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CONVERGENCE IN L^p FOR FEYNMAN PATH INTEGRALS

FABIO NICOLA

Abstract. We consider a class of Schrödinger equations with time-dependent smooth magnetic and electric potentials having a growth at infinity at most linear and quadratic, respectively. We study the convergence in L^p with loss of derivatives, $1 < p < \infty$, of the time slicing approximations of the corresponding Feynman path integral. The results are completely sharp and hold for long time, where no smoothing effect is available. The techniques are based on the decomposition and reconstruction of functions and operators with respect to certain wave packets in phase space.

1. Introduction

Feynman path integrals were introduced in 1948 [20, 21] to provide a new formulation of Quantum Mechanics and nowadays represent a fundamental tool in most branches of modern Physics. In particular, R. Feynman suggested the construction of the integral kernel $K(t, s, x, y)$ of the Schrödinger propagator as a suggestive sum-over-histories, in the following sense. First of all the kernel $K(t, s, x, y)$ itself is interpreted as the probability amplitude for a particle to be at the point x at time t provided it was at y at time s ($x, y \in \mathbb{R}^d$). Now, in the computation of this quantity every path γ joining y and x, therefore satisfying $\gamma(s) = y$, $\gamma(t) = x$, carries a contribution which is proportional to $e^{i\hbar^{-1}S[\gamma]}$, where $S[\gamma]$ is the action along the path γ:

$$S[\gamma] = \int_s^t \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau), \tau)\, d\tau,$$

$\mathcal{L}$ being the Lagrangian of the corresponding classical system. The total amplitude is finally obtained by superposition and can be written symbolically as an integral

$$K(t, s, x, y) = \int e^{i\hbar^{-1}S[\gamma]} \mathcal{D}[\gamma]$$

over the space of paths satisfying the above boundary conditions. Although a suitable measure on this space does not exist in the measure theoretic sense (cf. [7]), several rigorous justifications have been proposed by many authors and from different

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viewpoints (analytic continuation of the parabolic propagator, infinite dimensional oscillatory integrals, stochastic integrals, etc.). The literature is enormous and we refer to the books [1,49,51,53] and the references therein. Instead here we focus on the original approach by Feynman [20, 21] via time slicing approximations, which was carried on in a rigorous way in the papers [22,23,26–30,32,33,38,39,43–47,57] (see also [24, 25, 56]). Briefly one argues as follows. Suppose that for $|t-s|$ small enough there is only one classical path γ (i.e. a path satisfying the Euler-Lagrange equation) satisfying the boundary condition $\gamma(s)=y$, $\gamma(t)=x$. Define then the action

$$S(t,s,x,y)=\int_s^t \mathcal{L}(\gamma(\tau),\dot\gamma(\tau),\tau)\,d\tau, \tag{1}$$

along that path.

Consider the operator $E^{(0)}(t,s)$ defined by

$$E^{(0)}(t,s)f(x)=\frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)}f(y)\,dy. \tag{2}$$

The idea is that this operator should represent a good approximation of the actual propagator when $|t-s|$ is small (in fact, for the free particle $E^{(0)}(t,s)$ coincides with the exact propagator). In general one then considers a subdivision $\Omega: s=t_0<t_1<\ldots<t_L=t$ of the interval $[s,t]$ and the composition

$$E^{(0)}(\Omega,t,s)=E^{(0)}(t,t_{L-1})E^{(0)}(t_{L-1},t_{L-2})\ldots E^{(0)}(t_1,s), \tag{3}$$

which has integral kernel

$$\begin{aligned} &K^{(0)}(\Omega,t,s,x,y)\\ &=\prod_{j=1}^L \frac{1}{(2\pi i(t_j-t_{j-1})\hbar)^{d/2}}\int_{\mathbb{R}^{d(L-1)}}\exp\Big(i\hbar^{-1}\sum_{j=1}^L S(t_j,t_{j-1},x_j,x_{j-1})\Big)\prod_{j=1}^{L-1}dx_j, \end{aligned} \tag{4}$$

with $x=x_L$ and $y=x_0$; see Figure 1.

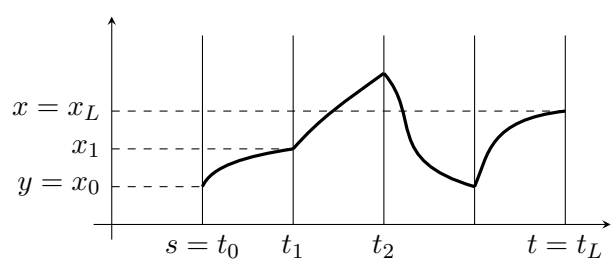

FIGURE 1. A piecewise classical path in spacetime.

Feynman stated that the operator $E^{(0)}(\Omega,t,s)$ converges (in some sense) to the actual propagator as $\omega(\Omega):=\sup\{t_j-t_{j-1}:j=1,\dots,L\}\to 0$. In view of the semi-classical approximation one also introduces higher order parametrices $E^{(N)}(t,s)$, $N=1,2\dots$, and the corresponding time slicing approximations $E^{(N)}(\Omega,t,s)$ (see Section 2 below), and set the same convergence problem. In the present note we investigate this issue for the uniform topology of bounded operators on L^p-based Sobolev spaces.

We consider the Schrödinger equation

$$i\hbar\partial_t u=-\frac{1}{2}\hbar^2\Delta u+V(t,x)u \tag{5}$$

where $0<\hbar\le 1$ and the potential $V(t,x)$, $t\in\mathbb{R}$, $x\in\mathbb{R}^d$, satisfies the following hypothesis.

Assumption (A) *$V(t,x)$ is a real function of $(t,x)\in\mathbb{R}\times\mathbb{R}^d$ with $\partial_x^\alpha V(t,x)$ continuous in $(t,x)\in\mathbb{R}\times\mathbb{R}^d$, for every*[1] *$\alpha\in\mathbb{N}_0^d$ and satisfying*

$$|\partial_x^\alpha V(t,x)|\le C_\alpha,\quad |\alpha|\ge 2,\ (t,x)\in\mathbb{R}\times\mathbb{R}^d.$$

Under this hypothesis the exact propagator $U(t,s)$ was constructed in [22, 23]: it turns out that, *for $|t-s|$ small enough*, $U(t,s)$ is an oscillatory integral operator of the form

$$U(t,s)f(x)=\frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d}e^{i\hbar^{-1}S(t,s,x,y)}b(\hbar,t,s,x,y)f(y)\,dy \tag{6}$$

for some amplitude b such that $\partial_x^\alpha\partial_y^\beta b(\hbar,t,s,x,y)$ is bounded, for every $\alpha,\beta\in\mathbb{N}_0^d$, $0<\hbar\le 1$. Instead, for $|t-s|$ large, $U(t,s)$ can be written as the composition of a finite number of oscillatory integral operators as above.

Moreover $E^{(N)}(\Omega,t,s)$ (for fixed N) was also shown to converge in a quite strong topology to the actual propagator as $\omega(\Omega)\to 0$ when $|t-s|$ is small enough, which implies the convergence in the uniform topology of bounded operators on $L^2(\mathbb{R}^d)$. While we are not aware of similar results in $L^p(\mathbb{R}^d)$ for $p\ne 2$, it is clear that in that case the propagator is in general not even bounded on L^p but a loss of derivatives occurs. For example, for the free propagator we have

$$\|e^{i\hbar\Delta}f\|_{L^p}\le C\|(1-\hbar\Delta)^{k/2}f\|_{L^p},\quad k=2d|1/2-1/p|,\ 1<p<\infty.$$

This estimate was proved in [50] in the case $\hbar=1$, whereas the general case follows at once by a scaling argument. Generalizations to different classes of potentials were proved, except for the endpoint, in [4, 6, 8, 16, 34, 35], exploiting in an essential way some smoothing effect (Gaussian estimates for the heat kernel). The above

[1] We set $\mathbb{N}_0=\{0,1,2\dots\}$, containing the number 0.

loss of derivatives is optimal [5] and is explained by the fact that the characteristic manifold of the Schrödinger operator, that is a paraboloid, has non-vanishing Gaussian curvature. For hyperbolic equations [18, 52, 54] we have instead the loss $k = (d-1)|1/2 - 1/p|$, because there is one flat direction in that case.

