]^4_0, \boldsymbol{m})$, spanned by the basis in Equation (32)) enjoy minor symmetry.

At this point, one may wonder how to treat transversely isotropic fourth-order tensors of type other than those in $([T\mathcal{B}]^4_0, \boldsymbol{m})$, for instance the tensors in $([T\mathcal{B}]^0_4, \boldsymbol{m})$, among which there are the inverses (when they exist) of those in $([T\mathcal{B}]^4_0, \boldsymbol{m})$. Fortunately, the representation with Walpole's array of Equation (39) is *independent* of the type of fourth-order tensor at hand. For instance, if we transform a tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ into its "covariant" counterpart $\mathbb{T}^\flat$ in $([T\mathcal{B}]^0_4, \boldsymbol{m})$, we have

$$\mathbb{T} = \overline{\mathbb{T}}^{\alpha\beta}\mathbb{U}_{\alpha\beta} + \overline{\mathbb{T}}^{\gamma}\mathbb{V}_{\gamma} \mapsto \mathbb{T}^\flat = \overline{\mathbb{T}}^{\alpha\beta}\mathbb{U}^\flat_{\alpha\beta} + \overline{\mathbb{T}}^{\gamma}\mathbb{V}^\flat_{\gamma}, \tag{36}$$

i.e., the transformation takes place on the basis tensors, leaving the Walpole components *untouched*. Thus, the double contraction of, e.g., a tensor $\mathbb{T}$ in $([T\mathcal{B}]^2{}_2, \boldsymbol{m})$ and a tensor $\mathbb{Z}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ is obtained by keeping in mind that the resulting tensor belongs to $([T\mathcal{B}]^4_0, \boldsymbol{m})$, and by performing the ordinary row-by-column product of the two matrices, and the multiplication of the homologous scalars, i.e.,

$$\overline{\mathbb{T} : \mathbb{Z}} = \left\{ \begin{bmatrix} \overline{\mathbb{T}}^{11} & \overline{\mathbb{T}}^{12} \\ \overline{\mathbb{T}}^{21} & \overline{\mathbb{T}}^{22} \end{bmatrix} \begin{bmatrix} \overline{\mathbb{Z}}^{11} & \overline{\mathbb{Z}}^{12} \\ \overline{\mathbb{Z}}^{21} & \overline{\mathbb{Z}}^{22} \end{bmatrix}, \overline{\mathbb{T}}^1\overline{\mathbb{Z}}^1, \overline{\mathbb{T}}^2\overline{\mathbb{Z}}^2 \right\}. \tag{37}$$

Also, it is now clear how to represent the inverse (when it exists) of a tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$. Indeed, the inverse of an invertible tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \boldsymbol{m})$ is the tensor $\mathbb{T}^{-1}$ in $([T\mathcal{B}]^0_4, \boldsymbol{m})$ such that $\mathbb{T} : \mathbb{T}^{-1} = \mathbb{I}$ and $\mathbb{T}^{-1} : \mathbb{T} = \mathbb{I}^T$, and has Walpole array representation

$$\overline{\mathbb{T}^{-1}} \equiv \overline{\mathbb{T}}^{-1} = \left\{ \begin{bmatrix} \overline{\mathbb{T}}^{11} & \overline{\mathbb{T}}^{12} \\ \overline{\mathbb{T}}^{21} & \overline{\mathbb{T}}^{22} \end{bmatrix}^{-1}, \frac{1}{\overline{\mathbb{T}}^1}, \frac{1}{\overline{\mathbb{T}}^2} \right\}. \tag{38}$$

In an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^3$, such that $\boldsymbol{e}_1 = \boldsymbol{m}$, the components of the Walpole array $\overline{\mathbb{T}} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^{\gamma}\}$ of a tensor $\mathbb{T}$ are related to the conventional components T^{ijkl} by

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} \mathrm{T}^{1111} & \sqrt{2}\,\mathrm{T}^{1122} \\ \sqrt{2}\,\mathrm{T}^{2211} & 2\,\mathrm{T}^{2222} - 2\,\mathrm{T}^{2323} \end{bmatrix}, 2\,\mathrm{T}^{2323}, 2\,\mathrm{T}^{1212} \right\}. \tag{39}$$

Since an isotropic tensor is transversely isotropic with respect to any direction $\boldsymbol{m}$, it is possible to express it in Walpole's transversely isotropic representation. In particular, the "contravariant"

[4] In our original work (see text immediately following Equation (20) in Federico et al., 2004), we had stated that $\mathbb{U}_{11}$, $\mathbb{U}_{22}$, $\mathbb{V}_1$ and $\mathbb{V}_2$ (called $\mathbb{B}_2$, $\mathbb{B}_1$, $\mathbb{B}_3$ and $\mathbb{B}_4$, respectively, in Federico et al., 2004) span the whole space of major- and minor-symmetric (transversely isotropic) tensors, which is of course incorrect, as we should have added also $\frac{1}{2}(\mathbb{U}_{12} + \mathbb{U}_{21})$ (corresponding to $\frac{1}{2}(\mathbb{B}_5 + \mathbb{B}_6)$ in Federico et al., 2004).

fourth-order identity, and the spherical and deviatoric operators in $[T\mathcal{B}]^4_0$, defined in Equation (24), have Walpole array representations

$$\overline{\mathbb{I}^\sharp} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1,\, 1 \right\}, \tag{40a}$$

$$\overline{\mathbb{K}^\sharp} = \left\{ \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix}, 0,\, 0 \right\}, \tag{40b}$$

$$\overline{\mathbb{M}^\sharp} = \left\{ \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}, 1,\, 1 \right\}. \tag{40c}$$

Thus, a generic symmetric isotropic tensor $\mathbb{T}$ in $([T\mathcal{B}]^4_0, \mathrm{Sym}, \mathrm{Iso})$, which is written as

$$\mathbb{T} = 3\,k\,\mathbb{K}^\sharp + 2\,m\,\mathbb{M}^\sharp, \tag{41}$$

in the symmetric isotropic basis $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ admits the representation

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} k + \frac{4}{3}\,m & \sqrt{2}\,(k - \frac{2}{3}\,m) \\ \sqrt{2}\,(k - \frac{2}{3}\,m) & 2\,(k + \frac{4}{3}\,m) - 2\,m \end{bmatrix}, 2\,m,\, 2\,m \right\}, \tag{42}$$

where the coefficients $3\,k$ and $2\,m$ in Equation (41) echo those typical of isotropic linear elasticity (see Equation (54)), and are found as shown in Equation (26).

2.8 Basic Relations of the Theory of Linear Elasticity

Linear Elasticity can be developed as an independent branch of Mathematical Physics (see, e.g., the text by Gurtin, 1972), or can be retrieved by linearising the general Theory of (Non-Linear) Elasticity (a covariant procedure is presented in the text by Marsden and Hughes, 1983). Linear Elasticity has a strong pedagogical character. Indeed, it often allows to find either analytical solutions or solutions in closed form to many problems of engineering relevance. Moreover, in many circumstances, it suffices to determine first-order approximations that, with relatively low computational costs, provide solutions to real-world problems even in the cases in which engineering materials undergo finite deformations. Perhaps because of these advantages, Linear Elasticity is what is usually taught to the vast majority of the students in structural/mechanical Engineering or Physics during their undergraduate studies. Linear Elasticity is so diffused that some call it "Classical Elasticity" and that, still today, quite many understand "Linear Elasticity", when they hear the word "Elasticity". One can choose to present the linear theory of elasticity either, and equivalently, by starting from stress or from energy. We choose the latter and we present this approach after having briefly introduced displacement, strain and stress.

In a body $\mathcal{B}$, the *displacement* is the vector field

$$\boldsymbol{u} : \mathcal{B} \to T\mathcal{B} : x \mapsto \boldsymbol{u}(x) \in T_x\mathcal{B}, \tag{43}$$

whose gradient (also called *covariant derivative*) is called *displacement gradient*,

$$\boldsymbol{h} = \mathrm{grad}\,\boldsymbol{u}, \qquad h^i{}_j = u^i{}_{|j}, \tag{44}$$

where the vertical bar denotes covariant differentiation.[5] The *infinitesimal strain* is the "covariant" second-order tensor field

$$\boldsymbol{\epsilon} : \mathcal{B} \to [T\mathcal{B}]^0_2 : x \mapsto \boldsymbol{\epsilon}(x) \in [T_x\mathcal{B}]^0_2, \tag{45}$$

defined as the symmetric part of the "covariant" displacement gradient tensor $\boldsymbol{h}^\flat = \boldsymbol{g}\,\boldsymbol{h}$ (with components $h_{ij} = g_{ip}h^p{}_j$), i.e.,

$$\boldsymbol{\epsilon} = \tfrac{1}{2}\,(\boldsymbol{h}^\flat + \boldsymbol{h}^{\flat T}), \qquad \epsilon_{ij} = \tfrac{1}{2}\,(h_{ij} + h_{ji}). \tag{46}$$

The *Cauchy stress* is defined as a "contravariant" second-order tensor field[6]

$$\boldsymbol{\sigma} : \mathcal{B} \to [T\mathcal{B}]^2_0 : x \mapsto \boldsymbol{\sigma}(x) \in [T_x\mathcal{B}]^2_0. \tag{47}$$

In the absence of external body forces and neglecting inertia, the balance of linear momentum reduces to the vanishing of the divergence of the Cauchy stress, i.e.,

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \qquad \sigma^{ij}{}_{|j} = 0. \tag{48}$$

Often, the balance of angular momentum is invoked to obtain the condition of symmetry of the Cauchy stress:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \qquad \sigma^{ij} = \sigma^{ji}. \tag{49}$$

Balance of linear and angular momentum constitute a system of 6 equations in 12 unknowns (the 3 components of the displacement and the 9 components of the Cauchy stress) or, equivalently, of 3 independent equations in 9 unknowns (the 3 components of the displacement and the 6 independent components of the Cauchy stress). In order to be able to close the system, the need arises for 6 additional relations, called *constitutive laws*, expressing the stress tensor as a function of the strain tensor.

