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The Spherical Design Algorithm in the numerical simulation of biological tissues with statistical fibre-reinforcement

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Abstract Nowadays, the description of complex phys-5 ical systems, such as biological tissues, calls for highly 6 detailed and accurate mathematical models. These, in 7 turn, necessitate increasingly elaborate numerical meth-8 ods as well as dedicated algorithms capable of resolv-9 ing each detail which they account for. Especially when 10 commercial software is used, the performance of the al-11 gorithms coded by the user must be tested and carefully 12 assessed. In Computational Biomechanics, the Spher-13 ical Design Algorithm (SDA) is a widely used algo-14 rithm to model biological tissues that, like articular 15 cartilage, are described as composites reinforced by sta-16 tistically oriented collagen fibres. The purpose of the 17 present work is to analyse the performances of the SDA, 18 which we implement in a commercial software for sev-19 eral sets of integration points (referred to as "spherical 20 designs"), and compare the results with those deter-21 mined by using an appropriate set of points proposed in 22 this manuscript. As terms for comparison we take the 23

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results obtained by employing the integration scheme 24 Integral, available in Matlab[®]. For the numerical sim-25 ulations, we study a well-documented benchmark test 26 on articular cartilage, known as 'unconfined compres-27 sion test'. The reported numerical results highlight the 28 influence of the fibres on the elasticity and permeabil-29 ity of this tissue. Moreover, some technical issues of the 30 SDA (such as the choice of the quadrature points and 31 their position in the integration domain) are proposed 32 and discussed. 33

Keywords Spherical Design Algorithm · Quadrature Methods · Fibre-reinforced Materials · Finite Element Method

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1 Introduction

Soft biological tissues are often described as complex 38 porous media filled with an interstitial fluid. The com-39 plexity of these media, formed by the co-existence of 40 several structural units with different physical and chem-41 ical properties, is related to their internal structure and 42 composition, which render them highly heterogeneous 43 and anisotropic materials. Several multiphasic models 44 of biological tissues have been developed (cf. e.g. [3, 45 13, 23, 26, 39, 41, 45), in which the Theory of Mixtures 46 is employed to account for the fact that the intersti-47 tial fluid comprises several constituents (like, for exam-48 ple, ionic species, nutrients for the cells and byprod-49 ucts of cellular metabolic reactions), and that differ-50 ent types of solid materials (e.g. biological proteins, 51 extra-cellular matrix, and collagen fibres) characterise 52 the overall properties of the "solid phase" of a tissue. 53 Among the solid constituents of biological tissues, col-54 lagen fibres provide a reinforcing function and their dis-55 tribution and orientation make the tissues anisotropic. 56

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For example, this is the case of tendons, arterial walls
[24,37,49], articular cartilage [17,41,46], and heart [52].

It is believed that a different arrangement of the fi-59 bre network, which may result to be either structurally 60 organised or not, corresponds to a different function-61 ality of the tissue [5,60]. In fact, this is particularly 62 the case of articular cartilage (a sheath of soft con-63 nective tissue covering the opposing ends of bone in 64 diarthrodial joints [46]), in which the collagen fibres 65 are concentrated throughout the tissue in a nonuni-66 form way and are oriented statistically according to 67 some point-dependent probability density distribution 68 [19,58]. More precisely, histological experiments per-69 formed on articular cartilage show that three (some-70 times four [46]) zones can be distinguished in the tis-71 sue, roughly layered along the direction of the tissue's 72 depth and determined by the characteristic alignment 73 of the collagen fibres in addition to other properties of 74 the tissue. In the so-called "deep zone", which is close 75 to the interface, also referred to as "tidemark", that 76 separates the cartilage from the subchondral bone, the 77 fibres are principally oriented perpendicularly to the 78 tidemark; in the "middle zone" the fibres are oriented 79 almost randomly; finally, in the "upper zone" the fi-80 bres tend to be parallel to the articular surface, i.e., 81 the surface on which the sheaths of cartilage cover-82 ing the opposing bones of the diarthrodial joint ex-83 change loads reciprocally. The pattern of fibre orien-84 tation depicted above influences the stiffness of the tis-85 sue as well as its capability of conveying the interstitial 86 fluid throughout its pores. In many cases of interest, a 87 fair approximation of the material behaviour of artic-88 ular cartilage can be achieved by regarding the tissue 89 as transversely isotropic [17, 58], which means that its 90 material properties are invariant under rotations about 91 92 a given symmetry axis. Moreover, the solid phase of articular cartilage is often assumed to be hyperelastic 93 with respect to some undeformed configuration, and its 94 principal solid and fluid constituents are regarded as 95 intrinsically incompressible. The latter hypothesis im-96 plies that the overall tissue's compressibility is related 97 to the evolution of its porosity, which accompanies the 98 flow of the interstitial fluid when the tissue is com-99 pressed. Hereafter, the dynamics of the interstitial fluid 100 is assumed to be governed by Darcy's law. Recently, 101 however, possible deviations from Darcy's regime have 102 been investigated in [28], on the basis of the theory re-103 ported in [10]. In the sequel, the hyperelastic, inhomo-104 geneous, and transversely isotropic material model of 105 articular cartilage studied in [58] will be adopted. This 106 model considers a statistical distribution of collagen fi-107 108 bres and, thus, introduces directional averaging operators to determine the overall tissue's properties from 109

those associated with one family of fibres aligned along 110 a given direction of space. More specifically, if the distri-111 bution of orientations is assumed to be continuous, the 112 averaging operators are integrals defined over the set 113 of all possible spatial directions. In the context of this 114 work, such integrals will be used to obtain the overall 115 strain energy density and the overall permeability of the 116 considered tissue sample. The fibre orientation pattern 117 is described by a point-dependent probability density 118 distribution, which serves as a weight for the averag-119 ing integrals, thereby modulating the influence of each 120 family of fibres. 121

The description of the mechanical response of artic-122 ular cartilage, which includes the study of the flow of 123 the interstitial fluid, is often referred to as the "bipha-124 sic model of cartilage" [46]. Within a purely mechanical 125 framework, and in its standard formulation, the bipha-126 sic model consists of two coupled equations, which rep-127 resent the mass and momentum balance laws for the 128 tissue as a whole. The model equations are obtained 129 by means of closure conditions that relate the tissue's 130 porosity with the volumetric deformation, and the fluid 131 filtration velocity with the pressure gradient inside the 132 tissue. The first relation is a consequence of incompress-133 ibility, which for porous media does not necessarily re-134 strict the deformation to be isochoric, whereas the sec-135 ond relation stems from Darcy's law and, thus, intro-136 duces the tissue's permeability tensor. 137

When the biphasic model of articular cartilage ac-138 counts for the inhomogeneity and anisotropy of the tis-139 sue, and the statistical orientation of the collagen fibres 140 is considered in the constitutive relations of the stress 141 and permeability, the model equations become highly 142 nonlinear and coupled with one another. In such situa-143 tions, Finite Element (FE) methods are usually adopted 144 to search for numerical solutions. Still, besides the tech-145 niques elaborated for solving the model equations nu-146 merically, either problem-dependent approximation cri-147 teria [34] or dedicated quadrature methods must be de-148 veloped for computing the integrals that determine the 149 directional averages of the constitutive functions defin-150 ing the mechanical stress and tissue's permeability. In 151 general, indeed, these quantities depend on the orien-152 tation of the fibres and on the deformation in a non-153 separable way [34], so that it is not possible to compute 154 exactly the averaging integrals before starting the FE 155 method. A rather largely employed algorithm, referred 156 to as Spherical Design Algorithm (SDA), is usually cou-157 pled with the FE method to solve the averaging inte-158 grals at each time step and linearisation iteration of 159 the solution procedure. The SDA treats the averaging 160 integrals determining the stress and the permeability 161 of articular cartilage as integrals defined over the sur-162 face of a unit sphere centred at each point of the region
of space occupied by the tissue. The implementation
of the SDA is explained in detail and its reliability is
tested against a different quadrature scheme, which is
available in Matlab.

It must be recalled that the issue of spherical in-168 tegration is "quite old" (see e.g. [6, 44]), and it has 169 been recently used for fibre-reinforced biological tis-170 sues (see, e.g., [18,57]). Our work aims to contribute, in 171 the framework of the constitutive modelling of tissues, 172 to the long-standing problem of the integration over 173 the surface of the sphere. This is done with the pur-174 pose of determining the constitutive information sup-175 plied by the statistical orientation of the reinforcing 176 collagen fibres. However, in other contexts, several au-177 thors have proposed both analytical and numerical ar-178 guments for providing accurate and efficient numeri-179 cal integration schemes [6, 44, 14, 57]. Among those, a 180 general method for obtaining conceptually integration 181 formulae was demonstrated by Bažant and Oh [6]. Nev-182 ertheless, to our understanding, what in other circum-183 stances makes a numerical scheme better than others 184 finds only practical and fortuitous evidences. In [6], it 185 is remarked that the integration points should be cho-186 sen in a way to maintain their symmetric and regular 187 distribution on the surface of the sphere. For instance, 188 a particularly entangling choice of the set of points can 189 be obtained by means of a projection of vertices, and, 190 more generally, of points lying on the edges of poly-191 hedra inscribed in the sphere [6]. Some collections of 192 Spherical *t*-Designs generated with the aid of this pro-193 jection method are reported also in [38]. In [57], the 194 weights and quadrature points obtained by Lebedev 195 [44] have been selected as the best choice for perform-196 ing angular integrations, among other quadrature meth-197 198 ods. Thus, unluckily, as also pointed out in [57], the best set of quadrature points for the SDA is extremely 199 problem-dependent, possibly related to the symmetries 200 of the function to be integrated, and, in general, the 201 degree of the formula, i.e., the degree of the polyno-202 mials that can be resolved exactly, is not a sufficient 203 condition to ensure the accuracy of the numerical inte-204 gration for a less smooth and regular function. In par-205 ticular, two different Spherical *t*-Designs with the same 206 accuracy could lead to extremely different numerical 207 performances when applied to a particular benchmark. 208 As we show in the following, the SDA accuracy strongly 209 depends not only on the number of points (representing 210 the directions of the fibres) by which we approximate 211 the integration, but also on the particular way in which 212 they are arranged on the domain of integration, whereas 213 214 the Matlab algorithm that has been used for comparison only depends on the user-defined degree of approx-215

imation, and adaptively chooses nodes and points for the quadrature.

The paper is organised as follows: The biphasic model 218 of articular cartilage, as formulated in [58], is sum-219 marised in Section 2. The probability density distri-220 butions and the constitutive laws specifically adopted 221 for articular cartilage are reported in Section 3. The 222 structure of the considered quadrature schemes, and in 223 particular of the SDA, is presented in Section 4. The 224 integration procedures are tested in Section 5, in which 225 a well-established benchmark problem is analysed. The 226 results of the benchmark test are commented in Sec-227 tion 6. Finally, some concluding remarks and plans for 228 future work are outlined in Section 7. 229

2 Biphasic Model of Articular Cartilage

The non-linear, poroelastic biphasic model of articular 231 cartilage (AC) has been investigated in several publi-232 cations with different level of complexity, depending on 233 whether the anisotropy and inhomogeneity of the tissue 234 are accounted for (cf., e.g. [28,29,31,58]). Since a de-235 tailed presentation of the model has been recently given 236 in [58] (although the presence of ions and their influ-237 ence on the tissue mechanics were neglected), we recall 238 here for the sake of completeness the main hypotheses 239 and logical steps leading to the equations that have to 240 be ultimately solved. Consistently with the approach 241 followed in other articles, and for the sake of generality, 242 we employ the covariant formalism of Continuum Me-243 chanics [47], along with the modifications put forward 244 in [19]. The theoretical apparatus, on which the bipha-245 sic model of AC is developed, rests on two pillars: (i) 246 Hybrid Mixture Theory [9,35], and (ii) Poroelasticity. 247

2.1 Microstructure

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In this section, we summarise the description of the mi-249 crostructure of AC, as presented in [58]. Within the ap-250 proximation presented in the following, the ionic phase 251 of AC is not accounted for (cf. e.g. [45,46] for a re-252 view on the role of electric charges on the mechanics 253 of AC). At a sufficiently coarse level of description, the 254 main constituents of AC are represented by a matrix 255 of proteoglycans, chondrocytes (i.e., cells secreting the 256 extracellular matrix), collagen fibres, and an intersti-257 tial fluid capable of flowing throughout the tissue [46]. 258 It is assumed that the fluid saturates completely the 259 tissue and, since the focus of the present study is not 260 on the chondrocytes, it is also hypothesised that ma-261 trix and chondrocytes can be regarded as a single en-262 tity. Moreover, in the sense explained in [58], the ex-263

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istence of a Representative Elementary Volume (REV) 264 is claimed, in which all considered tissue's constituents 265 coexist. The portion of REV occupied by the fluid is 266 described by the volumetric fraction $\phi_{\rm f}$. Due to the as-267 sumption of saturation, the volumetric fraction of the 268 solid constituents is defined by $\phi_s := 1 - \phi_f$. As in [58], 269 $\phi_{\rm s}$ is given by the sum $\phi_{\rm s} = \phi_{0\rm s} + \phi_{1\rm s}$, where $\phi_{0\rm s}$ and $\phi_{1\rm s}$ 270 denote, respectively, the volumetric fractions of matrix 271 and fibres, expressed per unit solid volume of the REV. 272 Since the tissue is inhomogeneous and deforms, the vol-273 umetric fractions are generally function of space and 274 time. From here one, the complex "matrix-and-fibres" 275 existing at each point of the tissue will be referred to 276 as "the solid phase" of AC. Analogously, the fluid can 277 also be referred to as the "fluid phase" of AC. 278

279 2.2 Kinematics

Within the framework of Mixture Theory, the kinematics of AC can be formulated as in [54,55], and subsequently adapted in [58] to fibre-reinforced porous media. The major hypothesis is that the solid phase of AC can be associated with a *reference configuration* \mathscr{B} . The motion of the solid phase is described by the oneparameter family of smooth mappings

$$\chi(\cdot, t): \mathscr{B} \to \mathbb{R}^3, \tag{1}$$

which map the point $X \in \mathscr{B}$ into $x = \chi(X, t) \in \mathbb{R}^3$, at each time $t \in \mathscr{I}$ of the time interval \mathscr{I} .

To define the motion of the fluid, the 3D material manifold \mathscr{M}_{f} is introduced, whose elements, i.e., the fluid particles $\mathscr{X}_{\mathrm{f}} \in \mathscr{M}_{\mathrm{f}}$, are embedded into \mathbb{R}^3 by means of the one-parameter family of mappings

$$\mathfrak{f}(\,\cdot\,,t):\mathscr{M}_{\mathrm{f}}\to\mathbb{R}^3.\tag{2}$$

²⁹³ The subset of \mathbb{R}^3 defined by

$$\mathscr{B}_t := \chi(\mathscr{B}, t) \cap \mathfrak{f}(\mathscr{M}_{\mathrm{f}}, t) \tag{3}$$

is the region of space occupied at time $t \in \mathscr{I}$ by the tis-294 sue, viewed as a solid-fluid mixture. In (3), $\chi(\mathcal{B}, t)$ and 295 $\mathfrak{f}(\mathscr{M}_{\mathfrak{f}},t)$ are the images of $\chi(\cdot,t)$ and $\mathfrak{f}(\cdot,t)$ at $t\in\mathscr{I}$, 296 respectively. The volumetric fractions $\phi_{\rm f}(\cdot, t), \phi_{\rm 0s}(\cdot, t),$ 297 and $\phi_{1s}(\cdot, t)$ are defined as scalar fields over \mathscr{B}_t . More-298 over, at each $x \in \mathscr{B}_t$, the vectors $\boldsymbol{v}_{s}(x,t) = \dot{\chi}(X,t)$ 299 and $\boldsymbol{v}_{\mathrm{f}}(x,t) = \dot{\mathfrak{f}}(\mathscr{X}_{\mathrm{f}},t)$ denote the velocities of the solid 300 phase and of the fluid particle \mathscr{X}_{f} passing through x 301 at time t. The superimposed dot means partial differ-302 entiation with respect to time. The two-point, second-303 order tensor F(X,t), definable through the directional 304 derivative of χ at $(X, t) \in \mathscr{B} \times \mathscr{I}$, i.e., 305

$$(\partial_{\boldsymbol{U}}\chi)(X,t) = \boldsymbol{F}(X,t)\boldsymbol{U},\tag{4}$$

with \boldsymbol{U} being a vector attached at X, is commonly re-306 ferred to as the deformation gradient of the solid phase 307 [47]. With respect to an appropriate tensor basis, its 308 components read $F^a{}_A = \partial \chi^a / \partial X^A$, with a, A = 1, 2, 3. 309 It is important to remark that, as in [58], also in this 310 work matrix and fibres share the same velocity, $\boldsymbol{v}_{s}(x,t)$, 311 at all times and at all points. To complete the kinematic 312 picture, the volumetric ratio $J = \det(\mathbf{F}) > 0$, and the 313 right Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{g}\boldsymbol{F}$ 314 are introduced. Here, g denotes the metric tensor as-315 sociated with \mathbb{R}^3 . For future use, we also introduce the 316 metric tensor associated with the undeformed configu-317 ration, G. 318

In the sequel, we adopt a rather standard notation 319 in Continuum Mechanics (cf., e.g., [47]). Let f be a 320 function of space and time expressing a given physical 321 quantity. Without loss of generality, we may assume 322 that f(x,t), with $x \in \mathbb{R}^3$ and $t \in \mathscr{I}$, is a scalar, but 323 it could generally represent a vector or a tensor of any 324 order. A function f of this type is sometimes said to pro-325 vide the "Eulerian description" of the physical quantity 326 with which it is associated. Since for each $X \in \mathscr{B}$ and 327 $t \in \mathscr{I}$ there exists $x \in \mathscr{B}_t \subset \mathbb{R}^3$ such that $x = \chi(X, t)$, 328 the composition $f_{(L)}(\cdot, t) = f(\cdot, t) \circ \chi(\cdot, t) : \mathscr{B} \to \mathbb{R}$ 329 is introduced, which determines the "Lagrangian de-330 scription" of the considered physical quantity. Hence, 331 it holds that $f_{(L)}(X,t) = f(x,t)$, with $x = \chi(X,t)$. The 332 partial derivative of $f_{(L)}$ with respect to time equals the 333 substantial derivative of f following the solid motion: 334

$$f_{(L)}(X,t) = D_{s}f(x,t)$$

= $\partial_{t}f(x,t) + (\operatorname{grad} f(x,t))\boldsymbol{v}_{s}(x,t),$ (5)

where grad f is said to be the *spatial gradient* of f, and $x = \chi(X, t)$. The *material gradient* of $f_{(L)}$, denoted by Grad $f_{(L)}$, is related to grad f through 337

grad
$$f(x,t) = \mathbf{F}^{-\mathrm{T}}(X,t)$$
Grad $f_{(L)}(X,t)$. (6)

If q is a spatial vector field, then the divergence of q is 338 given by 339

div
$$\boldsymbol{q}(x,t) = \boldsymbol{F}^{-\mathrm{T}}(X,t)$$
: Grad $\boldsymbol{q}_{(L)}(X,t)$
= tr $\left[\operatorname{Grad} \boldsymbol{q}_{(L)}(X,t) \boldsymbol{F}^{-1}(x,t) \right],$ (7)

and the symbol ":" stands for "double contraction" between second-order tensors. For the particular case of the solid phase velocity, it holds that 342

div
$$\boldsymbol{v}_{s}(x,t) = \boldsymbol{F}^{-T}(X,t)$$
: Grad $\boldsymbol{v}_{s(L)}(X,t)$
= $\dot{J}(X,t)/J(X,t)$, (8)

where $v_{s(L)}$ is the Lagrangian counterpart of v_s . When $_{343}$ q denotes the flux vector associated with a given phys- $_{344}$ ical quantity (for example, mass), the material vector $_{345}$



Fig. 1 Graphical representation of a longitudinal section of articular cartilage. The most probable angle of orientation Q, acting as a mean value for the probability density distribution, and the variance have been reported on the right [17]. For both the graphs, the ordinate represents a normalised depth $\xi = X^3/L$.

field Q, defined by the equality

$$\boldsymbol{Q}(X,t) = J(X,t)\boldsymbol{F}^{-1}(\chi(X,t),t)\boldsymbol{q}(\chi(X,t),t), \qquad (9)$$

³⁴⁷ is said to be the Piola transformation of \boldsymbol{q} through the ³⁴⁸ motion χ . Hence, recalling Piola's identity $\text{Div}(J\boldsymbol{F}^{-T}) =$ ³⁴⁹ $\boldsymbol{0}$ [47], we obtain the relation

$$J(X,t)\operatorname{div} \boldsymbol{q}(\chi(X,t),t) = \operatorname{Div} \boldsymbol{Q}(X,t)$$
(10)

between the spatial divergence of q and the material divergence of Q. For example, in a given Cartesian coordinate system, the material divergence Div Q reads

$$\operatorname{Div} \boldsymbol{Q} = \frac{\partial Q^A}{\partial X^A} \tag{11}$$

(here, Einstein's convention on repeated indices applies).