Motivated by this model, we will focus on the convergence of the time slicing approximations in the following Sobolev spaces.

For $1 < p < \infty$, $k \in \mathbb{R}$, define the space

$$\tilde{L}^p_k = \tilde{L}^p_k(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\tilde{L}^p_k} = \|(1-\hbar\Delta)^{k/2} f\|_{L^p} < \infty\}. \tag{7}$$

Hence, $\tilde{L}^p_k$ is the usual Sobolev space as a vector space, but with a rescaled norm at the Planck scale. Here is our main result.

Theorem 1.1. *Assume the condition in Assumption* (A). *Let* $1 < p < \infty$, $k = 2d|1/2 - 1/p|$.

a) *For every* $T_0 > 0$ *there exists a constant* $C(T_0) > 0$ *such that, for all* $f \in \mathcal{S}(\mathbb{R}^d)$, $|s-t| \le T_0$, $0 < \hbar \le 1$,

$$\|U(t,s)f\|_{L^p} \le C(T_0)\|f\|_{\tilde{L}^p_k}, \quad 1 < p \le 2, \tag{8}$$

$$\|U(t,s)f\|_{\tilde{L}^p_{-k}} \le C(T_0)\|f\|_{L^p}, \quad 2 \le p < \infty. \tag{9}$$

b) *For every* $T_0 > 0$, $N = 0, 1, 2, \ldots$, *there exists a constant* $C(T_0) > 0$ *such that, for* $0 < t - s \le T_0$ *and any sufficiently fine subdivision* Ω *of the interval* $[s,t]$, $f \in \mathcal{S}(\mathbb{R}^d)$, $0 < \hbar \le 1$, *we have*

$$\|\big(E^{(N)}(\Omega,t,s) - U(t,s)\big)f\|_{L^p} \le C(T_0)\hbar^N \omega(\Omega)^{N+1}(t-s)\|f\|_{\tilde{L}^p_k},\ 1 < p \le 2, \tag{10}$$

$$\|\big(E^{(N)}(\Omega,t,s) - U(t,s)\big)f\|_{\tilde{L}^p_{-k}} \le C(T_0)\hbar^N \omega(\Omega)^{N+1}(t-s)\|f\|_{L^p},\ 2 \le p < \infty. \tag{11}$$

We will show in the last section that these are, in general, all possible L^p-estimates for Schrödinger propagators of the above type. Extensions to the case of magnetic fields (cf. [26, 56, 57]) or even rough potentials will be discussed below.

Let us now say a few words about the strategy of the proof. There are two main issues. First, for *long time* the propagator does not have the integral representation (6) anymore and in general no smoothing effect is available. For example, for the quadratic potential $V(t,x) = |x|^2/2$ (and $\hbar = 1$ for simplicity), the propagator $U(t,0)$ at time $t = k\pi$, $k \in \mathbb{Z}$, has distribution kernel $e^{-ik\pi/2}\delta((-1)^k x - y)$ (see e.g. [37]). Secondly, the space of bounded operators $\tilde{L}^p_k \to L^p$ is of course not an algebra, and hence it does not behave nicely with respect to compositions as those in (20). To overcome these problems we lift the whole analysis to the phase space, using ideas originated by the seminal papers [19, 42, 48, 55] and developed by the

author et al. in [9, 11] (see also [13]). The key tool is a space of operators highly sparse with respect to certain wave packet decompositions. To be precise, consider a fixed Schwartz function $g \not\equiv 0$, and the wave packet

$$\pi(z)g(y) = e^{iy\xi}g(y-x), \quad z = (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$$

($y\xi = y \cdot \xi = \sum_{k=1}^d y_k\xi_k$) which is highly concentrated, in phase space, near (x,ξ). We will consider operators T satisfying (when $\hbar = 1$)

$$|\langle T\pi(z)g, \pi(w)g\rangle| \le C_m(1+|w-\chi(z)|)^{-m}, \quad z,w \in \mathbb{R}^{2d},$$

for every $m \in \mathbb{N}_0$ and for some canonical transformation $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ ($\langle\cdot,\cdot\rangle$ being the inner product in $L^2(\mathbb{R}^d)$). Composition of operators corresponds to the composition of the corresponding maps χ's. Moreover these operators are trivially bounded on the so-called modulation spaces M^p, $1 \le p \le \infty$ (see [17, 31] and Section 3 below), which measure the phase space concentration of a function. We finally come back to L^p spaces by means of a nontrivial embedding recently obtained in [41] (see also [40]).

While endpoint continuity results for $p \neq 2$ typically require a painful analysis in some version of the Hardy space (see e.g. [19, 50, 52, 54]), one of the novelties of this approach is that such technicalities are hidden under the above mentioned embedding and the analysis is focused on how the operators move the wave packets in phase space, which makes matters remarkably simple and close to intuition. Moreover, in view of the results in [10, 14, 48], we believe that the present approach could also extend to the case of rough potentials (and degenerate operators), which would provide an interesting low regularity version of the results of [23, 57]. We plan to investigate these issues in a subsequent paper (see also the last section for some partial result in this connection).

Briefly, the paper is organized as follows. In Section 2 we recall the short-time analysis of [22, 23, 57]. Section 3 is devoted to the phase space methods used in the following and, in particular, to the class of operators, closed under composition, mentioned above. In Section 4 we will show that the short-time propagators $U(t,s)$ and approximations $E^{(N)}(\Omega,t,s)$ belong to such operator class. Theorem 1.1 will be proved in Section 5. Section 6 treats a generalization concerning the convergence of $E^N(\Omega,t,s)$ with $N \ge 1$ in the presence of a magnetic field satisfying the same assumptions as in [57]. Finally in Section 7 we show the sharpness of the results and provide an extension of the first part of Theorem 1.1 to a class of rough potentials.

2. The Schrödinger flow: short time analysis

In this section we recall the main results obtained in [22, 23]; see also [57]. We will always assume the hypothesis in Assumption (A) in Introduction.

First of all we fix the following notation. We denote by $S^0_{0,0}$ the class of smooth function $a(x,y)$ in $\mathbb{R}^{2d}$ which are bounded together with their derivatives, endowed with the seminorms

$$\|a\|_m = \sup_{|\alpha|+|\beta|\le m} \|\partial_x^\alpha \partial_y^\beta a\|_{L^\infty(\mathbb{R}^{2d})}, \quad m \in \mathbb{N}_0 = \{0,1,\dots\}.$$

Consider now the Hamiltonian

$$H(t,x,\xi) = \frac{1}{2}|\xi|^2 + V(t,x).$$

Denote by $(x(t,s,y,\eta),\xi(t,s,y,\eta))$ $(s,t\in\mathbb{R},\ y,\eta\in\mathbb{R}^d)$, the solution of the corresponding system

$$\dot{x} = \xi, \quad \dot{\xi} = -\nabla_x V(t,x)$$

with initial condition at time $t=s$ given by $x(s,s,y,\eta)=y$, $\xi(s,s,y,\eta)=\eta$. The flow

$$(x(t,s,y,\eta),\xi(t,s,y,\eta)) = \chi(t,s)(y,\eta) \tag{12}$$

defines a smooth canonical transformation $\chi(t,s):\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ satisfying for every $T_0>0$ the estimates

$$|\partial_y^\alpha\partial_\eta^\beta x(t,s,y,\eta)| + |\partial_y^\alpha\partial_\eta^\beta \xi(t,s,y,\eta)| \le C_{\alpha,\beta}(T_0), \quad y,\eta\in\mathbb{R}^d \tag{13}$$

for some constant $C_{\alpha,\beta}(T_0)>0$, if $|t-s|\le T_0$ (see [23, Proposition 1.1]).

Moreover, there exists $\delta>0$ such that for $0<|t-s|\le\delta$ and every $x,y\in\mathbb{R}^d$, there exists only one classical path γ such that $\gamma(s)=y$, $\gamma(t)=x$. By computing the action along this path γ, as in (1), we define the generating function $S(t,s,x,y)$ for $0<|t-s|\le\delta$. It satisfies the estimates

$$|t-s|\,|\partial_x^\alpha\partial_y^\beta S(t,s,x,y)| \le C_{\alpha,\beta}, \quad |\alpha|+|\beta|\ge 2 \tag{14}$$

and

$$|t-s|\left|\det\Big(\frac{\partial^2 S(t,s,x,y)}{\partial y^2}\Big)\right| \ge \tilde{\delta}, \tag{15}$$

for some $\tilde{\delta}>0$ and every $x,y\in\mathbb{R}^d$, always for $0<|t-s|\le\delta$.

