Exact and approximate shell geometry in the free vibration analysis of one-layered and multilayered structures

**Original**

Exact and approximate shell geometry in the free vibration analysis of one-layered and multilayered structures / Brischetto, Salvatore. - In: INTERNATIONAL JOURNAL OF MECHANICAL SCIENCES. - ISSN 0020-7403. - 113:(2016), pp. 81-93. [10.1016/j.ijmecsci.2016.04.010]

**Availability:**
This version is available at: 11583/2641422 since: 2020-06-04T00:14:07Z

**Publisher:**
Elsevier

**Published**
DOI:10.1016/j.ijmecsci.2016.04.010

**Terms of use:**
This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

(Article begins on next page)
Exact and approximate shell geometry in the free vibration analysis of one-layered and multilayered structures

Salvatore Brischetto

Abstract
The present paper proposes the study of the approximation of the curvature terms in the three-dimensional (3D) equilibrium shell equations used for the free vibration analysis of one-layered and multilayered composite and sandwich structures. 3D equilibrium equations written for spherical shells degenerate into 3D equilibrium equations for cylindrical shells and plates considering one of the two radii of curvature or both as infinite, respectively. The approximation of curvature terms has been introduced in 3D equilibrium equations in order to study its effects in terms of frequency values. This study has been conducted by means of a comparison between 3D equilibrium equation results and 3D approximate curvature equilibrium equation results. These effects depend on the thickness and curvature of the considered structure, on the embedded material and lamination sequence, on the frequency order and vibration mode. The 3D equations have been considered in exact form for simply supported structures. The system of partial differential equations has been solved by means of the exponential matrix method. A layer wise approach is considered for multilayered structures. The approximation of the curvature has been introduced in the 3D equilibrium shell equations and not in the interlaminar continuity conditions and in the top and bottom boundary and loading conditions. This choice has been made for numerical reasons. The investigation of curvature approximation effects in the equilibrium equations allows an exhaustive analysis to understand the importance of curvature terms in the free vibration problems.

Keywords: three-dimensional exact solution; shell geometry; free vibrations; vibration modes; parametric coefficients; curvature effects; curvature approximation.

1 Introduction
Shells are common structural elements in many engineering applications such as concrete roofs, exteriors of rockets, ship hulls, automobile tires, containers of liquids, oil tanks, pipes, aerospace structures and so on. A shell is defined as a curved, thin-walled structure. It can be single- or multi-layer embedding isotropic or anisotropic materials. Shells can be classified according to their curvatures. Shallow shells have rise of not more than one fifth the smallest planform dimension of the shell [1], [2]. Shells are three-dimensional (3D) bodies bounded by two relatively close, curved surfaces. In the case of shell geometries, the 3D equations of elasticity are complicated. For this reason, all shell theories reduce the 3D elasticity
problem into a two-dimensional (2D) one for thin or thick and shallow or deep structures. This 2D simplification is usually made using Kirchhoff-Love hypotheses or their developments or refinements.

The pioneering work about the shell theory is due to Love in 1888 [3], it is considered as the first paper containing a complete and general linear theory of thin elastic shells. As discussed in the review paper by Wan and Weinitschke [4], further two important earlier publications should be considered. The first one due to Aron in 1874 [5] and the second one due to Lord Rayleigh in 1881 [6]. Aron [5] presented a set of equations for bending of thin shells to derive equations for small strains. The errors in this work were corrected by Love in his paper published in 1888 [3]. Lord Rayleigh [6] proposed a theory for the vibration of shells assuming the midsurface as unstretched. Love derived the linear equations of motion and the boundary conditions for shells, in the case of infinitesimal extensional and bending strains, using the thin plate theory assumptions by Kirchhoff [7] and the thin shell approximation conditions (e.g., as summarized in [8]). These last conditions neglect terms of the order of the thickness-to-radius of curvature ratio compared to the unit. The resulting linear shell theory was extensively used in engineering field for several decades [9] until the advent of the higher order two-dimensional shell theories that overcome some of the limitations that were intrinsic in the simplified models. The hypotheses used in the classical linear shell theories are known in the literature as Kirchhoff-Love hypotheses. The pioneering work by Love has been improved and developed in two main directions as discussed in [4]. The first direction concerns the solution techniques developed to consider different boundary and loading conditions. The second direction concerns the derivation of thin and thick shell theories from three-dimensional elasticity theory removing the Kirchhoff-Love hypotheses (e.g., asymptotic and/or iterative methods, formulation of nonlinear shell theories and their applications to finite deformation problems). One of the main problems in the development of classical and refined two-dimensional shell theories is the difficulty to understand the adequacy and accuracy of the proposed shell theory solution as approximation of the three-dimensional elasticity problem.

A number of theories exist for layered shells. Many of these theories, defined as classical ones, were developed originally for thin shells and were based on the Kirchhoff-Love kinematic hypotheses where straight lines normal to the un-deformed mid-surface remain straight and normal to the middle surface after deformation. Some of the most important classical shell theories were classified in [10]. They were named as Donnell-Mushtari, Love-Timoshenko, Arnold-Warburton, Houghton-Johns, Flügge-Byrne-Lurye, Reissner-Naghdı-Berry, Sanders, Vlasov, Kennard-Simplifi ed and Soedel. These shell theories were also described in details in [11]. Further theories are defined as higher order ones and they overcome the main limitations connected with the simple Kirchhoff-Love hypotheses. These theories for multilayered structures can be developed in the framework of an Equivalent Single Layer (ESL) approach or a Layer Wise (LW) approach [12]. In ESL models, multilayered structures are considered as only one equivalent layer with a global stiffness that is a weighted summation of the single stiffness values of each layer. In LW models, each layer of the multilayer configuration is separately considered from the kinematic point of view.

One of the recent trends in shell analysis is the use of more rigorous shell theories, in conjunction with the numerical techniques, in order to improve the accuracy and to allow the study of various refinements in shell theories for vibration, bending and buckling analyses. As shown in [13] and [14], shells can be classified as shallow or deep (depending on the values of the curve side length of the panel-radius of curvature ratio \((a/R)\)) and as thin or thick (depending on the value of the radius of curvature-thickness ratio \((R/h)\) or the curve side length of the panel-thickness ratio \((a/h)\)). Examples of thin and thick shells, and shallow and deep shells are given in Figure 1. Depending on the \(a/R\) and \(R/h\) ratios, the classical and refined 2D shell theories can give different approximation levels. For thick and/or deep shells, the use of refined 2D shell theories is mandatory. In these cases, the three-dimensional solutions could be appropriate to give correct analyses and also to propose reference solutions. However, the theory of 3D elasticity for shells is a cumbersome problem and its developing and solution procedure are usually proposed in the literature for several geometries separately (plates, circular plates, cylinders,
cylindrical and spherical shells), see interesting papers [15]-[24]. These works separately analyze shell or plate geometries and they do not give a general overview for both structures. Further papers about 3D vibration analysis of shells including different boundary conditions are [25]-[27]. The exact 3D model proposed in the present paper uses a general formulation for several geometries (square and rectangular plates, open and closed cylindrical shells and spherical shell panels). The equations of motion for the dynamic case, based on the 3D elasticity theory, are written in general orthogonal curvilinear coordinates using an exact geometry for multilayered shells. The system of second order differential equations is reduced to a system of first order differential equations, and subsequently exactly solved using the exponential matrix method and the Navier-type solution. The approach is developed in a layer-wise form imposing the continuity of displacements and transverse shear/normal stresses at each interface. The exponential matrix method was already used in [28] for the three-dimensional analysis of plates in rectilinear orthogonal coordinates and in [19] for an exact, three-dimensional, free vibration analysis of angle-ply laminated cylinders in cylindrical coordinates. In the present paper, the equations of motion written in orthogonal curvilinear coordinates are a general form of the equations of motion written in rectilinear orthogonal coordinates in [28] and in cylindrical coordinates in [19]. This general exact 3D shell solution has been used by the author for the free vibration analysis of one-layered, composite, sandwich, multilayered and FGM plates, cylinders, cylindrical panels, spherical panels and carbon nanotubes [29]-[37].

In this work, the approximation of curvature terms in parametric coefficients of the 3D shell equations has been studied for thin and thick structures and for deep and shallow structures. Isotropic one-layered and orthotropic one-layered and multilayered configurations have been investigated with particular attention to the order of frequencies and the vibration modes. The introduction of a curvature approximation is a prerogative of 2D shell models. The investigation of curvature approximation effects in 3D equilibrium equations has been proposed to see when such cumbersome equations can be simplified. These curvature approximations have been introduced only in the equilibrium equations and not in interlaminar, boundary and loading condition equations in order to avoid numerical problems in the solution procedure. However, in spite of this choice, the proposed models are sufficient to give an exhaustive analysis of the importance of curvature terms in the free vibrations of shells. They allow to understand when the use of an exact shell geometry is mandatory.

2 Three-dimensional equilibrium equations

This section introduces the equilibrium equations for spherical/cylindrical shell panels and for plates. Approximation for geometry is also considered. Some of the missed equations and mathematical steps can be found in [38]-[40] where similar formulations were proposed.