³⁵⁴ 2.3 Statistically oriented reinforcing fibres

The tissues addressed in this work are modelled as fibrereinforced composite materials in which the fibres are oriented statistically. To model the arrangement of the fibres, one introduces the tangent space of \mathscr{B} at X, $T_X \mathscr{B}$, and considers the set

$$\mathbb{S}_X^2 \mathscr{B} := \{ \boldsymbol{M} \in T_X \mathscr{B} : \| \boldsymbol{M} \| = 1 \}$$

$$(12)$$

of all unit vectors (directions) M emanating from X. 360 This set can be taken as representation of the unit 361 sphere centred at X. The probability density distribu-362 tion of finding a fibre aligned along $M \in \mathbb{S}^2_X \mathscr{B}$ at $X \in$ 363 \mathscr{B} is denoted by $\wp : \mathbb{S}_X^2 \mathscr{B} \to \mathbb{R}_0^+$, and must satisfy the normalisation condition $\int_{\mathbb{S}_X^2 \mathscr{B}} \wp(\mathbf{M}) = 1$. Furthermore, 364 365 since the phenomena considered in the present context 366 are insensitive to reflections of the unit vectors M, \wp 367 must also satisfy the parity condition $\wp(M) = \wp(-M)$. 368 We emphasise that \wp is also a function of X. However, 369 in order to keep the notation as light as possible, this 370

dependence will be omitted but understood throughout this work, unless otherwise specified.

Articular cartilage is often modelled as a transversely 373 isotropic material. In particular, in many cases of inter-374 est, a global symmetry axis, $\boldsymbol{\xi}$, can be detected, and 375 the material properties of the tissue are invariant un-376 der rotations about $\boldsymbol{\xi}$ [58]. Each plane orthogonal to $\boldsymbol{\xi}$ 377 is referred to as transverse plane. If the orthonormal 378 vector basis $\{\mathcal{E}_A\}_{A=1}^3 \subset T_X \mathscr{B}$ is attached to a given 379 $X \in \mathscr{B}$ in such a way that \mathcal{E}_3 is parallel to $\boldsymbol{\xi}$, the unit 380 vector M can be written as 381

$$M = \hat{M}(\Theta, \Phi)$$

= sin \(\Theta\) cos \(\Phi\) \(\mathcal{E}_1\) + sin \(\Phi\) sin \(\Phi\) \(\mathcal{E}_2\) + cos \(\Phi\) \(\mathcal{E}_3\), (13)

where the angles $\Theta \in [0, \pi]$ and $\Phi \in [0, 2\pi[$ are the colatitude and longitude, respectively, and the map \hat{M} associates pairs $(\Theta, \Phi) \in [0, \pi] \times [0, 2\pi[$ with unit vectors of $\mathbb{S}^2_X \mathscr{B}$. This permits to reinterpret the probability density distribution as a function $\hat{\wp} : [0, \pi] \times [0, 2\pi[\to \mathbb{R}^+_0]$ such that

$$\wp(\boldsymbol{M}) = \wp(\hat{\boldsymbol{M}}(\boldsymbol{\Theta}, \boldsymbol{\Phi})) = \hat{\wp}(\boldsymbol{\Theta}, \boldsymbol{\Phi}). \tag{14}$$

The functional dependence of the probability density distribution on M must be consistent with the symmetries attributed to the material under study. This leads to some restrictions. In particular, due to the transverse isotropy, $\hat{\wp}$ should be independent of the longitude Φ , i.e., $\hat{\wp}(\Theta, \Phi) \equiv \hat{\wp}(\Theta) : [0, \pi] \to \mathbb{R}_0^+$.

By introducing a function f over $\mathbb{S}^2_X \mathscr{B}$, the directional average of f is defined by (see [15] for the notation) 396

$$\langle\!\!\langle f \rangle\!\!\rangle = \int_{\mathbb{S}^2_X \mathscr{B}} \wp(\mathbf{M}) f(\mathbf{M}).$$
 (15)

In general, f may represent a scalar-, a vector-, or a $_{397}$ tensor-valued quantity. However, for the sake of simplicity, we will consider only scalars in the remainder of $_{399}$

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this section. In the following, f shall identify a consti-400 tutive function describing, for instance, the elasticity or 401 the permeability of articular cartilage. Moreover, as we 402 will see in Section 3, the dependence of such constitu-403 tive functions on the fibre orientation requires f to be 404 an even function of M, i.e., it holds f(M) = f(-M), 405 for all $M \in \mathbb{S}^2_X \mathscr{B}$. Since the probability density enjoys 406 the same property, the product $\wp f$ is itself an even func-407 tion of $M \in \mathbb{S}^2_X \mathscr{B}$, which means that the directional 408 average of f can also be performed by integrating on a 409 half of $\mathbb{S}^2_X \mathscr{B}$ only. For instance, if \wp and f are restricted 410 to the northern hemisphere 411

$$\mathbb{S}_X^{2+}\mathscr{B} := \{ \boldsymbol{M} \in T_X \mathscr{B} : \|\boldsymbol{M}\| = 1, \ \boldsymbol{M} \cdot \boldsymbol{\mathcal{E}}_3 \ge 0 \}, \quad (16)$$

 $_{412}$ then the directional average of f can be computed as

$$\langle\!\!\langle f \rangle\!\!\rangle = 2 \int_{\mathbb{S}^{2+}_X \mathscr{B}} \wp(\mathbf{M}) f(\mathbf{M}).$$
 (17)

⁴¹³ In order to be compliant with the normalisation condi-⁴¹⁴ tion, \wp is usually written as

$$\wp = \frac{\mathfrak{p}}{\mathcal{Z}}, \quad \mathcal{Z} = \int_{\mathbb{S}^2_X \mathscr{B}} \mathfrak{p}(\boldsymbol{M}),$$
 (18)

where $\mathfrak{p} : \mathbb{S}_X^2 \mathscr{B} \to \mathbb{R}_0^+$ is the non-normalised density, and \mathcal{Z} is referred to as the normalisation factor. In this case, by restricting \mathfrak{p} to $\mathbb{S}_X^{2+} \mathscr{B}$, and introducing the additional quantities

$$\gamma := \mathfrak{p}_{|\mathbb{S}_X^{2+}\mathscr{B}} : \mathbb{S}_X^{2+}\mathscr{B} \to \mathbb{R}_0^+, \tag{19a}$$

$$\mathcal{Z}_{+} := \int_{\mathbb{S}^{2+}_{X} \mathscr{B}} \gamma(\boldsymbol{M}) = \frac{1}{2} \mathcal{Z}, \qquad (19b)$$

419 one can define the new probability density

$$\Psi: \mathbb{S}_X^{2+}\mathscr{B} \to \mathbb{R}_0^+, \quad \Psi = \frac{\gamma}{\mathcal{Z}_+} = 2\,\wp_{|\mathbb{S}_X^{2+}\mathscr{B}},\tag{20}$$

 $_{420}$ and reformulate the average (17) as

$$\langle\!\!\langle f \rangle\!\!\rangle = \int_{\mathbb{S}^{2+}_X \mathscr{B}} \Psi(\boldsymbol{M}) f(\boldsymbol{M}).$$
 (21)

We emphasise that the procedure leading to (21) from 421 (15) can be applied only when a given probability den-422 sity \wp , that is naturally defined over $\mathbb{S}^2_X \mathscr{B}$, can be ap-423 propriately renormalised onto $\mathbb{S}^{2+}_X \mathscr{B}$ to obtain Ψ . For 424 example, this is the case of the von Mises distribution 425 introduced in Section 3. However, we will also consider 426 the case in which the probability density is defined only 427 on $\mathbb{S}^{2+}_X \mathscr{B}$, and is not prolongated to $\mathbb{S}^2_X \mathscr{B}$ (this oc-428 curs, for example, when the pseudo-Gaussian distribu-429 tion (35b) is used). In these situations, the directional 430 average of a given physical quantity is defined by (21), 431 which cannot be deduced from (15), and the probabil-432 433 ity density, being defined only on a hemisphere, cannot be claimed to enjoy any parity symmetry. 434

2.4 Balance laws and constitutive relations

In this section, we review the balance laws that are rel-436 evant for the poroelastic model of articular cartilage 437 and the hypotheses on which it relies. The presentation 438 follows that reported in [58]. The first assumption is 439 that the mass-exchange processes occurring among the 440 constituents of the tissue can be disregarded over the 441 time-scales that characterise the experiments investi-442 gated in this work. Accordingly, in local form, the mass 443 balance laws of the tissue's constituents are written as 444

$$D_{\rm s}(\phi_{0{\rm s}}\rho_0) + \phi_{0{\rm s}}\rho_0\,{\rm div}\,\boldsymbol{v}_{\rm s} = 0,\tag{22a}$$

$$D_{\rm s}(\phi_{\rm 1s}\rho_1) + \phi_{\rm 1s}\rho_1 \operatorname{div} \boldsymbol{v}_{\rm s} = 0, \qquad (22b)$$

$$D_{\rm s}(\phi_{\rm f}\rho_{\rm f}) + \phi_{\rm f}\rho_{\rm f}\operatorname{div}\boldsymbol{v}_{\rm s} + \operatorname{div}(\rho_{\rm f}\boldsymbol{q}) = 0, \qquad (22c)$$

where ρ_0 , ρ_1 , and ρ_f are the mass densities of the matrix, fibres, and interstitial fluid, respectively, and $\boldsymbol{q} = {}^{445}_{\phi_f}(\boldsymbol{v}_f - \boldsymbol{v}_s)$ is the filtration velocity of the fluid.

The second hypothesis is that ρ_0 , ρ_1 , and ρ_f are constant in space and time, which allows to divide the balance law of each constituent by the corresponding mass density. By multiplying the resulting expressions by J, and using (8)–(10), (22a)–(22c) become

$$\dot{\phi}_{0sR} = 0, \tag{23a}$$

$$\phi_{1sR} = 0, \tag{23b}$$

$$\dot{\phi}_{\rm fR} + {\rm Div}\,\boldsymbol{Q} = 0.$$
 (23c)

In (23a)–(23c), ϕ_{0sR} , ϕ_{1sR} , and ϕ_{fR} are the Piola transforms of ϕ_{0s} , ϕ_{1s} , and ϕ_{f} , respectively, and read

$$\phi_{\alpha sR}(X) = J(X,t)\phi_{\alpha s}(\chi(X,t),t), \quad \alpha \in \{0,1\}, \quad (24a)$$

$$\phi_{fR}(X,t) = J(X,t)\phi_{f}(\chi(X,t),t). \quad (24b)$$

Equations (23a) and (23b) imply that ϕ_{0sR} and ϕ_{1sR} are constant in time, which means that the Piola transform of the volumetric fraction of the solid phase as a whole, $\phi_{sR} = \phi_{0sR} + \phi_{1sR}$, is constant in time, too. In particular, coherently with (24a), ϕ_{sR} is given by 459

$$\phi_{\rm sR}(X) = J(X,t)\phi_{\rm s}(\chi(X,t),t). \tag{25}$$

The results (24a) and (25) are equivalent to (22a) and 460 (22b), respectively, and constrain ϕ_{0s} , ϕ_{1s} , and ϕ_{s} to 461 vary as the reciprocal of the volumetric ratio. This means 462 that the mass density of the solid phase, defined by 463 the equality $\phi_{\rm s}\rho_{\rm s} = \phi_{0\rm s}\rho_0 + \phi_{1\rm s}\rho_1$, has zero substan-464 tial derivative. Consequently, the solid phase behaves 465 as an incompressible material, even though J need not 466 be constrained to be equal to 1. Moreover, by enforc-467 ing the saturation condition into (24b), $\phi_{\rm fR}$ can be ex-468 pressed as $\phi_{fR} = J - \phi_{sR}$, which yields $\phi_{fR} = J$ and, 469 from (23c), 470

$$\dot{J} + \text{Div}\,\boldsymbol{Q} = 0. \tag{26}$$

The third hypothesis is that both inertial forces and
external body forces applied to the tissue can be disregarded. Consequently, the momentum balance laws for
the solid and fluid phase read

$$\operatorname{div} \boldsymbol{\sigma}_{\mathrm{s}} + \boldsymbol{m}_{\mathrm{s}} = \boldsymbol{0}, \tag{27a}$$

$$\operatorname{div} \boldsymbol{\sigma}_{\mathrm{f}} + \boldsymbol{m}_{\mathrm{f}} = \boldsymbol{0}, \tag{27b}$$

where $\sigma_{\rm s}$ and $\sigma_{\rm f}$ are the Cauchy stress tensors of the 475 solid and fluid phase, respectively, and the internal body 476 force \boldsymbol{m}_k , with k = f, s, represents the gain or loss of 477 momentum of the α th phase due to the interactions 478 with the other one. It should be noted that, since the 479 matrix and the fibres share the same velocity, only one 480 equation is written for the solid phase as a whole. Since 481 the tissue as a whole experiences neither production nor 482 loss of linear momentum, the body forces $m_{\rm s}$ and $m_{\rm f}$ 483 must satisfy the closure condition $m_{\rm f} + m_{\rm s} = 0$. Thus, 484 by summing together (27a) and (27b), one obtains 485

$$\operatorname{div}(\boldsymbol{\sigma}_{\mathrm{f}} + \boldsymbol{\sigma}_{\mathrm{s}}) = \mathbf{0}, \tag{28a}$$

$$\operatorname{div} \boldsymbol{\sigma}_{\mathrm{f}} + \boldsymbol{m}_{\mathrm{f}} = \boldsymbol{0}. \tag{28b}$$

⁴⁸⁶ The Piola transform of (28a) and (28b) is given by

$$\operatorname{Div}(\boldsymbol{P}_{\mathrm{f}} + \boldsymbol{P}_{\mathrm{s}}) = \boldsymbol{0},\tag{29a}$$

$$\operatorname{Div} \boldsymbol{P}_{\mathrm{f}} + J\boldsymbol{m}_{\mathrm{f}} = \boldsymbol{0},\tag{29b}$$

where P_k , k = f, s, is the first Piola-Kirchhoff stress tensor of the kth phase and, with abuse of notation, m_f in (29b) is expressed as a function of time and $X \in \mathcal{B}$. Equations (29a) and (29b) are obtained by multiplying (28a) and (28b) by J, and using the identities [47]

$$J(X,t)\operatorname{div}\boldsymbol{\sigma}_{k}(\chi(X,t),t) = \operatorname{Div}\boldsymbol{P}_{k}(X,t), \qquad (30a)$$

$$\boldsymbol{P}_{k}(X,t) = J(X,t)\boldsymbol{\sigma}_{k}(\chi(X,t),t)\boldsymbol{F}^{-\mathrm{T}}(X,t).$$
(30b)

⁴⁹² Hereafter, it is assumed that the fluid is macroscopically ⁴⁹³ inviscid and that the solid phase is hyperelastic with ⁴⁹⁴ respect to the configuration \mathscr{B} . Accordingly, $\sigma_{\rm f}$ and $\sigma_{\rm s}$ ⁴⁹⁵ are expressed as [9,35]

$$\boldsymbol{\sigma}_{\rm f} = -\phi_{\rm f} p \, \boldsymbol{g}^{-1},\tag{31a}$$

$$\boldsymbol{\sigma}_{\rm s} = -\phi_{\rm s} p \, \boldsymbol{g}^{-1} + \boldsymbol{\sigma}_{\rm sc}. \tag{31b}$$

⁴⁹⁶ In (31a) and (31b), p is the pore pressure, whereas $\sigma_{\rm sc}$ ⁴⁹⁷ is said to be the "constitutive part" of $\sigma_{\rm s}$, since it is ⁴⁹⁸ determined by the hyperelastic constitutive law

$$\boldsymbol{\sigma}_{\rm sc} = \hat{\boldsymbol{\sigma}}_{\rm sc}(\boldsymbol{F}) = \frac{1}{J} \boldsymbol{F} \left(2 \frac{\partial \hat{W}}{\partial \boldsymbol{C}}(\boldsymbol{C}) \right) \boldsymbol{F}^{\rm T}, \tag{32}$$

⁴⁹⁹ with the superimposed hat standing for "constitutive ⁵⁰⁰ function", and \hat{W} being the hyperelastic strain energy ⁵⁰¹ density function of the solid phase [31,58]. In (32) and in the following, $\sigma_{\rm f}$, $\sigma_{\rm s}$, and $\sigma_{\rm sc}$ are regarded as a func-502 tions of time and points $X \in \mathcal{B}$. It should be noted 503 that the pore pressure, p, cannot be determined con-504 stitutively, because the mass density of the fluid, $\rho_{\rm f}$, is 505 assumed to be constant, which implies that the fluid is 506 regarded as incompressible. Rather, p constitutes, to-507 gether with the motion of the solid phase, χ , one of the 508 unknowns of the problem. 509

It can be proven that the momentum exchange rate, 510 $\boldsymbol{m}_{\mathrm{f}}$, splits additively as $\boldsymbol{m}_{\mathrm{f}} = p \, \boldsymbol{g}^{-1} \mathrm{grad} \, \phi_{\mathrm{f}} + \boldsymbol{m}_{\mathrm{fd}}$, where 511 $m_{\rm fd}$ is the "dissipative part" of $m_{\rm f}$ [9,35]. The term 512 $m_{
m fd}$ can be expressed constitutively by supposing that 513 Darcy's law is valid. Hence, $m_{\rm fd}$ is written as a func-514 tion of F and the filtration velocity q. In particular, this 515 function is assumed to be nonlinear in F and linear in q. 516 We set, thus, $\boldsymbol{m}_{\rm fd} = \hat{\boldsymbol{m}}_{\rm fd}(\boldsymbol{F}, \boldsymbol{q}) = -\phi_{\rm f} \boldsymbol{g}^{-1} \boldsymbol{k}^{-1} \boldsymbol{q}$, where 517 \boldsymbol{k} is the permeability tensor of the medium. These re-518 sults allow to rephrase the momentum balance law (28b) 519 in terms of Darcy's law: 520

$$\boldsymbol{q} = -\boldsymbol{k} \operatorname{grad} \boldsymbol{p}. \tag{33}$$

In the literature on hydrogeological porous media, \boldsymbol{k} is 521 usually referred to as *hydraulic conductivity*, and it is 522 expressed as the ratio of the medium's *permeability* (in 523 fact, a quantity depending on the structure of the pore 524 space) to the fluid's viscosity [7]. In the present con-525 text, however, we prefer to stick to the nomenclature 526 adopted in Biomechanics, in which k is known as "tis-527 sue's permeability", and depends both on the tissue's 528 microstructure and the flow properties of the interstitial 529 fluid. 530

By performing the Piola transform of (31a), (31b), $_{531}$ and (33), the first Piola-Kirchhoff stress tensors $P_{\rm f}$ and $_{532}$ $P_{\rm s}$, and the filtration velocity Q take on the form $_{533}$

$$\boldsymbol{P}_{\rm f} = -(J - \phi_{\rm sR}) p \, \boldsymbol{g}^{-1} \boldsymbol{F}^{-\rm T},\tag{34a}$$

$$\boldsymbol{P}_{\rm s} = -\phi_{\rm sR} p \, \boldsymbol{g}^{-1} \boldsymbol{F}^{-\rm T} + \boldsymbol{P}_{\rm sc}, \qquad (34b)$$

$$\boldsymbol{Q} = -\boldsymbol{K} \operatorname{Grad} \boldsymbol{p},\tag{34c}$$

where $P_{\rm sc} = J \sigma_{\rm sc} F^{-T}$ is the constitutive part of the first Piola-Kirchhoff stress and $K = J F^{-1} k F^{-T}$ is referred to as the tissue's "material permeability" [4,25].