We now come to the construction of the parametrices. For $0<|t-s|\le\delta$, define the operator $E^{(0)}(t,s)$ as in (2).

For $N=1,2,\dots$ we define $E^{(N)}(t,s)$ as

$$E^{(N)}(t,s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)} e_N(\hbar,t,s,x,y) f(y)\,dy, \tag{16}$$

with $e_N(\hbar,t,s,x,y) = \sum_{k=1}^N (i\hbar)^{k-1} a_k(t,s,x,y)$, where the amplitudes a_k satisfy the transport equations

$$\frac{\partial a_k}{\partial t} + \sum_{j=1}^d \frac{\partial S}{\partial x_j}\frac{\partial a_k}{\partial x_j} + \frac{1}{2}\Big(\Delta_x S - \frac{d}{t-s}\Big) a_k = \frac{1}{2}\Delta_x a_{k-1},$$

with $a_0(t,s,x,y) \equiv 0$ and the initial conditions $a_1(s,s,x,y) = 1$ and $a_k(s,s,x,y) = 0$ for $k = 1, 2, \dots$

We have, for every $m \in \mathbb{N}_0$,

$$\|a_k(t,s,\cdot,\cdot)\|_m \le C_m \quad \text{for } 0 < |t-s| \le \delta. \tag{17}$$

The operators $E^{(N)}(t,s)$ are parametrices in the sense that, for $N = 0, 1, \dots,$

$$\big(i\hbar\partial_t + \frac{1}{2}\hbar^2\Delta - V(t,x)\big) E^{(N)}(t,s) f = G^{(N)}(t,s) f$$

with

$$G^{(N)}(t,s)f = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)} g_N(\hbar,t,s,x,y) f(y)\, dy, \tag{18}$$

where g_N satisfies the estimates [23, Propositions 1.5, 1.6]

$$\|g_N(\hbar,t,s,\cdot,\cdot)\|_m \le C_m \hbar^{N+1} |t-s|^{N+1}. \tag{19}$$

Moreover, for a subdivision $\Omega : s = t_0 < t_1 < \dots < t_L = t$ we define

$$E^{(N)}(\Omega,t,s) = E^{(N)}(t,t_{L-1}) E^{(N)}(t_{L-1},t_{L-2}) \dots E^{(N)}(t_1,s). \tag{20}$$

As already observed in the introduction, if δ is small enough, for $0 < |t-s| \le \delta$ the propagator has the form

$$U(t,s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)} b(\hbar,t,s,x,y) f(y)\, dy \tag{21}$$

for an amplitude b such that $\partial_x^\alpha \partial_y^\beta b(\hbar,t,s,x,y)$ is of class C^1 in t, s and satisfying

$$\|b(\hbar,t,s,\cdot,\cdot)\|_m \le C_m \tag{22}$$

for $0 < |t-s| \le \delta$, $0 < \hbar \le 1$, $m \in \mathbb{N}_0$.

When $|t-s|$ is large, $U(t,s)$ is the composition of a finite number of such oscillatory integral operators.

3. Phase space methods

We recall here some basic facts of phase space analysis used in the sequel; see [31] and especially [17] for a Mathematical Physics perspective. Then we report on some results from [9, 11, 55].

3.1. **Phase space concepts.** For a point $z=(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$ and a function f on $\mathbb{R}^d$, we denote the phase-space shifts by

$$\pi(z)f(y)=e^{iy\xi}f(y-x),\quad \text{where}\quad y\xi=y\cdot\xi=\sum_{k=1}^d y_k\xi_k\,.$$

The short-time Fourier transform (STFT), or Bargmann transform, or FBI transform, of a function/distribution f on $\mathbb{R}^d$ with respect to a Schwartz window function $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ is defined by

$$V_gf(z)=\langle f,\pi(z)g\rangle=\int_{\mathbb{R}^d}f(y)\overline{g(y-x)}e^{-iy\xi}\,dy, \tag{23}$$

for $z=(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$. Its adjoint is given by

$$V_g^*F(y)=\int_{\mathbb{R}^{2d}}F(z)\pi(z)g(y)\,dz,\quad y\in\mathbb{R}^d,$$

and we have the inversion formula (see e.g. [31, Corollary 3.2.3])

$$f=\frac{1}{(2\pi)^d\|g\|_{L^2}^2}V_g^*V_gf=\frac{1}{(2\pi)^d\|g\|_{L^2}^2}\int_{\mathbb{R}^{2d}}V_gf(z)\pi(z)g\,dz. \tag{24}$$

For $1\le p\le\infty$ the modulation space $M^p=M^p(\mathbb{R}^d)$ is the space of distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that their STFTs belong to the space $L^p(\mathbb{R}^{2d})$ with norm

$$\|f\|_{M^p}:=\|V_gf\|_{L^p(\mathbb{R}^{2d})}.$$

This definition does not depend on the choice of the window $g\in\mathcal{S}(\mathbb{R}^d), g\neq 0$, and different windows yield equivalent norms on M^p [31, Theorem 11.3.7]. We also have

$$V_g:M^p(\mathbb{R}^d)\to L^p(\mathbb{R}^{2d}),\quad V_g^*:L^p(\mathbb{R}^{2d})\to M^p(\mathbb{R}^d)$$

as bounded operators.

The following embedding results (see [41, Theorems 1.3, 1.4] and also [40]) are crucial in the sequel.

For $1<p<\infty$, $k\in\mathbb{R}$, let $L^p_k=L^p_k(\mathbb{R}^d)$ be the space of distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that $(1-\Delta)^{k/2}f\in L^p$, with the norm

$$\|f\|_{L^p_k}=\|(1-\Delta)^{k/2}f\|_{L^p}.$$

Theorem 3.1. *Let $1<p<\infty$ and $k=2d|1/2-1/p|$. Then we have*

$$M^p\hookrightarrow L^p,\quad L^p_k\hookrightarrow M^p,\quad 1<p\le 2 \tag{25}$$

as well as

$$L^p\hookrightarrow M^p,\quad M^p\hookrightarrow L^p_{-k},\quad 2\le p<\infty.$$

3.2. **Canonical transformations and the associated operators.** We now introduce a special class of operators, defined in terms of their sparsity with respect to Gabor wave packets and a given canonical transformation.

A canonical transformation $(x, \xi) = \chi(y, \eta)$ will be called *tame* if it enjoys the following two properties:

A1. $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is smooth, invertible, and preserves the symplectic form in $\mathbb{R}^{2d}$, i.e., $dx \wedge d\xi = dy \wedge d\eta$; χ is a *symplectomorphism.*

A2. We have

$$|\partial_y^\alpha \partial_\eta^\beta \chi(y, \eta)| \leq C_{\alpha,\beta}, \quad |\alpha| + |\beta| \geq 1, \ y, \eta \in \mathbb{R}^d. \tag{26}$$

For example, the canonical transformation $\chi(t, s)$ in (12) satisfies *A1* and *A2*, with constants $C_{\alpha,\beta} = C_{\alpha,\beta}(T_0)$, provided $|t - s| \leq T_0$ ($T_0 > 0$ being arbitrary), in view of (13).

Let us observe that *A1* and *A2* imply that χ and χ^{-1} are globally Lipschitz. With the notation $\langle z \rangle = (1 + |z|^2)^{1/2}$ (Japanese bracket), this property implies that

$$C^{-1} \langle w - \chi(z) \rangle \leq \langle \chi^{-1}(w) - z \rangle \leq C \langle w - \chi(z) \rangle \qquad w, z \in \mathbb{R}^{2d},$$

for some constant $C > 0$ *depending only on an upper bound for the first derivatives of* χ.

The following class of operators was introduced in [55].