2.1 Exact geometry

The three differential equations of equilibrium written for the case of free vibration analysis of multilayered spherical shells made of $N_L$ layers with constant radii of curvature $R_\alpha$ and $R_\beta$ are here given (the most general form for variable radii of curvature can be found in [41]-[42]):

$$H_\beta \frac{\partial \sigma_{k\alpha}}{\partial \alpha} + H_\alpha \frac{\partial \sigma_{k\beta}}{\partial \beta} + H_\alpha H_\beta \frac{\partial \sigma_{kz}}{\partial z} + \left( \frac{2H_\beta}{R_\alpha} + \frac{H_\alpha}{R_\beta} \right) \sigma_{kz} = \rho^k H_\alpha H_\beta \ddot{u}^k ,$$  \hspace{1cm} (1)

$$H_\beta \frac{\partial \sigma_{k\alpha}}{\partial \alpha} + H_\alpha \frac{\partial \sigma_{k\beta}}{\partial \beta} + H_\alpha H_\beta \frac{\partial \sigma_{kz}}{\partial z} + \left( \frac{2H_\alpha}{R_\beta} + \frac{H_\beta}{R_\alpha} \right) \sigma_{kz} = \rho^k H_\alpha H_\beta \ddot{v}^k ,$$  \hspace{1cm} (2)

$$H_\beta \frac{\partial \sigma_{k\alpha}}{\partial \alpha} + H_\alpha \frac{\partial \sigma_{k\beta}}{\partial \beta} + H_\alpha H_\beta \frac{\partial \sigma_{kz}}{\partial z} - \frac{H_\beta}{R_\alpha} \sigma_{k\alpha} - \frac{H_\alpha}{R_\beta} \sigma_{k\beta} + \left( \frac{H_\beta}{R_\alpha} + \frac{H_\alpha}{R_\beta} \right) \sigma_{zz} = \rho^k H_\alpha H_\beta \ddot{w}^k ,$$  \hspace{1cm} (3)
where $\rho^k$ is the mass density, $(\sigma_{\alpha\alpha}^k, \sigma_{\alpha\beta}^k, \sigma_{zz}^k, \sigma_{\beta\alpha}^k, \sigma_{\alpha z}^k, \sigma_{\beta z}^k)$ are the six stress components and $\ddot{u}^k, \ddot{v}^k$ and $\ddot{w}^k$ indicate the second temporal derivative of the three displacement components. Each quantity depends on the $k$ layer. $R_\alpha$ and $R_\beta$ are referred to the mid-surface $\Omega_0$ of the whole multilayered shell. $H_\alpha$ and $H_\beta$ continuously vary through the thickness of the multilayered shell and depend on the thickness coordinate. The middle surface $\Omega_0$ of the shell is the locus of points which lie midway between these surfaces. Geometry and the curvilinear orthogonal reference system $(\alpha, \beta, z)$ are shown in Figure 2. Displacement components are $u$, $v$, and $w$ in $\alpha$, $\beta$ and $z$ directions, respectively [43]. The parametric coefficients for shells with constant radii of curvature are:

$$H_\alpha = (1 + \frac{z}{R_\alpha}) = (1 + \frac{\bar{z} - h/2}{R_\alpha}), \quad H_\beta = (1 + \frac{z}{R_\beta}) = (1 + \frac{\bar{z} - h/2}{R_\beta}), \quad H_z = 1,$$

(4)

$H_\alpha$ and $H_\beta$ depend on $z$ or $\bar{z}$ coordinate (see Figure 3).

The strain-displacement relations of three-dimensional theory of elasticity in orthogonal curvilinear coordinates, as shown in [41] and [44], are written for the generic $k$ layer of the multilayered spherical shell with constant radii of curvature:

$$
\begin{align*}
\epsilon_{k\alpha\alpha}^k &= \frac{1}{H_\alpha} \frac{\partial u^k}{\partial \alpha} + \frac{w^k}{H_\alpha R_\alpha}, \\
\epsilon_{k\beta\beta}^k &= \frac{1}{H_\beta} \frac{\partial v^k}{\partial \beta} + \frac{w^k}{H_\beta R_\beta}, \\
\epsilon_{zz}^k &= \frac{1}{\partial z} \frac{\partial w^k}{\partial z}, \\
\gamma_{k\alpha\beta}^k &= \frac{1}{H_\alpha} \frac{\partial u^k}{\partial \alpha} + \frac{1}{H_\beta} \frac{\partial v^k}{\partial \beta}, \\
\gamma_{k\alpha z}^k &= \frac{1}{H_\alpha} \frac{\partial u^k}{\partial \alpha} + \frac{u^k}{H_\alpha R_\alpha}, \\
\gamma_{k\beta z}^k &= \frac{1}{H_\beta} \frac{\partial v^k}{\partial \beta} + \frac{v^k}{H_\beta R_\beta},
\end{align*}
$$

(5)-(10)

where symbol $\partial$ indicates the partial derivatives.

Eqs. (5)-(10) and constitutive equations in orthogonal curvilinear coordinates $(\alpha, \beta, z)$ for orthotropic material in the structural reference system are introduced in eqs.(1)-(3) in order to obtain the equilibrium equations written in displacement form:

$$
\begin{align*}
&\left(\frac{H_\beta C_{55}^k}{H_\alpha R_\alpha^2} - \frac{C_{55}^k}{R_\alpha R_\beta}\right)u^k + \left(\frac{C_{55}^k H_\alpha^3}{R_\alpha} + \frac{C_{55}^k H_\beta^3}{R_\beta}\right)\bar{z} + \left(\frac{C_{11}^k H_\beta}{H_\alpha}\right)w^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta}\right)\bar{z}w^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha}\right)\bar{z}w^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta}\right)\bar{z}w^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha}\right)\bar{z}w^k = \rho^k H_\alpha H_\beta \ddot{u}^k, \\
&(C_{11}^k + C_{66}^k)\ddot{w}^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)u^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z} + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)v^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z}v^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)\bar{z}v^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z}v^k = \rho^k H_\alpha H_\beta \ddot{v}^k, \\
&(C_{11}^k + C_{66}^k)\ddot{v}^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)u^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)v^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)v^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z}v^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z}v^k + \left(\frac{C_{11}^k H_\alpha}{H_\beta R_\beta}\right)\bar{z}v^k + \left(\frac{C_{11}^k H_\beta}{H_\alpha R_\alpha}\right)\bar{z}v^k = \rho^k H_\alpha H_\beta \ddot{w}^k.
\end{align*}
$$

(11)-(13)
Coefficients $C_k$ are the elastic coefficients of constitutive equations for each k layer. The differential equations (11)-(13) will be solved in exact form in Section 3. The cylinder and cylindrical shell panel cases are obtained considering an infinite radius of curvature $R_\alpha$ (which means $H_\alpha$ equals 1) in equilibrium equations (1)-(3), geometrical equations (5)-(10) and displacement form of equilibrium equations (11)-(13). Plate cases are obtained considering both radii of curvature $R_\alpha$ and $R_\beta$ as infinite (which mean $H_\alpha$=H_\alpha=1) in Eqs.(1)-(13). In the results proposed in Section 4, the 3D exact theory based on the complete equations for spherical shells will be indicated as 3D.

2.2 Approximation for geometry

In the case of thin and/or shallow shells the parametric coefficients in Eqs.(4) can be set to 1 ($H_\alpha = H_\beta = 1$) because the hypotheses $\frac{1}{R_\alpha} \simeq 0$ and $\frac{1}{R_\beta} \simeq 0$ are valid. Eqs.(1)-(3) are simplified as:

\[
\frac{\partial \sigma_{\alpha\alpha}^k}{\partial \alpha} + \frac{\partial \sigma_{\alpha\beta}^k}{\partial \beta} + \frac{\partial \sigma_{\alpha z}^k}{\partial z} + \left( \frac{2}{R_\alpha} + \frac{1}{R_\beta} \right) \sigma_{\alpha z}^k = \rho^k \ddot{u}^k, \tag{14}
\]

\[
\frac{\partial \sigma_{\alpha\beta}^k}{\partial \alpha} + \frac{\partial \sigma_{\beta\beta}^k}{\partial \beta} + \frac{\partial \sigma_{\beta z}^k}{\partial z} + \left( \frac{2}{R_\beta} + \frac{1}{R_\alpha} \right) \sigma_{\beta z}^k = \rho^k \ddot{v}^k, \tag{15}
\]

\[
\frac{\partial \sigma_{\alpha z}^k}{\partial \alpha} + \frac{\partial \sigma_{\beta z}^k}{\partial \beta} + \frac{\partial \sigma_{zz}^k}{\partial z} - \frac{1}{R_\alpha} \sigma_{\alpha z}^k - \frac{1}{R_\beta} \sigma_{\beta z}^k + \left( \frac{1}{R_\alpha} + \frac{1}{R_\beta} \right) \sigma_{zz}^k = \rho^k \ddot{w}^k, \tag{16}
\]

The strain-displacement relations in eqs.(5)-(10) are simplified as:

\[
e_{\alpha\alpha}^k = \frac{\partial u^k}{\partial \alpha} + \frac{w^k}{R_\alpha}, \tag{17}
\]

\[
e_{\beta\beta}^k = \frac{\partial v^k}{\partial \beta} + \frac{w^k}{R_\beta}, \tag{18}
\]

\[
e_{zz}^k = \frac{\partial w^k}{\partial z}, \tag{19}
\]

\[
\gamma_{\alpha\beta}^k = \frac{\partial u^k}{\partial \alpha} + \frac{\partial v^k}{\partial \beta} + \frac{u^k}{R_\alpha}, \tag{20}
\]

\[
\gamma_{\alpha z}^k = \frac{\partial u^k}{\partial \alpha} + \frac{\partial v^k}{\partial z} - \frac{u^k}{R_\alpha}, \tag{21}
\]

\[
\gamma_{\beta z}^k = \frac{\partial u^k}{\partial \beta} + \frac{\partial v^k}{\partial z} - \frac{v^k}{R_\beta}. \tag{22}
\]