To close the model, constitutive expressions for $P_{\rm sc}$ 537 and K are sought for. Since $P_{\rm sc}$ is defined through the relation $P_{\rm sc} = F(2\partial_C \hat{W}(C))$, it can be determined by prescribing the strain energy density function $\hat{W}(C)$. 540

3 Constitutive model of Articular Cartilage 541

In the sequel, a cylindric sample of AC of height L and circular cross section will be used for the benchmark tests. The upper and lower boundaries of the sample 542 543

represent, respectively, the articular surface of cartilage, 545 and the surface at which it is attached to the subchon-546 dral bone [46]. The specimen is assumed to be trans-547 versely isotropic with respect to the axis of the cylinder, 548 which thus coincides with $\boldsymbol{\xi}$. The direction of $\boldsymbol{\xi}$ is also 549 said to be the "direction of the tissue depth" [58]. If 550 the cartesian basis of orthonormal vectors $\{E_A\}_{A=1}^3$ is 551 chosen as the global reference frame for the whole sam-552 ple, with its unit vectors emanating from the centre of 553 the lower boundary, and $E_3 \equiv \boldsymbol{\xi}$, then the coordinate 554 associated with $\boldsymbol{\xi}, X^3$, can be normalised as $\boldsymbol{\xi} = X^3/L$, 555 thereby identifying the lower boundary with $\xi = 0$ and 556 the upper one with $\xi = 1$. 557

Histological studies performed by Guilak et al. [32] 558 show that AC can be roughly divided into three zones, 559 which can be related to the variability of the fibre ori-560 entation with the axial (normalised) coordinate ξ [2, 561 50. More specifically, the fibres appear to be almost 562 parallel to the symmetry axis in the deep zone, ran-563 domly oriented in the middle zone, and parallel to the 564 articular surface in the upper zone. This feature of the 565 tissue's microstructure suggests the approximation ac-566 cording to which the material properties of AC vary 567 with ξ , but are constant on each transverse plane. Con-568 sequently, we write $\hat{\Psi}(X,\Theta) \equiv \hat{\Psi}(\xi,\Theta)$ where, with a 569 slight abuse of notation, we refer to $\hat{\Psi}$ as to a function 570 of the axial coordinate ξ and the co-latitude Θ . Among 571 other possible choices, we employ the pseudo-Gaussian 572 distribution [17, 30]573

$$\hat{\Psi}_1(\xi,\,\cdot\,):[0,\frac{\pi}{2}]\to\mathbb{R}_0^+,\tag{35a}$$

$$\hat{\Psi}_1(\xi,\Theta) = \frac{1}{\mathcal{Z}_{1+}(\xi)} \exp\left(-\frac{[\Theta - Q(\xi)]^2}{2[\sigma(\xi)]^2}\right),\tag{35b}$$

574 and the von Mises distribution

$$\hat{\Psi}_2(\xi, \cdot) : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}_0^+,\tag{36a}$$

$$\hat{\Psi}_2(\xi,\Theta) = \frac{2}{\pi} \sqrt{\frac{b(\xi)}{2\pi}} \frac{\exp(b(\xi)[\cos(2\Theta) + 1])}{\operatorname{erfi}(\sqrt{2b(\xi)})}.$$
 (36b)

Note that the von Mises distribution (36b) is already normalised over the hemisphere $\mathbb{S}_X^{2+}\mathscr{B}$. Indeed, given the probability density $\Psi_2(\xi, \cdot) : \mathbb{S}_X^{2+}\mathscr{B} \to \mathbb{R}_0^+$, such that $\Psi_2(\xi, \mathbf{M}) = \hat{\Psi}_2(\xi, \Theta)$, it holds that

$$1 = \int_{\mathbb{S}_X^{2+} \mathscr{B}} \Psi_2(\xi, \boldsymbol{M})$$

= $2\pi \int_0^{\pi/2} \hat{\Psi}_2(\xi, \Theta) \sin \Theta \,\mathrm{d}\Theta$. (37)

⁵⁷⁹ In (35b), $\mathcal{Z}_{1+}(\xi)$ is the normalisation factor, $Q(\xi)$ is ⁵⁸⁰ the most probable angle, and $[\sigma(\xi)]^2$ is the variance, ⁵⁸¹ whereas the parameter $b(\xi)$ in (36b) is referred to as ⁵⁸² concentration parameter. Here, we prescribe $Q(\xi)$ and $\sigma(\xi)$ as in [17], and we take $b(\xi)$ as an affine function of ξ , i.e., ξ

$$Q(\xi) = \frac{\pi}{2} \left[1 - \cos\left(\frac{\pi}{2} \left(-\frac{2}{3}\xi^2 + \frac{5}{3}\xi\right)\right) \right],$$
 (38a)

$$\sigma(\xi) = 10^3 [\xi(1-\xi)]^4 + 3 \cdot 10^{-3}, \qquad (38b)$$

$$b(\xi) = -16\xi + 8. \tag{38c}$$

We remark that the general theory exposed in the following and the proposed algorithms, A1 and A2, hold for both choices of the probability density. 587

As shown in Figure 1, $Q(\xi)$ grows monotonically 588 from Q(0) = 0 to $Q(1) = \pi/2$. Indeed, at the bone-589 cartilage interface, the most probable fibre alignment is 590 along the direction of the symmetry axis, while at the 591 articular surface fibres lie in the transverse plane and, 592 due to transverse isotropy, they are randomly oriented 593 on it. In terms of fibre alignment, the tissue experiences 594 a "transition" between two ordered configurations (the 595 one at $\xi = 0$ being more ordered than that at $\xi = 1$), 596 passing through a highly disordered configuration. This 597 is reflected by the standard deviation, σ , which tends 598 towards zero (for simulation purposes, it is kept "small 599 enough") for ξ approaching $\xi = 0$ and $\xi = 1$, and at-600 tains a global maximum at $\xi = 1/2$, where the ran-601 domness in the fibre orientation is maximal (i.e., in the 602 limit of perfectly randomly oriented fibres, the variance 603 should tend towards infinity). In the case of the von 604 Mises probability density, the dispersion in the fibres' 605 alignment is represented by the depth-dependent pa-606 rameter $b(\xi)$, which is prescribed in (38c) to capture 607 the arrangement of the collagen fibres sketched in Fig-608 ure 1. In particular, since the concentration parameter 609 describes fibres aligned vertically when it diverges pos-610 itively, a random arrangement of fibres when it is zero, 611 and fibres oriented horizontally when it diverges nega-612 tively, we take $b(\xi)$ as a monotonously increasing func-613 tion of $\xi \in [0, 1]$ and, as done in previous works [58, 34], 614 we assume that its range is [-8, 8]. Hereafter, we shall 615 indicate the probability density by $\hat{\Psi}$ whenever there is 616 no need to specify whether it has to be $\hat{\Psi}_1$ or $\hat{\Psi}_2$. 617

3.1 Constitutive Model 618

Coherently with the model put forward in [18, 19, 21, 24, ⁶¹⁹ 43], \hat{W} is specified by superimposing the strain energy ⁶²⁰ density of the matrix, \hat{W}_0 , with that of the fibres, \hat{W}_e . ⁶²¹ Moreover, following [19], a penalty term, is added to ⁶²² account for the incompressibility of the solid phase at ⁶²³ compaction: ⁶²⁴

$$\hat{W}(\boldsymbol{C}) = \phi_{\mathrm{sR}} \hat{U}(J) + \phi_{0\mathrm{sR}} \hat{W}_0(\boldsymbol{C}) + \phi_{1\mathrm{sR}} \hat{W}_{\mathrm{e}}(\boldsymbol{C}).$$
(39)

The term \hat{U} depends on \boldsymbol{C} only through the volumetric 625 ratio, $J = \sqrt{\det(\mathbf{C})}$, and takes on the form [19] 626

$$\hat{U}(J) = \mathcal{H}(J_{\rm crit} - J) \frac{(J - J_{\rm crit})^{2q}}{(J - \phi_{\rm sR})^r},\tag{40}$$

where \mathcal{H} is the Heaviside function, $J_{\text{crit}} \in]\phi_{\text{sR}}, 1]$ is a 627 "critical" value of the volumetric ratio, below which the 628 potential \hat{U} is switched on, while $q \geq 2$ and $r \in [0, 1]$ are 629 material parameters (q is an integer). In this work, the 630 strain energy density of the matrix, W_0 , is assumed to 631 be of Neo-Hookean type [11], i.e., 632

$$\hat{W}_{0}(\boldsymbol{C}) = \frac{1}{8}\lambda_{0} \left(\ln[\det(\boldsymbol{C})]\right)^{2} - \frac{1}{2}\mu_{0}\ln[\det(\boldsymbol{C})] \\ + \frac{1}{2}\mu_{0} \left(\boldsymbol{G}^{-1}:\boldsymbol{C}-3\right), \qquad (41)$$

with λ_0 and μ_0 being the Lamé first modulus and the 633 shear modulus of the matrix, respectively. Nevertheless, 634 other choices of \hat{W}_0 are possible. Usually, for articular 635 cartilage, the Holmes-Mow strain energy is used [36]. 636 The strain energy density associated with the fibres, 637 \hat{W}_{e} , is said to be the "ensemble fibre potential" [20], 638 and may be written as (see [18-20, 22] for details) 639

$$W_{\rm e}(\boldsymbol{C}) = W_{\rm 1i}(\boldsymbol{C}) + W_{\rm 1a}(\boldsymbol{C})$$
$$= \hat{W}_{\rm 1i}(\boldsymbol{C}) + \int_{\mathbb{S}^{2+}_{X}\mathscr{B}} \Psi(\boldsymbol{M}) \hat{w}_{\rm 1a}(\boldsymbol{C}, \boldsymbol{M}).$$
(42)

In (42), \hat{W}_{1i} and \hat{W}_{1a} are the isotropic and the anisotropic 640 contributions of the fibres to the overall strain energy 641 density. More specifically, \hat{W}_{1i} is assumed to have the 642 same functional form as \hat{W}_0 , the only difference being 643 in the elastic constants, which are given by λ_1 and μ_1 in 644 the case of \hat{W}_{1i} . The summand \hat{W}_{1a} , instead, is defined 645 by the integral on the right-hand-side of (42), and is 646 constructed in two steps: For each $X \in \mathcal{B}$, one consid-647 ers the anisotropic strain energy density $\hat{w}_{1a}(C, M)$, 648 which is associated with the fibres oriented along the 649 unit vector M emanating from X. Then, $\hat{w}_{1a}(C, M)$ 650 is multiplied by the probability density $\Psi(\mathbf{M})$ of find-651 ing a family of fibres aligned along M, and the result 652 is integrated over the material unit hemisphere $\mathbb{S}^{2+}_X \mathscr{B}$. 653 Here, $\hat{w}_{1a}(\boldsymbol{C}, \boldsymbol{M})$ is chosen as 654

$$\hat{w}_{1a}(\boldsymbol{C}, \boldsymbol{M}) = \mathcal{H}(I_4 - 1)\frac{c}{2}[I_4 - 1]^2,$$
(43)

where the short-hand notation 655

$$I_4 \equiv I_4(\boldsymbol{C}, \boldsymbol{M}) := \operatorname{tr}[\boldsymbol{C}(\boldsymbol{M} \otimes \boldsymbol{M})] = \operatorname{tr}[\boldsymbol{C}\boldsymbol{A}]$$
(44)

has been introduced to express the fourth invariant of 656 the deformation, c is a material parameter, and \mathcal{H} is the 657 Heaviside function. The fourth invariant extracts the 658 component of C in the direction of M, and expresses 659 whether the fibre oriented along M is stretched, i.e., 660 $I_4 > 1$, contracted, i.e., $I_4 < 1$, or neutral, in which case 661

 $I_4 = 1$. The Heaviside function selects only stretched 662 fibres as contributors to the ensemble fibre potential. 663 The tensor $A = M \otimes M$ is referred to as structure 664 tensor. 665

When the penalty term $\hat{U}(J)$ is not active, the con-666 stitutive part of the first Piola-Kirchhoff stress tensor, 667 $\boldsymbol{P}_{\rm sc} = \hat{\boldsymbol{P}}_{\rm sc}(\boldsymbol{F})$, is given by

$$\hat{\boldsymbol{P}}_{\rm sc}(\boldsymbol{F}) = \frac{1}{2}\lambda \ln[\det(\boldsymbol{C})]\boldsymbol{g}^{-1}\boldsymbol{F}^{-\mathrm{T}} + \mu \boldsymbol{F} \left(\boldsymbol{G}^{-1} - \boldsymbol{C}^{-1}\right) \\ + \boldsymbol{F}\boldsymbol{S}_{\rm a}, \tag{45}$$

where $\lambda := \phi_{0sR}\lambda_0 + \phi_{1sR}\lambda_1$ and $\mu := \phi_{0sR}\mu_0 + \phi_{1sR}\mu_1$ 669 are the elastic moduli of the solid phase, and $\boldsymbol{S}_{\mathrm{a}}$ is 670 the anisotropic part of the constitutive second Piola-671 Kirchhoff stress tensor, i.e., 672

$$\boldsymbol{S}_{\mathrm{a}} = 2\phi_{1\mathrm{sR}}c \int_{\mathbb{S}_{X}^{2+}\mathscr{B}} \boldsymbol{\Psi}(\boldsymbol{M}) \mathcal{H}(I_{4}-1)[I_{4}-1]\boldsymbol{A}.$$
(46)

To complete the constitutive framework, the tissue's 673 permeability has to be specified. In this work, we use 674 the model presented in [19, 58], which extends the re-675 sults obtained in [21, 22] to the case of finite deforma-676 tions. In [21,22], a Representative Elementary Volume 677 (REV) is claimed to exist, which consists of one segment 678 of fibre, the matrix, and the fluid. The REV perme-679 ability is determined through an upscaling procedure, 680 based on techniques put forward in [42, 48, 51, 53]. By 681 construction, it depends on the direction of the fibre in 682 whose neighbourhood the REV is constructed. Thus, 683 the permeability of the tissue, k, is computed by per-684 forming the directional average of the REV permeabil-685 ity. Hence, the Piola transform of k yields the "material" 686 permeability", K, whose constitutive form is given by 687

$$\boldsymbol{K} = \hat{\boldsymbol{K}}(\boldsymbol{F}) = \hat{k}_0(J) \frac{(J - \phi_{1\mathrm{sR}})^2}{J} \boldsymbol{C}^{-1} + \hat{k}_0(J) \frac{(J - \phi_{1\mathrm{sR}})\phi_{1\mathrm{sR}}}{J} \hat{\boldsymbol{Z}}(\boldsymbol{C}), \quad (47)$$

where the scalar permeability $\hat{k}_0(J)$ is defined on the 688 basis of experimental data, and 689

$$\mathbf{Z} = \hat{\mathbf{Z}}(\mathbf{C}) := \int_{\mathbb{S}_X^{2+}\mathscr{B}} \Psi(\mathbf{M}) \frac{\mathbf{M} \otimes \mathbf{M}}{I_4}.$$
 (48)

According to the Holmes-Mow permeability model, $k_0(J)$ 690 reads 691

$$\hat{k}_0(J) = k_{0\mathrm{R}} \left[\frac{J - \phi_{\mathrm{sR}}}{1 - \phi_{\mathrm{sR}}} \right]^\kappa \exp\left(\frac{M}{2} [J^2 - 1]\right),$$
 (49)

where k_{0R} , κ , and M are material parameters gener-692 ally depending on the point of the tissue at which they 693 are evaluated. Note that a different formulation of the 694 deformation-dependent permeability in anisotropic me-695 dia was proposed in [4]. 696

⁶⁹⁸ By exploiting the constitutive expressions (45) and (47), ⁶⁹⁹ the balance laws (26) and (29a) become

$$\dot{J} + \operatorname{Div}\left(-\hat{K}(F)\operatorname{Grad} p\right) = 0,$$
 (50a)

Div
$$\left(-Jp \, \boldsymbol{g}^{-1} \boldsymbol{F}^{-\mathrm{T}} + \hat{\boldsymbol{P}}_{\mathrm{sc}}(\boldsymbol{F})\right) = \boldsymbol{0}.$$
 (50b)

Equations (50a) and (50b) constitute the set of coupled 700 partial differential equations that have to be solved to 701 determine the motion of the solid phase, χ , and the 702 pressure, p. We remark that the terms in parentheses 703 in (50b) express the overall first Piola-Kirchhoff stress 704 tensor $\hat{\hat{P}}(p, F) := -Jpg^{-1}F^{-T} + \hat{P}_{sc}(F)$ as a function 705 of pressure and deformation. We emphasise that, since 706 $\boldsymbol{P}_{\mathrm{sc}} = \hat{\boldsymbol{P}}_{\mathrm{sc}}(\boldsymbol{F})$ and $\boldsymbol{K} = \hat{\boldsymbol{K}}(\boldsymbol{F})$ depend solely on \boldsymbol{F} , 707 $\hat{\boldsymbol{P}}(p, \boldsymbol{F})$ is an affine function of p, and Darcy's law (34c) 708 constitutes a linear relation between Q and p, through 709 Grad p. Moreover, the model equations (50a) and (50b) 710 are nonlinear in χ because they feature $J = \det(\mathbf{F})$, 711 and also because $\hat{\pmb{P}}_{\mathrm{sc}}(\pmb{F})$ and $\hat{\pmb{K}}(\pmb{F})$ depend on \pmb{F} in 712 nonlinear way. 713

The approximated solution to (50a) and (50b) is usually sought for by having recourse to Finite Element methods. For example, in the case of zero Neumann boundary conditions, and non-zero Dirichlet boundary conditions, the following weak formulation [27,31] applies

$$\mathfrak{F}_p := -\int_{\mathscr{B}} \left\{ (\operatorname{Grad} \tilde{p}) \hat{\boldsymbol{K}}(\boldsymbol{F}) \operatorname{Grad} p + \tilde{p} \dot{J} \right\} = 0, \quad (51a)$$

$$\mathfrak{F}_{\chi} := \int_{\mathscr{B}} \hat{\boldsymbol{P}}(p, \boldsymbol{F}) : \boldsymbol{g} \operatorname{Grad} \tilde{\boldsymbol{u}} = 0, \qquad (51b)$$

where \tilde{p} and \tilde{u} are the test functions of the problem, and belong to the sets

$$\tilde{\mathscr{P}} = \{ \tilde{p} \in H^1_0(\mathscr{B}) : \quad \tilde{p}_{|\Gamma^p_{\mathcal{D}}} = 0 \},$$
(52a)

$$\tilde{\mathscr{V}} = \{ \tilde{\boldsymbol{u}} \in \boldsymbol{H}_0^1(\mathscr{B}) : \ \tilde{\boldsymbol{u}}_{|\Gamma_D^{\chi}} = \boldsymbol{0} \},$$
(52b)

with Γ_p^p and Γ_{Σ}^{χ} being, respectively, the portions of the boundary of \mathscr{B} on which Dirichlet conditions are imposed. Physically, \tilde{p} and \tilde{u} represent a virtual pressure and a virtual velocity, respectively. Note that the functionals $\mathfrak{F}_p =: \hat{\mathfrak{F}}_p(\chi, p, \tilde{p})$ and $\mathfrak{F}_{\chi} := \hat{\mathfrak{F}}_{\chi}(\chi, p, \tilde{u})$ are linear in \tilde{p} and \tilde{u} , affine in p, and highly nonlinear in χ .