Definition 3.2. *Let χ be a tame canonical transformation. Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. We denote by $FIO(\chi)$ the class of operators $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ such that, for every*[2] *$m \geq 0$,*

$$|\langle T\pi(z)g, \pi(w)g \rangle| \leq C_m \langle w - \chi(z) \rangle^{-m}, \quad z, w \in \mathbb{R}^{2d}. \tag{27}$$

We endow this space with the seminorms

$$\|T\|_{m,\chi} = \sup_{z,w \in \mathbb{R}^{2d}} \langle w - \chi(z) \rangle^m |\langle T\pi(z)g, \pi(w)g \rangle|.$$

It is proved in [9] that the definition of $FIO(\chi)$ does not depend on the window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

The following two theorems can also be found in [9], but we provide here the proof for the benefit of the reader, because they were proved there in a discrete framework; moreover here we need some further information on the uniformity of the constants.

[2]In this definition we could of course take m integer; we take m real in view of possible extensions to the case of rough potentials, where the propagators should satisfy the same estimates as in (27) but for a *fixed* $m > 2d$. In that case it is important to allow m to be as small as possible, in order to reach minimal regularity.

Theorem 3.3. *Let $T \in FIO(\chi)$. Then T extends to a bounded operator on $M^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ (and in particular on $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$). Moreover, for every $m > 2d$ there exists a constant $C > 0$ depending only on m and the dimension d such that*

$$\|T\|_{M^p \to M^p} \leq C\|T\|_{m,\chi}.$$

Proof. By the inversion formula for the STFT in (24) we have, if $\|g\|_{L^2} = (2\pi)^{-d/2}$,

$$T = V_g^* V_g T V_g^* V_g.$$

Since $V_g : M^p(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$ and $V_g^* : L^p(\mathbb{R}^{2d}) \to M^p(\mathbb{R}^d)$, it suffices to prove that the operator $V_g T V_g^*$ is bounded on $L^p(\mathbb{R}^{2d})$. Now, this in an integral operator in $\mathbb{R}^{2d}$ with integral kernel

$$K(w, z) = \langle T\pi(z)g, \pi(w)g\rangle.$$

Since $T \in FIO(\chi)$ we have, for every $m \in \mathbb{N}_0$,

$$|V_g T V_g^* F(w)| \leq \|T\|_{m,\chi} \int_{\mathbb{R}^{2d}} \langle w - \chi(z)\rangle^{-m} |F(z)|\, dz.$$

If we take $m > 2d$, the desired continuity on $L^p(\mathbb{R}^{2d})$ follows at once from Schur's test and the fact that the Jacobian determinant of χ is $= 1$, χ being symplectic. □

Theorem 3.4. *If $T^{(i)} \in FIO(\chi_i)$, $i = 1, 2$, then the composition $T^{(1)}T^{(2)}$ is in $FIO(\chi_1 \circ \chi_2)$. Moreover for every $m > 2d$ there exists a constant $C > 0$ depending only on m, the dimension d, and upper bounds for the first derivatives of χ_1 such that*

$$\|T^{(1)}T^{(2)}\|_{m,\chi_1\circ\chi_2} \leq C\|T^{(1)}\|_{m,\chi_1}\|T^{(2)}\|_{m,\chi_2}.$$

Proof. We have to estimate the integral kernel of $V_g T^{(1)} T^{(2)} V_g^*$. We write

$$V_g T^{(1)} T^{(2)} V_g^* = V_g T^{(1)} V_g^* V_g T^{(2)} V_g^*,$$

so that for $m > 2d$

$$\begin{aligned}|\langle T^{(1)}T^{(2)}\pi(z)g, \pi(w)g\rangle| &\leq \|T^{(1)}\|_{m,\chi_1}\|T^{(2)}\|_{m,\chi_2} \int_{\mathbb{R}^{2d}} \langle w - \chi_1(\zeta)\rangle^{-m} \langle \zeta - \chi_2(z)\rangle^{-m} d\zeta \\ &\leq C\|T^{(1)}\|_{m,\chi_1}\|T^{(2)}\|_{m,\chi_2} \int_{\mathbb{R}^{2d}} \langle w - \chi_1(\zeta)\rangle^{-m} \langle \chi_1(\zeta) - \chi_1 \circ \chi_2(z)\rangle^{-m} d\zeta.\end{aligned}$$

The change of variable $\tilde{\zeta} = \chi_1(\zeta) - \chi_1 \circ \chi_2(z)$ and the convolution property

$$\int_{\mathbb{R}^{2d}} \langle w - \zeta\rangle^{-m} \langle \zeta\rangle^{-m}\, d\zeta \leq C_{m,d} \langle w\rangle^{-m}$$

(valid for $m > 2d$, see e.g. [31, Formula (11.5)]) then give the desired estimate. □

The reader may want to compare the simplicity of this result with the similar composition formula obtained in [23, Theorem A.2] for oscillatory integral operators,

whose proof represented really a tour de force and moreover was limited to short time.

We now show that a familiar class of Fourier integral operators is in fact contained in $FIO(\chi)$ for a suitable χ.

A real phase function Φ on $\mathbb{R}^{2d}$ will be called *tame* if the following three properties are satisfied:

B1. $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$;

B2. We have

$$|\partial_x^\alpha \partial_\eta^\beta \Phi(x,\eta)| \le C_{\alpha,\beta}, \quad |\alpha|+|\beta| \ge 2,\ x,\eta \in \mathbb{R}^d; \tag{28}$$

B3. There exists $\tilde{\delta} > 0$ such that

$$|\det \partial^2_{x,\eta}\Phi(x,\eta)| \ge \tilde{\delta}, \quad x,\eta \in \mathbb{R}^d. \tag{29}$$

If we set

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta) \\ \xi = \nabla_x \Phi(x,\eta), \end{cases} \tag{30}$$

we can solve with respect to (x,ξ) by the global inverse function theorem and obtain a mapping χ defined by $(x,\xi) = \chi(y,\eta)$ and satisfying *A1, A2* above as well as the following property:

A3 There exists $\tilde{\delta} > 0$ such that,

$$|\det \frac{\partial x}{\partial y}(y,\eta)| \ge \tilde{\delta}, \quad y,\eta \in \mathbb{R}^d. \tag{31}$$

Conversely, to every transformation χ satisfying *A1*, *A2* and *A3* corresponds a tame phase Φ, uniquely determined up to a constant (see e.g. [9]).

The following result was proved in [11, Theorem 3.3].

Theorem 3.5. *Let $\Phi(x,\eta)$ be a tame phase, and let χ be the corresponding canonical transformation. Let $a(x,\eta)$ be a function in $S^0_{0,0}$. The Fourier integral operator*

$$Tf(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\Phi(x,\eta)} a(x,\eta) \widehat{f}(\eta)\, d\eta$$

belongs to $FIO(\chi)$. Moreover for every $m \in \mathbb{N}_0$ there exists $m' \in \mathbb{N}_0$ such that

$$\|T\|_{m,\chi} \le C\|a\|_{m'}$$

for some constant C depending only on m, the dimension d, upper bounds for a certain number of the derivatives of Φ in (28) *and the lower bound constant $\tilde{\delta}$ in* (29) *(recall $\|a\|_{m'} = \sup_{|\alpha|+|\beta|\le m'} \|\partial_x^\alpha \partial_\eta^\beta a\|_{L^\infty(\mathbb{R}^{2d})}$).*

3.3. **Semiclassical phase space methods.** Here we rephrase in the semiclassical regime the concepts appeared above in this section. Basically we have to rescale function spaces and operators with the dilation operators (unitary on $L^2(\mathbb{R}^d)$)

$$D_{\hbar^{-1/2}}f(x) = \hbar^{d/4}f(\hbar^{1/2}x), \quad D_{\hbar^{1/2}}f(x) = \hbar^{-d/4}f(\hbar^{-1/2}x).$$

We already defined in Introduction the spaces $\tilde{L}^p_k$, which are a rescaled version of the Sobolev spaces L^p_k. For $1 \le p \le \infty$, consider also the modulation spaces

$$M^p_\hbar = M^p_\hbar(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{M^p_\hbar} = \|D_{\hbar^{-1/2}}f\|_{M^p} < +\infty\}.$$

We have the following embeddings.