Constitutive relations do not change. The substitution of constitutive equations and eqs.(17)-(22) in equilibrium relations of eqs.(14)-(16) gives the simplified version of eqs.(11)-(13). This simplified version can directly be obtained substituting the hypotheses $H_\alpha = H_\beta = 1$ in eqs.(11)-(13):

\[
\left( -\frac{C_{15}^k}{R_\alpha^2} - \frac{C_{55}^k}{R_\alpha R_\beta} \right) u^k + \left( \frac{C_{15}^k}{R_\alpha} + \frac{C_{55}^k}{R_\beta} \right) u^k_{\alpha z} + \left( C_{11}^k \right) u^k_{\alpha\alpha} + \left( C_{66}^k \right) u^k_{\beta\beta} + \left( C_{55}^k \right) u^k_{\beta z} + \left( C_{12}^k + C_{66}^k \right) v_{\alpha\beta}^k + \\
\left( C_{11}^k \right) + \left( C_{12}^k \right) \frac{C_{55}^k}{R_\beta} + \frac{C_{55}^k}{R_\alpha} w^k_{\alpha z} + \left( C_{13}^k + C_{55}^k \right) w^k_{\alpha z} = \rho^k \ddot{u}^k, \tag{23}
\]

\[
\left( -\frac{C_{44}^k}{R_\beta} - \frac{C_{44}^k}{R_\alpha \beta} \right) v^k + \left( \frac{C_{44}^k}{R_\beta} + \frac{C_{44}^k}{R_\alpha} \right) v^k_{\beta z} + \left( C_{66}^k \right) v^k_{\alpha\alpha} + \left( C_{22}^k \right) v^k_{\beta\beta} + \left( C_{44}^k \right) v^k_{\beta z} + \left( C_{12}^k + C_{66}^k \right) v_{\alpha\beta}^k + \\
\left( C_{44}^k \right) + \left( C_{44}^k \right) \frac{C_{44}^k}{R_\alpha} + \frac{C_{44}^k}{R_\beta} w^k_{\beta z} + \left( C_{44}^k + C_{22}^k \right) w_{\beta z} = \rho^k \ddot{v}^k, \tag{24}
\]
The differential equations (23)-(25) will be solved in exact form in Section 3. The cylinder and cylindrical shell panel cases are obtained considering an infinite radius of curvature \( R_\alpha \) (which means \( H_\alpha \) equals 1) in equilibrium equations (14)-(16), geometrical equations (17)-(22) and displacement form of equilibrium equations (23)-(25). Plate cases are obtained considering both radii of curvature \( R_\alpha \) and \( R_\beta \) as infinite (which means \( H_\alpha = H_\beta = 1 \)) in Eqs.(14)-(25). In the results proposed in Section 4, the 3D exact theory based on these equations for shells simplified by means of the hypotheses \( H_\alpha = H_\beta = 1 \) will be indicated as 3D\((H_\alpha, H_\beta = 1)\).

### 3 Three-dimensional exact solution

The exact closed form solution for equilibrium equations detailed in Sections 2.1 and 2.2 for shells and shells with simplified geometry, respectively, will be obtained by means of the exponential matrix method and in layer-wise form. This method has been described in details in previous author’s works [29]- [37] and it was also successfully applied by Messina [28] for the case of plates in rectilinear orthogonal coordinates \((x,y,z)\) and by Soldatos and Ye [19] for the case of closed cylinders in cylindrical coordinates \((\rho, \theta)\). The equilibrium equations for shells do not have constant coefficients because of the parametric coefficients \( H_\alpha \) and/or \( H_\beta \) which depend on the \( z \) coordinate. In order to obtain differential equations with constant coefficients, each \( k \) layer is divided in \( l \) mathematical layers where the parametric coefficients \( H_\alpha \) and \( H_\beta \) can be easily calculated in the middle of each fictitious layer. Equilibrium equations are rewritten using \( j = k \times l \) mathematical layers that allow constant coefficients to be considered (see [29] for further details).

Simply supported shells and plates are analyzed. In these cases, the three displacement components have the following harmonic form:

\[
\begin{align*}
\alpha^j(\alpha, \beta, z, t) &= U^j(z)e^{i\alpha t} \cos(\bar{\alpha} \alpha) \sin(\bar{\beta} \beta), \\
\beta^j(\alpha, \beta, z, t) &= V^j(z)e^{i\alpha t} \sin(\bar{\alpha} \alpha) \cos(\bar{\beta} \beta), \\
\gamma^j(\alpha, \beta, z, t) &= W^j(z)e^{i\alpha t} \sin(\bar{\alpha} \alpha) \sin(\bar{\beta} \beta)
\end{align*}
\]

where \( U^j(z) \), \( V^j(z) \) and \( W^j(z) \) are the displacement amplitudes in \( \alpha, \beta \) and \( z \) directions, respectively. \( i \) is the coefficient of the imaginary unit. \( \omega = 2\pi f \) is the circular frequency where \( f \) is the frequency value, \( t \) is the time. In coefficients \( \bar{\alpha} = \frac{mn}{a} \) and \( \bar{\beta} = \frac{mn}{b} \), \( m \) and \( n \) are the half-wave numbers and \( a \) and \( b \) are the shell dimensions in \( \alpha \) and \( \beta \) directions, respectively (calculated in the mid-surface \( \Omega_0 \)).

The system of second order differential equations is reduced to a system of first order differential equations in analogy with the method described in [45] and [46]. A compact form for the system of first order differential equations is:

\[
D^j \frac{\partial U^j}{\partial z} = A^j U^j, \tag{29}
\]

where \( \frac{\partial U^j}{\partial z} = U^j \) and \( U^j = [U^j V^j W^j U^j V^j W^j] \). Eq.(29) can be written as:

\[
\begin{align*}
D^j U^j &= A^j U^j, \\
U^j &= D^{-1} A^j U^j, \\
U^j &= A^j U^j
\end{align*}
\]
with $A_j^* = D_j^{-1} A_j$.

The solution of Eq.(32) can be written as (see [46]):

$$U_j^j(\tilde{z}^j) = \exp(A_j^* \tilde{z}^j)U_j^j(0) \quad \text{with} \quad \tilde{z}^j \in [0, h^j],$$

where $\tilde{z}^j$ is the thickness coordinate of each $j$ layer from 0 at the bottom to $h^j$ at the top (see Figure 3). The exponential matrix is calculated with $\tilde{z}^j = h^j$ for each $j$ layer as:

$$A_j^{**} = \exp(A_j^* h^j) = I + A_j^* h^j + \frac{A_j^{*2}}{2!} h^j 2 + \frac{A_j^{*3}}{3!} h^j 3 + \ldots + \frac{A_j^{*N}}{N!} h^j N,$$

where $I$ is the $6 \times 6$ identity matrix. This expansion has a fast convergence and it is not time consuming from the computational point of view.

Considering $j = M$ mathematical layers, $M - 1$ transfer matrices must be calculated using for each interface the interlaminar continuity conditions of displacements and transverse shear/normal stresses. Moreover, the structures must be considered as simply supported and free stresses at the top and at the bottom. All these conditions allow the final system to be obtained:

$$E U^1(0) = 0,$$

where matrix $E$ has always $(6 \times 6)$ dimension, independently from the number of layers $M$, even if the method uses a layer-wise approach. $U^1(0)$ means $U$ calculated at the bottom of the whole multilayered shell (first layer 1 with $\tilde{z}^1 = 0$). Further details about this procedure, and all the steps missed in this paper can be found in [29], [30] and [31] where the extensions of this 3D exact method have been made for the first time.

The free vibration analysis means to find the non-trivial solution of $U^1(0)$ in Eq.(35) imposing the determinant of matrix $E$ equals zero:

$$\det[E] = 0,$$

Eq.(36) allows to calculate the roots of an higher order polynomial in $\lambda = \omega^2$. For each pair of half-wave numbers $(m, n)$ a certain number of circular frequencies (from 1 to $\infty$) are obtained. This number depends on the order $N$ chosen for each exponential matrix $A_j^{**}$ and the number $M$ of mathematical layers.

A certain number of circular frequencies $\omega_s$ are found when half-wave numbers $m$ and $n$ are imposed in the structures. For each frequency $\omega_s$, it is possible to find the vibration mode through the thickness direction $z$ in terms of the three displacement components $u, v$ and $w$.

4 Results

Before the results proposed to analyze the curvature approximation effects in several shell geometries embedding different materials, the present 3D exact shell solution must be validated. It is important to notice that this 3D model has been successfully applied by the author for the free vibration analysis of one-layered, composite, sandwich, multilayered and FGM plates, cylinders, cylindrical panels, spherical panels and carbon nanotubes; see papers [29]- [37] for these validations and to see the choice made for the number of mathematical layers and the order of expansion for the exponential matrix.