As pointed out in [16,34], the main difficulty in solv-728 ing (51a) and (51b) is due to the constitutive laws ex-729 pressing $\hat{P}_{sc}(F)$ and $\hat{K}(F)$, which necessitate the so-730 lution of integrals of functions, defined over $\mathbb{S}^{2+}_X \mathscr{B}$, for 731 which the deformation cannot be factorised (cf. (45)732 and (47)). This is due to the presence of the Heavi-733 side function of $(I_4(\boldsymbol{C}, \boldsymbol{M}) - 1)$ in the case of $\hat{\boldsymbol{P}}_{sc}(\boldsymbol{F})$, 734 and to the division by $I_4(C, M)$ in the case of $\hat{K}(F)$. 735

To circumvent these problems, dedicated algorithms are

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4 The Spherical Design Algorithm (SDA)

required. The rest of this work is devoted to a com-

parative study of some of these algorithms and to the

evaluation of their features.

Let f be any (scalar, vector, or tensor) function defined over $\mathbb{S}_X^{2+}\mathscr{B}$. By using the identification (13), it holds that $f(\mathbf{M}) = \hat{f}(\Theta, \Phi)$ for all $\mathbf{M} \in \mathbb{S}_X^{2+}\mathscr{B}$, and $(\Theta, \Phi) \in$ $\mathscr{D} = [0, \pi/2] \times [0, 2\pi]$. If f is integrable over $\mathbb{S}_X^{2+}\mathscr{B}$, one can write 745

$$\int_{\mathbb{S}_X^{2+}\mathscr{B}} f(\boldsymbol{M}) = \iint_{\mathscr{D}} \hat{f}(\boldsymbol{\Theta}, \boldsymbol{\Phi}) \sin(\boldsymbol{\Theta}) \mathrm{d}\boldsymbol{\Theta} \mathrm{d}\boldsymbol{\Phi}.$$
 (53)

The Spherical *t*-Design [33] is a numerical method used ⁷⁴⁶ to solve integrals of the same type as (53). After selecting two positive integers, *m* and *n*, and a proper ⁷⁴⁸ set of pairs $\mathcal{X}_{ij} = (\Theta_i, \Phi_j) \in \mathcal{D}$, with $i = 1, \ldots, m$ and ⁷⁴⁹ $j = 1, \ldots, n$, the integrals in (53) are approximated by ⁷⁵⁰

$$\iint_{\mathscr{D}} \hat{f}(\Theta, \Phi) \sin(\Theta) \mathrm{d}\Theta \mathrm{d}\Phi \simeq \sum_{i=1}^{m} \sum_{j=1}^{n} w(\mathcal{X}_{ij}) \hat{f}(\mathcal{X}_{ij})$$
$$= \frac{2\pi}{N} \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{f}(\mathcal{X}_{ij}), \qquad (54)$$

with N = mn. Formula (54) is exact when $f(\mathbf{M})$ is a 751 polynomial of degree t and N = mn is a sufficiently high 752 number of integration points [38]. Here, *m* denotes the 753 dimension of the set of co-latitudes, whereas n stands 754 for the dimension of the set of longitudes. It is worth-755 while to notice that the factor $2\pi/N$ is the Nth part of 756 the area of the surface of the unit hemisphere, and rep-757 resents the area of a generic element of the grid covering 758 \mathscr{D} . Since, for every $X \in \mathscr{B}$, each $\mathcal{X}_{ij} \in \mathscr{D}$ corresponds 759 univocally to the unit vector $M_{ij} = \hat{M}(\mathcal{X}_{ij}) \in \mathbb{S}^{2+}_X \mathscr{B}$ 760 and, thus, to the point Y_{ij} on the surface of the unit 761 hemisphere, such that $Y_{ij} - X = M_{ij}$, the grid covering 762 \mathscr{D} is mapped onto the surface of the unit hemisphere. 763

We remark that the SDA relies on the fact that, if the integration points are properly chosen, the weights $w(\mathcal{X}_{ij})$ in (54) are all equal to $\frac{2\pi}{N}$ for each point \mathcal{X}_{ij} , with $i = 1, \ldots, m$ and $j = 1, \ldots, n$. For other choices of the integration points, instead, non-trivial weights should be determined [8,12].

4.1 Proper choice of the integration points

The first important numerical issue related to the SDA $_{771}$ is the appropriate choice of the discrete set of points $_{772}$ in the domain \mathscr{D} . As shown in Figure 2, if we pick the $_{773}$

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⁷⁷⁴ points \mathcal{X}_{ij} , i = 1, ..., m, j = 1, ..., n, homogeneously ⁷⁷⁵ in \mathscr{D} , they cluster close to the poles, as the integral ⁷⁷⁶ measure depends on $\sin(\Theta)$. To skip this problem, we ⁷⁷⁷ perform the change of variables [59],

$$\Theta = \operatorname{acos}(2v - 1), \quad \Phi = 2\pi u, \quad (v, u) \in \mathscr{E}, \tag{55}$$

with $\mathscr{E} = \left[\frac{1}{2}, 1\right] \times [0, 1]$. Clearly, the area of the surface of the unit hemisphere is preserved by (55), i.e.,

$$|\mathbb{S}_X^{2+}\mathscr{B}| = \iint_{\mathscr{D}} \sin(\Theta) \mathrm{d}\Theta \mathrm{d}\Phi = 4\pi \iint_{\mathscr{E}} \mathrm{d}v \mathrm{d}u = 2\pi, \quad (56)$$

and the integrals (53) and (54) can be rephrased as

$$\int_{\mathbb{S}_X^{2+}\mathscr{B}} f(\boldsymbol{M}) = 4\pi \iint_{\mathscr{E}} \tilde{f}(v, u) \, \mathrm{d}v \mathrm{d}u$$
$$\simeq \frac{2\pi}{N} \sum_{i=1}^m \sum_{j=1}^n \tilde{f}(v_i, u_j).$$
(57)

Hereafter, we say that points chosen according to (55)781 are "equidistributed" in the sense that, quoting from 782 [59], "any small area on the [hemi]sphere is expected 783 to contain the same number of points". A number of 784 suitable sets of points for the Spherical t-Design has 785 been produced by Sloane [38], and is freely available 786 online. We refer to such integration points as to "Sloane 787 points". We remark, however, that any set of "Sloane 788 points", hereafter denoted by \mathcal{S} , is defined as a discrete 789 subset of $[0,\pi] \times [0,2\pi]$. Thus, to apply the approxima-790 tion formula (54) (or (57), in the case of equidistributed 791 points), it is necessary to determine a subset $\mathcal{S}' \subset \mathcal{S}$ of 792 Sloane points such that 793

$$\mathcal{S}' = \{ \mathcal{X}_{ij} \in \mathcal{S} \mid 0 \le \Theta_i \le \frac{\pi}{2}, \\ i = 1, ..., m, \ j = 1, ..., n \}$$

$$(58)$$

(and its counterpart in \mathscr{E} , if (57) is invoked). In the following, we shall calculate the directional averages featuring in the expressions of Z and $S_{\rm a}$ by using \mathcal{S}' in (54). For example, in the case of Z, we obtain

$$Z = \int_{\mathbb{S}_{X}^{2+}\mathscr{B}} \Psi(\boldsymbol{M}) \frac{\boldsymbol{M} \otimes \boldsymbol{M}}{I_{4}}$$
$$\approx \frac{2\pi}{N} \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{\Psi}(\mathcal{X}_{ij}) \frac{\hat{\boldsymbol{M}}(\mathcal{X}_{ij}) \otimes \hat{\boldsymbol{M}}(\mathcal{X}_{ij})}{I_{4}(\boldsymbol{C}, \hat{\boldsymbol{M}}(\mathcal{X}_{ij}))},$$
(59)

where $\mathcal{X}_{ij} \in \mathcal{S}'$ for all $i = 1, \ldots, m$, and $j = 1, \ldots, n$, and $\hat{\Psi}$ can be either the pseudo-Gaussian distribution $\hat{\Psi}_1$ in (35b) or the von Mises distribution $\hat{\Psi}_2$ in (36b). As we will show in discussing the results of the present work, depending on the type of function to be averaged, a given set of points of \mathcal{S}' may, or may not, deliver acceptable results.

As it will be seen in the computation of the tissue's permeability (cf. Figure 3), an appropriate choice of the integration points strongly influences the evaluation of the integrals over $\mathbb{S}_{X}^{2+}\mathscr{B}$ by means of the SDA.



Fig. 2 Points $\mathcal{X}_{ij} \in \mathcal{D}$ mapped onto the surface of the sphere (blue dots). (a) Points corresponding to homogeneously chosen \mathcal{X}_{ij} . (b) Points corresponding to \mathcal{X}_{ij} chosen according to (55).

4.2 Preliminary test of the SDA

The SDA, as sketched in algorithm A1, is often used 810 for the numerical evaluation of the integrals over $\mathbb{S}^{2+}_X \mathscr{B}$ 811 that appear in the constitutive expressions of stress and 812 permeability (cf. (45) and (47), respectively). It comes 813 into play after (51a) and (51b) are discretised in time 814 and space, and is invoked within each time step and 815 each iteration of the adopted linearisation procedure. 816 As shown in A1, three nested loops have to be per-817 formed to run the SDA: One of them is on the chosen 818 grid vertices, while the second two refer to the set of 819 points \mathcal{X}_{ij} selected in \mathcal{D} . In this work, the SDA is com-820 pared with an integration scheme available in Matlab, 821 which has been sketched in the algorithms A2 and A3. 822 In particular, we specify in A3 the functions used for 823 the implementation of A2. 824

The Heaviside function in (46) is taken into account by means of an If cycle both in the SDA routine (line 18 of algorithm A1) and in the Matlab subroutine "STRESS" (line 11 of algorithm A2). Indeed, since for a given Cauchy-Green deformation tensor C, the integrand of (46) is different from zero only if $I_4 > 1$, only stretched fibres contribute to the stress S_a , which can thus be computed as (recall that $A = M \otimes M$)

$$\boldsymbol{S}_{\mathrm{a}} = 2\phi_{1\mathrm{sR}}c \int_{\mathbb{H}_{X}(\boldsymbol{C})} \Psi(\boldsymbol{M})[I_{4}-1]\boldsymbol{A}, \tag{60}$$

where the integration domain $\mathbb{H}_X(C)$ is given by

$$\mathbb{H}_X(\boldsymbol{C}) = \left\{ \boldsymbol{M} \in \mathbb{S}_X^{2+} \mathscr{B} | \; \boldsymbol{C} : (\boldsymbol{M} \otimes \boldsymbol{M}) > 1 \right\}.$$
(61)

We remark that the integrand of (60) is bounded and $C^{\infty}(\mathbb{H}_X(\mathbf{C}))$, for all \mathbf{C} . The numerical evaluation of (60), performed by means of the SDA, yields

$$\boldsymbol{S}_{a} = 2\phi_{1sR}c \int_{\mathbb{H}_{X}(\boldsymbol{C})} \boldsymbol{\Psi}(\boldsymbol{M})[I_{4}-1]\boldsymbol{A}$$
$$\approx 2\phi_{1sR}c \left(\frac{2\pi}{N}\sum_{ij}^{\prime} \hat{\boldsymbol{\Psi}}(\boldsymbol{\mathcal{X}}_{ij})[\hat{I}_{4}(\boldsymbol{\mathcal{X}}_{ij})-1]\hat{\boldsymbol{A}}(\boldsymbol{\mathcal{X}}_{ij})\right), \quad (62)$$

where $\hat{A}(\mathcal{X}_{ij}) = \hat{M}(\mathcal{X}_{ij}) \otimes \hat{M}(\mathcal{X}_{ij})$, and \sum_{ij}' means that the sum is performed only for the values of i and j such that $C: \hat{A}(\mathcal{X}_{ij}) > 1$.

In this preliminary set of tests that we present, we 840 investigate the reliability and the convergence of the 841 SDA in response to the chosen set of integration points 842 only, i.e., without implementing it in any Finite Ele-843 ment software. For this purpose, we consider the un-844 deformed state of the tissue, in which it holds that 845 $C = G, Z = Z_0 \equiv \hat{Z}(G)$ (see (48)), and the stress 846 tensor $S_{\rm a}$ vanishes identically. Indeed, since M is a 847 unit vector in the norm induced by G, it holds that 848 $I_4(\boldsymbol{G}, \boldsymbol{M}) = \operatorname{tr}[\boldsymbol{G}(\boldsymbol{M} \otimes \boldsymbol{M})] = 1$. Hence, \boldsymbol{Z}_0 can be 849 evaluated as 850

$$\boldsymbol{Z}_{0} = \iint_{\mathscr{D}} \hat{\boldsymbol{\Psi}}(\boldsymbol{\xi}, \boldsymbol{\Theta}) \hat{\boldsymbol{A}}(\boldsymbol{\Theta}, \boldsymbol{\Phi}) \sin(\boldsymbol{\Theta}) \mathrm{d}\boldsymbol{\Theta} \mathrm{d}\boldsymbol{\Phi}, \tag{63}$$

with $\hat{A}(\Theta, \Phi) = \hat{M}(\Theta, \Phi) \otimes \hat{M}(\Theta, \Phi)$. We remark that Z_0 is the averaged structure tensor. By expressing Mas in (13), the components of Z_0 become

$$(\mathbf{Z}_0)^{11} = (\mathbf{Z}_0)^{22} = \pi \int_0^{\pi/2} \hat{\Psi}(\xi, \Theta) [\sin(\Theta)]^3 \mathrm{d}\Theta, \quad (64a)$$

$$(\mathbf{Z}_0)^{12} = (\mathbf{Z}_0)^{13} = (\mathbf{Z}_0)^{23} = 0,$$
 (64b)

$$(\mathbf{Z}_0)^{33} = 1 - 2(\mathbf{Z}_0)^{11}.$$
 (64c)

Since the deformation is not involved in this calcula-854 tion, the evaluation of $(\mathbf{Z}_0)^{11}$ need not be coupled with 855 the FE code, and serves as a preliminary analysis of the 856 reliability of the SDA, thereby helping understand how 857 the considered quadrature methods work. We compute 858 $(\mathbf{Z}_0)^{11}$ by using both the SDA and the algorithm A2, 859 and compare the results delivered by the two proce-860 dures. 861

It must be noticed that, in (64a), the integration with respect to Φ is computed exactly. Thus, only the 864

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866

integral with respect to Θ needs to be approximated. This is done by invoking the SDA and the Matlab routine.

As anticipated in section 4.1, and shown in Figure 3, 867 if the values of Θ and Φ are taken homogeneously, the 868 corresponding points on the unit hemisphere gather in 869 the neighbourhood of the poles, and the radial compo-870 nent $(\mathbf{Z}_0)^{11}$ of the averaged structure tensor produced 871 by the SDA (cf. algorithm A1) does not converge to 872 the effective value of the integral (black, bold curve in 873 Figure 3, evaluated by means of A2). The same con-874 siderations apply to the axial component $(\mathbf{Z}_0)^{33}$. Note 875 that the points on the hemisphere corresponding to a 876 homogeneous distribution of pairs $(\Theta, \Phi) \in \mathscr{D}$ are also 877 said to be equispaced. We remark that the arrow cross-878 ing the curves in Figure 3 indicates the direction of 879 ascending m. We see that, by increasing the value of 880 m, the result of the numerical computation of (64a), 881 done by means of the algorithm A1, converges to the 882 one obtained with the algorithm A2, if equidistributed 883 points are used to discretise the integration domain. 884



Fig. 3 Preliminary results of the algorithms A1 and A2 for the undeformed configuration. The solutions obtained by means of equispaced and equidistributed points are compared with the numerical outcome of the routine A2 (see also [22] for comparison).

In Figure 4, we show the results of the calculation 885 of $(\mathbf{Z}_0)^{11}$ as in (64a) according to the numerical inte-886 gration scheme A2 and to different sets of integration 887 points used for the SDA. The first two sets are $\mathcal{S}'_{120} \subset$ 888 \mathcal{S}_{240} and $\mathcal{S}'_{21} \subset \mathcal{S}_{41}$. They represent the subsets of the 889 spherical designs S_{240} and S_{41} comprising 240 and 41 890 points, respectively [38], and obtained by selecting the 891 pairs $\mathcal{X}_{ij} = (\Theta_i, \Phi_j)$, with $\Theta_i \in [0, \frac{\pi}{2}]$ and $\Phi_j \in [0, 2\pi]$. 892 In fact, the spherical designs in [38] are conceived to 893 perform the quadrature over $\mathbb{S}^2_X \mathscr{B}$, whereas we need to 894 integrate over $\mathbb{S}^{2+}_X \mathscr{B}$, only. Three sets of equidistributed 895 ⁸⁹⁶ points, with m = 10, 30, 50, respectively, have been con-⁸⁹⁷ sidered. Moreover, in this work, we propose and test a ⁸⁹⁸ further set of points distributed as shown below

$$\Theta \in \mathcal{I} = \left\{ 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{2} \right\},\tag{65a}$$

$$\Phi \in \mathcal{J} = \left\{ 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \right\}.$$
 (65b)

The set $\mathcal{I} \times \mathcal{J} \subset \mathscr{D}$ consists of 48 points. However, by combining each Θ in (65a) with all the longitudes Φ in (65b), and noticing that for $\Theta = 0$ all directions defined by varying Φ "condense" in the north pole, the set $\mathcal{I} \times \mathcal{J}$ determines only 41 different elements of $\mathbb{S}_X^{2+} \mathscr{B}$.