Theorem 3.6. *Let $1 < p < \infty$ and $k = 2d|1/2 - 1/p|$. There exists a constant $C > 0$ such that*

$$\|f\|_{L^p} \le C\hbar^{d(1/p-1/2)/2}\|f\|_{M^p_\hbar}, \quad \|f\|_{M^p_\hbar} \le C\hbar^{d(1/2-1/p)/2}\|f\|_{\tilde{L}^p_k}, \quad 1 < p \le 2 \tag{32}$$

as well as

$$\|f\|_{M^p_\hbar} \le C\hbar^{d(1/2-1/p)/2}\|f\|_{L^p}, \quad \|f\|_{\tilde{L}^p_{-k}} \le C\hbar^{d(1/p-1/2)/2}\|f\|_{M^p_\hbar}, \quad 2 \le p < \infty. \tag{33}$$

Proof. By (25) we have, for $1 < p \le 2$ and $k = 2d|1/p - 1/2|$,

$$\|f\|_{L^p} \le C\|f\|_{M^p}, \quad \|f\|_{M^p} \le C\|f\|_{L^p_k}.$$

Now we replace f by $D_{\hbar^{-1/2}}f$ and we obtain (32), because

$$\|D_{\hbar^{-1/2}}f\|_{L^p} = \hbar^{d(1/2-1/p)/2}\|f\|_{L^p} \quad \text{and} \quad \|D_{\hbar^{-1/2}}f\|_{L^p_k} = \hbar^{d(1/2-1/p)/2}\|f\|_{\tilde{L}^p_k}.$$

Similarly one deduces (33). ∎

Definition 3.7. *Let χ be a tame canonical transformation. We denote by $FIO_\hbar(\chi)$ the space of linear operators $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ such that $D_{\hbar^{-1/2}}TD_{\hbar^{1/2}} \in FIO(\chi)$, endowed with the seminorms*

$$\|T\|^\hbar_{m,\chi} = \|D_{\hbar^{-1/2}}TD_{\hbar^{1/2}}\|_{m,\chi}.$$

From Theorems 3.3 we obtain at once the following result.

Theorem 3.8. *Let $T \in FIO_\hbar(\chi)$. Then T extends to a bounded operator on $M^p_\hbar(\mathbb{R}^d)$, $1 \le p \le \infty$. Moreover, for every $m > 2d$ there exists a constant $C > 0$ depending only on m and the dimension d such that*

$$\|T\|_{M^p_\hbar \to M^p_\hbar} \le C\|T\|^\hbar_{m,\chi}.$$

Corollary 3.9. *Let $T \in FIO_\hbar(\chi)$, $1 < p < \infty$ and $k = 2d|1/p - 1/2|$. Then T extends to a bounded operator $T : \tilde{L}^p_k \to L^p$ if $1 < p \le 2$ and $T : \tilde{L}^p \to \tilde{L}^p_{-k}$ for $2 \le p < \infty$. Moreover for every $m > 2d$ there exists a constant $C > 0$ depending on m, d, p such that*

$$\|T\|_{\tilde{L}^p_k \to L^p} \le C\|T\|^\hbar_{m,\chi} \quad \text{for } 1 < p \le 2$$

and

$$\|T\|_{L^p \to \tilde{L}^p_{-k}} \le C\|T\|^\hbar_{m,\chi} \quad \text{for } 2 \le p < \infty.$$

Proof. The results follow at once from Theorem 3.8 and Theorem 3.6. For example, for $1 < p \le 2$ we have

$$\begin{aligned}\|Tf\|_{L^p} &\le C_1 \hbar^{d(1/p-1/2)/2}\|Tf\|_{M^p_\hbar} \le C_1 C_2 \hbar^{d(1/p-1/2)/2}\|T\|^\hbar_{m,\chi}\|f\|_{M^p_\hbar} \\ &\le C_1 C_2 C_3 \|T\|^\hbar_{m,\chi}\|f\|_{\tilde{L}^p_k}.\end{aligned}$$

□

The following result will be crucial in the sequel.

Theorem 3.10. *If $T^{(i)} \in FIO_\hbar(\chi_i)$, $i = 1, 2$, then the composition $T^{(1)}T^{(2)}$ is in $FIO_\hbar(\chi_1 \circ \chi_2)$. Moreover for every $m > 2d$ there exists a constant $C > 0$ depending only on m, the dimension d, and upper bounds for the first derivatives of χ_1 such that*

$$\|T^{(1)}T^{(2)}\|^\hbar_{m,\chi_1\circ\chi_2} \le C\|T^{(1)}\|^\hbar_{m,\chi_1}\|T^{(2)}\|^\hbar_{m,\chi_2}$$

Proof. We write

$$D_{\hbar^{-1/2}}T^{(1)}T^{(2)}D_{\hbar^{1/2}} = D_{\hbar^{-1/2}}T^{(1)}D_{\hbar^{1/2}}D_{\hbar^{-1/2}}T^{(2)}D_{\hbar^{1/2}}$$

and apply Theorem 3.4. □

4. The Schrödinger flow: wave packet analysis

In this section we prove that the oscillatory integral operators appearing in Section 2 belong to the class $FIO_\hbar(\chi^\hbar(t,s))$ for a suitable canonical transformation $\chi^\hbar$ defined in Corollary 4.2 below. Then we will use Theorem 3.10 to treat their composition for large time.

We adopt the notation of Section 2. In particular the oscillatory integral operators are well defined for $0 < |t - s| \le \delta$.

Proposition 4.1. *Let $0 < |t - s| \le \delta$ and $a \in S^0_{0,0}$. Consider the operator*

$$Tf(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)} a(x,y) f(y)\, dy.$$

Then T can be written as a Fourier integral operators

$$Tf(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\Phi(t,s,x,\hbar\eta)} b(\hbar, t, s, x, \hbar\eta) \widehat{f}(\eta)\, d\eta \tag{34}$$

where

$$\Phi(t, s, x, \eta) = y\eta + S(t, s, x, y)$$

is the generating function in the coordinates[3] *x, η, and $a \in S^0_{0,0}$. Moreover for every $m \in \mathbb{N}_0$ there exists $m' \in \mathbb{N}_0$ such that*

$$\|b(\hbar, t, s, \cdot, \cdot)\|_m \leq C\|a\|_{m'}$$

for some constant $C > 0$, depending only on m, the dimension d, upper bounds for a certain number of the derivatives of $(t - s)S(t, s, x, y)$ in (14) and the lower bound constant $\tilde{\delta}$ in (15).

Proof. Set $\tilde{S}(t, s, x, y) = (t - s)S(t, s, x, y)$ and $\tilde{\eta} = \hbar(t - s)\eta$. By the Plancherel theorem we can write T in the form

$$\begin{aligned} &Tf(x) \\ &= \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \exp\Big(i(\hbar(t-s))^{-1}[y\tilde{\eta} + \tilde{S}(t, s, x, y)] \Big) a(x, y)\, dy \Big) \widehat{f}(\eta)\, d\eta. \end{aligned}$$

We now apply the stationary phase principle in the form proved in [2, page 320 and Lemma 3.2]; the assumptions are satisfied because for $0 < |t - s| \leq \delta$ we have, by (14) and (15),

$$|\partial_x^\alpha \partial_y^\beta \tilde{S}(t, s, x, y)| \leq C_{\alpha,\beta}, \quad |\alpha| + |\beta| \geq 2,$$

and

$$\Big| \det \Big(\frac{\partial^2 \tilde{S}(t, s, x, y)}{\partial y^2} \Big) \Big| \geq \tilde{\delta} > 0.$$

Moreover, since $y = y(t, s, x, \eta)$ is the unique solution of $-\eta = \partial S(t, s, x, y)/\partial y$, the function $y = y(t, s, x, \tilde{\eta}/(t-s))$ will be the unique solution to $-\tilde{\eta} = \partial \tilde{S}(t, s, x, y)/\partial y$, and we obtain

$$\begin{aligned} &\frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} \exp\Big(i(\hbar(t-s))^{-1}[y\tilde{\eta} + \tilde{S}(t, s, x, y)] \Big) a(x, y)\, dy \\ &\quad = \exp\Big(i(\hbar(t-s))^{-1}[y(t, s, x, \tilde{\eta}/(t-s))\tilde{\eta} + \tilde{S}(t, s, x, y(t, s, x, \tilde{\eta}/(t-s)))] \Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times b(\hbar, t, s, x, \tilde{\eta})) \end{aligned}$$

[3]Namely, here $y = y(t, s, x, \eta)$ is the unique solution to $-\eta = \partial S(t, s, x, y)/\partial y$.

for some amplitude $b(\hbar, t, s, \cdot, \cdot)$ belonging to some bounded subset of $S^0_{0,0}$, when $0 < |t-s| \leq \delta$, $0 < \hbar \leq 1$. Again, we can write the last expression as

$$\exp\Big(i\hbar^{-1}[y(t,s,x,\hbar\eta)\hbar\eta + S(t,s,x,y(t,s,x,\hbar\eta))]\Big) b(\hbar, t, s, x, (t-s)\hbar\eta),$$

which gives the desired expression for Tf.