The proposed 3D solution was successfully validated for cylindrical and spherical shell panels [29]- [37]. Several lamination sequences, thickness ratios, vibration modes and imposed half-wave numbers have been investigated. Therefore, the 3D model can be used with confidence to study the curvature approximation effects in one-layered and multilayered structures. Four different simply supported geometries are considered in the present analysis. 1) A closed cylinder with radius of curvature in $\alpha$ direction $R_\alpha = 10m$ and infinite radius of curvature $R_\beta$ in $\beta$ direction, dimensions $a = 2\pi R_\alpha$ and
b = 20m, and thickness ratios $R_a/h$ equal 5, 10, 100 and 1000. 2) A cylindrical panel with radius of curvature in $\alpha$ direction $R_\alpha = 10m$ and infinite radius of curvature $R_\beta$ in $\beta$ direction, dimensions $a = \frac{3}{4}R_\alpha$ and $b = 20m$, and thickness ratios $R_a/h$ equal 5, 10, 100 and 1000. 3) A spherical panel with radii of curvature in $\alpha$ and $\beta$ directions $R = R_\alpha = R_\beta = 10m$, dimensions $a = b = \frac{3}{4}R$, and thickness ratios $R_a/h$ equal 5, 10, 100 and 1000. 4) A spherical panel with in-plane dimensions $a = b = 1m$, the curve side length of the panel-radius of curvature ratio $a/R_\alpha$ equals 0, 0.010, 0.025, 0.050, 0.100, 0.200, 0.400, 0.500, 1.0, and the curve side length of the panel-thickness ratio $a/h$ equals 100, 50, 20, 10 and 5. The four described geometries consider three different material configurations. 1) One-layered isotropic configuration embedding an Aluminium Alloy with Young modulus $E = 73GPa$, Poisson ratio $\nu = 0.3$ and mass density $\rho = 2800kg/m^3$. 2) A three-layered composite $90^o/0^o/90^o$ configuration where the three layers have thickness $h/3 = h_1 = h_2 = h_3$. The composite material properties are Young modulii $E_1 = 132.38GPa$ and $E_2 = E_3 = 10.756GPa$, shear modulii $G_{12} = G_{13} = 5.6537GPa$ and $G_{23} = 3.603GPa$, Poisson ratios $\nu_{12} = \nu_{13} = 0.24$ and $\nu_{23} = 0.49$, mass density $\rho = 1600kg/m^3$. 3) A sandwich configuration with external skins made of Aluminium Alloy and thickness values $h_1 = h_3 = 0.2h$ and an internal soft core made of PVC with thickness value $h_2 = 0.6h$. The PVC properties are Young modulus $E = 0.18GPa$, Poisson ratio $\nu = 0.37$ and mass density $\rho = 50kg/m^3$. The plate cases are not investigated because no curvature approximation studies can be conducted in these cases.

Tables 1, 2 and 3 show the free frequencies for isotropic one-layered, composite three-layered and sandwich cylinders, respectively. The first six frequencies have been organized in according with the results proposed by the 2D numerical models in [34]. The first column indicates the half-wave numbers $m$ and $n$ imposed in $\alpha$ and $\beta$ direction to obtain the 3D exact solution. The second column indicates the 3D frequency order (from I to $\infty$) for the imposed ($m,n$) values. The frequency results given in the third column are obtained by means of the 3D equilibrium equations for cylinders proposed in Section 2.1. The frequencies in the fourth column are calculated by means of the 3D($H_{\alpha,\beta}=1$) model of Section 2.2 obtained considering the approximation $H_{\alpha} = 1$ due to the hypothesis $z/R_\alpha \simeq 0$ ratio. The differences in percentage between the 3D and the 3D($H_{\alpha,\beta}=1$) models are calculated for several thickness ratios. This error is small for thin shells and it increases for thick shells. Moreover, it also depends on the considered mode: in general this error is smaller for small values of the circumferential half-wave number $m$. The error due to the curvature approximation also depends on the material configuration, the error is bigger for less rigid structures and/or structures with a bigger transverse anisotropy. This feature is confirmed by the fact that the biggest errors are shown for the sandwich cylinder in Table 3.

Isotropic one-layered, composite three-layered and sandwich cylindrical shell panels have been investigated in Tables 4, 5 and 6, respectively, in terms of frequency values. Each table gives the first six frequencies, for different thickness ratios, as organized in [35] by means of the 2D numerical models. The same comments and conclusions already seen for the cylinders of Tables 1, 2 and 3 are here confirmed for the cylindrical panel. The error is acceptable for thin shells and it increases for thick shells. There is a dependence on the half-wave numbers. In general, $m=n=1$ condition for thick shells gives the biggest errors. The sandwich configuration, which has the biggest transverse anisotropy, gives the largest errors. On the contrary, the composite configuration, which has the biggest rigidity, gives the smallest errors. In the case of in-plane vibration modes ($w=0$), the errors are small because the curvature effects through the thickness direction are negligible.

Tables 7, 8 and 9 show the frequency values and errors due to the curvature approximation in the free vibration analysis of isotropic one-layered, composite three-layered and sandwich spherical shell panels, respectively. Each table gives the first six frequencies organized by means of the 2D numerical models seen in [35]. All the conclusions and features seen in Tables 1-3 for cylinders and in Tables 4-6 for cylindrical shells are here confirmed for the spherical shell geometries. The only difference is due to the presence of two radii of curvature ($R_\alpha = R_\beta$) which makes more rigid the structure and allows a perfect geometrical symmetry. For these two reasons, the errors due to the curvature approximation
are smaller than cylindrical (open and closed) cases. In particular, such errors are very small for thin spherical shells. It is confirmed that sandwich configurations show the biggest errors.

Tables 1-9 give an important information about the errors due to the curvature approximation when the thickness of the structure changes. Moreover, the relation between such errors and geometry, material configuration and frequency order has also been investigated. In shell structures the frequency values do not increase in a monotonic way when half-wave numbers increase because of the coupling between the three displacement components due to the curvature. In order to better understand the effects of the half-wave numbers in the errors due to the curvature approximation, the same results already seen in Tables 1-9 have been proposed in Tables 10-18 when the half-wave numbers \( m \) and \( n \) increase in a monotonic way independently by the order of frequency. For the cylinder cases of Tables 10-12 the longitudinal half-wave number \( n \) is fixed to 1 and the circumferential half-wave number \( m \) has been set to 2, 4, 6 and 8 (only even values because the cylinder is a closed structure in the \( \alpha \) direction). In general, the error increases with the increasing of the \( m \) value. All the other features for material configuration and thickness ratio are confirmed (see Tables 1-3). The cylindrical panel cases are investigated in Tables 13-15. In these cases the values \( m = 1, 2 \) and \( n = 1, 2 \) (all the combinations) have been considered. For this geometry the relation between the errors due to the curvature approximation and the half-wave numbers \( m \) and \( n \) is not so clear as in the cylinder case. In general, the biggest error is for the couple \( m=1 \) and \( n=1 \) when the shell is thick (\( R_\alpha/h \) equals 10 and 5) and for the couple \( m=2 \) and \( n=1 \) when the shell is thin (\( R_\alpha/h \) equals 1000 and 100). The other comments for the material configuration and the thickness ratio are the same already seen in Tables 4-6. The spherical panel cases are investigated in Tables 16-18 for all the possible combinations of \( m=1,2 \) and \( n=1,2 \). Conclusions and comments similar to the cylindrical panel are obtained: the error is quite zero for each half-wave number combination when the shell is thin (\( R_\alpha/h \) equals 1000 and 100). For thick spherical shells, the relation between the error due to the curvature approximation and the half-wave numbers \( m \) and \( n \) are not a priori predictable. The other comments for material configuration and thickness ratio are confirmed (see Tables 7-9).

All the results seen in Tables 1-18 show the relations between the errors due to the curvature approximation and several parameters such as thickness ratio, frequency order, half-wave numbers, geometry and material configuration. The last parameter to be considered is the curve side length of the panel-radius of curvature ratio \( a/R \) to see the effects for both shallow and deep shells. For this aim, a spherical shell panel has been investigated in Tables 19-21. Table 19 shows the isotropic one-layered spherical shell investigation when imposed half-wave numbers are \( m=n=1 \). Table 20 shows the composite three-layered spherical shell investigation when imposed half-wave numbers are \( m=n=1 \). Table 21 shows the sandwich spherical shell investigation when imposed half-wave numbers are \( m=n=1 \), \( m=n=2 \) and \( m=n=3 \), respectively. In Tables 19-21, thin and thick shells have been considered varying the curve side length of the panel-thickness ratio \( a/h \) from 100 to 5. Shallow and deep shells have been considered varying the curve side length of the panel-radius of curvature ratio \( a/R \) from 0.000 to 1.000. The errors due to the curvature approximation increases when the shell is thicker (smaller \( a/h \) values) and/or the shell is deeper (bigger \( a/R \) values). These features are general for all the material configurations and for each couple of half-wave numbers \( m \) and \( n \). For the sandwich configuration, the \((m,n)\) effect investigation has been added. The biggest errors are obtained for the couples \( m=n=2 \) and \( m=n=3 \).

5 Conclusions

The paper proposed the analysis of the approximation of the curvature terms in the free vibrations of one-layered and multilayered isotropic, composite and sandwich cylindrical (open and closed) and spherical shells. A three-dimensional exact solution for shell structures has been proposed in the framework of a layer-wise approach. The system of differential equations has been solved by means of the
exponential matrix method. For numerical reasons, the approximation of the curvature terms has been considered only in the 3D equilibrium shell equations and not in the interlaminar continuity conditions and in the top and bottom boundary and loading conditions. However, exhaustive conclusions have been obtained. The errors in terms of frequency value due to the curvature approximation depend on the geometry, lamination and material, frequency order, vibration mode, thickness ratio and curve length of the shell-radius of curvature ratio. The curvature approximation is valid for thin and/or shallow shells. Structures including sandwich configurations usually show bigger errors because of their bigger transverse anisotropy. There is also a dependence on the half-wave numbers. For the cylinders, the error increases with the increasing of the circumferential half-wave numbers. In the cases of cylindrical and spherical shell panels, there is a dependence on the half-wave numbers but it is not a priori predictable. The error also depends on the considered vibration mode. In general, the approximation of the curvature terms does not give important errors in the case of in-plane vibration modes.