A material is said to obey a linear elastic constitutive law if there exists a quadratic function of the infinitesimal strain,

$$W(x) = \hat{W}(\boldsymbol{\epsilon}(x), x) = \tfrac{1}{2}\,\boldsymbol{\epsilon}(x) : \mathbb{L}(x) : \boldsymbol{\epsilon}(x) = \tfrac{1}{2}\,\epsilon_{ij}(x)\,\mathrm{L}^{ijkl}(x)\,\epsilon_{kl}(x), \tag{50}$$

called (quadratic) *elastic potential* ($W : \mathcal{B} \to \mathbb{R} : x \mapsto W(x)$ denotes the *scalar field*, while $\hat{W}$ denotes the corresponding *constitutive function*), such that the stress can be obtained as the derivative of $\hat{W}$ with respect to the strain, i.e.,

$$\boldsymbol{\sigma}(x) = \frac{\partial \hat{W}}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(x), x) = \mathbb{L}(x) : \boldsymbol{\epsilon}(x), \qquad \sigma^{ij}(x) = \frac{\partial \hat{W}}{\partial \epsilon_{ij}}(\boldsymbol{\epsilon}(x), x) = \mathrm{L}^{ijkl}(x)\,\epsilon_{kl}(x). \tag{51}$$

The "contravariant" fourth-order tensor field

$$\mathbb{L} : \mathcal{B} \to [T\mathcal{B}]^4_0 : x \mapsto \mathbb{L}(x) \in [T_x\mathcal{B}]^4_0 \tag{52}$$

[5] In Cartesian coordinates, covariant differentiation of a vector or tensor field reduces to the regular partial derivative and one writes, e.g., for a vector field, $u^i{}_{,j}$.

[6] In the general, large-deformation setting of Continuum Mechanics, the Cauchy stress is defined as a *spatial* tensor field, valued in $[T\mathcal{S}]^2_0$. In the small-deformation theory, however, the distinction between reference configuration (or body $\mathcal{B}$) and current configuration fades out, and it is legitimate to define also tensors, which by their nature would be spatial, in the body $\mathcal{B}$ rather than in the space $\mathcal{S}$. In contrast, it is natural to define the infinitesimal strain $\boldsymbol{\epsilon}$ as a tensor field valued in $[T\mathcal{B}]^0_2$, since it can be thought of as the linearisation of the *material* Green-Lagrange strain $\boldsymbol{E}$.

is called the *linear elasticity tensor* and does *not* depend on x if the body is *homogeneous*. Equation (50) and the symmetry of $\boldsymbol{\epsilon}$ imply that the elasticity tensor $\mathbb{L}$ enjoys both major and minor symmetry. In order to guarantee the positiveness of the elastic potential, which implies the positivity of the internal work (or deformation work), one normally requires the positive definiteness of the elasticity tensor $\mathbb{L}$. The positive definiteness of $\mathbb{L}$ in turn implies its invertibility. Note that the inverse of the "contravariant" elasticity tensor $\mathbb{L}$, which is also called *stiffness elasticity tensor*, is the "covariant" *compliance elasticity tensor* $\mathbb{L}^{-1}$, which is a tensor field valued in $[T\mathcal{B}]^0_4$.

For a transversely isotropic elasticity tensor $\mathbb{L}$, Walpole's representation takes the form (see Equation (39))

$$\overline{\mathbb{L}} = \left\{ \begin{bmatrix} n & \sqrt{2}\,\ell \\ \sqrt{2}\,\ell & 2\,c \end{bmatrix}, 2\,\mu_t,\, 2\,\mu_a \right\}, \tag{53}$$

where (see Hill, 1964) n is the *modulus in uniaxial strain* (also called *p-wave modulus* or also, in the literature on articular cartilage, *aggregate modulus*: see Holmes and Mow, 1990), c is the *plane-strain bulk modulus* (in the transverse plane of transverse isotropy), ℓ is the *cross modulus* (transversely isotropic analogue of the first Lamé's constant $\lambda = \kappa - \frac{2}{3}\,\mu$ of isotropic linear elasticity: see Spencer, 1984), μ_t is the shear modulus in the transverse plane, and μ_a is the shear modulus in any plane containing the axis of symmetry $\boldsymbol{m}$ of transverse isotropy.

An isotropic elasticity tensor $\mathbb{L}$ can be represented in the basis $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ of symmetric isotropic tensors in $[T\mathcal{B}]^4_0$ as (see Equation (41))

$$\mathbb{L} = 3\,\kappa\,\mathbb{K}^\sharp + 2\,\mu\,\mathbb{M}^\sharp, \tag{54}$$

where κ and μ are the bulk modulus and shear modulus, respectively. Using Equation (42) for the expression of an isotropic tensor in Walpole's transversely isotropic array, we can represent the isotropic elasticity tensor of Equation (54) as

$$\overline{\mathbb{L}} = \left\{ \begin{bmatrix} \kappa + \frac{4}{3}\,\mu & \sqrt{2}\,(\kappa - \frac{2}{3}\,\mu) \\ \sqrt{2}\,(\kappa - \frac{2}{3}\,\mu) & 2\,(\kappa + \frac{4}{3}\,\mu) - 2\,\mu \end{bmatrix}, 2\,\mu,\, 2\,\mu \right\}, \tag{55}$$

in which it is possible to recognise the first Lamé's constant $\lambda = \kappa - \frac{2}{3}\,\mu$ and the modulus in uniaxial strain $n = \kappa + \frac{4}{3}\,\mu = \lambda + 2\,\mu$. In terms of the Lamé's moduli λ and μ, the Walpole array reads

$$\overline{\mathbb{L}} = \left\{ \begin{bmatrix} \lambda + 2\,\mu & \sqrt{2}\,\lambda \\ \sqrt{2}\,\lambda & 2(\lambda + 2\,\mu) - 2\,\mu \end{bmatrix}, 2\,\mu,\, 2\,\mu \right\}, \tag{56}$$

where $(\lambda + 2\,\mu) - \mu = \lambda + \mu$ is the isotropic equivalent of the plane-strain bulk modulus c of transverse isotropy (Equation (53)).

3 Composite Materials with Aligned Inclusions

We first recall the definitions of *Eshelby's fourth-order tensor* $\mathbb{S}$ introduced by Eshelby (1957) and of the closely related *strain concentration tensor* $\mathbb{A}$, which arises in the case of inclusions with material properties different from those of the matrix. Finally, we introduce composite materials with inclusions as described by the works of Hill (1963, 1965) and Walpole (1966a,b, 1969), and focus on the case of aligned inclusions.

3.1 Eshelby's Inclusion and Fourth-Order Tensor

Eshelby (1957) studied the problem of an inclusion in an infinite matrix, and in particular the case of an ellipsoidal inclusion. Eshelby constructed the inclusion problem in several steps (Eshelby, 1957, last paragraph of page 376), which we report in our own words, following a previous work (Alhasadi and Federico, 2017).

1) A cavity is cut in a body $\mathcal{B}$ and a *transformation strain* $\boldsymbol{\epsilon}^*$ is applied to the geometry of the region $\mathcal{D}$ occupied by the cavity, which is thus mapped into the new region $\mathcal{D}^*$; the remaining region $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ is called *matrix*;

2) The transformed region $\mathcal{D}^*$ is now "filled" with a material, which could be the same as that of the matrix, with elasticity tensor $\mathbb{L}_0$, or another one, with elasticity tensor $\mathbb{L}_1$. The transformed region $\mathcal{D}^*$, now assigned with certain elastic properties, constitutes the *inclusion*, which *no longer fits* the original cavity $\mathcal{D}$;

3) In order to make the inclusion occupying the transformed region $\mathcal{D}^*$ fit again into the original cavity $\mathcal{D}$, tractions are applied on the boundary of the inclusion, so that it attains a strain $-\boldsymbol{\epsilon}^*$, and then it is put back into the cavity;

4) Once the inclusion is back in place, the tractions on the boundary are removed, and so the inclusion and the surrounding matrix relax, causing a *cancelling strain* or *constrained strain* $\boldsymbol{\epsilon}^c$, which is *discontinuous* across the boundary of the inclusion.

At the end of this sequence of operations, and in the absence of external tractions applied on the boundary of the body $\mathcal{B}$, the *residual* strain due to the *geometrical misfit* is

$$\boldsymbol{\epsilon}^b = \boldsymbol{\epsilon}^c, \qquad \text{in } \mathcal{M}, \tag{57a}$$

$$\boldsymbol{\epsilon}^b = \boldsymbol{\epsilon}^c - \boldsymbol{\epsilon}^*, \qquad \text{in } \mathcal{D}, \tag{57b}$$

where we emphasise again that the cancelling strain $\boldsymbol{\epsilon}^c$ is *discontinuous* across the boundary of $\mathcal{D}$ and thus must be studied and described *piecewise*.

In the absence of the inclusion, i.e., if the body $\mathcal{B}$ were perfectly homogeneous (elasticity tensor equal to $\mathbb{L}_0$ everywhere) and without any region with geometrical misfit (identically vanishing transformation strain $\boldsymbol{\epsilon}^*$), the cancelling strain would vanish identically, and the application of traction forces on the boundary of $\mathcal{B}$ would cause a stress state described by the continuous field $\boldsymbol{\sigma}^a$ everywhere in $\mathcal{B}$, which in turn would cause the continuous strain field

$$\boldsymbol{\epsilon}^a = \mathbb{L}_0^{-1} : \boldsymbol{\sigma}^a, \quad \text{everywhere in } \mathcal{B}. \tag{58}$$

In the presence of both inclusion and external tractions, the linearity of the problem allows to write the total strain as the superposition of that in Equation (57), which was obtained in the absence of external applied tractions, and of that in Equation (58), which was obtained in the absence of inclusion, as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^b = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^c, \qquad \text{in } \mathcal{M}, \tag{59a}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^b = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^c - \boldsymbol{\epsilon}^* \qquad \text{in } \mathcal{D}. \tag{59b}$$

For the case of an ellipsoidal inclusion, it is clear that it remains an ellipsoid if, and only if, the transformation strain $\boldsymbol{\epsilon}^*$ is uniform (Eshelby, 1957). In this case, also the cancelling strain $\boldsymbol{\epsilon}^c$

in the inclusion is uniform, and it is possible to relate it to the transformation strain $\boldsymbol{\epsilon}^*$ by means of the relation (Eshelby, 1957)

$$\boldsymbol{\epsilon}^c = \mathbb{S} : \boldsymbol{\epsilon}^*, \quad \text{in } \mathcal{D}, \tag{60}$$

where the tensor $\mathbb{S}$ in $[T\mathcal{B}]_2{}^2$ is the *fourth-order Eshelby tensor*, which depends on ratios of the elastic constants of the matrix and on the geometry of the inclusion. For an isotropic matrix, there is only one independent geometrical ratio of elastic constants, which is usually chosen to be the Poisson's ratio ν_0 (e.g., Qiu and Weng, 1990). For the case of spheroidal inclusions, i.e., revolution ellipsoids, the only independent geometrical ratio is that of the major to the minor semi-axis (e.g., Qiu and Weng, 1990). Note that the fourth-order Eshelby tensor can be seen as the linear operator $\mathbb{S} : [T\mathcal{B}]_2^0 \to [T\mathcal{B}]_2^0$ mapping the transformation strain into the cancelling strain. In components, Equation (60) reads

$$\epsilon^c_{ij} = \mathrm{S}_{ij}{}^{kl}\, \epsilon^*_{kl}, \quad \text{in } \mathcal{D}. \tag{61}$$

We remark that the Eshelby tensor $\mathbb{S}$ has minor symmetry on each pair of legs, but it lacks major symmetry (i.e., its "contravariant" counterpart $\mathbb{S}^\sharp$, with components $(\mathbb{S}^\sharp)^{ijkl} \equiv \mathrm{S}^{ijkl} = g^{ip}\, g^{jq}\, \mathrm{S}_{pq}{}^{kl}$, and its "covariant" counterpart $\mathbb{S}^\flat$, with components $(\mathbb{S}^\flat)_{ijkl} \equiv \mathrm{S}_{ijkl} = \mathrm{S}_{ij}{}^{rs}\, g_{rk}\, g_{sl}$, lack major symmetry). Therefore, assuming an isotropic matrix, for the case of an ellipsoidal inclusion with three distinct semi-axes, $\mathbb{S}$ is a non-major-symmetric orthotropic tensor with 12 independent components and, for the case of a spheroidal inclusion with two equal semi-axes (i.e., a revolution ellipsoid), it is a non-major-symmetric transversely isotropic tensor with 6 independent components.