The estimates of the seminorms of b in terms of those of a follow from the proof of the stationary phase principle. ∎

Corollary 4.2. *With the notation of Proposition 4.1, we have $T \in FIO_\hbar(\chi^\hbar(t,s))$, with*

$$\chi^\hbar(t,s)(y,\eta) = \hbar^{-1/2}\chi(t,s)(\hbar^{1/2}y, \hbar^{1/2}\eta).$$

Moreover, for every $m \in \mathbb{N}_0$ there exists $m' \in \mathbb{N}_0$ such that, for $0 < |t-s| \leq \delta$,

$$\|T\|^\hbar_{m,\chi^\hbar(t,s)} \leq C\|a\|_{m'}$$

for some constant $C > 0$ independent of a and $\hbar$, t, s ($0 < \hbar \leq 1$, $0 < |t-s| \leq \delta$).

Proof. We have to prove that $D_{\hbar^{-1/2}} T D_{\hbar^{1/2}} \in FIO(\chi^\hbar(t,s))$. Using (34) and a scaling argument we can write

$$D_{\hbar^{-1/2}} T D_{\hbar^{1/2}} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\Phi(\hbar^{1/2}x, \hbar^{1/2}\eta)} b(\hbar, t, s, \hbar^{1/2}x, \hbar^{1/2}\eta) \widehat{f}(\eta)\, d\eta.$$

Now, the phase $\hbar^{-1}\Phi(t, s, \hbar^{1/2}x, \hbar^{1/2}\eta)$ generates the canonical transformation $\chi^\hbar(t,s)$ as in the statement. Moreover $\hbar^{-1}\Phi(t, s, \hbar^{1/2}x, \hbar^{1/2}\eta)$ is tame *uniformly* with respect to $\hbar$ and t, s, for $0 < \hbar \leq 1$ and $0 < |t-s| \leq \delta$, in the sense that the required bounds hold with constants independent of these parameters, or equivalently $\chi^\hbar(t,s)$ satisfies the properties *A1,A2,A3* in Section 3.2 above, with uniform bounds. This is clear, because $\chi(t,s)$ satisfies *A1,A2,A3* uniformly with respect to t, s for $0 < |t-s| \leq \delta$ (possibly for a smaller value of δ): *A1,A2* follow from (13), whereas *A3* holds (even in the presence of a magnetic field as in Section 6 below) by [57, Proposition 2.3'] applied with $\alpha = \beta = 0$.

Finally, the symbol $b(\hbar, t, s, \hbar^{1/2}x, \hbar^{1/2}\eta)$ has seminorms in $S^0_{0,0}$ dominated by those of $b(\hbar, t, s, \cdot, \cdot)$. Hence the desired result follows from Theorem 3.5. ∎

Remark 4.3. *With the notation of the previous corollary, notice that for any $s, \tau, t \in \mathbb{R}$ we have*

$$\chi^\hbar(t,s) = \chi^\hbar(t,\tau) \circ \chi^\hbar(\tau,s).$$

Indeed, this follows at once from the case $\hbar = 1$ and the definition of $\chi^\hbar$.

5. Proof of the main result (Theorem 1.1)

Let us first prove (8) and (9). We know from Section 2 that the propagator $U(t,s)$ is an oscillatory integral operator of the form (21) for $0<|t-s|\leq\delta$, and therefore by Corollary 4.2 we have $U(t,s)\in FIO_\hbar(\chi^\hbar(t,s))$, with seminorms $\|U(t,s)\|^\hbar_{m,\chi^\hbar(t,s)}$, $m\in\mathbb{N}_0$, uniformly bounded with respect to $\hbar,t,s$ for $0<|t-s|\leq\delta$. Using the evolution properties of the propagator, Remark 4.3 and Theorem 3.10 we see that, for any $T_0>0$ we have $U(t,s)\in FIO_\hbar(\chi^\hbar(t,s))$ with seminorms

$$\|U(t,s)\|^\hbar_{m,\chi^\hbar(t,s)}\leq C_0 \tag{35}$$

for a constant C_0 independent of $\hbar,t,s$, for $|t-s|\leq T_0$ (but depending on T_0); here we are using the fact that the constant C in Theorem 3.10 can be chosen independent of such parameters, being $\chi^\hbar(t,s)$ uniformly tame. We then deduce (8) and (9) from Corollary 3.9.

We now prove (10) and (11). By arguing as above it suffices to prove that

$$\|E^{(N)}(\Omega,t,s)-U(t,s)\|^\hbar_{m,\chi^\hbar(t,s)}\leq C(T_0)\hbar^N\omega(\Omega)^{N+1}(t-s) \tag{36}$$

for $0<t-s\leq T_0$ and for some $m>2d$.

Now, by Corollary 4.2 and (19) the operator $G^{(N)}(t,s)$ defined in (18) belongs to $FIO_\hbar(\chi^\hbar(t,s))$, with seminorms

$$\|G^{(N)}(t,s)\|^\hbar_{m,\chi^\hbar(t,s)}\leq C_m\hbar^{N+1}(t-s)^{N+1} \tag{37}$$

for a constant C_m independent of $\hbar,t,s$, for $0<|t-s|\leq\delta$. On the other hand we have

$$R^{(N)}(t,s)f:=E^{(N)}(t,s)f-U(t,s)f=-i\hbar^{-1}\int_s^t U(t,\tau)G^{(N)}(\tau,s)f\,d\tau$$

so that by (35), (37) and Theorem 3.10 we obtain

$$\|R^{(N)}(t,s)\|^\hbar_{m,\chi^\hbar(t,s)}\leq C_m\hbar^N(t-s)^{N+2}. \tag{38}$$

We then can write

$$E^{(N)}(\Omega,t,s)-U(t,s)=\big(U(t,t_{L-1})+R^{(N)}(t,t_{L-1})\big)\dots\big(U(t_1,s)+R^{(N)}(t_1,s)\big)-U(t,s)$$

and argue as in [23, Lemma 3.2], applying the composition estimate in Theorem 3.10. We sketch the argument for the benefit of the reader.

One expands the above product and obtains a sum of ordered products of operators, where each product has the following structure: *from right to left* we have, say, q_1 factors of type U, p_1 factors of type $R^{(N)}$, q_2 factors of type U, p_2 factors of type $R^{(N)}$, etc., up to q_k factors of type U, p_k factors of type $R^{(N)}$, to finish with q_{k+1} factors of type U. Here $p_1,\dots,p_k,q_1,\dots q_k,q_{k+1}$ are non negative integers whose sum is L, with $p_j>0$ and we can of course group together the consecutive factors of type U, using the evolution property of the propagator. Now, we estimate the

seminorm $\|\cdot\|^{\hbar}_{m,\chi^{\hbar}(t,s)}$ of such an ordered product, for some fixed $m > 2d$, using Theorem 3.10 combined with the known estimates for each factor, namely (35) and (38), which we rephrase here as

$$\|R^{(N)}(t_j, t_{j-1})\|^{\hbar}_{m,\chi^{\hbar}(t_j,t_{j-1})} \leq \tilde{C}_m \hbar^N (t_j - t_{j-1})^{N+2}. \tag{39}$$

If C denotes the constant in Theorem 3.10 we then dominate the seminorm $\|\cdot\|^{\hbar}_{m,\chi^{\hbar}(t,s)}$ of the above ordered product by

$$C^{p_1+\ldots+p_k+k} C_0^{k+1} \prod_{j=1}^{k} \prod_{i=1}^{p_j} \tilde{C}_m \hbar^N (t_{J_j+i} - t_{J_j+i-1})^{N+2}$$

with C_0 of (35), where $J_j = p_1 + \ldots + p_{j-1} + q_1 + \ldots + q_j$ for $j \geq 2$ and $J_1 = q_1$. Now, it was proved in the last part of the proof of [23, Lemma 3.2] that the sum over $p_1, \ldots, p_k, q_1, \ldots, q_{k+1}$ of these terms is in turn dominated by

$$C(T_0)\hbar^N \omega(\Omega)^{N+1}(t-s)$$

for $0 < t - s \leq T_0$. This gives (36) and concludes the proof.