References


<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$3D$</th>
<th>$3D(H_{α,β}=1)$</th>
<th>$∆(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_α/h=1000$</td>
<td></td>
</tr>
<tr>
<td>18,1</td>
<td>I</td>
<td>3.1235</td>
<td>3.1359</td>
<td>0.40</td>
</tr>
<tr>
<td>20,1</td>
<td>I</td>
<td>3.1543</td>
<td>3.1695</td>
<td>0.48</td>
</tr>
<tr>
<td>16,1</td>
<td>I</td>
<td>3.3911</td>
<td>3.4002</td>
<td>0.27</td>
</tr>
<tr>
<td>22,1</td>
<td>I</td>
<td>3.4068</td>
<td>3.4238</td>
<td>0.50</td>
</tr>
<tr>
<td>24,1</td>
<td>I</td>
<td>3.8164</td>
<td>3.8345</td>
<td>0.47</td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>4.0450</td>
<td>4.0508</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_α/h=100$</td>
<td></td>
</tr>
<tr>
<td>10,1</td>
<td>I</td>
<td>9.5578</td>
<td>9.6803</td>
<td>1.28</td>
</tr>
<tr>
<td>12,1</td>
<td>I</td>
<td>10.413</td>
<td>10.576</td>
<td>1.56</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>11.280</td>
<td>11.345</td>
<td>0.58</td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>12.848</td>
<td>13.029</td>
<td>1.41</td>
</tr>
<tr>
<td>16,1</td>
<td>I</td>
<td>16.220</td>
<td>16.408</td>
<td>1.16</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.676</td>
<td>16.700</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_α/h=10$</td>
<td></td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>28.721</td>
<td>29.968</td>
<td>4.34</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>30.189</td>
<td>30.665</td>
<td>1.58</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>41.416</td>
<td>42.942</td>
<td>3.68</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>49.084</td>
<td>49.098</td>
<td>0.03</td>
</tr>
<tr>
<td>2,0</td>
<td>I(w=0)</td>
<td>50.419</td>
<td>50.398</td>
<td>-0.04</td>
</tr>
<tr>
<td>6,2</td>
<td>I</td>
<td>57.691</td>
<td>58.615</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_α/h=5$</td>
<td></td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>35.847</td>
<td>37.226</td>
<td>3.85</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>47.658</td>
<td>50.003</td>
<td>4.92</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>49.918</td>
<td>49.956</td>
<td>0.08</td>
</tr>
<tr>
<td>2,0</td>
<td>I(w=0)</td>
<td>50.481</td>
<td>50.398</td>
<td>-0.16</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>74.926</td>
<td>77.216</td>
<td>3.06</td>
</tr>
<tr>
<td>4,2</td>
<td>I</td>
<td>78.468</td>
<td>79.720</td>
<td>1.59</td>
</tr>
</tbody>
</table>

Table 1: Simply supported isotropic one-layered cylinder. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_α/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{α,β}=1)-3D}{3D} \times 100.$
<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$R_{\alpha}/h=1000$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22,1</td>
<td>I</td>
<td>2.7432</td>
<td>2.7489</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>24,1</td>
<td>I</td>
<td>2.7518</td>
<td>2.7587</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>26,1</td>
<td>I</td>
<td>2.8868</td>
<td>2.8945</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>28,1</td>
<td>I</td>
<td>3.1219</td>
<td>3.1301</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>20,1</td>
<td>I</td>
<td>2.8874</td>
<td>2.8919</td>
<td>0.16</td>
<td></td>
</tr>
<tr>
<td>18,1</td>
<td>I</td>
<td>3.2095</td>
<td>3.2128</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_{\alpha}/h=100$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12,1</td>
<td>I</td>
<td>7.7168</td>
<td>7.7759</td>
<td>0.77</td>
<td></td>
</tr>
<tr>
<td>10,1</td>
<td>I</td>
<td>8.1510</td>
<td>8.1889</td>
<td>0.46</td>
<td></td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>8.4061</td>
<td>8.4088</td>
<td>0.29</td>
<td></td>
</tr>
<tr>
<td>16,1</td>
<td>I</td>
<td>9.9229</td>
<td>10.006</td>
<td>0.84</td>
<td></td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>9.9378</td>
<td>9.9566</td>
<td>0.19</td>
<td></td>
</tr>
<tr>
<td>18,1</td>
<td>I</td>
<td>12.017</td>
<td>12.104</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_{\alpha}/h=10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>20.815</td>
<td>21.227</td>
<td>1.98</td>
<td></td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>22.378</td>
<td>22.484</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>26.274</td>
<td>26.921</td>
<td>2.46</td>
<td></td>
</tr>
<tr>
<td>2,0</td>
<td>I($w=0$)</td>
<td>29.930</td>
<td>29.918</td>
<td>-0.04</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>32.637</td>
<td>32.616</td>
<td>-0.06</td>
<td></td>
</tr>
<tr>
<td>10,1</td>
<td>I</td>
<td>36.389</td>
<td>37.133</td>
<td>2.04</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_{\alpha}/h=5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.649</td>
<td>27.975</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>2,0</td>
<td>I($w=0$)</td>
<td>29.967</td>
<td>29.918</td>
<td>-0.16</td>
<td></td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>32.175</td>
<td>33.107</td>
<td>2.90</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>34.562</td>
<td>34.492</td>
<td>-0.20</td>
<td></td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>45.115</td>
<td>46.335</td>
<td>2.70</td>
<td></td>
</tr>
<tr>
<td>0,1</td>
<td>I($w=0$)</td>
<td>46.994</td>
<td>47.035</td>
<td>0.09</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Simply supported composite $90^\circ/0^\circ/90^\circ$ cylinder. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)-3D}{3D} \times 100$. 
<table>
<thead>
<tr>
<th>m,n</th>
<th>Mode</th>
<th>3D</th>
<th>3D(H_{\alpha,\beta}=1)</th>
<th>\Delta(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R_{\alpha}/h=1000</td>
</tr>
<tr>
<td>18,1</td>
<td>I</td>
<td>3.6442</td>
<td>3.6645</td>
<td>0.56</td>
</tr>
<tr>
<td>16,1</td>
<td>I</td>
<td>3.6871</td>
<td>3.7030</td>
<td>0.43</td>
</tr>
<tr>
<td>20,1</td>
<td>I</td>
<td>3.9222</td>
<td>3.9455</td>
<td>0.59</td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>4.1699</td>
<td>4.1806</td>
<td>0.26</td>
</tr>
<tr>
<td>22,1</td>
<td>I</td>
<td>4.4253</td>
<td>4.4502</td>
<td>0.56</td>
</tr>
<tr>
<td>24,1</td>
<td>I</td>
<td>5.0823</td>
<td>5.1082</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R_{\alpha}/h=100</td>
</tr>
<tr>
<td>10,1</td>
<td>I</td>
<td>10.686</td>
<td>10.872</td>
<td>1.74</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>11.679</td>
<td>11.791</td>
<td>0.96</td>
</tr>
<tr>
<td>12,1</td>
<td>I</td>
<td>12.195</td>
<td>12.423</td>
<td>1.87</td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>14.983</td>
<td>15.228</td>
<td>1.63</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.623</td>
<td>16.666</td>
<td>0.26</td>
</tr>
<tr>
<td>16,1</td>
<td>I</td>
<td>18.402</td>
<td>18.652</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R_{\alpha}/h=5</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>24.681</td>
<td>27.451</td>
<td>11.2</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>21.934</td>
<td>24.830</td>
<td>13.2</td>
</tr>
<tr>
<td>10,1</td>
<td>I</td>
<td>32.499</td>
<td>32.664</td>
<td>0.51</td>
</tr>
<tr>
<td>12,1</td>
<td>I</td>
<td>42.964</td>
<td>38.831</td>
<td>-9.62</td>
</tr>
<tr>
<td>14,1</td>
<td>I</td>
<td>55.277</td>
<td>46.088</td>
<td>-16.6</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>28.835</td>
<td>29.966</td>
<td>3.92</td>
</tr>
</tbody>
</table>

Table 3: Simply supported sandwich cylinder. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)-3D}{3D} \times 100$. 
<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$R_{\alpha}/h=1000$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,1</td>
<td>I</td>
<td>3.1235</td>
<td>3.1359</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>3.8164</td>
<td>3.8345</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>5.2156</td>
<td>5.2189</td>
<td>0.06</td>
<td></td>
</tr>
<tr>
<td>5,1</td>
<td>I</td>
<td>5.6255</td>
<td>5.6448</td>
<td>0.34</td>
<td></td>
</tr>
<tr>
<td>4,2</td>
<td>I</td>
<td>6.4020</td>
<td>6.4132</td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td>5,2</td>
<td>I</td>
<td>6.6721</td>
<td>6.6888</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$R_{\alpha}/h=100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>10.413</td>
<td>10.576</td>
<td>1.56</td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>16.676</td>
<td>16.700</td>
<td>0.14</td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>20.256</td>
<td>20.446</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>20.342</td>
<td>20.439</td>
<td>0.48</td>
<td></td>
</tr>
<tr>
<td>3,2</td>
<td>I</td>
<td>23.583</td>
<td>23.758</td>
<td>0.74</td>
<td></td>
</tr>
<tr>
<td>3,3</td>
<td>I</td>
<td>30.344</td>
<td>30.494</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td>$R_{\alpha}/h=10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>28.721</td>
<td>29.968</td>
<td>4.34</td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>57.691</td>
<td>58.615</td>
<td>1.60</td>
<td></td>
</tr>
<tr>
<td>0,1</td>
<td>I($w=0$)</td>
<td>79.165</td>
<td>79.192</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>85.462</td>
<td>86.945</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>89.460</td>
<td>90.385</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>102.21</td>
<td>103.56</td>
<td>1.32</td>
<td></td>
</tr>
<tr>
<td>$R_{\alpha}/h=5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>47.658</td>
<td>50.003</td>
<td>4.92</td>
<td></td>
</tr>
<tr>
<td>0,1</td>
<td>I($w=0$)</td>
<td>79.165</td>
<td>79.271</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>85.354</td>
<td>87.045</td>
<td>1.98</td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>134.81</td>
<td>136.25</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>148.53</td>
<td>150.09</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>1,0</td>
<td>I($w=0$)</td>
<td>151.41</td>
<td>151.19</td>
<td>-0.14</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Simply supported isotropic one-layered cylindrical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100$. 