In Figure 4a we can see that, if the pseudo-Gaussian 904 distribution is used, i.e., if it holds $\hat{\Psi} = \hat{\Psi}_1$ for all the 905 sets of points discretising the hemisphere, the results of 906 the numerical integration of (64a) are in good agree-907 ment with those of the Matlab integration. In Figure 908 4b, we show the same results as in Figure 4a, but ob-909 tained by employing the von Mises probability density 910 distribution, i.e., for $\hat{\Psi} = \hat{\Psi}_2$. In this case, we see that 911 the set S'_{120} captures better the Matlab curve, and that, 912 in general, the other sets of points fail for positive and 913 increasing values of the concentration parameter b (we 914 recall that b describes the dispersion of the fibres from 915 the direction of most probable alignment). For exam-916 ple, this can be deduced by inspecting Figure 4b for 917 $\xi = 0$, which corresponds to b(0) = 8. This inaccuracy 918 could be attributed to the fact that the performance of 919 some spherical designs deteriorates when the denomi-920 nator of Ψ_2 is sufficiently big. For instance, at b(0) = 8, 921 $\operatorname{erfi}(\sqrt{2b(0)})$ returns a value that is about $1 \cdot 10^6$. 922

⁹²³ In Figure 5, we reported the integral error of the ⁹²⁴ SDA with respect to the Matlab integration

$$err_{int} = \int_0^1 \left| \left[(\boldsymbol{Z}_0)^{11} \right]_{SDA} - \left[(\boldsymbol{Z}_0)^{11} \right]_{Matlab} \right| d\xi,$$
 (66)

which is computed numerically by means of a trape-925 zoidal quadrature formula. The orange, green, and ma-926 genta circles in the figure represent err_{int} in the discreti-927 sation of (64a) by means of \mathcal{S}'_{21} , \mathcal{S}'_{120} , and the new set 928 $\mathcal{I} \times \mathcal{J}$, respectively, and for $\hat{\Psi} = \hat{\Psi}_1$. From Figure 5, we 929 notice that the error in the computation with equidis-930 tributed points decreases exponentially with increasing 931 m (we recall that m is the number of values of Θ taken 932 in $[0, \frac{\pi}{2}]$). If the integration with respect to the variable 933 Φ is performed analytically (as is the case in (64a)), the 934 degree of accuracy of the sets \mathcal{S}'_{21} and \mathcal{S}'_{120} , in this pre-935 liminary analysis, is almost equal to the one obtained 936 with m = 21 and m = 120 equidistributed points, re-937 spectively, whereas the outcomes of the new set $\mathcal{I} \times \mathcal{J}$ 938 is less accurate than the one obtainable with the same 939 number of equidistributed points. The coloured squares 940

represent, respectively, the errors $err_{\rm int}$ computed by 941 employing $\mathcal{I} \times \mathcal{J}$ (magenta), \mathcal{S}'_{21} (orange), and \mathcal{S}'_{120} 942 (green), for $\hat{\Psi} = \hat{\Psi}_2$. Differently from the case of the 943 pseudo-Gaussian distribution, the magenta square rep-944 resenting err_{int} for the new set $\mathcal{I} \times \mathcal{J}$ lies under the 945 blue, squared curve obtained for the case of equidis-946 tributed points, thereby producing a better result in the 947 evaluation of $(\mathbf{Z}_0)^{11}$. Concerning \mathcal{S}'_{21} , we obtain a less 948 accurate result than in the case of 21 equidistributed 949 points, whereas the result obtained by employing S'_{120} 950 is almost exact, as the corresponding value of the error 951 is smaller than the one obtained with the same num-952 ber of equidistributed points. We can notice that, for 953 both the probability densities considered in the present 954 study, the new set of points returns almost the same 955 degree of accuracy, whereas the two considered sets of 956 Sloane points (\mathcal{S}'_{21} and \mathcal{S}'_{120} , respectively) are more in-957 fluenced by the nature of the integrand. 958

Finally, we compare the computational time required 959 for the algorithms A1 and A2. We see from Figure (6)960 that, by increasing the number of integration points, 961 the SDA necessitates linearly increasing computational 962 time, while the elapsed time for algorithm A2 increases 963 almost linearly with the refinement of the grid ver-964 tices (which, at this stage, consist of the points needed 965 to evaluate the depth dependent probability density 966 distribution). We remark that the computational time 967 reported in Figure 6 has been determined for a one-968 dimensional grid, represented by a given discrete set of 969 values of the normalised axial coordinate ξ . Thus, it is 970 not the overall time required for a full simulation. In-971 deed, in the full FEM model of (51a) and (51b), the 972 routines shown in the algorithms A1 and A2 will be 973 called for each point of the computational grid, and for 974 each of the six independent components of Z and S_{a} . 975 Thus, we may conclude from this first analysis that less, 976 but properly chosen, integration points are preferable to 977 an arbitrary big set of equidistributed points. The latter 978 ones, however, lead to a solution that converges to the 979 exact one, whereas the equispaced points do not, un-980 less non-trivial weights are determined (see Figure 2). 981 In the following, we present results of the routines A1 982 and A2. Afterwards, an *internal* implementation of the 983 SDA is presented for the set of equidistributed points, 984 for \mathcal{S}'_{21} and \mathcal{S}'_{120} , and for the set of points proposed in 985 $\mathcal{I} \times \mathcal{J}$. We will say that the implementation is *internal*, 986 if no call to a Matlab routine is needed. 987



Fig. 4 Results of the evaluation of $(\mathbf{Z}_0)^{11}$ with different spherical designs. (a) Pseudo-Gaussian probability density distribution. (b) von Mises probability density distribution. The curves corresponding to the 41 and 240 Sloane points have been found by using, respectively, 21 and 120 points on the hemisphere $\mathbb{S}_X^{2+} \mathscr{B}$. All other sets of points refer to integrations on $\mathbb{S}_X^{2+} \mathscr{B}$.

Algorithm 1 – A1– SDA (Spherical Design Algorithm)

- 1: procedure SDA
- 2: for k = 1, ..., M do (M is the number of grid vertices)
- 3: Initialise $(S_a)_k = 0$, $Z_k = 0$, and $\mathcal{Z}_k = 0$ (partial sums)
- 4: Load the point set $\{\mathcal{X}_{ij} \in \mathcal{D}\}_{i,j=1}^{N=mn}$ (chosen in a proper way)
- 5: Load $Q(\xi_k)$ and $\sigma(\xi_k)$
- 6: for i = 1, ..., m do (inner cycle to evaluate the normalisation factor)

7: Evaluate
$$\hat{\gamma}_1(\xi_k, \Theta_i) = \exp\left(\frac{-(\Theta_i - Q(\xi_k))^2}{2[\sigma(\xi_k)]^2}\right)$$

8:
$$(\mathcal{Z}_1)_k = (\mathcal{Z}_1)_k + \frac{2\pi}{N} \hat{\gamma}_1(\xi_k, \Theta_i)$$

- 9: end for
- 10: Calculate

$$(\hat{\Psi}_1)_{ik} = \hat{\Psi}_1(\xi_k, \Theta_i) = \frac{\hat{\gamma}_1(\xi_k, \Theta_i)}{(\mathcal{Z}_1)_k}, \ i = 1, \dots, m$$

11: Given C_k at time t: for i = 1, ..., m do 12:for $j = 1, \ldots, n$ do 13:Evaluate $(I_4)_{ijk} = \operatorname{tr}(C_k A_{ij})$, and 14:15: $A_{ij} = M_{ij} \otimes M_{ij}$, with $M_{ij} = \hat{M}(\Theta_i, \Phi_j)$ 16:if $(\hat{\Psi}_1)_{ik} > \operatorname{tol}_{(\Psi)}$ then $(Z_{\text{par}})_{ijk} = (\hat{\Psi}_1)_{ik} \frac{A_{ij}}{(I_4)_{ijk}}$ 17: $Z_k = Z_k + \frac{2\pi}{N} (Z_{\text{par}})_{ijk}$ (Partial sum 18:has to be uploaded) if $(I_4)_{ijk} > 1$ then $(S_{par}^{(0)})_{ijk} = (\hat{\Psi}_1)_{ik} [(I_4)_{ijk} - 1] A_{ij}$ $(S_{par})_{ijk} = 2\phi_{1sRc} (S_{par}^{(0)})_{ijk}$ $(S_a)_k = (S_a)_k + \frac{2\pi}{N} (S_{par})_{ijk}$ (Par-19:20:21: 22: tial sum has to be uploaded) 23:end if 24:end if 25:end for 26end for

28: end procedure



Fig. 5 Discrepancy err_{int} (see equation (66)), expressed as a function of the increasing number of equidistributed values of Θ_i , between the values of $(\mathbb{Z}_0)^{11}$ corresponding to the Matlab integration and the values of $(\mathbb{Z}_0)^{11}$ corresponding to a set of equidistributed points. The green, yellow, and magenta dots and squares correspond to the discrepancy obtained between the Matlab and the SDA outcomes for 21 and 120 Sloane points, and for the new set $\mathcal{I} \times \mathcal{J}$, respectively.

Algorithm	2	-A2-	Matlab	Integration	Algorithm	

- 1: **procedure** MATLAB INTEGRATION ALGORITHM (needs the call to the functions in algorithm (A3))
- 2: **for** k = 1, ... M **do** (M is the number of grid vertices) 3: $Z_k = \text{Integral}(@(\Theta)\hat{\gamma}_1(\Theta, \xi_k), 0, \pi)$

4: Calculate
$$\hat{\Psi}_1(\Theta, \xi_k) = \frac{1}{(\mathcal{Z}_1)_k} \hat{\gamma}_1(\Theta, \xi_k).$$

if
$$\hat{\Psi}_1(\Theta, \xi_k) > \operatorname{tol}_{(\Psi)}$$
 then

6:
$$\boldsymbol{Z}_k = \text{Integral2}(@(\Theta, \Phi)\boldsymbol{Z}_{\text{par}}(\boldsymbol{C}_k, \Theta, \Phi), 0, \frac{\pi}{2}, 0, 2\pi)$$

7:
$$(\mathbf{S}_{\mathrm{a}})_{k} = \mathrm{Integral2}(@(\Theta, \Phi)\mathbf{S}_{\mathrm{par}}(\mathbf{C}_{k}, \Theta, \Phi), \bar{0}, \frac{\pi}{2}, 0, 2\pi)$$

5:

- 9: end for10: end procedure
- (Note that, here, we employ the standard Matlab notation. In particular, the symbol @ represents the function handle constructor.)



Fig. 6 Required computational time for the SDA algorithm (a) and for the algorithm A2 (b).



15: end if

16: (the "If" condition on the fourth invariant should be considered in the function that evaluates S_{par}. The present code returns a tensor S_{par} whose components for which (Î₄)_k ≤ 1 are set equal to zero by means of the term [(Î₄)_k > 1] in square brackets)
17: end procedure

5 Solution of a benchmark test

⁹⁸⁹ 5.1 The unconfined compression test

Equations (51a) and (51b) are now solved for a specific benchmark test, along with appropriate boundary and initial conditions, and the constitutive functions defined in (45)–(49). In the inhomogeneous model of articular cartilage considered hereafter, these depend on material points through the volumetric fractions ϕ_{0sR} and ϕ_{1sR} , and the referential permeability k_{0R} , which are assumed



Fig. 7 Computational grid and boundary conditions of the unconfined compression benchmark.

to vary with the normalised axial coordinate ξ as [58] 997

$$\phi_{0sR}(\xi) = -0.062\xi^2 + 0.038\xi + 0.046, \tag{67a}$$

$$\phi_{1sR}(\xi) = +0.062\xi^2 - 0.138\xi + 0.404, \tag{67b}$$

$$\frac{k_{0\mathrm{R}}(\xi)}{k_{0\mathrm{R,hom}}} = \left(\frac{e_{\mathrm{R}}(\xi)}{e_{\mathrm{R,hom}}}\right)^{\kappa} \mathrm{e}^{\left(\frac{M}{2}\left[\left(\frac{1+e_{\mathrm{R}}(\xi)}{1+e_{\mathrm{R,hom}}}\right)^{2}-1\right]\right)}, \qquad (67\mathrm{c})$$

where $e_{\rm R} := (1 - \phi_{\rm sR})/\phi_{\rm sR}$, with $\phi_{\rm sR} = \phi_{\rm 0sR} + \phi_{\rm 1sR}$, 998 is referred to as "referential void ratio" (i.e., the ratio 999 between the referential volumetric fraction of the voids, 1000 $1 - \phi_{\rm sR}$, and the referential volumetric fraction of the 1001 solid phase as a whole, ϕ_{sR}), and $e_{R,hom}$ is a homo-1002 geneous value of the void ratio, taken as reference for 1003 the simulations. All the model parameters have been 1004 defined in Table 1. 1005

The simulated benchmark problem is an unconfined 1006 compression test, performed on a cylindric sample of 1007

Name	Description	Value
μ_0	Matrix Shear Modulus	0.2716 [MPa]
μ_1	Fibres Shear Modulus	7.2727 [MPa]
λ_0	Matrix First Lamé's Constant	$0.0556 [{ m MPa}]$
λ_1	Fibres First Lamé's Constant	2.4828 [MPa]
c	Material Parameter in $\hat{W}_{1a}(C)$	7.5062 [MPa]
$k_{0\mathrm{R,hom}}$	Reference scalar hydraulic conductivity	$3 \cdot 10^{-15} [\text{m}^4/(\text{Ns})]$
κ	Material parameter in (67c)	0.0848
M	Material parameter in (67c)	4.638
$e_{ m R,hom}$	Homogeneous void ratio	4
R_0	Radius of the cylinder	0.5 [mm]
L	Height of the cylinder	1.0 [mm]
tol_{arPsi}	Tolerance on the probability density	$5 \cdot 10^{-5}$

 Table 1
 Constant values in the considered benchmark.

articular cartilage of height L and circular cross sec-1008 tion of radius R_0 in the undeformed configuration \mathscr{B} . 1009 The specimen is put between two parallel, impermeable 1010 plates, which are kept in contact with its top and bot-1011 tom surfaces. During compression, the interstitial fluid 1012 flows out of the specimen through its lateral wall. The 1013 lower plate is held fixed and, in this work, the specimen 1014 is assumed to be clamped at this end in order to sim-1015 ulate the attachment of the tissue to the subchondral 1016 bone. The upper plate is moved axially either in force 1017 or in displacement control to mimic the load that the 1018 tissue has to bear in physiological or pathological con-1019 ditions (for testing purposes, the experimental protocol 1020 sometimes imposes loading conditions far beyond the 1021 physiological range). In this work, a linearly increasing 1022 axial displacement w is applied to the upper plate by 1023 means of a loading ramp ending at 20% compression of 1024 the sample after T = 20 s, i.e., 1025

$$w(t) = -0.2L\frac{t}{T}.$$
(68)

¹⁰²⁶ In formulae, the boundary conditions describing the ¹⁰²⁷ simulated experimental setting are given by:

$$\chi^3 = L + w,$$
 $Q.E_3 = 0,$ $\forall X \in \Gamma_u,$
(69a)

-p=0, $\boldsymbol{P}.\boldsymbol{N}=\boldsymbol{0},$ $\forall X\in \Gamma_{\mathrm{w}},$

$$\chi(X,t) - \chi(X,0) = \mathbf{0}, \quad \mathbf{Q}.(-\mathbf{E}_3) = 0, \quad \forall X \in \Gamma_1,$$
(69b)
(69b)
(69b)
(69c)

where $\Gamma_{\rm u}$, $\Gamma_{\rm l}$, and $\Gamma_{\rm w}$ denote the upper, lower and lat-1028 eral boundaries of the specimen, respectively, and the 1029 conditions (69a)–(69c) are intended to apply at all times 1030 $t \in [0,T]$. The unit vectors **N** and **E**₃ represent the 1031 radial and axial directions of the sample, respectively. 1032 Usually, the height of the tissue at the end of the load-1033 ing ramp, i.e., $L_T = L + w(T) = 0.8L \ (w(T) \le 0)$, is 1034 maintained constant over a given time interval, during 1035

which the fluid filtration velocity and the pore pressure 1036 relax, while the constitutive part of the stress reaches 1037 a constant value that depends on the degree of com-1038 pression to be maintained. Since in this work we are 1039 interested in the numerical performances of the SDA, 1040 and not in the way in which the system relaxes towards 1041 its stationary state, we present simulations referred to 1042 the loading ramp only. 1043

1044

5.2 The Finite Element setting

A schematic representation of the chosen mesh, and the 1045 above described boundary conditions are reported in 1046 Figure 7. Equations (51a) and (51b) were solved nu-1047 merically, along with the boundary conditions (69a)-1048 (69c) and the initial condition J(X,0) = 1, by means 1049 of a damped Newton method and a Backward Differen-1050 tiation Formula (BDF) for the time discretisation. The 1051 BDF is a multi-step generalisation of the Backward Eu-1052 ler scheme. It is characterised by an integer number s, 1053 normally ranging between 1 and 6, and thereby deter-1054 mining the order of the scheme (for further details, the 1055 Reader is referred to [1,40]). When s = 1, the BDF 1056 coincides with the Backward Euler method. In our sim-1057 ulations, s ranges between 1 and 5, and the damping 1058 coefficient used in the Newton method is automatically 1059 chosen by the solvers of our commercial software. 1060

In the remainder of this section, we describe how 1061 the quadrature method used to determine the direc-1062 tional averages couples with the FE implementation of 1063 the model equations. To this end, we sketch the time 1064 and space discretisation of the functionals (51a) and 1065 (51b). This is, in fact, an extension of the discretisation 1066 presented in [27,31] to the case of anisotropic, biphasic 1067 media. Notice that, from here on, the quantities with 1068 a superimposed tilde pertain to the test functions as-1069 socited with the sets defined in (52a) and (52b). More-1070 over, for any function f defined on $\mathscr{B} \times \mathscr{I}$, the notation 1071