6. The case of magnetic fields

In this section we extend some of the previous results in the presence of a magnetic field. Hence, consider the Schrödinger equation

$$i\hbar\partial_t u = \frac{1}{2}\big(-i\hbar\nabla - A(t,x)\big)^2 u + V(t,x)u \tag{40}$$

where $V(t,x)$ and $A(t,x) = (A_1(t,x), \ldots, A_d(t,x))$ are electric scalar and magnetic vector potential of the field, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

Assume the following hypothesis (cf. [26, 56, 57]).

Assumption (B)

a) *For $j = 1, \ldots, d$, $A_j(t,x)$ is a real function of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ and $\partial_x^\alpha A_j(t,x)$ is C^1 in $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, for every $\alpha \in \mathbb{N}_0^d$. Moreover there exists $\epsilon > 0$ such that*

$$|\partial_x^\alpha B(t,x)| \leq C_\alpha (1+|x|)^{-1-\epsilon}, \quad |\alpha| \geq 1,$$

$$|\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \leq C_\alpha, \quad |\alpha| \geq 1,$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, where $B(t,x)$ is the magnetic field, i.e. the skew-symmetric matrix with entries $B_{j,k}(t,x) = (\partial A_k/\partial x_j - \partial A_j/\partial x_k)(t,x)$.

b) *$V(t,x)$ is a real function of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ with $\partial_x^\alpha V(t,x)$ continuous in $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, for every $\alpha \in \mathbb{N}_0^d$ and satisfying*

$$|\partial_x^\alpha V(t,x)| \leq C_\alpha, \quad |\alpha| \geq 2, \ (t,x) \in \mathbb{R}\times\mathbb{R}^d.$$

It was proved in [57, Sections 2,3] that if $V=0$, under Assumption (B), all the short time results about the propagator, as well as the construction of the operators $E^{(N)}(t,s)$, $G^{(N)}(t,s)$ summarized in Section 2 continue to hold if $N\geq 1$. To be precise, the same formulas for $E^{(N)}(t,s)$, $G^{(N)}(t,s)$ and $E^{(N)}(\Omega,t,s)$ hold, where now the amplitudes a_k satisfy the transport equations

$$\frac{\partial a_k}{\partial t}+\sum_{j=1}^d\Big(\frac{\partial S}{\partial x_j}(t,s,x,y)-A_j(t,x)\Big)\frac{\partial a_k}{\partial x_j} + \frac{1}{2}\Big(\Delta_x S(t,s,x,y)-\frac{d}{t-s}-\mathrm{div}_x\,A(t,x)\Big)a_k=\frac{1}{2}\Delta_x a_{k-1}, \tag{41}$$

with $a_0(t,s,x,y)\equiv 0$ and the initial conditions $a_1(s,s,x,y)=1$ and $a_k(s,s,x,y)=0$ for $k=1,2,\dots$ (these transport equations hold even when the electric potential is present[4]). The key estimates (17)–(22) are valid too, for $N\geq 1$.

We have therefore the following result.

Theorem 6.1. *Under the above Assumption (B), the conclusions of Theorem 1.1 still hold, at least for $N\geq 1$.*

Proof. As we have already observed, the results in Section 2 keep valid under the Assumption (B), if $V=0$ and $N\geq 1$, therefore the whole subsequent analysis applies, giving the desired result if $V=0$.

Consider now the case when a potential $V(t,x)$ is present. Let us prove just part b) of Theorem 1.1 (the first part is similar and easier). We apply the gauge transformation

$$\mathcal{G}(t)u(t,x)=\exp(i\hbar^{-1}G(t,x))u(t,x)$$

which maps the solution of the equation with potentials A,V into the solution of the equation with potentials $A+\nabla_x G$, $V-\partial_t G$, cf. [57]. More precisely, if we denote by $U(t,s)$ and $U'(t,s)$ the propagators, with the prime denoting that of the new equation, we have

$$U(t,s)=\mathcal{G}(t)^{-1}U'(t,s)\mathcal{G}(s).$$

In particular if we choose $G(t,x)=\int_0^t V(\tau,x)\,d\tau$, in the new equation the electric potential is absent and and magnetic potential is $A'=A+\int_0^t\partial_x V(\tau,x)d\tau$, still satisfying the above Assumption (B). Since the Lagrangian function $\mathcal{L}(x,v,t)=$

[4]Indeed, since S satisfies the Hamilton-Jacobi equation $\partial_t S+\frac{1}{2}(\nabla_x S-A)^2+V=0$, we have

$$\Big(i\hbar\partial_t-\frac{1}{2}(-i\hbar\nabla_x-A)^2-V(t,x)\Big)\frac{1}{(2\pi i\hbar(t-s))^{d/2}}\int_{\mathbb{R}^d}e^{i\hbar^{-1}S(t,s,x,y)}a(x,y)\,dy$$
$$=\frac{1}{(2\pi i\hbar(t-s))^{d/2}}\int_{\mathbb{R}^d}e^{i\hbar^{-1}S(t,s,x,y)}i\hbar\Big[\partial_t a+(\nabla_x S-A)\cdot\nabla_x a+\frac{1}{2}(\Delta_x S-\frac{d}{t-s}-\mathrm{div}_x\,A)-\frac{i\hbar}{2}\Delta_x a\Big]\,dy.$$

$\frac{1}{2}|v|^2 + A(t,x)\cdot v - V(t,x)$ changes to $\mathcal{L}' = \mathcal{L} + dG(t,x(t))/dt$, the flow remains the same and the new generating function is

$$S'(t,s,x,y) = S(t,s,x,y) + \int_0^t V(\tau,x)\,d\tau - \int_0^s V(\tau,y)\,d\tau$$

where $S(t,s,x,y)$ is the generating function for the original equation. The amplitudes a_k for the new equation are the same as those for the original one, because the transport equations (41) do not change under the substitution $A \to A + \int_0^t \nabla_x V(\tau,x)\,d\tau$, $S \to S'$, $V \to 0$. As a consequence we have

$$E^{(N)}(t,s) = \mathcal{G}^{-1}(t)E^{(N)'}(t,s)\mathcal{G}(s)$$

and therefore

$$E^{(N)}(\Omega,t,s) = \mathcal{G}^{-1}(t)E^{(N)'}(\Omega,t,s)\mathcal{G}(s)$$

where the the primes denote the corresponding operators for the new equation. As a consequence

$$E^{(N)}(\Omega,t,s) - U(t,s) = \mathcal{G}^{-1}(t)\Big(E^{(N)'}(t,s) - U'(t,s)\Big)\mathcal{G}(s).$$

Now, we know that (36) holds for the difference $E^{(N)'}(\Omega,t,s) - U'(t,s)$ and we want to prove that the same is true for $E^{(N)}(\Omega,t,s) - U(t,s)$. Using Theorem 3.10, it is then sufficient to prove that $\mathcal{G}(t) \in FIO_\hbar(\tilde{\chi}^\hbar)$ for some tame canonical transformation $\tilde{\chi}$. This follows by Corollary 4.2, because $\mathcal{G}(t)$ can be written in the form (34) with $\Phi(t,x,\eta) = x\eta + \int_0^t V(\tau,x)\,d\tau$, $b \equiv 1$, and

$$\tilde{\chi}(y,\eta) = \Big(y, \eta + \int_0^t \nabla_x V(\tau,x)\,d\tau\Big).$$

□

Remark 6.2. *We do not know whether the conclusion of Theorem 1.1 holds for $N = 0$ in the presence of a magnetic field as above (in fact, in [57, Theorem 5] the case $N = 0$ is excluded as well). Indeed, the approach in [23] requires very precise estimates on the generating function $S(t,s,x,y)$, which seem to be quite tricky when a magnetic field is present.*

7. Concluding remarks

7.1. **Sharpness of the results.** In general we cannot avoid the dichotomy $p > 2$, $p < 2$ in Theorem 1.1. Consider, for example, the case of the harmonic oscillator,

$$i\partial_t u = -\frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u \tag{42}$$

(we take $\hbar = 1$ for simplicity, so that $\tilde{L}^p_k = L^p_k$). By the Mehler formula (see e.g. [37, formula (5a)]), the propagator $U(t,0)$ at $t = \pi/2$ is (up to constant factor) just the Fourier transform $f \mapsto \widehat{f}$. Now, we have the following result.