Table 5: Simply supported composite 90°/0°/90° cylindrical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_\alpha/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_\alpha,\beta=1)-3D}{3D} \times 100$. 

<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$R_\alpha/h=1000$</th>
<th>$R_\alpha/h=100$</th>
<th>$R_\alpha/h=10$</th>
<th>$R_\alpha/h=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$3D$</td>
<td>$3D(H_\alpha,\beta=1)$</td>
<td>$\Delta(%)$</td>
<td>$\Delta(%)$</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>2.7518</td>
<td>2.7587</td>
<td>0.25</td>
<td>7.7168</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>3.2095</td>
<td>3.2128</td>
<td>0.10</td>
<td>12.017</td>
</tr>
<tr>
<td>5,1</td>
<td>I</td>
<td>3.4347</td>
<td>3.4432</td>
<td>0.25</td>
<td>15.595</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>4.6996</td>
<td>4.7086</td>
<td>0.19</td>
<td>15.615</td>
</tr>
<tr>
<td>6,2</td>
<td>I</td>
<td>5.6024</td>
<td>5.6101</td>
<td>0.14</td>
<td>20.620</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>7.7168</td>
<td>7.7759</td>
<td>0.77</td>
<td>20.815</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>12.017</td>
<td>12.104</td>
<td>0.72</td>
<td>49.513</td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>13.428</td>
<td>13.434</td>
<td>0.04</td>
<td>65.527</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>15.595</td>
<td>15.643</td>
<td>0.44</td>
<td>82.058</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>20.815</td>
<td>21.227</td>
<td>1.98</td>
<td>32.175</td>
</tr>
<tr>
<td>0,1</td>
<td>I(w=0)</td>
<td>46.994</td>
<td>47.005</td>
<td>0.02</td>
<td>46.994</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>47.285</td>
<td>47.432</td>
<td>0.31</td>
<td>66.117</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>49.513</td>
<td>50.286</td>
<td>1.56</td>
<td>83.548</td>
</tr>
<tr>
<td>0,1</td>
<td>I(w=0)</td>
<td>66.117</td>
<td>66.499</td>
<td>0.58</td>
<td>89.874</td>
</tr>
<tr>
<td>0,2</td>
<td>I(w=0)</td>
<td>93.989</td>
<td>94.009</td>
<td>0.02</td>
<td>93.989</td>
</tr>
<tr>
<td>m,n</td>
<td>Mode</td>
<td>3D</td>
<td>3D($H_{\alpha,\beta}=1$)</td>
<td>$\Delta(%)$</td>
<td></td>
</tr>
<tr>
<td>-----</td>
<td>------</td>
<td>----</td>
<td>-----------------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>3.6442</td>
<td>3.6645</td>
<td>0.56</td>
<td></td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>5.0823</td>
<td>5.1082</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>5.2301</td>
<td>5.2364</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>4,2</td>
<td>I</td>
<td>7.2691</td>
<td>7.2880</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>5,1</td>
<td>I</td>
<td>7.6669</td>
<td>7.6936</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>5,2</td>
<td>I</td>
<td>8.5537</td>
<td>8.5783</td>
<td>0.29</td>
<td></td>
</tr>
</tbody>
</table>

$R_\alpha/h=1000$

<table>
<thead>
<tr>
<th>m,n</th>
<th>Mode</th>
<th>3D</th>
<th>3D($H_{\alpha,\beta}=1$)</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,1</td>
<td>I</td>
<td>12.195</td>
<td>12.423</td>
<td>1.87</td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>16.623</td>
<td>16.666</td>
<td>0.26</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>21.415</td>
<td>21.560</td>
<td>0.68</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>22.173</td>
<td>22.426</td>
<td>1.14</td>
</tr>
<tr>
<td>3,2</td>
<td>I</td>
<td>25.159</td>
<td>25.392</td>
<td>0.93</td>
</tr>
<tr>
<td>3,3</td>
<td>I</td>
<td>31.304</td>
<td>31.502</td>
<td>0.63</td>
</tr>
</tbody>
</table>

$R_\alpha/h=100$

<table>
<thead>
<tr>
<th>m,n</th>
<th>Mode</th>
<th>3D</th>
<th>3D($H_{\alpha,\beta}=1$)</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1</td>
<td>I</td>
<td>20.044</td>
<td>21.412</td>
<td>6.82</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>29.481</td>
<td>32.292</td>
<td>9.54</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>37.098</td>
<td>39.451</td>
<td>6.34</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>43.940</td>
<td>44.668</td>
<td>1.66</td>
</tr>
<tr>
<td>2,3</td>
<td>I</td>
<td>49.736</td>
<td>51.608</td>
<td>3.76</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>52.796</td>
<td>54.181</td>
<td>2.62</td>
</tr>
</tbody>
</table>

$R_\alpha/h=10$

<table>
<thead>
<tr>
<th>m,n</th>
<th>Mode</th>
<th>3D</th>
<th>3D($H_{\alpha,\beta}=1$)</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1</td>
<td>I</td>
<td>21.934</td>
<td>24.830</td>
<td>13.2</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>42.964</td>
<td>38.831</td>
<td>-9.62</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>46.274</td>
<td>48.094</td>
<td>3.93</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>52.383</td>
<td>49.084</td>
<td>-6.30</td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>66.237</td>
<td>67.749</td>
<td>2.28</td>
</tr>
<tr>
<td>2,3</td>
<td>I</td>
<td>68.193</td>
<td>65.776</td>
<td>-3.54</td>
</tr>
</tbody>
</table>

$R_\alpha/h=5$

Table 6: Simply supported sandwich cylindrical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_\alpha/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)-3D}{3D} \times 100$. 

18
Table 7: Simply supported isotropic one-layered spherical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_\alpha/h$. Error in percentage calculated as $\Delta(\%) = \frac{|3D(H_{\alpha,\beta}=1) - 3D|}{3D} \times 100$. 

<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>3D</th>
<th>3D($H_{\alpha,\beta}=1$)</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$R_\alpha/h=1000$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>77.540</td>
<td>77.540</td>
<td>0.00</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>79.759</td>
<td>79.759</td>
<td>0.00</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>79.759</td>
<td>79.759</td>
<td>0.00</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>80.334</td>
<td>80.334</td>
<td>0.00</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>80.533</td>
<td>80.533</td>
<td>0.00</td>
</tr>
<tr>
<td>3,2</td>
<td>I</td>
<td>80.728</td>
<td>80.728</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_\alpha/h=100$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>77.629</td>
<td>77.637</td>
<td>0.01</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>80.420</td>
<td>80.446</td>
<td>0.03</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>80.420</td>
<td>80.446</td>
<td>0.03</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>82.088</td>
<td>82.131</td>
<td>0.05</td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>83.295</td>
<td>83.349</td>
<td>0.06</td>
</tr>
<tr>
<td>3,2</td>
<td>I</td>
<td>83.295</td>
<td>83.349</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_\alpha/h=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>85.696</td>
<td>86.320</td>
<td>0.73</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>125.28</td>
<td>126.45</td>
<td>0.93</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>125.28</td>
<td>126.45</td>
<td>0.93</td>
</tr>
<tr>
<td>0,1</td>
<td>II($w=0$)</td>
<td>150.88</td>
<td>151.24</td>
<td>0.24</td>
</tr>
<tr>
<td>1,0</td>
<td>II($w=0$)</td>
<td>150.88</td>
<td>151.24</td>
<td>0.24</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>172.46</td>
<td>173.65</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_\alpha/h=5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>103.08</td>
<td>104.42</td>
<td>1.30</td>
</tr>
<tr>
<td>0,1</td>
<td>II($w=0$)</td>
<td>149.92</td>
<td>151.36</td>
<td>0.96</td>
</tr>
<tr>
<td>1,0</td>
<td>II($w=0$)</td>
<td>149.92</td>
<td>151.36</td>
<td>0.96</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>183.78</td>
<td>184.98</td>
<td>0.65</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>183.78</td>
<td>184.98</td>
<td>0.65</td>
</tr>
<tr>
<td>1,1</td>
<td>II($w=0$)</td>
<td>211.97</td>
<td>213.94</td>
<td>0.93</td>
</tr>
</tbody>
</table>
Table 8: Simply supported composite $90^\circ/0^\circ/90^\circ$ spherical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_\alpha/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100$. 