There are three possible inclusion problems:

- The "homogeneous inclusion", with geometrical misfit caused by a transformation strain $\boldsymbol{\epsilon}^*$, but material properties identical to those of the matrix, i.e., $\mathbb{L}_0 = \mathbb{L}_1$;
- The "inhomogeneous inclusion", with no geometrical misfit, i.e., $\boldsymbol{\epsilon}^* = \mathbf{0}$, but material properties different from those of matrix, i.e., $\mathbb{L}_0 \neq \mathbb{L}_1$;
- The "general inclusion", with both geometrical misfit, i.e., $\boldsymbol{\epsilon}^* \neq \mathbf{0}$, and material properties different from those of matrix, i.e., $\mathbb{L}_0 \neq \mathbb{L}_1$.

The "homogeneous" case is the fundamental one, and indeed the "inhomogeneous" and the "general" cases are solved by reducing the effect of the different material properties to an *equivalent transformation strain* (Eshelby, 1957; Mura, 1987; Alhasadi and Federico, 2017). In this work, we shall restrict our attention to the "inhomogeneous" case. Thus, we shall exclusively deal with inclusions with no geometrical misfit with the matrix, but with material properties different from those of the matrix.

3.2 Strain Concentration Tensor

The *strain concentration tensor* arises in the cases of the "inhomogeneous" inclusion and "general" inclusion, and is the object that captures the difference in material properties between matrix and inclusion within the method of the *equivalent transformation strain*, which is that fictitious transformation strain that has the same effect on the stress and strain fields that the mismatch in material properties has. This method is, again, described in detail by Eshelby (1957), and we also mention the classical book by Mura (1987).

The standard derivation of the strain concentration tensor is done in the case of the "inhomogeneous inclusion", and we report its expression (for the details, see, e.g., Weng, 1984, 1990; Alhasadi and Federico, 2017)

$$\mathbb{A} = \left[\mathbb{I}^T + \mathbb{S} : [\mathbb{L}_0^{-1} : \mathbb{L}_1 - \mathbb{I}^T]\right]^{-1}, \quad \text{in } \mathcal{D}, \tag{62}$$

which clearly depends on the Eshelby tensor, $\mathbb{S}$, and the elasticity tensors of matrix and inclusion, $\mathbb{L}_0$ and $\mathbb{L}_1$. Like $\mathbb{S}$, tensor $\mathbb{A}$ is in $[T\mathcal{B}]_2{}^2$ and is endowed with minor, but not major symmetry. Its component expression is

$$(\mathbb{A}^{-1})_{ij}{}^{kl} = (\mathbb{I}^T)_{ij}{}^{kl} + \mathrm{S}_{ij}{}^{pq}\,[(\mathbb{L}_0^{-1})_{pqrs}\,(\mathbb{L}_1)^{rskl} - (\mathbb{I}^T)_{pq}{}^{kl}], \quad \text{in } \mathcal{D}. \tag{63}$$

Note the use of the *transpose* of the symmetric identity $\mathbb{I}$. Indeed, since the symmetric identity $\mathbb{I}$ belongs to $[T\mathcal{B}]^2{}_2$, it is necessary here to use its transpose $\mathbb{I}^T$, which belongs to $[T\mathcal{B}]_2{}^2$, in order to be able to sum it to the other tensors. This distinction is unnecessary in Cartesian coordinates, and indeed in all papers and books we are aware of, including our own past works, one finds the expression in Equation (62) written with $\mathbb{I}$.

The strain concentration tensor gives the cancelling strain in the inclusion as

$$\boldsymbol{\epsilon}^c = \left(\mathbb{A} - \mathbb{I}^T\right) : \boldsymbol{\epsilon}^a, \quad \text{in } \mathcal{D}, \tag{64}$$

with components

$$\epsilon^c_{ij} = \left(\mathrm{A}_{ij}{}^{kl} - (\mathbb{I}^T)_{ij}{}^{kl}\right)\,\epsilon^a_{kl}, \quad \text{in } \mathcal{D}. \tag{65}$$

It is important to note the structural similarity between Equation (60) and Equation (64). However, it is even more important to emphasise that, while in the case of the "homogeneous inclusion" (Equation (60)), the cancelling strain $\boldsymbol{\epsilon}^c$ is a constant once the transformation strain $\boldsymbol{\epsilon}^*$ is assigned, in the case of the "inhomogeneous inclusion", the cancelling strain $\boldsymbol{\epsilon}^c$ is *linearly related* to the applied strain $\boldsymbol{\epsilon}^a$. Indeed, since there is no geometrical misfit ($\boldsymbol{\epsilon}^* = \mathbf{0}$), the cancelling strain $\boldsymbol{\epsilon}^c$ is identically zero when no tractions are applied, i.e., when the applied strain $\boldsymbol{\epsilon}^a$ is zero.

By adding the applied strain $\boldsymbol{\epsilon}^a$ to either side of Equation (64), we obtain the total strain in the "inhomogeneous" inclusion as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^c = \mathbb{A} : \boldsymbol{\epsilon}^a, \quad \text{in } \mathcal{D}, \tag{66}$$

with component expression

$$\epsilon_{ij} = \epsilon^a_{ij} + \epsilon^c_{ij} = \mathrm{A}_{ij}{}^{kl}\,\epsilon^a_{kl}, \quad \text{in } \mathcal{D}. \tag{67}$$

Equation (66) gives $\mathbb{A}$ in $[T\mathcal{B}]_2{}^2$ its physical meaning of *strain concentration tensor*: it can indeed be seen as the linear operator $\mathbb{A} : [T\mathcal{B}]^0_2 \to [T\mathcal{B}]^0_2$ that maps the applied strain $\boldsymbol{\epsilon}^a$ that would be attained in the absence of inclusion into the strain $\boldsymbol{\epsilon}$ *actually attained* by the inclusion.

3.3 Composites with Spheroidal Inclusions, and the Aligned Case

In the 1060s, Hill (1963, 1965), Hashin (1963) and Walpole (1966a,b, 1969) gave fundamental contributions to the development of techniques for the evaluation of the overall elasticity tensor (overall elastic moduli, in the terminology of the time) of a composite starting from the elasticity

tensors of the individual constituents of the composite. A composite differs from the system of Eshelby's inclusion problem in that an inclusion is no longer a solitary *singularity* in an infinite matrix, but is one of many other inclusions, which could be of the same or of different type.

The problem of the evaluation of the overall elasticity tensor for a composite with one or more families of inclusions is therefore tackled by means of the concept of *representative element of volume* or, with the customary acronym, REV. The REV could be defined as the smallest region whose material properties are equivalent to those of the whole composite. We quote the definition of REV given by Hill (1963):

> "*This phrase* [representative element of volume] *will be used when referring to a sample that* (a) *is structurally entirely typical of the whole mixture on average, and* (b) *contains a sufficient number of inclusions for the apparent overall moduli to be effectively independent of the surface values of traction and displacement, so long as these values are "macroscopically uniform". That is, they fluctuate about a mean with a wavelength small compared with the dimensions of the sample, and the effects of such fluctuations become insignificant within a few wavelengths of the surface. The contribution of this surface layer to any average can be made negligible by taking the sample large enough.*"

Thus, the REV that we consider must contain a sufficient number of inclusions for the overall elasticity tensor (or the collection of the "apparent elastic moduli", in Hill's words) to be representative of that of the whole composite.

The composite is assumed to be comprised of $N+1$ phases, with phase 0 referring to the matrix, and phases $r \in \{1, ..., N\}$ referring to the r-th inclusion families. The inclusions are assumed to be *perfectly fitted* (Walpole, 1966a) in the matrix, which, in Eshelby's terminology, means that they are "inhomogeneous inclusions", i.e., inclusions with *no geometrical misfit* with the matrix, but with elastic properties *different* from those of the matrix. It is important to emphasise that, in order to be able to apply Eshelby's theory as described in Sections 3.1 and 3.2, which is based on an *inclusion in an infinite matrix*, we must make sure that each inclusion is far enough from its prime neighbours and the interactions among inclusions can be neglected. This is achieved by imposing a reasonably low *volumetric fraction* for each of the inclusion phases. The volumetric fraction of each phase is defined as

$$\phi_r = \frac{\Omega_r}{\Omega}, \tag{68}$$

where Ω is the volume of the REV, and Ω_r is the volume of the portion of the REV occupied by phase r. The volumetric fractions obey the constraint

$$\sum\nolimits_{r=0}^{N} \phi_r = 1. \tag{69}$$

The strain concentration tensor seen in Section 3.2 has been extensively used in the determination of the overall elastic properties of composite materials with inclusions (see, e.g., Hill, 1963; Walpole, 1966a,b, 1969; Weng, 1984, 1990; Qiu and Weng, 1990). In the formalism introduced by Walpole (1966a,b, 1969) and Weng (1990), the overall elasticity tensor $\mathbb{L}$ reads

$$\mathbb{L} = \left[\sum\nolimits_{r=0}^{N} \phi_r \, \mathbb{L}_r : \mathbb{A}_r\right] : \left[\sum\nolimits_{r=0}^{N} \phi_r \, \mathbb{A}_r\right]^{-1}, \tag{70}$$

where $\mathbb{L}_r$ is the elasticity tensor of phase r and

$$\mathbb{A}_r = \left[\mathbb{I}^T + \mathbb{S}_r : [\mathbb{L}_0^{-1} : \mathbb{L}_r - \mathbb{I}^T]\right]^{-1} \tag{71}$$

is the strain concentration tensor of the r-th phase, in which $\mathbb{S}_r$ is the Eshelby fourth-order tensor relative to the r-th phase, depending on the shape of the inclusions of phase r and the elastic constants of the matrix. Note that the strain concentration tensor $\mathbb{A}_0$ of the matrix is identically equal to the transpose $\mathbb{I}^T$ of the symmetric identity. Indeed, *since the matrix is not an inclusion embedded in itself*, from Equation (71) we have

$$\mathbb{A}_0 = \left[\mathbb{I}^T + \mathbb{S}_0 : [\mathbb{L}_0^{-1} : \mathbb{L}_0 - \mathbb{I}^T]\right]^{-1} = \left[\mathbb{I}^T + \mathbb{S}_0 : \mathbb{O}\right]^{-1} = \mathbb{I}^T, \tag{72}$$

where $\mathbb{O}$ is the zero tensor, regardless of the value of the tensor $\mathbb{S}_0$.[7]

We remark that Equation (70) is analogical to that of the centre of mass of a system of particles, i.e.,

$$x_G = \frac{\sum_{r=0}^{N} m_r\, x_r}{\sum_{r=0}^{N} m_r}. \tag{73}$$

In this analogy, Equation (70) provides the "barycentric elasticity tensor" of a composite, in which the "masses" are the products $\phi_r\, \mathbb{A}_r$ and the "moment arms" are the elasticity tensors $\mathbb{L}_r$.