6

 $f^{j} \equiv f(X, t^{j})$ applies, for all *j*. With reference to (51a), in which the time-derivative of *J* features explicitly, we suppose for demonstrational purposes that the time discretisation reduces to a first order BDF, and we write

$$\dot{J} \approx \frac{J^j - J^{j-1}}{\tau_j}, \quad \tau_j = t_j - t_{j-1}, \quad j \ge 1, \ j \in \mathbb{N},$$
(70)

with $\tau_j > 0$ being the width of the *j*th time step. At the *j*th time step, the time-discrete version of the functionals (51a) and (51b) takes on the form

$$0 = \mathfrak{F}^{j}_{\chi} = \int_{\mathscr{B}} \boldsymbol{P}^{j} : \boldsymbol{g} \operatorname{Grad} \boldsymbol{\tilde{u}},$$
(71)

$$0 = \mathfrak{F}_{p}^{j} = -\int_{\mathscr{B}} (\operatorname{Grad} \tilde{p}) \boldsymbol{K}^{j} (\operatorname{Grad} p^{j}) -\int_{\mathscr{B}} \tilde{p} \frac{J^{j} - J^{j-1}}{\tau^{j}},$$
(72)

with $P^{j} \equiv \hat{P}(p^{j}, F^{j})$, and $K^{j} \equiv \hat{K}(F^{j})$. Next, we consider the overall first Piola-Kirchhoff stress tensor, $\hat{P}(p, F)$, whose expression we rearrange as

$$\hat{\boldsymbol{P}}(p,\boldsymbol{F}) = -Jp\boldsymbol{g}^{-1}\boldsymbol{F}^{-\mathrm{T}} + \hat{\boldsymbol{P}}_{0}(\boldsymbol{F}) + \hat{\boldsymbol{P}}_{\mathrm{a}}(\boldsymbol{F}), \qquad (73)$$

1082 with

$$\hat{\boldsymbol{P}}_{0}(\boldsymbol{F}) = \lambda \ln(J)\boldsymbol{g}^{-1}\boldsymbol{F}^{-\mathrm{T}} + \mu \boldsymbol{F} \left(\boldsymbol{G}^{-1} - \boldsymbol{C}^{-1}\right), \quad (74)$$

$$\hat{\boldsymbol{P}}_{\mathrm{a}}(\boldsymbol{F}) = \boldsymbol{F}\hat{\boldsymbol{S}}_{\mathrm{a}}(\boldsymbol{C}),\tag{75}$$

and $\hat{S}_{a}(C)$ being defined in (46). Analogously, by invoking the expression of the material permeability in (47), we split $\hat{K}(F)$ as

$$\hat{\boldsymbol{K}}(\boldsymbol{F}) = \hat{\boldsymbol{K}}_0(\boldsymbol{F}) + \hat{\kappa}(\boldsymbol{F})\hat{\boldsymbol{Z}}(\boldsymbol{C}), \qquad (76)$$

1086 where we have set

$$\hat{K}_{0}(F) = \hat{k}_{0}(J) \frac{(J - \phi_{1\text{sR}})^{2}}{J} C^{-1}, \qquad (77a)$$

$$\hat{\kappa}(\boldsymbol{F}) = \hat{k}_0(J) \frac{(J - \phi_{1\mathrm{sR}})\phi_{1\mathrm{sR}}}{J}, \quad J = \det(\boldsymbol{F}).$$
(77b)

The decompositions (73) and (76) are done in order to highlight the presence of the anisotropic summands, $\hat{P}_{a}(F)$ and $\hat{\kappa}(F)\hat{Z}(C)$, and their implementation in the Finite Element method. Indeed, by accounting for (73) and (76), the time-discrete version of the functionals \mathfrak{F}_{χ} and \mathfrak{F}_{p} at the instant of time t_{j} reads

$$0 = \mathfrak{F}_{\chi}^{j} = \int_{\mathscr{B}} \left\{ -J^{j} p^{j} \boldsymbol{g}^{-1} (\boldsymbol{F}^{j})^{-\mathrm{T}} : \boldsymbol{g} \tilde{\boldsymbol{H}} \right\} + \int_{\mathscr{B}} \boldsymbol{P}_{0}^{j} : \boldsymbol{g} \tilde{\boldsymbol{H}} + \int_{\mathscr{B}} \boldsymbol{F}^{j} \boldsymbol{S}_{\mathrm{a}}^{j} : \boldsymbol{g} \tilde{\boldsymbol{H}}, \qquad (78a)$$
$$0 = \mathfrak{F}_{p}^{j} = -\int_{\mathscr{A}} (\operatorname{Grad} \tilde{p}) \boldsymbol{K}_{0}^{j} (\operatorname{Grad} p^{j})$$

$$-\int_{\mathscr{B}} (\operatorname{Grad} \tilde{p}) \kappa^{j} \mathbf{Z}^{j} (\operatorname{Grad} p^{j}) -\int_{\mathscr{B}} \tilde{p} \frac{J^{j} - J^{j-1}}{\tau^{j}},$$
(78b)

in which
$$\tilde{\boldsymbol{H}} \equiv \operatorname{Grad} \tilde{\boldsymbol{u}}$$
, and $\boldsymbol{P}_0^j \equiv \hat{\boldsymbol{P}}_0(\boldsymbol{F}^j)$, $\boldsymbol{K}_0^j = \hat{\boldsymbol{K}}_0(\boldsymbol{F}^j)$, 1092
 $\kappa^j = \hat{\kappa}(\boldsymbol{F}^j)$, and 1094

$$\begin{split} S_{\mathbf{a}}^{j} &\equiv \widehat{\boldsymbol{S}}_{\mathbf{a}}(\boldsymbol{C}^{j}) \\ &= 2\phi_{1\mathrm{sR}} c \int_{\mathbb{S}_{Y}^{2+} \mathscr{B}} \Psi(\boldsymbol{M}) \mathcal{H}(I_{4}^{j}-1)[I_{4}^{j}-1]\boldsymbol{A}, \end{split}$$
(79a)

$$\boldsymbol{Z}^{j} \equiv \hat{\boldsymbol{Z}}(\boldsymbol{C}^{j}) = \int_{\mathbb{S}^{2+}_{X}\mathscr{B}} \Psi(\boldsymbol{M}) \frac{\boldsymbol{M} \otimes \boldsymbol{M}}{I_{4}^{j}}.$$
 (79b)

It should be noticed that, because of the integral over 1095 $\mathbb{S}^{2+}_X \mathscr{B}$ in (79a) and (79b), the third summand on the 1096 right-hand-side of (78a) and the second summand on 1097 the right-hand-side of (78b) feature two integrations. 1098 which have to be performed hierarchically: First, one 1099 has to solve at each time step the integral over $\mathbb{S}_X^{2+}\mathscr{B}$, 1100 and then the integral over \mathscr{B} . Since the problem is 1101 nonlinear, we look for solutions by invoking Newton 1102 method. Hence, at the jth time step, and within the 1103 kth Newton iteration $(k \ge 1, k \in \mathbb{N})$, we solve the lin-1104 earised equations 1105

$$\mathfrak{F}_{\chi}^{j,k-1} + \mathfrak{D}_{\chi}\mathfrak{F}_{\chi}^{j,k-1}[\boldsymbol{h}^{j,k}] + \mathfrak{D}_{p}\mathfrak{F}_{\chi}^{j,k-1}[\vartheta^{j,k}] = 0, \qquad (80a)$$
$$\mathfrak{F}_{p}^{j,k-1} + \mathfrak{D}_{\chi}\mathfrak{F}_{p}^{j,k-1}[\boldsymbol{h}^{j,k}] + \mathfrak{D}_{p}\mathfrak{F}_{p}^{j,k-1}[\vartheta^{j,k}] = 0, \qquad (80b)$$

in which we adopted the notation

$$\mathfrak{F}_{\chi}^{j,k-1} = \mathfrak{F}_{\chi}(\chi^{j,k-1}, p^{j,k-1}), \tag{81a}$$

$$\mathfrak{F}_p^{j,k-1} = \mathfrak{F}_p(\chi^{j,k-1}, p^{j,k-1}), \tag{81b}$$

 \mathfrak{D}_{χ} and \mathfrak{D}_{p} indicate Gâteaux differentiation with respect to χ and p, respectively, and $h^{j,k}$ and $\vartheta^{j,k}$ are the sure, respectively, along which the Gâteaux derivatives are computed, i.e., 1107

$$\chi^{j,k} = \chi^{j,k-1} + \boldsymbol{h}^{j,k}, \tag{82a}$$

$$p^{j,k} = p^{j,k-1} + \vartheta^{j,k}. \tag{82b}$$

The explicit computation of the Gâteaux derivatives 1112 in (80a) and (80b) trasforms the linearised variational 1113 problem (78a)–(78b) into the abstract form [31] 1114

$$A(\boldsymbol{h}^{j,k}, \tilde{\boldsymbol{u}}) - B(\vartheta^{j,k}, \tilde{\boldsymbol{u}}) = -\mathfrak{F}_{\chi}^{j,k-1}, \quad \forall \; \tilde{\boldsymbol{u}} \in \tilde{\mathscr{V}}, \quad (83a)$$
$$-C(\boldsymbol{h}^{j,k}, \tilde{p}) - D(\vartheta^{j,k}, \tilde{p}) = -\mathfrak{F}_{p}^{j,k-1}, \quad \forall \; \tilde{p} \in \tilde{\mathscr{P}}, \quad (83b)$$

17

which should be solved for the increments $h^{j,k}$ and $\theta^{j,k}$. In (83a) and (83b) we define the bilinear forms [31]

. .

$$A(\boldsymbol{h}^{j,k}, \tilde{\boldsymbol{u}}) = \mathfrak{D}_{\chi} \mathfrak{F}_{\chi}^{j,k-1}[\boldsymbol{h}^{j,k}]$$
$$= \int_{\mathscr{B}} \boldsymbol{g} \tilde{\boldsymbol{H}} : \mathbb{A}^{j,k-1} : \boldsymbol{H}^{j,k}, \qquad (84a)$$

$$-B(\vartheta^{j,k}, \tilde{\boldsymbol{u}}) = \mathfrak{D}_{p}\mathfrak{F}_{\chi}^{j,k-1}[\vartheta^{j,k}]$$

$$= \int_{\mathscr{B}} \left\{ -J^{j,k-1}\vartheta^{j,k}\boldsymbol{g}^{-1}(\boldsymbol{F}^{j,k-1})^{-\mathrm{T}}:\boldsymbol{g}\tilde{\boldsymbol{H}} \right\}, \quad (84\mathrm{b})$$

$$-C(\boldsymbol{h}^{j,k}, \tilde{p}) = \mathfrak{D}_{\chi}\mathfrak{F}_{p}^{j,k-1}[\boldsymbol{h}^{j,k}]$$

$$= -\frac{B(\boldsymbol{h}^{j,k}, \tilde{p})}{\tau_{j}}$$

$$-\int_{\mathscr{B}} (\operatorname{Grad} \tilde{p}) \left(\mathbb{K}^{j,k-1}:\boldsymbol{H}^{j,k} \right) (\operatorname{Grad} p^{j,k-1}), \quad (84\mathrm{c})$$

$$-D(\vartheta^{j,k}, \tilde{p}) = \mathfrak{D}_{p}\mathfrak{F}_{p}^{j,k-1}[\vartheta^{j,k}]$$

$$= -\int_{\mathscr{B}} (\operatorname{Grad} \tilde{p}) \boldsymbol{K}^{j,k-1} (\operatorname{Grad} \vartheta^{j,k}), \quad (84\mathrm{d})$$

where $\boldsymbol{H}^{j,k} = \operatorname{Grad} \boldsymbol{h}^{j,k}$, $\mathbb{A}^{j,k-1}$ is the algorithmic first elasticity tensor [47], and $\mathbb{K}^{j,k-1}$ in (84c) is given by

$$\mathbb{K}^{j,k-1} \equiv \frac{\partial \hat{K}}{\partial F} (F^{j,k-1}). \tag{85}$$

The SDA (or the Matlab integration) comes into play in the computation of $\mathbb{A}^{j,k-1}$, $\mathbf{K}^{j,k-1}$, and $\mathbb{K}^{j,k-1}$. Let us consider, for instance, the expression of the elasticity tensor, and let us split it into a "standard", isotropic part, $\mathbb{A}_{st}^{j,k-1}$, and a "non-standard", anisotropic part pertaining to the fibres, $\mathbb{A}_{e}^{j,k-1}$, i.e.,

$$\mathbb{A}^{j,k-1} = \mathbb{A}_{\mathrm{st}}^{j,k-1} + \frac{\partial \hat{P}_{\mathrm{a}}}{\partial F} (F^{j,k-1})$$
$$= \mathbb{A}_{\mathrm{st}}^{j,k-1} + \mathbb{A}_{\mathrm{e}}^{j,k-1}.$$
(86)

¹¹²⁵ The standard elasticity tensor $\mathbb{A}_{st}^{j,k-1}$ contains all the ¹¹²⁶ terms stemming from the incompressibility constraint, ¹¹²⁷ the energetic contribution of the penalty term (40), and ¹¹²⁸ the terms pertaining to the isotropic part of the model, ¹¹²⁹ whereas the non standard contribution is such that the ¹¹³⁰ following identity holds

$$\boldsymbol{g}\tilde{\boldsymbol{H}}: \mathbb{A}_{e}^{j,k-1}: \boldsymbol{H}^{j,k} = \operatorname{sym}(\tilde{\boldsymbol{H}}^{\mathrm{T}}.\boldsymbol{H}^{j,k}): \boldsymbol{S}_{a}^{j,k-1} + [(\boldsymbol{F}^{j,k-1})^{\mathrm{T}}.\tilde{\boldsymbol{H}}]: \mathbb{C}_{e}^{j,k-1}: [(\boldsymbol{F}^{j,k-1})^{\mathrm{T}}.\boldsymbol{H}^{j,k}], \qquad (87)$$

where $S_{a}^{j,k-1}$ is obtained from (79a) by substituting I_{4}^{j} with the value of the invariant at the (k-1)th Newton iteration, i.e., with $I_{4}^{j,k-1}$, and

$$\mathbb{C}_{\mathrm{e}}^{j,k-1} = 4\phi_{1\mathrm{sR}}c \int_{\mathbb{S}_{X}^{2+}\mathscr{B}} \mathcal{\Psi}(\boldsymbol{M})\mathcal{H}(I_{4}^{j,k-1}-1)\boldsymbol{A}\otimes\boldsymbol{A}.$$
 (88)

1134 Analogously, $oldsymbol{K}^{j,k-1} = \hat{oldsymbol{K}}(oldsymbol{F}^{j,k-1})$ splits as follows

$$\boldsymbol{K}^{j,k-1} = \boldsymbol{K}_{0}^{j,k-1} + \kappa^{j,k-1} \boldsymbol{Z}^{j,k-1},$$
(89)

where

$$\hat{k}^{j,k-1} \equiv \hat{k}_0(J^{j,k-1}) \frac{(J^{j,k-1} - \phi_{1\mathrm{sR}})\phi_{1\mathrm{sR}}}{J^{j,k-1}},$$
 (90a)

$$\begin{aligned} \mathbf{K}_{0}^{j,k-1} &= \hat{k}_{0}(J^{j,k-1}) \frac{(J^{j,k-1} - \phi_{1\mathrm{sR}})^{2}}{J^{j,k-1}} (\mathbf{C}^{j,k-1})^{-1}, \ (90\mathrm{b}) \\ \mathbf{Z}^{j,k-1} &= \int_{\mathbb{S}^{2+}} \Psi(\mathbf{M}) \frac{\mathbf{M} \otimes \mathbf{M}}{I_{4}^{j,k-1}}, \end{aligned}$$
(90c)

and
$$\mathbb{K}^{j,k-1}$$
 is such that

$$\mathbb{K}^{j,k-1}: \boldsymbol{H}^{j,k} = \mathbb{K}_{0}^{j,k-1}: \boldsymbol{H}^{j,k} + \left[\frac{\partial \hat{\kappa}}{\partial \boldsymbol{F}}(\boldsymbol{F}^{j,k-1}): \boldsymbol{H}^{j,k}\right] \boldsymbol{Z}^{j,k-1} + \kappa^{j,k-1} \left[\frac{\partial \hat{\boldsymbol{Z}}}{\partial \boldsymbol{F}}(\boldsymbol{F}^{j,k-1}): \boldsymbol{H}^{j,k}\right], \quad (91)$$

with

К

$$\mathbb{K}_{0}^{j,k-1} = \frac{\partial \hat{\boldsymbol{K}}_{0}}{\partial \boldsymbol{F}}(\boldsymbol{F}^{j,k-1}).$$
(92)

Note, in particular, that the term between brackets in 1138 the third summand on the right-hand-side of (91) is 1139 given by 1140

$$\left[\frac{\partial \hat{\boldsymbol{Z}}}{\partial \boldsymbol{F}}(\boldsymbol{F}^{j,k-1}):\boldsymbol{H}^{j,k}\right] = -2 \int_{\mathbb{S}_{X}^{2^{+}}\mathscr{B}} \Psi(\boldsymbol{M}) \frac{\boldsymbol{A}}{(I_{4}^{j,k-1})^{2}} \left[\boldsymbol{A}:\left((\boldsymbol{F}^{j,k-1})^{\mathrm{T}}.\boldsymbol{H}^{j,k}\right)\right],$$
(93)

with $\boldsymbol{A} = \boldsymbol{M} \otimes \boldsymbol{M}$ and $I_4^{j,k-1} = \boldsymbol{A} : \boldsymbol{C}^{j,k-1}$. The integral in (93) is nontrivial due to the presence of highly 1142 oscillating functions, which manifest themselves when 1143 polar coordinates are used.

Both the SDA and the Matlab integration come into 1145 play at each time step and within each Newton itera-1146 tion performed to construct $A(\mathbf{h}^{j,k}, \tilde{\mathbf{u}}), C(\mathbf{h}^{j,k}, \tilde{p})$, and 1147 $D(\vartheta^{j,k}, \tilde{p})$ (cf. (84a), (84c), and (84d)). In particular, 1148 they are adopted to compute the integrals over $\mathbb{S}^{2+}_X \mathscr{B}$ 1149 that appear in the evaluation of $S_{a}^{j,k-1}$, $\mathbb{C}_{e}^{j,k-1}$, $Z^{j,k-1}$, 1150 and $\partial_{\mathbf{F}} \hat{\mathbf{Z}}(\mathbf{F}^{j,k-1})$. For this purpose, the quadrature 1151 methods are invoked at each integration point of each 1152 grid element, during the assembly of the "stiffness ma-1153 trix". The computational effort of this procedure is re-1154 lated to the number of integration points within each el-1155 ement of the FE discretisation. Thus, if, for instance (in 1156 a 2D mesh), each element features 4 integration points, 1157 then $4 \times (\text{number of elements})$ calls to the algorithms A1 1158 (or A2) per time step and per linearisation iteration are 1159 required. 1160

It is worthwhile to mention that the basis functions 1161 of the Finite Element discretisation are necessary to 1162 determine $S_{a}^{j,k-1}$, $\mathbb{C}_{e}^{j,k-1}$, $Z^{j,k-1}$, and $\partial_{F} \hat{Z}(F^{j,k-1})$. 1163

1135

1136

This occurs because each of these quantities depends on $I_4^{j,k-1}$, which, in turn, requires the interpolation of $C^{j,k-1} = (\mathbf{F}^{j,k-1})^{\mathrm{T}} \mathbf{g} \mathbf{F}^{j,k-1}$ and, thus, of $\mathbf{F}^{j,k-1}$. To highlight the relation between the basis functions and $I_{4}^{j,k-1}$, let us first introduce the grid function ${}^{h}\chi$ approximating the motion at the *j*th instant of time and at the (k-1)th Newton iteration:

$$({}^{h}\chi^{a})^{j,k-1} = \sum_{i=1}^{\mathcal{N}} (\chi^{a(i)})^{j,k-1} \varphi_{(i)},$$
 (94)

¹¹⁷¹ where $\{\varphi_{(i)}\}_{i=1}^{\mathscr{N}}$ is the set of basis functions, and \mathscr{N} is ¹¹⁷² the number of nodes. Consequently, the generic com-¹¹⁷³ ponents of the grid functions ${}^{h}F^{j,k-1}$ and ${}^{h}C^{j,k-1}$ are ¹¹⁷⁴ given by

$$({}^{h}F^{a}_{A})^{j,k-1} = \sum_{i=1}^{\mathscr{N}} (\chi^{a(i)})^{j,k-1} \frac{\partial \varphi_{(i)}}{\partial X^{A}},$$
(95a)

$$({}^{h}C_{BD})^{j,k-1} = g_{ab} ({}^{h}F^{a}_{\ B})^{j,k-1} ({}^{h}F^{b}_{\ D})^{j,k-1},$$
 (95b)

which implies that $I_4^{j,k-1}$ is approximated by the grid function

(note that in (95b) and (96) Einstein's convention on 1177 repeated indices is used to denote the summation over 1178 a, b = 1, 2, 3 and A, B = 1, 2, 3). It follows from (96) 1179 that $({}^{h}I_{4})^{j,k-1}$ necessitates the derivatives of the basis 1180 functions, as well as the coefficients $(\chi^{a(i)})^{j,k-1}$. These 1181 are, in fact, the inputs that a quadrature scheme (be it 1182 the SDA or the Matlab routine) receives from the FE 1183 discretisation at the (k-1)th Newton iteration within 1184 1185 the *j*th time step.