<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$R_\alpha/h=1000$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
<th>$R_\alpha/h=100$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
<th>$R_\alpha/h=10$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
<th>$R_\alpha/h=5$</th>
<th>$3D$</th>
<th>$3D(H_{\alpha,\beta}=1)$</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1</td>
<td>I</td>
<td>51.297</td>
<td>51.297</td>
<td>0.00</td>
<td></td>
<td>1,1</td>
<td>I</td>
<td>51.414</td>
<td>51.415</td>
<td>0.00</td>
<td></td>
<td>60.003</td>
<td>59.987</td>
<td>-0.03</td>
<td></td>
<td>71.499</td>
<td>71.159</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>54.349</td>
<td>54.350</td>
<td>0.00</td>
<td></td>
<td>2,2</td>
<td>I</td>
<td>56.685</td>
<td>56.693</td>
<td>0.01</td>
<td></td>
<td>88.903</td>
<td>88.759</td>
<td>-0.16</td>
<td></td>
<td>88.990</td>
<td>89.816</td>
</tr>
<tr>
<td>3,3</td>
<td>I</td>
<td>55.064</td>
<td>55.064</td>
<td>0.00</td>
<td></td>
<td>3,3</td>
<td>I</td>
<td>61.713</td>
<td>61.718</td>
<td>0.01</td>
<td></td>
<td>85.735</td>
<td>85.877</td>
<td>0.17</td>
<td></td>
<td>88.989</td>
<td>89.849</td>
</tr>
<tr>
<td>4,4</td>
<td>I</td>
<td>55.566</td>
<td>55.566</td>
<td>0.00</td>
<td></td>
<td>4,4</td>
<td>I</td>
<td>62.283</td>
<td>62.309</td>
<td>0.03</td>
<td></td>
<td>89.565</td>
<td>89.777</td>
<td>0.24</td>
<td></td>
<td>112.45</td>
<td>112.47</td>
</tr>
<tr>
<td>3,4</td>
<td>I</td>
<td>56.333</td>
<td>56.334</td>
<td>0.00</td>
<td></td>
<td>3,4</td>
<td>I</td>
<td>66.184</td>
<td>66.198</td>
<td>0.02</td>
<td></td>
<td>116.12</td>
<td>115.91</td>
<td>-0.18</td>
<td></td>
<td>141.53</td>
<td>140.62</td>
</tr>
<tr>
<td>4,5</td>
<td>I</td>
<td>56.548</td>
<td>56.548</td>
<td>0.00</td>
<td></td>
<td>4,5</td>
<td>I</td>
<td>66.361</td>
<td>66.379</td>
<td>0.03</td>
<td></td>
<td>127.95</td>
<td>127.79</td>
<td>-0.12</td>
<td></td>
<td>164.35</td>
<td>163.68</td>
</tr>
<tr>
<td>$m,n$</td>
<td>Mode</td>
<td>$3D$</td>
<td>$3D(H_{\alpha,\beta}=1)$</td>
<td>$\Delta(%)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>------</td>
<td>------------------</td>
<td>----------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_\alpha/h=1000$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>76.663</td>
<td>76.663</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>78.863</td>
<td>78.864</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>78.863</td>
<td>78.864</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>79.443</td>
<td>79.444</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>79.650</td>
<td>79.651</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>79.650</td>
<td>79.651</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_\alpha/h=100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>76.815</td>
<td>76.830</td>
<td>0.02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>79.826</td>
<td>79.867</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>79.826</td>
<td>79.867</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>81.670</td>
<td>81.735</td>
<td>0.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>82.884</td>
<td>82.961</td>
<td>0.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>82.884</td>
<td>82.961</td>
<td>0.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_\alpha/h=10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>78.270</td>
<td>78.742</td>
<td>0.60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>85.061</td>
<td>85.974</td>
<td>1.07</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>85.061</td>
<td>85.974</td>
<td>1.07</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>91.834</td>
<td>92.600</td>
<td>0.83</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>96.889</td>
<td>97.281</td>
<td>0.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>96.889</td>
<td>97.281</td>
<td>0.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_\alpha/h=5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>79.023</td>
<td>79.688</td>
<td>0.84</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>91.141</td>
<td>87.736</td>
<td>-3.74</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>91.141</td>
<td>87.736</td>
<td>-3.74</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>105.65</td>
<td>96.675</td>
<td>-8.49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,3</td>
<td>I</td>
<td>116.19</td>
<td>104.61</td>
<td>-9.97</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,1</td>
<td>I</td>
<td>116.19</td>
<td>104.61</td>
<td>-9.97</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9: Simply supported sandwich spherical shell panel. First six frequencies $f$ in Hz (cylindrical bending frequencies have not been included) for several thickness ratios $R_\alpha/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)-3D}{3D} \times 100$. 
Table 10: Simply supported isotropic one-layered cylinder. First mode for \( m=2, 4, 6 \) and 8 and \( n=1 \), frequencies \( f \) in Hz for several thickness ratios \( R_\alpha/h \). Error in percentage calculated as \( \Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100. \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>Mode</th>
<th>3D</th>
<th>3D(( H_{\alpha,\beta}=1 ))</th>
<th>( \Delta(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_\alpha/h=1000 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.792</td>
<td>48.792</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.869</td>
<td>27.869</td>
<td>0.00</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.505</td>
<td>16.505</td>
<td>0.00</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>10.506</td>
<td>10.507</td>
<td>0.01</td>
</tr>
<tr>
<td>( R_\alpha/h=100 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.796</td>
<td>48.795</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.893</td>
<td>27.899</td>
<td>0.02</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.676</td>
<td>16.700</td>
<td>0.14</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>11.280</td>
<td>11.345</td>
<td>0.58</td>
</tr>
<tr>
<td>( R_\alpha/h=10 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>49.084</td>
<td>49.098</td>
<td>0.03</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>30.189</td>
<td>30.665</td>
<td>1.58</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>28.721</td>
<td>29.968</td>
<td>4.34</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>41.416</td>
<td>42.942</td>
<td>3.68</td>
</tr>
<tr>
<td>( R_\alpha/h=5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>49.918</td>
<td>49.956</td>
<td>0.08</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>35.847</td>
<td>37.226</td>
<td>3.85</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>47.658</td>
<td>50.003</td>
<td>4.92</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>74.926</td>
<td>77.216</td>
<td>3.06</td>
</tr>
</tbody>
</table>

Table 11: Simply supported composite 90\(^\circ\)/0\(^\circ\)/90\(^\circ\) cylinder. First mode for \( m=2, 4, 6 \) and 8 and \( n=1 \), frequencies \( f \) in Hz for several thickness ratios \( R_\alpha/h \). Error in percentage calculated as \( \Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100. \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>Mode</th>
<th>3D</th>
<th>3D(( H_{\alpha,\beta}=1 ))</th>
<th>( \Delta(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_\alpha/h=1000 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>31.764</td>
<td>31.764</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>19.690</td>
<td>19.690</td>
<td>0.00</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>13.326</td>
<td>13.326</td>
<td>0.00</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>9.6086</td>
<td>9.6088</td>
<td>0.00</td>
</tr>
<tr>
<td>( R_\alpha/h=100 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>31.773</td>
<td>31.773</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>19.721</td>
<td>19.722</td>
<td>0.00</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>13.428</td>
<td>13.434</td>
<td>0.04</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>9.9378</td>
<td>9.9566</td>
<td>0.19</td>
</tr>
<tr>
<td>( R_\alpha/h=10 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>32.637</td>
<td>32.616</td>
<td>-0.06</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>22.378</td>
<td>22.484</td>
<td>0.47</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>20.815</td>
<td>21.227</td>
<td>1.98</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>26.274</td>
<td>26.921</td>
<td>2.46</td>
</tr>
<tr>
<td>( R_\alpha/h=5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>34.562</td>
<td>34.492</td>
<td>-0.20</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.649</td>
<td>27.975</td>
<td>1.18</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>32.175</td>
<td>33.107</td>
<td>2.90</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>45.115</td>
<td>46.335</td>
<td>2.70</td>
</tr>
<tr>
<td>$m,n$</td>
<td>Mode</td>
<td>$3D$</td>
<td>$3D(H_{\alpha, \beta}=1)$</td>
<td>$\Delta(%)$</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>------</td>
<td>-----------------</td>
<td>-----------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=1000$</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.240</td>
<td>48.240</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.554</td>
<td>27.554</td>
<td>0.00</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.320</td>
<td>16.320</td>
<td>0.00</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>10.395</td>
<td>10.397</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=100$</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.245</td>
<td>48.245</td>
<td>0.00</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>27.599</td>
<td>27.609</td>
<td>0.04</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>16.623</td>
<td>16.666</td>
<td>0.26</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>11.679</td>
<td>11.791</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=10$</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.446</td>
<td>48.421</td>
<td>-0.05</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>28.456</td>
<td>28.858</td>
<td>1.41</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>20.044</td>
<td>21.412</td>
<td>6.82</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>19.792</td>
<td>22.100</td>
<td>11.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=5$</td>
<td></td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>48.501</td>
<td>48.460</td>
<td>-0.08</td>
</tr>
<tr>
<td>4,1</td>
<td>I</td>
<td>28.835</td>
<td>29.966</td>
<td>3.92</td>
</tr>
<tr>
<td>6,1</td>
<td>I</td>
<td>21.934</td>
<td>24.830</td>
<td>13.2</td>
</tr>
<tr>
<td>8,1</td>
<td>I</td>
<td>24.681</td>
<td>27.451</td>
<td>11.2</td>
</tr>
</tbody>
</table>

Table 12: Simply supported sandwich cylinder. First mode for $m=2, 4, 6$ and $8$ and $n=1$, frequencies $f$ in $Hz$ for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha, \beta}=1)-3D}{3D} \times 100$.