In principle, Equation (70) applies to any composite with ellipsoidal inclusions. However, for ellipsoidal inclusion families with different semi-axis ratios, different alignment of the semi-axes, and different alignment of the directions or planes of material symmetry, the overall elasticity tensor could be completely anisotropic. Equation (70) becomes immediately usable in the case of transverse isotropy with respect to a given direction $\boldsymbol{m}_0$, which is obtained when:

(A1) The matrix is either isotropic or transversely isotropic with respect to direction $\boldsymbol{m}_0$;

(A2) All inclusions in all families have their axis of symmetry oriented in direction $\boldsymbol{m}_0$, are spheroidal (i.e., are revolution ellipsoids), and are either isotropic or transversely isotropic with respect to $\boldsymbol{m}_0$.

When the two conditions (A1) and (A2) are satisfied, all tensors featuring in Equation (70) are transversely isotropic in direction $\boldsymbol{m}_0$, and can be represented using Walpole's formalism (Walpole, 1981, 1984) presented in Section 2.7. This is the procedure followed by Weng (1990) and Qiu and Weng (1990), leading to the Walpole array representation

$$\overline{\mathbb{L}} = \left[\sum_{r=0}^{N} \phi_r\, \overline{\mathbb{L}_r : \mathbb{A}_r}\right] \left[\sum_{r=0}^{N} \phi_r\, \overline{\mathbb{A}_r}\right]^{-1}. \tag{74}$$

Qiu and Weng (1990) also noted that the lack of major symmetry of the strain concentration tensors $\mathbb{A}_r$ causes in general the lack of major symmetry of the overall elasticity tensor obtained via Equation (70), except in the perfectly isotropic case mentioned above and when the aligned inclusion

[7] On one occasion, we had stated that $\mathbb{S}_0$ reduces to the identity $\mathbb{I}^T$ (paragraph following Equation (35) in Federico et al., 2004) and, on another occasion, that it reduces to the zero tensor $\mathbb{O}$ (paragraph following Equation (12) in Federico, 2010a). Both statements are incorrect, as this $\mathbb{S}_0$ is really arbitrary. One can think to obtain $\mathbb{A}_0$ by imagining to have an inclusion with an arbitrary ellipsoidal shape and an elasticity tensor $\mathbb{L}_0'$, which defines a corresponding $\mathbb{S}_0$, and then by performing the limit $\mathbb{L}_0' \to \mathbb{L}_0$. This yields $\mathbb{A}_0 \to \mathbb{I}^T$ regardless of the value of $\mathbb{S}_0$.

phases have all the same shape. Thus, the transversely isotropic overall elasticity tensor found from Equation (70) when the conditions (A1) and (A2) are satisfied has Walpole representation

$$\overline{\mathbb{L}} = \left\{ \begin{bmatrix} n & \sqrt{2}\,\ell \\ \sqrt{2}\,\ell' & 2\,c \end{bmatrix}, 2\,\mu_t,\, 2\,\mu_a \right\}, \tag{75}$$

where, in contrast with Equation (53), $\ell' \neq \ell$. A "brute force" solution to this problem was proposed by Wu and Herzog (2002), who took the (major) symmetric part of the overall elasticity tensor $\mathbb{L}$ of Equation (75), i.e.,

$$\mathbb{L}_{\text{sym}} = \tfrac{1}{2}(\mathbb{L} + \mathbb{L}^T), \qquad (\mathbb{L}_{\text{sym}})^{ijkl} = \tfrac{1}{2}(\mathrm{L}^{ijkl} + \mathrm{L}^{klij}), \tag{76}$$

which, in terms of the symmetrised cross modulus to be used in the Walpole representation $\overline{\mathbb{L}_{\text{sym}}}$ of $\mathbb{L}_{\text{sym}}$, reads

$$\ell_{\text{sym}} = \tfrac{1}{2}(\ell + \ell'). \tag{77}$$

Finally, we note that, if the matrix is isotropic and all inclusions in all families are spherical and isotropic, isotropy is retrieved as a trivial particular case of transverse isotropy.

4 Composite Materials with Statistically Oriented Inclusions

In this section, we report, in the more recent notation presented in Section 2 (Federico, 2010a, 2015; Federico et al., 2015), our results for the general case of a composite with statistically oriented spheroidal inclusions, in which the orientation obeys a given probability density (Federico et al., 2004).

4.1 Generalised Walpole's Formula

In order to univocally identify the orientation of an ellipsoid, we need three parameters, e.g., the three Euler angles. In the case of a spheroid, by virtue of the rotational symmetry, only two parameters are required, and those could be, e.g., two Euler angles or, equivalently, the unit vector describing the direction of the axis of symmetry of the spheroid. We shall restrict our attention to the case of *spheroidal* inclusions. A phase of statistically oriented spheroidal inclusions (i.e., inclusions all sharing the same geometry and elastic properties, but having different orientations) can be thought of as an infinity of phases, each oriented in a certain direction, so that the summation in Equation (70) becomes an integral on the unit sphere $\mathbb{S}^2\mathcal{B}$. In this integral, at every point x in the body $\mathcal{B}$, the weighing function is a probability density

$$\psi : \mathbb{S}^2\mathcal{B} \to \mathbb{R}_0^+ : \boldsymbol{m} \mapsto \psi(\boldsymbol{m}), \tag{78}$$

which describes the probability to find, at each point x in $\mathcal{B}$, an inclusion oriented in direction $\boldsymbol{m}$. In the case of inhomogeneous bodies, ψ depends explicitly on the point x in the body. In the present formulation, however, for the sake of a lighter notation, this dependence is omitted but understood. The probability density ψ must be normalised over the sphere and must be invariant for reflections $\boldsymbol{m} \mapsto -\boldsymbol{m}$, i.e.,

$$\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m}) = 1, \tag{79a}$$

$$\psi(-\boldsymbol{m}) = \psi(\boldsymbol{m}). \tag{79b}$$

For any function f defined on the unit sphere $\mathbb{S}^2\mathcal{B}$ and valued in a tensor space of any order (including order zero, i.e., scalar functions), we denote by[8]

$$\langle\!\langle f \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, f(\boldsymbol{m}) \tag{80}$$

its *directional average*. Note that we do not explicitly indicate the area element (more precisely, the *area two-form*; see Epstein, 2010; Segev, 2013) in the integral. If the function f to be averaged enjoys the same symmetry as ψ, i.e., $f(-\boldsymbol{m}) = f(\boldsymbol{m})$, it is possible to restrict the integral to the *north hemisphere*, defined by

$$\mathbb{S}^{2+}\mathcal{B} = \{\boldsymbol{m} \in \mathbb{S}^2\mathcal{B} : \boldsymbol{m}.\boldsymbol{m}_0 \geq 0\}, \tag{81}$$

where $\boldsymbol{m}_0 \in \mathbb{S}^2\mathcal{B}$ is the chosen *polar direction*. Naturally, since the integral is performed on *half* the domain (the north hemisphere $\mathbb{S}^{2+}\mathcal{B}$ is *half* of the sphere $\mathbb{S}^2\mathcal{B}$), one has to take *twice* the value of the integral (alternatively, one could re-normalise the probability density).

When the sum in Equation (70) becomes an integral, we need to transform the variables according to

$$\phi_r \quad \mapsto \quad \phi_1\, \psi(\boldsymbol{m}), \tag{82a}$$
$$\mathbb{L}_r \quad \mapsto \quad \mathbb{L}_1(\boldsymbol{m}), \tag{82b}$$
$$\mathbb{A}_r \quad \mapsto \quad \mathbb{A}_1(\boldsymbol{m}), \tag{82c}$$

i.e., we can call the collection of all inclusions "phase 1", and identify all orientations in $\mathbb{S}^2\mathcal{B}$ by means of the probability density ψ. Equation (70) then becomes

$$\mathbb{L} = [\phi_0\, \mathbb{L}_0 + \phi_1 \langle\!\langle \mathbb{L}_1 : \mathbb{A}_1 \rangle\!\rangle] : \left[\phi_0\, \mathbb{I}^T + \phi_1 \langle\!\langle \mathbb{A}_1 \rangle\!\rangle\right]^{-1}, \tag{83}$$

where the terms relative to the matrix account for the fact that the strain concentration tensor $\mathbb{A}_0$ reduces to the transpose $\mathbb{I}^T$ of the symmetric identity (Equation (72)), and we used the definition (80) of directional average in

$$\langle\!\langle \mathbb{L}_1 : \mathbb{A}_1 \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{L}_1(\boldsymbol{m}) : \mathbb{A}_1(\boldsymbol{m}), \tag{84}$$
$$\langle\!\langle \mathbb{A}_1 \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{A}_1(\boldsymbol{m}). \tag{85}$$

In the most general case, the composite is comprised of matrix (subscript 0), N_a inclusion phases, all aligned in a definite direction (subscript $r \in \{1, ...N_a\}$), and N_p inclusion phases with statistical orientation (subscript $r \in \{1, ...N_p\}$). Thus, the overall elasticity tensor reads

$$\begin{aligned} \mathbb{L} = &\left[\phi_0\, \mathbb{L}_0 + \sum\nolimits_{r=1}^{N_a} \phi_r\, \mathbb{L}_r : \mathbb{A}_r + \sum\nolimits_{s=1}^{N_p} \phi_s \langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle\right] \\ &: \left[\phi_0\, \mathbb{I}^T + \sum\nolimits_{r=1}^{N_a} \phi_r\, \mathbb{A}_r + \sum\nolimits_{s=1}^{N_p} \phi_s \langle\!\langle \mathbb{A}_s \rangle\!\rangle\right]^{-1}, \end{aligned} \tag{86}$$

[8]Note that, in some previous works (Federico et al., 2004; Federico, 2010a), we used the symbol $\langle f \rangle$ for the integral in Equation (80) in the case of isotropic probability $\psi(\boldsymbol{m}) = 1/4\pi$, and called $\langle f \rangle$ the "average of f" . We do not adopt this meaning of "average" here and, much more generally, we use "directional average" for the integral in Equation (80) with *any* probability density ψ.

where, for each phase s,

$$\langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi_s(\boldsymbol{m})\, \mathbb{L}_s(\boldsymbol{m}) : \mathbb{A}_s(\boldsymbol{m}), \tag{87}$$

$$\langle\!\langle \mathbb{A}_s \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi_s(\boldsymbol{m})\, \mathbb{A}_s(\boldsymbol{m}) \tag{88}$$

are the directional averages of the product $\mathbb{L}_s : \mathbb{A}_s$ and of the strain concentration tensor $\mathbb{A}_s$, respectively, and ψ_s is the probability density describing the orientation.