Proposition 7.1. *Let $1 < p < \infty$, $k_1, k_2 \in \mathbb{R}$. The Fourier transform maps $L^p_{k_1} \to L^p_{k_2}$ continuously if and only if*

$$k_1 \geq 2d(1/p - 1/2) \quad \text{and} \quad k_2 \leq 0, \quad 1 < p \leq 2$$

$$k_1 \geq 0 \quad \text{and} \quad k_2 \leq -2d(1/2 - 1/p), \qquad 2 \leq p < \infty.$$

Proof. Sufficient conditions. The desired estimates follow by the inclusion relations of Sobolev spaces and duality if we prove the continuity of the Fourier transform $L^p_k \to L^p$ with $k = 2d(1/p - 1/2)$, for $1 < p \leq 2$. This is equivalent to the estimate

$$\|(1 + |\xi|^2)^{-k/2} \widehat{f}(\xi)\|_{L^p} \leq C\|f\|_{L^p},$$

which in turn is a consequence of the Hardy-Littlewood-Paley inequality [3, Theorem 1.4.1] (that is the same estimate with $(1 + |\xi|^2)^{-k/2}$ replaced by $|\xi|^{-k}$ in the left-hand side).

Of course the result also follows from Theorem 1.1 applied to the equation (42), with $s = 0$, $t = \pi/2$ ($\hbar = 1$).

Necessary conditions. Let us first prove the condition on k_2 when $2 \leq p < \infty$. By duality this will give the condition on k_1 for $1 < p \leq 2$ as well.

Consider the space

$$\mathcal{A} = \{f \in \mathcal{S}(\mathbb{R}^d) : \ \widehat{f}(\xi) = 0 \text{ for } |\xi| \geq 1\}.$$

By Bernstein inequalities (see e.g. [58, Proposition 5.3]) there exists a constant $C > 0$ such that

$$\|f\|_{L^p_{k_1}} \leq C\|f\|_{L^p}, \quad \forall f \in \mathcal{A},$$

even for $k_1 < 0$ (for $k_1 \geq 0$ this is trivially true for every $f \in \mathcal{S}(\mathbb{R}^d)$).

Hence, if the Fourier transform is bounded $L^p_{k_1} \to L^p_{k_2}$, we have

$$\|\widehat{f}\|_{L^p_{k_2}} \leq C\|f\|_{L^p}, \quad \forall f \in \mathcal{A}. \tag{43}$$

Now, suppose by contradiction that $k_2 > -2d(1/2 - 1/p)$. Fix $f \in \mathcal{A} \setminus \{0\}$ and test this estimate on $f(x/\lambda)$, with $\lambda \geq 1$, so that $f(x/\lambda)$ belongs to $\mathcal{A}$ too. We have

$$\|(1 - \Delta)^{k_2/2} \widehat{f(\cdot/\lambda)}\|_{L^p} \leq C\|f(\cdot/\lambda)\|_{L^p} \tag{44}$$

where

$$(1 - \Delta)^{k_2/2} \widehat{f(\cdot/\lambda)} = \lambda^{d+k_2}[(\lambda^{-2} - \Delta)^{k_2/2} \widehat{f}](\lambda \cdot).$$

By the dominated convergence theorem ($|x|^{k_2}$ is locally integrable, because $k_2 > -d$) we have $[(\lambda^{-2}-\Delta)^{k_2/2}\widehat{f}](x) \to |D|^{k_2}\widehat{f}(x)$ for every $x \in \mathbb{R}^d$, and therefore by the Fatou lemma,

$$0 \neq \||D|^{k_2}\widehat{f}\|_{L^p} \leq \liminf_{\lambda\to+\infty} \|(\lambda^{-2}-\Delta)^{k_2/2}\widehat{f}\|_{L^p}.$$

Hence, letting $\lambda \to +\infty$ in (44) we obtain $k_2 \leq -2d(1/2-1/p)$, which is a contradiction.

Let us now prove that $k_2 \leq 0$ if $1 < p \leq 2$. This will give the condition on k_1 for $2 \leq p < \infty$ as well.

Fix $f \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, and let $f_\lambda(x) = f(x_1-\lambda, x_2, \ldots, x_d)$, $\lambda > 0$. Suppose the following estimate holds:

$$\|\widehat{f_\lambda}\|_{L^p_{k_2}} \leq C\|f_\lambda\|_{L^p_{k_1}} = C\|f\|_{L^p}. \tag{45}$$

We have

$$\begin{aligned}(1-\Delta)^{k_2/2}\widehat{f_\lambda}(\xi) &= (1-\Delta)^{k_2/2}[e^{-i\lambda\xi_1}\widehat{f}(\xi)]\\ &= \int_{\mathbb{R}^d} e^{ix\xi}(1+|x|^2)^{k_2/2} f(-(x_1+\lambda), -x_2, \ldots, -x_d)\,dx\\ &= \lambda^{k_2}e^{-i\lambda\xi_1}\int_{\mathbb{R}^d} e^{-iy\xi}(\lambda^{-2}+|1+\lambda^{-1}y_1|^2+\lambda^{-2}|y_2|^2+\ldots+\lambda^{-2}|y_d|^2)^{k_2/2} f(y)\,dy.\end{aligned}$$

As $\lambda \to +\infty$ the last integral converges to $\widehat{f}(\xi)$, for every $\xi \in \mathbb{R}^d$, by the dominated convergence theorem. One can then conclude as above, by letting $\lambda \to +\infty$ in (45) and using the Fatou lemma.

□

7.2. **The case of rough potentials.** Here we present an extension of the first part of Theorem 1.1 to a class of potentials in the so-called Sjöstrand class

$$M^{\infty,1}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^{\infty,1}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \|V_g f(\cdot,\xi)\|_{L^\infty}\,d\xi < +\infty\}$$

where, as usual, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

Functions in this space are bounded in $\mathbb{R}^d$ and locally have the mild regularity of a function whose Fourier transform is in L^1. Hence, it is not required the existence of any derivative. We refer to the books [17, 31] for a detailed analysis of this function space and its role in Time-frequency Analysis and Mathematical Physics. Here we just need the notion of narrow continuity for a one-parameter family $t \mapsto f_t$ of functions in this space. In fact, it is quite a weak notion of continuity.

Definition 7.2. *Let $I \subset \mathbb{R}$ be an interval. We say that a map $I \ni t \mapsto f_t \in M^{\infty,1}(\mathbb{R}^d)$ is continuous for the narrow convergence if it is continuous in $\mathcal{S}'(\mathbb{R}^d)$*

(weakly) and if there exists a function $H \in L^1(\mathbb{R}^d)$ *such that* $\sup_{x\in\mathbb{R}^d} |V_g f_t(x,\xi)| \leq H(\xi)$ *for every* $t \in I$ *and almost every* $\xi \in \mathbb{R}^d$.

Consider now the equation (with $\hbar = 1$, for simplicity)

$$i\partial_t u = (-\Delta)^{\kappa/2} u + V_2(t,x)u + V_1(t,x)u + V_0(t,x)u$$

with $0 < \kappa \leq 2$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. Suppose

$$\partial_x^\alpha V_j(t,\cdot) \in M^{\infty,1}(\mathbb{R}^d) \text{ for } |\alpha| = j,\ j = 0,1,2,$$

with V_2 and V_1 real-valued. Assume moreover that the map $\mathbb{R} \ni t \mapsto \partial_x^\alpha V_j(t,\cdot) \in M^{\infty,1}(\mathbb{R}^d)$ for $|\alpha| = j$, $j = 0,1,2$, are continuous in the sense of the narrow convergence.

Under this assumption the propagator $U(t,s)$ was constructed in [14] (see also [10, 12] and [1, 36] for classes of potentials having Fourier transform in L^1) and shown to be bounded $M^p \to M^p$ for every $1 \leq p \leq \infty$. As a consequence of Theorem 3.1 we then see that $U(t,s)$ enjoys the continuity property

$$U(t,s) : L^p_k \to L^p, \quad 1 < p \leq 2, \quad U(t,s) : L^p \to L^p_{-k}, \quad 2 \leq p < \infty$$

with $k = 2d|1/2 - 1/p|$.

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