In our simulations, polynomial basis functions of de-1186 gree one, two, and three are tested. Due to the element-1187 wise smoothness of the basis functions, the presence of 1188 their derivatives in (96) does not worsen in a signifi-1189 cant way the integration procedure. However, varying 1190 the degree of the polynomials may yield to apprecia-1191 bly different performances, as we could see by running 1192 a FE simulation of a simplified, isotropic version of 1193 the model in which the fibres are absent (in this case, 1194 $K = K_0 = \hat{k}_0(J)JC^{-1}$ and $S_a = 0$, so that neither 1195 the SDA nor the Matlab integration scheme are nec-1196 essary). We found that quadratic Lagrangian polyno-1197 mials are the optimal choice for discretising both the 1198 displacements and the pressure. Indeed, if linear basis 1199 functions are used, an oscillatory solution is obtained 1200 for some components of the stress and the permeability 1201 of the system. 1202

Concerning the grid convergence of the mesh reported in Figure 7, we compared three different quadrangular meshes to a finer one (1250 elements).

In Table 2, the relative integral errors in the output $_{1206}$ curves representing the components of Z, and those of $_{1207}$ $S_{\rm a}$, are reported. To compute such errors, we applied $_{1208}$ the following formulae $_{1209}$

$$\operatorname{err}_{\xi} = \frac{\left|\int_{0}^{1} \mathcal{Q}_{i} \mathrm{d}\xi - \int_{0}^{1} \mathcal{Q}_{\mathrm{finer}} \mathrm{d}\xi\right|}{\left|\int_{0}^{1} \mathcal{Q}_{\mathrm{finer}} \mathrm{d}\xi\right|},\tag{97}$$

$$\operatorname{err}_{t} = \frac{\left|\int_{0}^{\mathrm{T}} \mathcal{Q}_{\mathrm{i}} \mathrm{d}t - \int_{0}^{\mathrm{T}} \mathcal{Q}_{\mathrm{finer}} \mathrm{d}t\right|}{\left|\int_{0}^{\mathrm{T}} \mathcal{Q}_{\mathrm{finer}} \mathrm{d}t\right|},\tag{98}$$

where the index i denotes the type of mesh, i.e., i = 1210Coarse, Intermediate, Fine, Q is one of the generic quantity of the model, and T represents the final instant of the loading ramp. 1211

The pressure p, the norm of the displacement \boldsymbol{u} , ¹²¹⁴ and the von Mises equivalent stress $\sigma_{\rm VM}$, to which Table 2 refers, are the results of a pointwise evaluation, ¹²¹⁶ in the point $X_1 = (0, 0, \frac{L}{2})$, for the pressure, and $X_2 =$ ¹²¹⁷ $(\frac{R}{2}, \frac{R}{2}, \frac{L}{2})$, for the velocity and the stress. The components of the tensors \boldsymbol{Z} and $\boldsymbol{S}_{\rm a}$ used to evaluate the ¹²¹⁹ errors are, instead, taken over the depth of the sample. ¹²²⁰

As visible from the reported errors, the discrepancy 1221 between the results obtained with progressively finer 1222 grids are small. Thus, with the grid represented in Fig-1223 ure 7 (fine grid of Table 2), we already reached the mesh 1224 convergence. Note that the integrations for computing 1225 the errors in (97) have been performed by means of a 1226 trapezoidal numerical integration. 1227

As previously specified, the simulations were run by 1228 adopting two different methods for the solution of the 1229 integrals over $\mathbb{S}_X^{2+}\mathscr{B}$. The first method, outlined in A1, 1230 is the SDA, while the second method is the integration 1231 routine A2. For the SDA (either *internal* or in Matlab), 1232 at each $X \in \mathcal{B}$, an appropriate set of points was cho-1233 sen on $\mathbb{S}^2_{\mathcal{X}}\mathscr{B}$. The algorithm A2 is performed with fixed 1234 absolute and relative tolerances in the Matlab quadra-1235 ture. 1236

6 Results

6.1 Matlab validation

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Table 2Grid Convergence

	Coarse (50 elem.)	Intermediate (120 elem.)	Fine (392 elem.)
< <u>-11</u> <			
$\operatorname{err}_{\xi}(Z_0^{11}(t=0))$	$7.5 \cdot 10^{-7}$	$3.8 \cdot 10^{-7}$	$4.9 \cdot 10^{-8}$
$\operatorname{err}_{\xi}(Z_0^{11}(t=20))$	$5.8 \cdot 10^{-4}$	$4.4 \cdot 10^{-4}$	$3.9 \cdot 10^{-4}$
$\operatorname{err}_{\xi}(Z_0^{33}(t=0))$	$4.7 \cdot 10^{-7}$	$1.1 \cdot 10^{-7}$	$1.6 \cdot 10^{-8}$
$\operatorname{err}_{\xi}(Z_0^{33}(t=20))$	$2.7 \cdot 10^{-4}$	$8.7 \cdot 10^{-5}$	$4.4 \cdot 10^{-4}$
$\operatorname{err}_{\boldsymbol{\xi}}((\boldsymbol{S}_{\mathrm{a}})^{11})$	$6.6 \cdot 10^{-3}$	$5.2 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$
$\operatorname{err}_{\boldsymbol{\xi}}((\boldsymbol{S}_{\mathrm{a}})^{33})$	$4.5 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	$5.5 \cdot 10^{-3}$
$\operatorname{err}_t(p)$	$6.2 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$	$3.0 \cdot 10^{-4}$
$\operatorname{err}_t(\boldsymbol{u})$	$7.1 \cdot 10^{-4}$	$9.0 \cdot 10^{-4}$	$3.0 \cdot 10^{-4}$
$\operatorname{err}_t(\sigma_{VM})$	$2.2 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$	$3.0 \cdot 10^{-4}$

with the SDA. More specifically, it has been checked 1245 for no error propagation in time due to the coupling 1246 between the FEM solver and the routine A2. We re-1247 mark that the Matlab routine "Integral" (cf. Algorithm 1248 A2) chooses the numerical quadrature method depend-1249 ing on the kind of function that has to be integrated. A 1250 control on the quality of the integration can be done by 1251 manipulating the absolute and the relative tolerances in 1252 the error associated with the quadrature results. Here, 1253 the considered default values for these tolerances, de-1254 noted in Matlab by AbsTol and RelTol, respectively, 1255 are AbsTol = $1 \cdot 10^{-10}$ and RelTol = $1 \cdot 10^{-6}$. In fact, 1256 although $(\mathbf{Z}_0)^{11}$ is decoupled from deformation, the in-1257 tegral in (64a) cannot be performed exactly. 1258

For the purposes outlined above, we consider in this 1259 section the simple case of a homogeneous and isotropic 1260 material model, with the probability density (35b). In 1261 particular, we set $\phi_{0sR} = 0.1$ and $\phi_{1sR} = 0.3$ everywhere 1262 in the computational domain, and we assume that the 1263 probability density distribution reduces to $\hat{\Psi}_1(\xi, \Theta) =$ 1264 $1/2\pi$ for all values of $\xi \in [0,1]$ and for all $\Theta \in [0,\frac{\pi}{2}]$. 1265 This means that the normalisation factor is $\mathcal{Z}_{1+} = 2\pi \approx$ 1266 6.2832. Note that the values $\phi_{0sR} = 0.1$ and $\phi_{1sR} =$ 1267 0.3 are not taken from (67a) and (67b), and do not 1268 correspond to a value of $\xi \in [0, 1]$. 1269

Furthermore, in order to make the integral defining Z in (48) exactly solvable, we enforce the (strong) assumption that Z, in a neighbourhood of $I_4(C, M) = 1$, reduces to

$$\boldsymbol{Z} = \int_{\mathbb{S}_X^{2+}\mathscr{B}} \Psi_1(\boldsymbol{M}) \frac{\boldsymbol{M} \otimes \boldsymbol{M}}{I_4(\boldsymbol{C}, \boldsymbol{M})} \approx \frac{1}{2\pi} \int_{\mathbb{S}_X^{2+}\mathscr{B}} \boldsymbol{M} \otimes \boldsymbol{M}.$$
(99)

1274 Accordingly, the components of Z are given by

$$Z^{12} = Z^{13} = Z^{23} = 0, (100a)$$

$$Z^{11} = Z^{22} = Z^{33} = Z, (100b)$$

¹²⁷⁵ in which all the three diagonal components of \mathbf{Z} are ¹²⁷⁶ set equal to $Z = \frac{1}{3}$, the material being isotropic in this

preliminary study. Finally, to solve exactly the integral 1277 in the definition (46) of $\boldsymbol{S}_{\mathrm{a}}$, we allow the fibres to con-1278 tribute to the stress even when $I_4(C, M)$ is smaller 1279 than unity. Note, however, that this simplification is 1280 sometimes physically sound in those situations in which 1281 the collagen fibres contribute to the overall compressive 1282 stiffness of cartilage [56]. In these cases, the Heaviside 1283 function can be eliminated from the expression of $S_{\rm a}$, 1284 which becomes 1285

$$\boldsymbol{S}_{\mathrm{a}} = \frac{2\phi_{1\mathrm{sR}}c}{2\pi} \int_{\mathbb{S}_X^{2+}\mathscr{B}} [I_4(\boldsymbol{C}, \boldsymbol{M}) - 1] \boldsymbol{M} \otimes \boldsymbol{M}.$$
(101)

Consequently, the diagonal components of $m{S}_{\mathrm{a}}$ are given 1286 by 1287

$$(\boldsymbol{S}_{a})^{11} = \frac{2\phi_{1sR}c}{15}(3C_{11} + C_{22} + C_{33} - 5), \qquad (102a)$$

$$(\boldsymbol{S}_{a})^{22} = \frac{2\phi_{1sR}c}{15}(C_{11} + 3C_{22} + C_{33} - 5), \qquad (102b)$$

$$(\mathbf{S}_{a})^{33} = \frac{2\phi_{1sR}c}{15}(C_{11} + C_{22} + 3C_{33} - 5).$$
 (102c)

We compare now the components of Z computed by 1288 means of the algorithm A2, Z_{A2} , with the components 1289 of Z determined in (100a) and (100b). We compute, 1290 thus, the mean relative deviation 1291

$$\operatorname{Err}(Z) = \frac{Z_{A2} - Z}{Z_{A2}},$$
 (103)

which turns out to be $\mathcal{O}(10^{-6})$. For the stress, we define 1292 the mean relative error associated with the diagonal 1293 components of $\boldsymbol{S}_{\mathrm{a}}$ 1294

$$\operatorname{Err}(\boldsymbol{S}_{\mathrm{a}}) = \max_{J=1,2,3} \left\{ \frac{1}{\operatorname{Vol}(\mathscr{B})} \int_{\mathscr{B}} \frac{(\boldsymbol{S}_{\mathrm{a}})_{\mathrm{A2}}^{JJ} - (\boldsymbol{S}_{\mathrm{a}})^{JJ}}{(\boldsymbol{S}_{\mathrm{a}})_{\mathrm{A2}}^{JJ}} \right\},$$
(104)

where $(\mathbf{S}_{a})_{A2}^{JJ}$ is the JJ-component of \mathbf{S}_{a} evaluated ¹²⁹⁵ with the aid of the algorithm A2. Also in this case, ¹²⁹⁶ we obtain $\operatorname{Err}(\mathbf{S}_{a}) = \mathcal{O}(10^{-6})$. We notice that both ¹²⁹⁷ $\operatorname{Err}(Z)$ and $\operatorname{Err}(\mathbf{S}_{a})$ are of the same order of magnitude as the Matlab input RelTol. We remark that, in ¹²⁹⁹ the simplified model discussed in this section, Z is constant in time and, since material inhomogeneities have been disregarded, it is constant also in space. The stress S_{a} , instead, depends on time through the deformation.

From the estimated errors we can see that the Matlab quadrature method defines a proper approximation
of the integrals. Thus, in the following, the results of
algorithm A2 can be used as a basis for comparison.

¹³⁰⁸ 6.2 Inhomogeneous and anisotropic model

In this section, the numerical simulations done with the
setting in Table 3 for the simulations with the pseudoGaussian distribution, and in Table 6, for the von Mises
distribution, are reported and compared one with each
other in terms of accuracy and computational time.

First, in order to understand the mechanical response to external stimuli of a biological tissue like articular cartilage, we propose here a Finite Element description of an unconfined compression test (a test that is widely used to estimate the material properties of soft tissues).

In Figure 8, the 2D snapshots of pore pressure, ra-1320 dial filtration velocity, and equivalent von Mises stress 1321 are reported in the case of an isotropic tissue without 1322 fibres (i.e., with $\phi_{1sR} = 0$ in (46) and (47)). The re-1323 quired computational effort of such FEM simulation, 1324 for which no fibres are considered, consists of a com-1325 putational time of about 13 seconds, and a memory 1326 allocation of about 1.44 Gb. The response of the tis-1327 sue to the applied incremental compression manifests 1328 itself with an increasing pore pressure, which assumes 1329 its higher value at the bottom of the sample, whereas 1330 the higher velocity is observed at the top external wall 1331 of the cylinder. The stress, finally, has its peak at the 1332 1333 bottom of the sample, where the tissue is constrained.

As we will see in Figure 12, the addition of fibres 1334 as a constituent of the solid phase has the consequence 1335 of lowering the maximum value of the pore pressure 1336 attained in the domain, and, on the other hand, of in-1337 creasing the solid stress at the bottom of the sample 1338 and the fluid flux at the escaping wall at the top zone 1339 of the tissue. The latter behaviour is due to the pres-1340 ence, in that zone, of horizontal fibres. The results of 1341 the simulations obtained for an inhomogeneous (both 1342 in material properties and probability density distribu-1343 tion) and fibre-reinforced sample of articular cartilage 1344 are reported. All the results were obtained from simu-1345 lations being equipped on an Intel Xeon E5-2620 pro-1346 cessor. 1347

In Tables 4 and 5, the computational time and the memory allocation required for the simulations listed in Table 3 and 6 are reported. A comparison of the ¹³⁵⁰ performances of the FEM simulations in Tables 3 and ¹³⁵¹ 6 with the computational effort required by the model ¹³⁵² with no fibres could give a rough estimate of the time ¹³⁵³ spent by each routine for the only integration. ¹³⁵⁴

6.2.1 Results with the pseudo-Gaussian distribution 1355

 Table 3 Numerical Tests -pseudo-Gaussian distribution

Name	Integration	N
Sim-1G	SDA, external $(A1)$	900 equidistributed points
Sim-2G	SDA, internal	200 equidistributed points
Sim-3G	SDA, internal	120 Sloane points
Sim-4G	SDA, internal	21 Sloane points
Sim-5G	SDA, internal	625 equidistributed points
Sim-6G	SDA, internal	41, $(\Theta, \Phi) \in \mathcal{I} \times \mathcal{J}$

In Table 4, the computational time, the memory 1356 allocation, and the absolute errors of the SDA simu-1357 lations, obtained by employing Ψ_1 in the calculations, 1358 (both with internal and external implementations) are 1359 reported. The absolute errors listed in the table refer 1360 to the maximum discrepancies between the values of a 1361 given physical quantity, evaluated by using the SDA, 1362 and the values of the same quantity obtained by using 1363 the algorithm A2. Moreover, the normalised coordinate 1364 along the symmetry axis, ξ , for which such absolute 1365 discrepancy is maximum is also reported. 1366

Firstly, we notice that both the computational time 1367 and the memory allocation are higher if the FEM solver 1368 needs to call the external Matlab function. Indeed, we 1369 can notice that, in the case of Sim-1G, both the com-1370 putational time and the memory allocation are higher 1371 than in the other five considered cases. In fact, as ar-1372 guable from the algorithm A1, in the SDA implementa-1373 tion, three nested cycles are required to perform the nu-1374 merical integration. The first one is on the nodes of the 1375 Finite Element mesh, the second and the third are, re-1376 spectively, on the pairs $(\Theta_i, \Phi_i) \in \mathscr{D}$, with $i = 1, \ldots, m$ 1377 and j = 1, ..., n. However, the simulations Sim-2G-1378 Sim-6G are referred to as *internal*, since the sums over 1379 the integration points $(\Theta_i, \Phi_j) \in \mathscr{D}$ are already per-1380 formed, and an explicit expression of them is provided 1381 to the FEM software. On the other hand, the simula-1382 tion Sim-1G is referred to as *external* because a Matlab 1383 code representing the algorithm A1 has been written. 1384

From Figures 9, 10, and 11, obtained by setting 1385 $\hat{\Psi} = \hat{\Psi}_1$, we notice that the Sloane sets (Sim-3G and 1386 Sim-4G) produce a less accurate result, especially at 1387 the top and the bottom zones of the sample, in which 1388



Fig. 8 2D plot of the pore pressure [MPa] (a), norm of the filtration velocity [m/s] (b), and equivalent von Mises stress [MPa] (c) for the model without fibres, respectively.