In both the case of Equation (83) with one phase of statistically oriented inclusions and the general case of Equation (86), it is important to remark that all inclusions in the same phase s have identical geometry and mechanical properties, which means that, for every pair of directions $\boldsymbol{m}_\alpha$ and $\boldsymbol{m}_\beta$, there are suitable rotation tensors $\boldsymbol{Q}$ and $\boldsymbol{R}$ (which coincide if the axes of geometrical symmetry coincide with the axes of material symmetry) such that

$$\mathbb{L}_s(\boldsymbol{m}_\alpha) = (\boldsymbol{Q} \underline{\otimes} \boldsymbol{Q}) : \mathbb{L}_s(\boldsymbol{m}_\beta) : (\boldsymbol{Q}^T \underline{\otimes} \boldsymbol{Q}^T), \tag{89a}$$

$$\mathbb{A}_s(\boldsymbol{m}_\alpha) = (\boldsymbol{R}^{-T} \underline{\otimes} \boldsymbol{R}^{-T}) : \mathbb{A}_s(\boldsymbol{m}_\beta) : (\boldsymbol{R}^T \underline{\otimes} \boldsymbol{R}^T). \tag{89b}$$

4.2 Transversely Isotropic Case: Preliminaries

In the general system with statistically oriented inclusions described by Equation (86), transverse isotropy in direction $\boldsymbol{m}_0$ is obtained with weaker conditions than those of the aligned case seen in Section 3.3. Specifically, while condition (S1) below is identical to condition (A1), condition (S2) below echoes (A2) but is valid only for the aligned phases $r \in \{1, ..., N_a\}$, and a new condition (S3) must be stated for the statistically oriented phases $s \in \{1, ..., N_p\}$

(S1) The matrix is either isotropic or transversely isotropic with respect to $\boldsymbol{m}_0$;

(S2) All inclusions in all aligned families $r \in \{1, ..., N_a\}$ have their axis of symmetry oriented in direction $\boldsymbol{m}_0$, are spheroidal (i.e., are revolution ellipsoids), and are either isotropic or transversely isotropic with respect to $\boldsymbol{m}_0$.

(S3) All inclusions in all statistically oriented families $s \in \{1, ..., N_p\}$ are spheroidal, are either isotropic or transversely isotropic with respect to their axis of geometrical symmetry, and the probability densities ψ_s are all transversely isotropic with respect to direction $\boldsymbol{m}_0$, i.e., $\psi_s(\boldsymbol{Q}\boldsymbol{m}) = \psi_s(\boldsymbol{m})$, where $\boldsymbol{Q}$ is an orthogonal tensor such that $\boldsymbol{Q}\boldsymbol{m}_0 = \pm\boldsymbol{m}_0$.

Indeed, under the hypotheses (S1), (S2) and (S3), all elasticity tensors $\mathbb{L}_0$ and $\mathbb{L}_r$, the transpose $\mathbb{I}^T$ of the symmetric identity, all strain concentration tensors $\mathbb{A}_r$, and all directional averages $\langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle$ and $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$ in Equation (86) are transversely isotropic with respect to $\boldsymbol{m}_0$, which implies the transverse isotropy of the overall elasticity tensor $\mathbb{L}$. Thus, all tensors in Equation (86) can be decomposed in Walpole's transversely isotropic basis of Equation (32) relative to direction $\boldsymbol{m}_0$, i.e., $\{\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0), \mathbb{V}_\gamma(\boldsymbol{m}_0)\}^2_{\alpha,\beta,\gamma=1}$, so that all tensor contractions and inversions can be performed conveniently exploiting Walpole's array formalism, in which Equation (86) takes the form

$$\overline{\mathbb{L}} = \left[\phi_0\, \overline{\mathbb{L}_0} + \sum\nolimits_{r=1}^{N_a} \phi_r\, \overline{\mathbb{L}_r : \mathbb{A}_r} + \sum\nolimits_{s=1}^{N_p} \phi_s \overline{\langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle}\right] \left[\phi_0\, \overline{\mathbb{I}^T} + \sum\nolimits_{r=1}^{N_a} \phi_r\, \overline{\mathbb{A}_r} + \sum\nolimits_{s=1}^{N_p} \phi_s \overline{\langle\!\langle \mathbb{A}_s \rangle\!\rangle}\right]^{-1}, \tag{90}$$

where the Walpole array $\overline{\mathbb{I}^T}$ is equal to $\bar{\mathbb{I}}$, identically.

The decomposition in the transversely isotropic basis $\{\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0), \mathbb{V}_\gamma(\boldsymbol{m}_0)\}^2_{\alpha,\beta,\gamma=1}$ is straightforward for $\mathbb{L}_0$, $\mathbb{I}^T$, $\mathbb{L}_r$ and $\mathbb{A}_r$, which are all transversely isotropic in direction $\boldsymbol{m}_0$ by hypothesis, but requires some additional work for the directional averages $\langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle$ and $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$. Even though Equation (90) requires the determination of the Walpole components of $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$ in $[T\mathcal{B}]_2{}^2$, we shall decompose its "contravariant" counterpart $\langle\!\langle \mathbb{A}_s^\sharp \rangle\!\rangle$ in $[T\mathcal{B}]^4_0$, for which the formulae (34) are applied. This can be done because, as seen in Section 4.3, $\langle\!\langle \mathbb{A}_s^\sharp \rangle\!\rangle$ and $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$ have the same Walpole array (see Section 2.7). We recall that $\langle\!\langle \mathbb{A}_s^\sharp \rangle\!\rangle$ is obtained from $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$ by raising the first pair of indices of $\mathbb{A}_s$ through $\mathbb{I}^\sharp$ (the "contravariant" symmetric identity defined in Equation (24a)), i.e., by computing $\mathbb{A}_s^\sharp = \mathbb{I}^\sharp : \mathbb{A}_s$, and averaging the resulting expression: $\langle\!\langle \mathbb{A}_s^\sharp \rangle\!\rangle = \langle\!\langle \mathbb{I}^\sharp : \mathbb{A}_s \rangle\!\rangle = \mathbb{I}^\sharp : \langle\!\langle \mathbb{A}_s \rangle\!\rangle = \langle\!\langle \mathbb{A}_s \rangle\!\rangle^\sharp$. Note that the equality $\langle\!\langle \mathbb{A}_s^\sharp \rangle\!\rangle = \langle\!\langle \mathbb{A}_s \rangle\!\rangle^\sharp$ stems from the fact that directional averaging and raising of indices commute with each other.

The whole problem of inclusions oriented according to a given transversely isotropic probability density reduces to the evaluation of the Walpole array of the directional average of the $2\,s$ tensors $\langle\!\langle \mathbb{L}_s : \mathbb{A}_s \rangle\!\rangle$ and $\langle\!\langle \mathbb{A}_s \rangle\!\rangle$. This latter problem is solved once we are able to evaluate the Walpole array of the directional average of a generic tensor $\mathbb{T}$, which is the topic of Section 4.3. We conclude this section by noting that isotropy is retrieved if the inclusions in the "aligned" phases are spherical and isotropic, and if the probability density is imposed to be isotropic, which means that the inclusions of the statistical phases are oriented randomly. Under this latter hypothesis, the inclusions of the statistically oriented phases are allowed to be of spheroidal shape and either isotropic or transversely isotropic with respect to the direction of their axis of geometrical symmetry.

4.3 Transversely Isotropic Case: Average of a Function of the Direction

Let

$$\mathbb{T} : \mathbb{S}^2\mathcal{B} \to [T\mathcal{B}]^4_0 : \boldsymbol{m} \mapsto \mathbb{T}_s(\boldsymbol{m}) \in ([T\mathcal{B}]^4_0, \boldsymbol{m}), \tag{91}$$

be a $([T\mathcal{B}]^4_0, \boldsymbol{m})$-valued function, $\boldsymbol{m}_0$ a direction in $\mathbb{S}^2\mathcal{B}$ and ψ a probability density with transverse isotropy with respect to $\boldsymbol{m}_0$. Our purpose is to study the directional average

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{T}(\boldsymbol{m}). \tag{92}$$

First, we note that, since the probability density ψ is transversely isotropic with respect to $\boldsymbol{m}_0$, then $\langle\!\langle \mathbb{T} \rangle\!\rangle$ belongs to $([T\mathcal{B}]^4_0, \boldsymbol{m}_0)$ and we have the identity

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \overline{\langle\!\langle \mathbb{T} \rangle\!\rangle}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\langle\!\langle \mathbb{T} \rangle\!\rangle}^{\gamma}\, \mathbb{V}_\gamma(\boldsymbol{m}_0), \tag{93}$$

where $\{\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0), \mathbb{V}_\gamma(\boldsymbol{m}_0)\}^2_{\alpha,\beta,\gamma=1}$ is the basis of $([T\mathcal{B}]^4_0, \boldsymbol{m}_0)$, and

$$\overline{\langle\!\langle \mathbb{T} \rangle\!\rangle}^{\alpha\beta} = \langle \langle\!\langle \mathbb{T} \rangle\!\rangle, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) \rangle, \qquad \overline{\langle\!\langle \mathbb{T} \rangle\!\rangle}^{\gamma} = \tfrac{1}{2} \langle \langle\!\langle \mathbb{T} \rangle\!\rangle, \mathbb{V}_\gamma(\boldsymbol{m}_0) \rangle, \tag{94}$$

according to Equation (34). Second, if we decompose $\mathbb{T}(\boldsymbol{m})$ in the basis $\{\mathbb{U}_{\mu\nu}(\boldsymbol{m}), \mathbb{V}_\pi(\boldsymbol{m})\}^2_{\mu,\nu,\pi=1}$ of the space $([T\mathcal{B}]^4_0, \boldsymbol{m})$ of transversely isotropic tensors with respect to $\boldsymbol{m}$, the directional average

in Equation (92) becomes

$$\begin{aligned}\langle\!\langle \mathbb{T} \rangle\!\rangle &= \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m}) \left[\overline{\mathbb{T}}^{\mu\nu}\, \mathbb{U}_{\mu\nu}(\boldsymbol{m}) + \overline{\mathbb{T}}^{\pi}\, \mathbb{V}_{\pi}(\boldsymbol{m})\right] \\ &= \overline{\mathbb{T}}^{\mu\nu} \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{U}_{\mu\nu}(\boldsymbol{m}) + \overline{\mathbb{T}}^{\pi} \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{V}_{\pi}(\boldsymbol{m}) \\ &= \overline{\mathbb{T}}^{\mu\nu}\, \langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle + \overline{\mathbb{T}}^{\pi}\, \langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle, \end{aligned} \tag{95}$$

where the Walpole components $\overline{\mathbb{T}}^{\mu\nu}$ and $\overline{\mathbb{T}}^{\pi}$ *do not* depend on the direction $\boldsymbol{m}$ and can be therefore factorised out of the integral.

Now, we note that the directional averages $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$ and $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$ are of the same type as the average $\langle\!\langle \mathbb{T} \rangle\!\rangle$ in Equation (92), and thus are transversely isotropic in direction $\boldsymbol{m}_0$. Therefore, we can use Equation (93) to write $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$ and $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$ as

$$\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle = \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma}\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0), \tag{96a}$$

$$\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle = \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma}\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0), \tag{96b}$$

In this way, the problem of evaluating the directional average $\langle\!\langle \mathbb{T} \rangle\!\rangle$ of the tensor-valued function $\mathbb{T}(\boldsymbol{m})$ is reduced to finding the averages $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$ and $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$ of $\mathbb{U}_{\mu\nu}(\boldsymbol{m})$ and $\mathbb{V}_{\pi}(\boldsymbol{m})$. Since the scalar product by a tensor that is independent of the direction $\boldsymbol{m}$ and integration over all directions $\boldsymbol{m}$ commute, we have

$$\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta} = \left\langle \left(\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{U}_{\mu\nu}(\boldsymbol{m})\right), \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) \right\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \langle \mathbb{U}_{\mu\nu}(\boldsymbol{m})\,,\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0)\rangle, \tag{97a}$$

$$\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma} = \frac{1}{2}\left\langle \left(\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{U}_{\mu\nu}(\boldsymbol{m})\right), \mathbb{V}_{\gamma}(\boldsymbol{m}_0) \right\rangle = \frac{1}{2}\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \langle \mathbb{U}_{\mu\nu}(\boldsymbol{m})\,,\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0)\rangle, \tag{97b}$$

$$\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta} = \left\langle \left(\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{V}_{\pi}(\boldsymbol{m})\right), \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) \right\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \langle \mathbb{V}_{\pi}(\boldsymbol{m})\,,\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0)\rangle, \tag{97c}$$

$$\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma} = \frac{1}{2}\left\langle \left(\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{V}_{\pi}(\boldsymbol{m})\right), \mathbb{V}_{\gamma}(\boldsymbol{m}_0) \right\rangle = \frac{1}{2}\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \langle \mathbb{V}_{\pi}(\boldsymbol{m})\,,\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0)\rangle. \tag{97d}$$

This procedure reduces the number of integrals to be evaluated from $6\times 81 = 486$ (indeed, each of the 6 tensors $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$, $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$, with $\mu, \nu, \pi \in \{1, 2\}$, has 81 components) to $6\times 6 = 36$ (or $16+8+8+4 = 36$, if one looks at the four Equations (97)), i.e., 6 independent components for each of the 6 averages $\{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle, \langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle\}_{\mu\nu\pi=1}^{2}$. Moreover, this procedure eliminates many integrals, which vanish because of the transverse isotropy of the system with respect to $\boldsymbol{m}_0$, and could give numerical problems as they could be highly oscillatory (Federico et al., 2004). As we shall show in Section 4.4, these integrals can be expressed in spherical coordinates as a function of the co-latitude and longitude angles taken from a reference frame in which the polar axis is the overall direction of symmetry $\boldsymbol{m}_0$.

Using Equations (93), (95) and (96), we obtain the expression of the directional average $\langle\!\langle \mathbb{T} \rangle\!\rangle$ in the basis $\{\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0), \mathbb{V}_{\gamma}(\boldsymbol{m}_0)\}_{\alpha,\beta,\gamma=1}^{2}$ as

$$\begin{aligned}\langle\!\langle \mathbb{T} \rangle\!\rangle = {}& \overline{\mathbb{T}}^{\mu\nu} \left(\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma}\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0)\right) \\ &+ \overline{\mathbb{T}}^{\pi} \left(\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma}\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0)\right), \end{aligned} \tag{98}$$

which can be rearranged into the final expression

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \left(\overline{\mathbb{T}}^{\mu\nu}\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta} + \overline{\mathbb{T}}^{\pi}\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta}\right) \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \left(\overline{\mathbb{T}}^{\mu\nu}\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma} + \overline{\mathbb{T}}^{\pi}\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma}\right) \mathbb{V}_{\gamma}(\boldsymbol{m}_0). \tag{99}$$

The Walpole array form of Equation (99) is

$$\overline{\langle\!\langle \mathbb{T} \rangle\!\rangle} = \left\{ \left[\overline{\mathbb{T}}^{\mu\nu} \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta} + \overline{\mathbb{T}}^{\pi} \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta} \right], \overline{\mathbb{T}}^{\mu\nu} \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma} + \overline{\mathbb{T}}^{\pi} \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma} \right\}. \tag{100}$$

4.4 Transversely Isotropic Case: Solution in the Polar Parametrisation

Assuming that the symmetry axis $\boldsymbol{m}_0$ of the transverse isotropy coincides with vector $\boldsymbol{e}_1$ of an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^3$, the generic direction $\boldsymbol{m}$ can be expressed as a function

$$\boldsymbol{m}(\theta, \varphi) = \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \cos\varphi\, \boldsymbol{e}_2 + \sin\theta\, \sin\varphi\, \boldsymbol{e}_3, \tag{101}$$

where θ is the *co-latitude*, measured from the polar direction $\boldsymbol{m}_0 \equiv \boldsymbol{e}_1$, and φ is the *longitude*, measured from the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ (see Figure 2). Using the polar parametrisation of the sphere, the probability density ψ can be written as

$$\varrho(\theta) = \psi(\boldsymbol{m}(\theta, \varphi)), \tag{102}$$

where the new function ϱ does *not* depend on the latitude angle φ because of the transverse isotropy of ψ. In the polar parametrisation, the directional average (80) of a generic function f becomes

$$\langle\!\langle f \rangle\!\rangle = \int_0^{2\pi} \left[\int_0^{\pi} \varrho(\theta)\, f(\boldsymbol{m}(\theta, \varphi))\, \sin\theta\, \mathrm{d}\theta \right] \mathrm{d}\varphi. \tag{103}$$

We recall the symmetry of ψ for reflections $\boldsymbol{m} \mapsto -\boldsymbol{m}$, and note that it is inherited by ϱ as a symmetry in θ about the value $\pi/2$, i.e., $\varrho(\theta) = \varrho(\pi - \theta)$. Thus, for functions f invariant under reflections $\boldsymbol{m} \mapsto -\boldsymbol{m}$ (such as all our fourth-order tensors, which depend on $\boldsymbol{m}$ via the structure tensor $\boldsymbol{a} = \boldsymbol{m} \otimes \boldsymbol{m}$), we can also write

$$\langle\!\langle f \rangle\!\rangle = 2 \int_0^{2\pi} \left[\int_0^{\pi/2} \varrho(\theta)\, f(\boldsymbol{m}(\theta, \varphi))\, \sin\theta\, \mathrm{d}\theta \right] \mathrm{d}\varphi, \tag{104}$$

which is equivalent to integrating ψ over the *north hemisphere* $\mathbb{S}^{2+}\mathcal{B}$ as in Equation (81). For our purposes, the generic function f has to be replaced by the integrands in Equation (97).

4.5 Some Relevant Particular Cases

The most "classical" particular cases of transversely isotropic probability density ψ are the case of probability density converging to the Dirac-delta, describing orientation in one direction, the case of in-plane random orientation, in which all directions within the same plane are equally probable, and the case of random orientation, yielding an isotropic solution. In the solutions that we report below, in order to minimise the possibility of making mistakes with the tedious integrals (97), we employed *Wolfram Mathematica*.

The case of alignment in one definite direction can be tackled by means of a parametric probability density peaked at $\theta = 0$, which can be made to converge, in the sense of distributions (see, e.g., Kolmogorov and Fomin, 1999) to a Dirac delta, which is the approach we used in the past (Federico et al., 2004). However, quite trivially, one can directly say that, if all inclusions are oriented in the same direction, Equation (86) reduces to the Walpole solution (70) which Weng

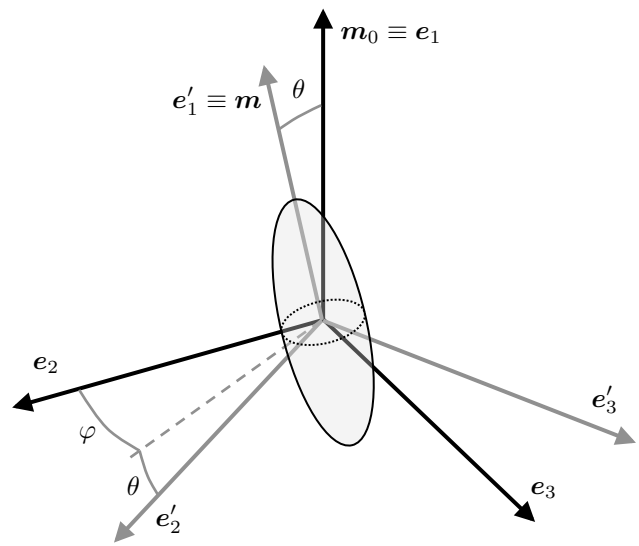

Figure 2: Representation of the generic direction $\boldsymbol{e}'_1 \equiv \boldsymbol{m}$ of the axis of symmetry of a spheroidal inclusion in terms of the colatitude angle θ and the longitude angle φ. The colatitude θ is calculated from the global direction of symmetry $\boldsymbol{e}_1 \equiv \boldsymbol{m}_0$, and the longitude φ is calculated from the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

(1990) and Qiu and Weng (1990) used for aligned inclusions (and which we reported in the form of a Walpole array in Equation (74)).