	Sim-1G	Sim-2G	Sim-3G	Sim-4G	Sim-5G	Sim-6G
Comp. Time	9 h 44 min	4 min 45 s	45 s	$2 \min 42 s$	$11 \min 30 s$	15 s
Memory [Gb]	3.75	2.27	2.53	1.97	3.67	1.47
$ Z_{\rm SDA}^{11} - Z_{\rm A2}^{11} \\ (t=0)$	0.0168 $\xi = 0.8884$	0.0502 $\xi = 1$	0.2103 $\xi = 1$	0.1246 $\xi = 1$	0.0202 $\xi = 0.8839$	0.0250 $\xi = 0.1473$
$ Z_{\text{SDA}}^{11} - Z_{\text{A2}}^{11} \\ (t = 20)$	0.0134 $\xi = 0.8661$	0.0398 $\xi = 1$	0.1639 $\xi = 1$	0.1007 $\xi = 1$	0.0161 $\xi = 0.8661$	0.0283 $\xi = 0.1473$
$ Z_{\text{SDA}}^{33} - Z_{\text{A2}}^{33} \\ (t = 0)$	0.0211 $\xi = 0.1473$	0.0318 $\xi = 0.1473$	0.9982 $\xi = 0$	0.0545 $\xi = 0.1205$	0.0253 $\xi = 0.1473$	0.0500 $\xi = 0.1473$
$ Z_{\rm SDA}^{33} - Z_{\rm A2}^{33} \\ (t = 20)$	0.0419 $\xi = 0.1384$	$\begin{vmatrix} 0.0618\\ \xi = 0.1384 \end{vmatrix}$	1.2370 $\xi = 0.0268$	0.1094 $\xi = 0.1250$	0.0419 $\xi = 0.1384$	0.0894 $\xi = 0.1473$
$ (S_{a})_{SDA}^{11} - (S_{a})_{A2}^{11} $ [MPa]	$0.01106 \\ \xi = 0.9821$	$\begin{vmatrix} 0.03165\\ \xi = 0.9911 \end{vmatrix}$	0.14390 $\xi = 1$	0.07200 $\xi = 1$	0.01292 $\xi = 0.9866$	$\begin{array}{c} 0.00885\\ \xi = 0.8527 \end{array}$
$ (S_{\rm a})_{ m SDA}^{ m 33} - (S_{\rm a})_{ m A2}^{ m 33} $ [MPa]	0.0003870 $\xi = 0.8795$	0.0005079 $\xi = 0.3214$	0.0006323 $\xi = 0.9107$	0.0029116 $\xi = 0.7902$	0.0004852 $\xi = 0.8750$	0.0040588 $\xi = 0.9196$

Table 4 Comparison of the performances, Full model - pseudo-Gaussian distribution

fibres are horizontally and vertically aligned, respec-1389 tively. This is possibly due to the fact that, to apply 1390 the SDA with such sets of points, we need to select a 1391 subset of each spherical design proposed by Sloane to 1392 restrict the integration to $\mathbb{S}^{2+}_X \mathscr{B}$ rather than to $\mathbb{S}^2_X \mathscr{B}$. 1393 Moreover, in contrast to what has been done in Section 1394 4.2, where the integration with respect to Φ was exact, 1395 here the SDA implementation inside the FE discretisa-1396 tion solves numerically also the integration with respect 1397 to Φ . Thus, the lack in accuracy of Sim-3G and Sim-1398 4G, which can be registered in Figures 9, 10, and 11, 1399 is influenced by the choice of the longitudes Φ_j in the 1400 pairs $(\Theta_i, \Phi_j) \in \mathcal{D}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$. 1401

In Sim-1G, Sim-2G and Sim-5G, which refer to sets of equidistributed points, it can be noticed that the computed discrepancy between Matlab and the SDA outcomes decreases, as expected, while increasing the number of points in \mathscr{D} . Sim-1G, in particular, has the highest number of points in the spherical design, ob-1407 tained with m = n = 30, and returns the smallest er-1408 rors (see Table 4) for all the required integrals. Unfor-1409 tunately, the computational effort that such simulation 1410 requires is three times higher than the one performed 1411 with the Matlab routine. Indeed, concerning the com-1412 putational time and the memory allocation required by 1413 the Matlab integration coupled with the FEM, we reg-1414 istered, on the same workstation and with the same 1415 numerical setting, an elapsed time of about 3 h and 40 1416 min, with a memory allocation of about 2.6 Gb. Also in 1417 this case, this was due to the external Matlab call. To 1418 obtain the curves in Sim-2G and Sim-5G, respectively, 1419 we set m = 20 and m = 25. 1420

Finally, the simulation performed with the set proposed in this manuscript, $\mathcal{I} \times \mathcal{J}$ (Sim-6G), can be expressed as 1422



Fig. 9 Radial component Z^{11} of the averaged structure tensor Z vs the normalised depth. (a) Evaluation at t = 0 s (see also [22] for comparison). (b) Evaluation at t = 20 s. The probability density is the pseudo-Gaussian distribution in both cases.

$$\int_{\mathbb{S}^2_+\mathscr{B}} f(\boldsymbol{M}) \simeq \frac{2\pi}{N} \left(\sum_{i=2}^6 \sum_{j=1}^8 f(\mathcal{X}_{ij}) + f(0, \Phi) \right). \quad (105)$$

We can notice from Table 4 that $\mathcal{I} \times \mathcal{J}$ produces er-1424 rors in capturing the radial and axial components of Z1425 that are comparable with the ones obtained with Sim-1426 1G and Sim-5G, and even a better estimate of $(\mathbf{S}_{a})^{11}$. 1427 Moreover, since $\mathcal{I} \times \mathcal{J}$ contains a smaller number of 1428 pairs \mathcal{X}_{ij} , its implementation produces the fastest re-1429 sults. Indeed, even if an internal implementation of the 1430 points employed in Sim-1G is performed, its computa-1431 tional effort would be greater than, or equal to, the one 1432 registered for Sim-5G. Thus, the set $\mathcal{I} \times \mathcal{J}$ gives us, for 1433 the considered set of numerical tests in Table 3, results 1434 that are comparable with those of Sim-1G, but with a 1435 faster and lighter implementation. A less acceptable re-1436 sult, however, is obtained for $(S_a)^{33}$ (see Figure 11b), 1437 for which the computed error is one order of magnitude 1438



Fig. 10 Axial component Z^{33} of Z vs the normalised depth. (a) Evaluation at t = 0 s (see also [22] for comparison). (b) Evaluation at t = 20 s. The probability density is the pseudo-Gaussian distribution in both cases.

bigger than the one obtained for Sim-1G. In particular, ¹⁴³⁹ for that component of the tensor $S_{\rm a}$, the smaller error is obtained with the set S'_{120} . Such set of points, as ¹⁴⁴¹ visible from Figures 13, 14, and 15, is the one that, coherently with the preliminary analysis done in Section ¹⁴⁴³ 4.2, returns the better result if $\hat{\Psi}_2$ is employed. ¹⁴⁴⁴

As a consequence of a different outcome of the SDA, 1445 depending on the choice of the spherical design, the 1446 characteristic values of pore pressure, fluid filtration ve-1447 locity and stress of the whole solid could change more or 1448 less remarkably. In analysing the mechanical response 1449 of articular cartilage, it is important to have a good es-1450 timate of the pressure and the stresses that accumulate 1451 in the tissue undergoing a finite deformation. In the fol-1452 lowing, the consequences of the addition of the fibres in 1453 the model, and the consequences that a less accurate 1454 SDA could yield, are discussed. 1455

In Figure 12 the time evolution of the pore pressure, ¹⁴⁵⁶ the filtration velocity (radial component) and the von ¹⁴⁵⁷ Mises equivalent stress have been reported, respectively, ¹⁴⁵⁸



Fig. 11 Radial (a) and axial (b) components of $S_{\rm a}$ vs the normalised depth. Evaluation at t = 20 s. Pseudo-Gaussian distribution.

from left to right. All these quantities are evaluated in 1459 those points of the domain in which they attain the 1460 maximum value. Thus, as visible also from Figures 8, 1461 for the pressure we take the origin of the coordinate 1462 system, for the filtration velocity we take the upper 1463 external point, and for the solid stress the maximum 1464 value is at the bottom external point, where the sample 1465 is fixed. 1466

As we can see from Figure 12, the pressure (Figure 1467 12a) obtained without adding the fibres in the model 1468 is greater than the one obtained by considering them. 1469 Conversely, the filtration velocity (Figure 12b) is in-1470 creased by the presence of horizontally aligned fibres. 1471 This is due to the fact that, in the top zone of the do-1472 main, the fibres facilitate the radial flow going outward 1473 the sample. Finally, the equivalent von Mises stress 1474 (Figure 12c) is amplified by the presence of the fibres. 1475 The artifacts arising due to unsatisfactory results of 1476 the SDA are visible particularly in Figure 12b. Indeed, 1477 since in the upper part of the domain the Sloane sets 1478 of points do not capture in a proper way the values of 1479

1494

 Z^{11} , we see that the profile of the filtration velocity ob-1480 tained as an outcome of Sim-3G and Sim-4G shows a 1481 strong discrepancy with respect to the other employed 1482 spherical designs and the Matlab outcomes. The inac-1483 curacy of the SDA is less evident in the plots of the 1484 pore pressure and the solid stress (Figure 12a and 12c, 1485 respectively), possibly because the order of magnitude 1486 of such quantities (MPa), evaluated in their maximum 1487 points, are greater than the ones registered for the re-1488 lated errors, as shown in Tables 4 and 5 for the solid 1489 stress. Finally, the pale blue, marked curve, which cor-1490 responds to Sim-6G, is almost overlapped to the black 1491 one, representing the Matlab outcome in all the three 1492 plots in Figure 12. 1493

6.2.2 Results with the von Mises distribution

The performed simulations with the von Mises distri-1495 bution aim principally to verify the goodness of the set 1496 \mathcal{S}'_{120} , and to compare the results obtained by employ-1497 ing it with the ones obtained with the entire set S_{240} , 1498 i.e., with the integration over $\mathbb{S}^2_X \mathscr{B}$. Indeed, differently 1499 from the pseudo-Gaussian distribution, it holds that 1500 $\hat{\Psi}_2(\xi,\Theta) = \hat{\Psi}_2(\xi,\Theta+\pi)$. Thus, first we aim to com-1501 pare the solutions obtained with 240 and 120 Sloane 1502 points in Sim-2VM and Sim-3VM, respectively. A set 1503 of equidistributed points is considered in Sim-1VM, to-1504 gether with the set \mathcal{S}'_{21} (Sim-4VM) and our set $\mathcal{I} \times \mathcal{J}$ 1505 (Sim-5VM). In Table 5, the computational time, allo-1506 cated memory, and the absolute errors in computing 1507 the values of Z^{11} , Z^{33} , $(\boldsymbol{S}_{\mathrm{a}})^{11}$ and $(\boldsymbol{S}_{\mathrm{a}})^{33}$ are reported. 1508

Due to the symmetry properties of $\hat{\Psi}_2$, the performances of Sim-2VM and Sim-3VM are perfectly equivalent. Indeed, as visible from Figures 13, 14, and 15, their results overlap completely.

Again, as visible from Figures 13, 14, and 15a, the 1513 set of points proposed in the present work performs well 1514 in capturing Z and the radial component of the stress. 1515 Indeed, for such curves, the outcomes of the spherical 1516 design $\mathcal{I} \times \mathcal{J}$ almost overlap, or are relatively less distant 1517 from the outcomes of the Matlab routine and the SDA 1518 with S'_{120} , or S_{240} , equivalently. However, the set of 1519 points is not properly capturing, also in the case of the 1520 von Mises distribution, the axial component of $S_{\rm a}$. The 1521 less accurate results are, instead, obtained by means of 1522 the 500 equidistributed points, which show the same 1523 degree of accuracy as the set \mathcal{S}'_{21} , but require a greater 1524 computational effort. 1525

We omit the graphs related to the pore pressure, velocity and stress in the present case, since the amount of error registered for the numerical tests here discussed, i.e., the ones in Table 6, are not sufficiently big, as in 1529



Fig. 12 (a) Pointwise evaluation (in the point P = (0, 0, 0) mm) of the pore pressure over the time. (b) point-wise evaluation (in the point P = (0.5, 0.5, 1) mm of the filtration velocity over the time. (c) Pointwise evaluation (in the point P = (0.5, 0.5, 0) mm) of the equivalent von Mises stress over the time. A zoom of the same curve is reported, due to the strong vicinity of each curve in the graph.

	Sim-1VM	Sim-2VM	Sim-3VM	Sim-4VM	Sim-5VM
	10 . 00			42	
Comp. Time	13 min 39 s	3 min 11 s	2 min 42 s	42 s	16 s
Memory [Gb]	3.45	2.71	2.06	1.66	1.61
$ Z_{\rm SDA}^{11} - Z_{\rm A2}^{11} $	0.0489	0.0046	0.0046	0.0487	0.0217
(t=0)	$\xi = 0.9464$	$\xi = 0.4062$	$\xi = 0.4062$	$\xi = 1.0000$	$\xi = 0.3125$
		•	1	•	
$ Z_{\rm SDA}^{11} - Z_{\rm A2}^{11} $	0.0401	0.0034	0.0034	0.0401	0.0220
(t = 20)	$\xi = 0.7500$	$\xi = 0$	$\xi = 0$	$\xi = 0.9911$	$\xi = 0.3214$
$ Z_{\rm SDA}^{33} - Z_{\rm A2}^{33} $	0.0455	0.0092	0.0092	0.0625	0.0433
(t=0)	$\xi = 0.3929$	$\xi = 0.4062$	$\xi = 0.4062$	$\xi = 0.0223$	$\xi = 0.3125$
$ Z_{\rm SDA}^{33} - Z_{\rm A2}^{33} $	0.1050	0.0233	0.0233	0.1339	0.0732
(t = 20)	$\xi = 0.2143$	$\xi = 0.1786$	$\xi = 0.1786$	$\xi = 0.1786$	$\xi = 0.0.3304$
$ (S_{\rm a})^{11}_{\rm SDA} - (S_{\rm a})^{11}_{\rm A2} $	0.0289	0.0051	0.0051	0.0265	0.0138
[MPa]	$\xi = 0.9821$	$\xi = 0.5357$	$\xi = 0.5357$	$\xi = 0.8973$	$\xi = 0.5670$
$ ({m S}_{ m a})^{33}_{ m SDA} - ({m S}_{ m a})^{33}_{ m A2} $	0.0013	0.0009	0.0009	0.0038	0.0043
[MPa]	$\xi = 0.5446$	$\xi = 0.5179$	$\xi = 0.5179$	$\xi = 0.5045$	$\xi = 0.8170$

Table 5 Comparison of the performances, Full model - von Mises distribution

 ${\bf Table \ 6} \ {\rm Numerical \ Tests \ - \ von \ Mises \ distribution}$

Name	Integration	N
Sim-1VM	SDA, internal	500 equidistributed points
Sim-2VM	SDA, internal	240 Sloane points
Sim-3VM	SDA, internal	120 Sloane points
Sim-4VM	SDA, internal	21 Sloane points
Sim-5VM	SDA, internal	41, $(\Theta, \Phi) \in \mathcal{I} \times \mathcal{J}$

the previous case, to produce strong discrepancies in 1530 such physical quantities of the model. 1531

7 Conclusions and future work

1532

In this work, we analysed the performances of a numerical procedure, the SDA, which is often employed to compute the permeability and the mechanical stresses in highly anisotropic, fibre-reinforced, composite materials. In particular, we addressed composite materials of 1537



Fig. 13 (a) Radial component Z^{11} and (b) axial component Z^{33} of Z versus the normalised depth at time t = 0 s. For these simulations, the probability density is the von Mises distribution.

biomechanical interest, in which the fibre-reinforcement 1538 is due to the presence of collagen fibres, as is the case 1539 for articular cartilage. The mathematical model, within 1540 which the SDA has been implemented, has been taken 1541 from [58], and leads to a set of coupled and highly non-1542 linear partial differential equations, whose weak form is 1543 written in (51a) and (51b) [31]. These equations have to 1544 be solved in conjunction with the constitutive expres-1545 sions defining the mechanical stress and the permeabil-1546 ity of the tissue. The core issue of the problem is given, 1547 in fact, by the integrals associated with the anisotropic 1548 parts of these two properties of the tissue (cf. (46) and 1549 (47)), in which the coupling between the directional-1550 ity of the material response, expressed by the structure 1551 tensor A, and its evolution with the deformation are 1552 accounted for by the invariant $I_4 = tr(CA)$. In the ma-1553 jority of the cases, these integrals cannot be evaluated 1554 analytically. More importantly, due to the coupling be-1555 tween A and C, they cannot be even solved numeri-1556 cally once for all, since they have to be updated at each 1557



Fig. 14 (a) Radial component Z^{11} and (b) axial component Z^{33} of Z versus the normalised depth at time t = 20 s. For these simulations, the probability density is the von Mises distribution.

step of the deformation evolution. This calls for the implementation in a FEM code of dedicated quadrature 1559 schemes, such as the SDA, on the fly. 1560

With the motivation outlined above, we underlined 1561 some problematic issues of the SDA, such as the strong 1562 dependence on the choice of both the number and the 1563 placement of the quadrature nodes that this scheme 1564 requires. After a first analysis on the performance of 1565 the SDA against a comparative quadrature method, 1566 the SDA (see algorithm A1) has been implemented in 1567 a way to be coupled with a commercial FEM software. 1568 This has been done with the aim of testing the software 1569 through the simulation of a well-established benchmark 1570 problem, employed for studying the hydraulic and me-1571 chanical properties of hydrated soft tissues: namely, the 1572 unconfined compression test in displacement control of 1573 a cartilage sample. The results of the FE simulations 1574 have been performed with the goal of comparing re-1575 ciprocally different point sets adopted for the SDA. 1576 Their reliability has been tested by comparison with 1577



Fig. 15 Radial (a) and axial (b) components of S_a vs the normalised depth. Evaluation at t = 20 s. The probability density is the von Mises distribution.

the quadrature scheme A2, available in Matlab. After such analysis, we designed a set of points, i.e., the one represented in (65a) and (65b), which, in this work, gave us the best feedback in computing the anisotropic part of permeability and mechanical stress.

From the analysis of the performances reported in 1583 Tables 4 and 5, we deduce that, if the quadrature points 1584 are properly chosen, an internal implementation of the 1585 SDA is in general preferable to an external Matlab call. 1586 However, the latter is faster and more accurate than 1587 the SDA, if the points on the hemisphere are chosen in 1588 a way to be equidistributed. We discuss here the perfor-1589 mances of the set of points $\mathcal{I} \times \mathcal{J}$ for which, as visible 1590 throughout the present work, we obtain a small com-1591 putational effort, with acceptable results in capturing 1592 the main quantities of interest for the problem at hand. 1593 The set of points $\mathcal{I} \times \mathcal{J}$ is easy to implement by writing 1594 a relatively small list of summands, and by perform-1595 ing the sums required in the SDA routine without a 1596 recursive and nested procedure. This is due to the fact 1597 that $\mathcal{I} \times \mathcal{J}$ has only 41 points, which in turn are con-1598

ceived for the integration over the hemisphere, thereby addressing the material and geometrical symmetries of the considered problem. As discussed in Section 6, the performances of such set are, in most of the cases, comparable to those returning acceptable, even if heavier, results.

To validate the set $\mathcal{I} \times \mathcal{J}$ for a more general compu-1605 tational and mathematical setting, we need to test it on 1606 a wider range of benchmark problems and constitutive 1607 laws. Moreover, we remark that there might be cases 1608 in which the choice of the point set, which is necessary 1609 to achieve the best approximation of a given physical 1610 quantity, has to be done adaptively. These tasks shall be 1611 the subject of our future investigations on this theme. 1612 For the sake of completeness, we finally notice that 1613 the reduced 2D model studied in this work (justified 1614 by the geometric and material symmetries of the con-1615 sidered medium) should be extended to realistic three-1616 dimensional geometries. This is important also in view 1617 of generalising the presented framework to biomechan-1618 ical problems in which the fluid flow may deviate from 1619 the Darcian regime or growth and remodelling occur. 1620

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