The case of random orientation in a plane can also be solved with a parametric probability density peaked at $\theta = \pi/2$ that converges, in the sense of distributions, to a Dirac delta (Federico et al., 2004). However, we can follow a simpler method. Rather than averaging the function $f(\boldsymbol{m}(\,\cdot\,,\,\cdot\,))$ with values $f(\boldsymbol{m}(\theta,\varphi))$ of Equation (103) on the whole unit sphere $\mathbb{S}^2\mathcal{B}$, we average the function $f(\boldsymbol{m}(\pi/2,\,\cdot\,))$ with values $f(\boldsymbol{m}(\pi/2,\varphi))$ on the equatorial unit circumference $(\mathbb{S}^1\mathcal{B}, \boldsymbol{m}_0)$, i.e., the circumference laying on the plane orthogonal to the direction of overall symmetry $\boldsymbol{m}_0$ (which is the plane spanned by $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ in Figure 2). This boils down to transforming the integral in Equation (103) into

$$\langle\!\langle f \rangle\!\rangle = \int_0^{2\pi} \frac{1}{2\pi}\, f(\boldsymbol{m}(\pi/2,\varphi))\, \sin(\pi/2)\, \mathrm{d}\varphi = \frac{1}{2\pi} \int_0^{2\pi} f(\boldsymbol{m}(\pi/2,\varphi))\, \mathrm{d}\varphi, \tag{105}$$

where $1/2\pi$ is the constant value of the probability density on the equatorial unit circumference $(\mathbb{S}^1\mathcal{B}, \boldsymbol{m}_0)$, and equals the reciprocal of the amplitude of the interval $[0, 2\pi]$ within which the longitude φ varies. Considering Equation (105), the components of the directional averages $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$ and $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$ in Equation (97) can be obtained and represented via the Walpole arrays

$$\overline{\langle\!\langle \mathbb{U}_{11} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, 2,\, 0\right\}, \qquad \overline{\langle\!\langle \mathbb{U}_{12} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 0 & 0 \\ 4 & 2\sqrt{2} \end{bmatrix}, -\sqrt{2},\, 0\right\}, \tag{106a}$$

$$\overline{\langle\!\langle \mathbb{U}_{21} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 0 & 4 \\ 0 & 2\sqrt{2} \end{bmatrix}, -\sqrt{2},\, 0\right\}, \qquad \overline{\langle\!\langle \mathbb{U}_{22} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{bmatrix}, 1,\, 0\right\}, \tag{106b}$$

$$\overline{\langle\!\langle \mathbb{V}_{1} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 1,\, 4\right\}, \qquad \overline{\langle\!\langle \mathbb{V}_{2} \rangle\!\rangle} = \frac{1}{8}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 4,\, 4\right\}. \tag{106c}$$

For the case of random orientation, we follow the approach we used in the past (Federico et al., 2004), with some minor notational and procedural differences. When the probability density

is given by $\psi(\boldsymbol{m}(\theta,\varphi)) = \varrho(\theta,\varphi) = 1/4\pi$ (random orientation), the integral in Equation (92) *must* coincide with its isotropic projection (see Equation (28)), i.e., must necessarily be isotropic. Thus, we have the identity

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{T}(\boldsymbol{m}) \equiv \left[\int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{T}(\boldsymbol{m})\right]_{\text{iso}} = \langle\!\langle \mathbb{T} \rangle\!\rangle_{\text{iso}}. \tag{107}$$

Since the operation of isotropic projection (28) and the averaging integral commute, we can also write

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, \mathbb{T}(\boldsymbol{m}) \equiv \int_{\mathbb{S}^2\mathcal{B}} \psi(\boldsymbol{m})\, [\mathbb{T}(\boldsymbol{m})]_{\text{iso}} = \langle\!\langle \mathbb{T}_{\text{iso}} \rangle\!\rangle. \tag{108}$$

Moreover, the dependence on $\boldsymbol{m}$ in $[\mathbb{T}(\boldsymbol{m})]_{\text{iso}}$ *must* disappear, as $[\mathbb{T}(\boldsymbol{m})]_{\text{iso}}$ is isotropic. Thus, we can replace $[\mathbb{T}(\boldsymbol{m})]_{\text{iso}}$ by $[\mathbb{T}(\boldsymbol{m}_0)]_{\text{iso}}$, where $\boldsymbol{m}_0$ is an *arbitrary* direction, and factorise $[\mathbb{T}(\boldsymbol{m}_0)]_{\text{iso}}$ *outside* of the integral sign, to obtain the final expression

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = [\mathbb{T}(\boldsymbol{m}_0)]_{\text{iso}}\,, \tag{109}$$

where we used the normalisation to one of the probability density. Since $\mathbb{T}(\boldsymbol{m}_0) = \overline{\mathbb{T}}^{\alpha\beta}\, \mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0) + \overline{\mathbb{T}}^{\gamma}\, \mathbb{V}_{\gamma}(\boldsymbol{m}_0)$, we can use linearity and write

$$\langle\!\langle \mathbb{T} \rangle\!\rangle = \overline{\mathbb{T}}^{\alpha\beta}\, [\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0)]_{\text{iso}} + \overline{\mathbb{T}}^{\gamma}\, [\mathbb{V}_{\gamma}(\boldsymbol{m}_0)]_{\text{iso}}\,. \tag{110}$$

The isotropic projections $[\mathbb{U}_{\alpha\beta}(\boldsymbol{m}_0)]_{\text{iso}}$ and $[\mathbb{V}_{\gamma}(\boldsymbol{m}_0)]_{\text{iso}}$ can be written either in the isotropic basis $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ or in Walpole's transversely isotropic representation as seen in Equation (42). In the former case, we have

$$[\mathbb{U}_{11}]_{\text{iso}} = \frac{1}{3}\,\mathbb{K}^\sharp + \frac{2}{15}\,\mathbb{M}^\sharp, \qquad [\mathbb{U}_{12}]_{\text{iso}} = \frac{\sqrt{2}}{3}\,\mathbb{K}^\sharp - \frac{\sqrt{2}}{15}\,\mathbb{M}^\sharp, \tag{111a}$$

$$[\mathbb{U}_{21}]_{\text{iso}} = \frac{\sqrt{2}}{3}\,\mathbb{K}^\sharp - \frac{\sqrt{2}}{15}\,\mathbb{M}^\sharp, \qquad [\mathbb{U}_{22}]_{\text{iso}} = \frac{2}{3}\,\mathbb{K}^\sharp + \frac{1}{15}\,\mathbb{M}^\sharp, \tag{111b}$$

$$[\mathbb{V}_{1}]_{\text{iso}} = 0\,\mathbb{K}^\sharp + \frac{2}{5}\,\mathbb{M}^\sharp, \qquad [\mathbb{V}_{2}]_{\text{iso}} = 0\,\mathbb{K}^\sharp + \frac{2}{5}\,\mathbb{M}^\sharp, \tag{111c}$$

and, in the latter, we have

$$\overline{[\mathbb{U}_{11}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}, 2,\, 2\right\}, \qquad \overline{[\mathbb{U}_{12}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} \sqrt{2} & 4 \\ 4 & 3\sqrt{2} \end{bmatrix}, -\sqrt{2},\, -\sqrt{2}\right\}, \tag{112a}$$

$$\overline{[\mathbb{U}_{21}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} \sqrt{2} & 4 \\ 4 & 3\sqrt{2} \end{bmatrix}, -\sqrt{2},\, -\sqrt{2}\right\}, \qquad \overline{[\mathbb{U}_{22}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} 4 & 3\sqrt{2} \\ 3\sqrt{2} & 7 \end{bmatrix}, 1,\, 1\right\}, \tag{112b}$$

$$\overline{[\mathbb{V}_{1}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 6,\, 6\right\}, \qquad \overline{[\mathbb{V}_{2}]_{\text{iso}}} = \frac{1}{15}\left\{\begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 6,\, 6\right\}. \tag{112c}$$

5 Discussion

In this chapter, we summarised and discussed in detail some selected results from previous studies of ours, with the purpose of rephrasing in a more efficient, consequent, and formally correct way the linear elastic formulation of our picture of composite materials with statistically oriented spheroidal inclusions.

After presenting the covariant formulation of the linear algebra of isotropic and transversely isotropic second- and fourth-order tensors, we addressed some fundamental aspects of composite materials with aligned inclusions, which required to review Eshelby's inclusion problem, Eshelby's fourth-order tensor $\mathbb{S}$, and the strain concentration tensor $\mathbb{A}$. Within this framework, we discussed the conditions, pertaining the geometry of the inclusions and their orientation as well as the material symmetries of the matrix, which lead to a globally transversely isotropic (or, in some special cases, isotropic) composite. Then, we considered composite materials with statistically oriented inclusions. To this end, we introduced the probability density describing the probability that the symmetry axis of an inclusion is in a given direction, and generalised Walpole's formula (70) (Walpole, 1966a,b, 1969; Weng, 1990) to the case of transversely isotropic materials with respect to a symmetry axis $\boldsymbol{m}_0$ (cf. Equation (99)). In order to achieve this and to minimise the number of integrals to be performed, we translated the directional averaging of tensor functions depending on the direction into Walpole's formalism, and obtained Equation (100), which determines the Walpole array of the directional average of a given fourth-order tensor $\mathbb{T}$. Finally, we showed some explicit calculations for the relevant cases of isotropy and transverse isotropy.

It is important to emphasise the difference in terms of conditions necessary to obtain transverse isotropy between Walpole's original formula and the generalised one. Walpole's original formula (70) necessitates only the two conditions (A1) and (A2) seen in Section 3.3 to be applicable to transversely isotropic materials, in the Walpole array form of Equation (74). In contrast, the generalised Walpole's formula (86) considers N_a families of aligned inclusions and N_p families of statistically oriented fibres, and the three conditions (S1), (S2) and (S3) of Section 4.2 are needed. While condition (S1) is identical to the "old" condition (A1), and (S2) echoes condition (A2), but only for the N_a aligned phases, a new condition (S3) needs to be stated for the N_p statistically oriented families (i.e., the N_p probability densities ψ_s must be transversely isotropic with respect to $\boldsymbol{m}_0$).

The theory of composite materials with statistical orientation of the inclusions is a rich research field in which very diverse scientific interests converge. The trigger of our studies has been the mechanical characterisation of soft biological tissues. These are highly organised media, endowed with a complex internal structure, whose mechanical properties are vastly influenced by the presence and orientation of collagen fibres. Tendons and ligaments are typical examples of tissues in which the collagen fibres are aligned, and blood vessels and articular cartilage are examples of tissues in which the collagen fibres have statistical orientation. Collagen fibres can indeed be viewed as inclusions that provide structural reinforcement to the non-fibrous extracellular matrix, and modulate several important bio-chemo-mechanical processes, which involve, for instance, the flow of interstitial fluids as well as the diffusive-reactive dynamics of the chemical species populating the tissues (nutrients and outputs of chemical reactions). These processes are associated with both second- and fourth-order tensor quantities that, depending on the (either statistical or not) arrangement of the fibres, can be represented by using the methods outlined in Sections 3 and 4. In the case of statistical orientation, the directional average defined in Equation (92) takes a tensor describing how a given quantity is associated with the spatial direction $\boldsymbol{m}$ of local fibre alignment, and returns the overall tensor quantity defined in one point of the tissue. This allows for obtaining microstructurally based constitutive laws and puts in evidence how the evolution of the tissue's internal structure yields an evolution of the averaged tensor quantity $\langle\!\langle \mathbb{T} \rangle\!\rangle$ associated with the considered material property.

In two previous papers of ours (Grillo et al., 2012, 2015), we proposed a theory of remodelling in fibre-reinforced materials, where by "remodelling" we mean here the structural reorganisation of a body, be it a tissue or a non-biological material. In this theory, the evolution of the internal

structure of a given medium was described by the time change of the probability density ψ featuring in the averaging integral (92). Under the hypothesis that the evolution of ψ does not modify the transverse isotropy of the material with respect to the direction $\boldsymbol{m}_0$ (in fact, this requires ψ to evolve by maintaining itself transversely isotropic, i.e., by maintaining itself independent of the longitude angle, in the spherical coordinate setting of Section 4.4), the use of Walpole's notation in Equation (100) makes it possible to isolate the effect of remodelling on the averaged tensor quantity $\langle\!\langle \mathbb{T} \rangle\!\rangle$, expressed in terms of the array $\overline{\langle\!\langle \mathbb{T} \rangle\!\rangle}$. Indeed, while the averaged tensors of the Walpole's basis for transverse isotropy with respect to the generic direction $\boldsymbol{m}$, i.e., $\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle$ and $\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle$, evolve in time as they are driven by the time change of ψ, the components $\overline{\mathbb{T}}^{\mu\nu}$ and $\overline{\mathbb{T}}^{\pi}$ do not. In turn, since the direction $\boldsymbol{m}_0$ is assumed to be preserved by the considered remodelling process, only the arrays $\{[\overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\alpha\beta}], \overline{\langle\!\langle \mathbb{U}_{\mu\nu} \rangle\!\rangle}^{\gamma}\}$, $\{[\overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\alpha\beta}], \overline{\langle\!\langle \mathbb{V}_{\pi} \rangle\!\rangle}^{\gamma}\}$ vary in time. In conclusion, by adopting Walpole's arrays, it is possible to study the influence of remodelling on a global property, expressed by the averaged fourth-order tensor $\langle\!\langle \mathbb{T} \rangle\!\rangle$ (for example, the fourth-order elasticity tensor of the considered medium), by looking at the evolution of the components of the averages of the Walpole's basis tensors $\mathbb{U}_{\mu\nu}$ and $\mathbb{V}_{\pi}$. This subject is among the topics of our current investigations.

Acknowledgements

At the time in which we originally conceived the methods presented in this chapter, back in 2003 (see Federico et al., 2004), we were quite younger and less experienced than we are now, and we received help from several senior colleagues, whom we would like to acknowledge again. We are grateful to Prof. L.J. Walpole (East Anglia University) and Prof. G.J. Weng (Rutgers University) for the precious suggestions and crucial references they provided, to Prof. F. Santoro (Liceo Scientifico Archimede, Messina) for the invaluable tutorials on the use of *Wolfram Mathematica*. We would also like to thank Prof. W. Herzog (The University of Calgary) for his guidance and contributions to the original work (Federico et al., 2004). Finally, we would like to acknowledge the contribution of Mr. M. F. Alhasadi (The University of Calgary) to the material contained in Sections 3.1 and 3.2, which is a revisitation of the classical work by Eshelby and of the derivation of the strain concentration tensor, developed as part of his PhD thesis work (see Alhasadi and Federico, 2017).

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In Memoriam

This work is dedicated to the memory of our Maestro and Mentor Prof. Gaetano Giaquinta (Catania, Italy, 25 November 1945 – 13 August 2016). Prof. Giaquinta was professor of Structure of Matter at the Università di Catania (Italy), and was our teacher of Physics and Alfio's PhD supervisor. He had a profound influence on our scientific as well as humanistic formation, as he taught us how to look at things in life. Among the many other things, his input for the original work that we are revisiting in this chapter has been of fundamental importance. It is not possible to properly express our gratitude to Prof. Giaquinta for all he has meant, means and will mean in our lives.

References

Alhasadi, M. F. and Federico, S. (2017). Relation between Eshelby stress and Eshelby fourth-order tensor within an ellipsoidal inclusion. *Acta Mech.*, 228:1045–1069.

Bhattacharyya, A. and Weng, G. J. (1994). Plasticity of isotropic composites with randomly oriented and packeted inclusions. *Int. J. Plasticity*, 10:553–578.

Bishop, R. L. and Goldberg, S. I. (1968). *Tensor Analysis on Manifolds*. Prentice-Hall, NJ, USA.

Bonet, J. and Wood, R. D. (2008). *Nonlinear Continuum Mechanics for Finite Element Analysis (Second Edition)*. Cambridge University Press, Cambridge, UK.

Chou, T. W. and Nomura, S. (1981). Fibre orientation effects on the thermoelastic properties of short-fibre composites. *Fibre Sci. Technol.*, 14:279–291.

Curnier, A., He, Q.-C., and Zysset, P. (1995). Conewise linear elastic materials. *J. Elasticity*, 37:1–38.

Epstein, M. (2010). *The Geometrical Language of Continuum Mechanics*. Cambridge University Press, Cambridge, UK.

Eshelby, J. D. (1951). The force on an elastic singularity. *Philos. T. Roy. Soc. A*, 244:87–112.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. Roy. Soc. Lond. A*, 241:376–396.

Eshelby, J. D. (1975). The elastic energy-momentum tensor. *J. Elasticity*, 5:321–335.

Federico, S. (2010a). On the linear elasticity of porous materials. *Int. J. Mech. Sci.*, 52:175–182.

Federico, S. (2010b). Volumetric-distortional decomposition of deformation and elasticity tensor. *Math. Mech. Solids*, 15:672–690.

Federico, S. (2012). Covariant formulation of the tensor algebra of non-linear elasticity. *Int. J. Non-Linear Mech.*, 47:273–284.

Federico, S. (2015). Porous materials with statistically oriented reinforcing fibres. In Dorfmann, L. and Ogden, R. W., editors, *Nonlinear Mechanics of Soft Fibrous Materials*, pages 49–120. Springer, Berlin. CISM Courses and Lectures No. 559, International Centre for Mechanical Sciences.

Federico, S. and Grillo, A. (2012). Elasticity and permeability of porous fibre-reinforced materials under large deformations. *Mech. Mat.*, 44:58–71.

Federico, S., Grillo, A., and Herzog, W. (2004). A transversely isotropic composite with a statistical distribution of spheroidal inclusions: a geometrical approach to overall properties. *J. Mech. Phys. Solids*, 52:2309–2327.

Federico, S., Grillo, A., and Imatani, S. (2015). The linear elasticity tensor of incompressible materials. *Math. Mech. Solids*, 20:643–662.

Federico, S., Grillo, A., La Rosa, G., Giaquinta, G., and Herzog, W. (2005). A transversely isotropic, transversely homogeneous microstructural-statistical model of articular cartilage. *J. Biomech.*, 38:2008–2018.

Grillo, A., Federico, S., and Wittum, G. (2012). Growth, mass transfer, and remodeling in fiber-reinforced, multi-constituent materials. *Int. J. Non-Linear Mech.*, 47:388–401.

Grillo, A., Wittum, G., Tomic, A., and Federico, S. (2015). Remodelling in statistically oriented fibre-reinforced composites and biological tissues. *Math. Mech. Solids*, 20:1107–1129.

Gurtin, M. E. (1972). The linear theory of elasticity. In Truesdell, C. A., editor, *Handbuch der Physik, Vol. IVa/2*. Springer.

Gurtin, M. E., Fried, E., and Anand, L. (2010). *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, Cambridge, UK.

Hashin, Z. (1963). Theory of mechanical behavior of heterogeneous media. *Appl. Mech. Rev.*, 17:1–9.

Hill, R. (1963). Elastic properties of reinforced solids: some theoretical principles. *J. Mech. Phys. Solids*, 11:357–372.

Hill, R. (1964). Theory of mechanical properties of fibre-strengthned materials - I. Elastic beheviour. *J. Mech. Phys. Solids*, 12:199–212.

Hill, R. (1965). A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids*, 13:213–222.

Holmes, M. H. and Mow, V. C. (1990). The nonlinear characteristics of soft gels and hydrated connective tissues in ultrafiltration. *J. Biomech.*, 23:1145–1156.

Jog, C. S. (2006). A concise proof of the representation theorem for fourth-order isotropic tensors. *J. Elasticity*, 85:119–124.

Kolmogorov, A. N. and Fomin, S. V. (1999). *Elements of the Theory of Functions and Functional Analysis*. Dover.

Marsden, J. E. and Hughes, T. J. R. (1983). *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliff, NJ, USA.

Mura, T. (1987). *Micromechanics of Defects in Solids*. The Hague: Martinus Nijhoff.

Qiu, Y. P. and Weng, G. J. (1990). On the application of Mori-Tanaka's theory involving transversely isotropic spheroidal inclusions. *Int. J. Eng. Sci.*, 28:1121–1137.

Segev, R. (2013). Notes on metric independent analysis of classical fields. *Math. Meth. Appl. Sci.*, 36:497–566.

Soltz, M. A. and Ateshian, G. A. (2000). A conewise linear elasticity mixture model for the analysis of tension-compression nonlineartiy in articular cartilage. *J. Biomech. Eng.*, 122:576–586.

Spencer, A. J. M. (1984). Constitutive theory for strongly anisotropic solids. In Spencer, A. J. M., editor, *Continuum Theory of the Mechanics of Fibre-Reinforced Composites*, pages 1–32. Springer-Verlag, Wien, Austria. CISM Courses and Lectures No. 282, International Centre for Mechanical Sciences.

Tandon, G. P. and Weng, G. J. (1986). Average stress in the matrix and effective moduli of randomly oriented composites. *Compos. Sci. Technol.*, 27:111–132.

Walpole, L. J. (1966a). On bounds for the overall elastic moduli of inhomogeneous systems - I. *J. Mech. Phys. Solids*, 14:151–162.

Walpole, L. J. (1966b). On bounds for the overall elastic moduli of inhomogeneous systems - II. *J. Mech. Phys. Solids*, 14:289–301.

Walpole, L. J. (1969). On the overall elastic moduli of composite materials. *J. Mech. Phys. Solids*, 17:235–251.

Walpole, L. J. (1981). Elastic behavior of composite materials: Theoretical foundations. *Adv. in Appl. Mech.*, 21:169–242.

Walpole, L. J. (1984). Fourth-rank tensors of the thirty-two crystal classes: Multiplication tables. *Proc. Roy. Soc. Lond. A*, 391:149–179.

Weng, G. J. (1984). Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions. *Int. J. Eng. Sci.*, 22:845–856.

Weng, G. J. (1990). The theoretical connection between Mori-Tanaka's theory and the Hashin-Shtrikman-Walpole bounds. *Int. J. Eng. Sci.*, 28:1111–1120.

Wu, J. Z. and Herzog, W. (2002). Elastic anisotropy of articular cartilage is associated with the micro-structures of collagen fibers and chondrocytes. *J. Biomech.*, 35:931–942.