<table>
<thead>
<tr>
<th>$m,n$</th>
<th>Mode</th>
<th>$3D$</th>
<th>$3D(H_{\alpha, \beta}=1)$</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=1000$</td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>16.505</td>
<td>16.505</td>
<td>0.00</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>40.972</td>
<td>40.972</td>
<td>0.00</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>5.2156</td>
<td>5.2189</td>
<td>0.06</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>17.259</td>
<td>17.260</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=100$</td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>16.676</td>
<td>16.700</td>
<td>0.14</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>41.183</td>
<td>41.198</td>
<td>0.04</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>10.413</td>
<td>10.576</td>
<td>1.56</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>20.342</td>
<td>20.439</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=10$</td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>28.721</td>
<td>29.968</td>
<td>4.34</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>57.691</td>
<td>58.615</td>
<td>1.60</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>85.462</td>
<td>86.945</td>
<td>1.73</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>102.21</td>
<td>103.56</td>
<td>1.32</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_{\alpha}/h=5$</td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>47.658</td>
<td>50.003</td>
<td>4.92</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>85.354</td>
<td>87.045</td>
<td>1.98</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>148.53</td>
<td>150.09</td>
<td>1.05</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>172.56</td>
<td>173.78</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Table 13: Simply supported isotropic one-layered cylindrical shell panel. First mode for $m=1, 2$ and $n=1, 2$, frequencies $f$ in $Hz$ for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha, \beta}=1)-3D}{3D} \times 100$. 
<table>
<thead>
<tr>
<th>m,n</th>
<th>Mode</th>
<th>3D</th>
<th>3D((H_{\alpha,\beta}=1))</th>
<th>∆(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 14: Simply supported composite 90°/0°/90° cylindrical shell panel. First mode for \(m=1,2\) and \(n=1,2\), frequencies \(f\) in Hz for several thickness ratios \(R_{\alpha}/h\). Error in percentage calculated as \(\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)−3D}{3D} \times 100\).
Table 16: Simply supported isotropic one-layered spherical shell panel. First mode for \( m=1,2 \) and \( n=1,2 \), frequencies \( f \) in Hz for several thickness ratios \( R_\alpha/h \). Error in percentage calculated as \( \Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100 \).

<table>
<thead>
<tr>
<th>( m,n )</th>
<th>Mode</th>
<th>3D</th>
<th>3D(( H_{\alpha,\beta}=1 ))</th>
<th>( \Delta(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>77.540</td>
<td>77.540</td>
<td>0.00</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>79.759</td>
<td>79.759</td>
<td>0.00</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>79.759</td>
<td>79.759</td>
<td>0.00</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>80.334</td>
<td>80.334</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>77.629</td>
<td>77.637</td>
<td>0.01</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>80.420</td>
<td>80.446</td>
<td>0.03</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>80.420</td>
<td>80.446</td>
<td>0.03</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>82.088</td>
<td>82.131</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>85.696</td>
<td>86.320</td>
<td>0.73</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>125.28</td>
<td>126.45</td>
<td>0.93</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>125.28</td>
<td>126.45</td>
<td>0.93</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>172.46</td>
<td>173.65</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>103.08</td>
<td>104.42</td>
<td>1.30</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>183.78</td>
<td>184.98</td>
<td>0.65</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>183.78</td>
<td>184.98</td>
<td>0.65</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>259.56</td>
<td>260.20</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 17: Simply supported composite 90°/0°/90° spherical shell panel. First mode for \( m=1,2 \) and \( n=1,2 \), frequencies \( f \) in Hz for several thickness ratios \( R_\alpha/h \). Error in percentage calculated as \( \Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100 \).

<table>
<thead>
<tr>
<th>( m,n )</th>
<th>Mode</th>
<th>3D</th>
<th>3D(( H_{\alpha,\beta}=1 ))</th>
<th>( \Delta(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>51.297</td>
<td>51.297</td>
<td>0.00</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>60.085</td>
<td>60.085</td>
<td>0.00</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>63.690</td>
<td>63.690</td>
<td>0.00</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>54.349</td>
<td>54.350</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>51.414</td>
<td>51.415</td>
<td>0.00</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>61.713</td>
<td>61.718</td>
<td>0.01</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>64.008</td>
<td>64.012</td>
<td>0.01</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>56.685</td>
<td>56.693</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>60.003</td>
<td>59.987</td>
<td>-0.03</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>116.12</td>
<td>115.91</td>
<td>-0.18</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>85.735</td>
<td>85.877</td>
<td>0.17</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>127.95</td>
<td>127.79</td>
<td>-0.12</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,1</td>
<td>I</td>
<td>71.499</td>
<td>71.159</td>
<td>-0.47</td>
</tr>
<tr>
<td>1,2</td>
<td>I</td>
<td>141.53</td>
<td>140.62</td>
<td>-0.64</td>
</tr>
<tr>
<td>2,1</td>
<td>I</td>
<td>112.45</td>
<td>112.47</td>
<td>0.02</td>
</tr>
<tr>
<td>2,2</td>
<td>I</td>
<td>164.35</td>
<td>163.68</td>
<td>-0.41</td>
</tr>
<tr>
<td>$m,n$</td>
<td>Mode</td>
<td>$3D(H_{\alpha,\beta}=1)$</td>
<td>$\Delta(%)$</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>----------------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R_{\alpha}/h=1000$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>I</td>
<td>76.663</td>
<td>76.663</td>
<td>0.00</td>
</tr>
<tr>
<td>1.2</td>
<td>I</td>
<td>78.863</td>
<td>78.864</td>
<td>0.00</td>
</tr>
<tr>
<td>2.1</td>
<td>I</td>
<td>78.863</td>
<td>78.864</td>
<td>0.00</td>
</tr>
<tr>
<td>2.2</td>
<td>I</td>
<td>79.443</td>
<td>79.444</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>$R_{\alpha}/h=100$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>I</td>
<td>76.815</td>
<td>76.830</td>
<td>0.02</td>
</tr>
<tr>
<td>1.2</td>
<td>I</td>
<td>79.826</td>
<td>79.867</td>
<td>0.05</td>
</tr>
<tr>
<td>2.1</td>
<td>I</td>
<td>79.826</td>
<td>79.867</td>
<td>0.05</td>
</tr>
<tr>
<td>2.2</td>
<td>I</td>
<td>81.670</td>
<td>81.735</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>$R_{\alpha}/h=10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>I</td>
<td>78.370</td>
<td>78.742</td>
<td>0.60</td>
</tr>
<tr>
<td>1.2</td>
<td>I</td>
<td>85.061</td>
<td>85.974</td>
<td>1.07</td>
</tr>
<tr>
<td>2.1</td>
<td>I</td>
<td>85.061</td>
<td>85.974</td>
<td>1.07</td>
</tr>
<tr>
<td>2.2</td>
<td>I</td>
<td>91.834</td>
<td>92.600</td>
<td>0.83</td>
</tr>
<tr>
<td></td>
<td>$R_{\alpha}/h=5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>I</td>
<td>79.023</td>
<td>79.688</td>
<td>0.84</td>
</tr>
<tr>
<td>1.2</td>
<td>I</td>
<td>91.141</td>
<td>87.736</td>
<td>-3.74</td>
</tr>
<tr>
<td>2.1</td>
<td>I</td>
<td>91.141</td>
<td>87.736</td>
<td>-3.74</td>
</tr>
<tr>
<td>2.2</td>
<td>I</td>
<td>105.65</td>
<td>96.675</td>
<td>-8.49</td>
</tr>
</tbody>
</table>

Table 18: Simply supported sandwich spherical shell panel. First mode for $m=1,2$ and $n=1,2$, frequencies $f$ in $Hz$ for several thickness ratios $R_{\alpha}/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100$. 

<table>
<thead>
<tr>
<th>$a/R_\alpha$</th>
<th>3D</th>
<th>$3D(H_{n,\beta}=1)$</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/h=100$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>48.525</td>
<td>48.525</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>49.200</td>
<td>49.201</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>52.602</td>
<td>52.603</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>63.274</td>
<td>63.278</td>
<td>0.01</td>
</tr>
<tr>
<td>0.100</td>
<td>94.583</td>
<td>94.594</td>
<td>0.01</td>
</tr>
<tr>
<td>0.200</td>
<td>169.27</td>
<td>169.29</td>
<td>0.01</td>
</tr>
<tr>
<td>0.400</td>
<td>326.29</td>
<td>326.35</td>
<td>0.02</td>
</tr>
<tr>
<td>0.500</td>
<td>404.70</td>
<td>404.76</td>
<td>0.01</td>
</tr>
<tr>
<td>1.000</td>
<td>779.71</td>
<td>779.80</td>
<td>0.01</td>
</tr>
<tr>
<td>$a/h=50$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>96.949</td>
<td>96.949</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>97.288</td>
<td>97.288</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>99.046</td>
<td>99.049</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>105.08</td>
<td>105.09</td>
<td>0.01</td>
</tr>
<tr>
<td>0.100</td>
<td>126.37</td>
<td>126.40</td>
<td>0.02</td>
</tr>
<tr>
<td>0.200</td>
<td>188.71</td>
<td>188.80</td>
<td>0.05</td>
</tr>
<tr>
<td>0.400</td>
<td>336.42</td>
<td>336.62</td>
<td>0.06</td>
</tr>
<tr>
<td>0.500</td>
<td>412.69</td>
<td>412.93</td>
<td>0.06</td>
</tr>
<tr>
<td>1.000</td>
<td>783.04</td>
<td>783.40</td>
<td>0.05</td>
</tr>
<tr>
<td>$a/h=20$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>240.62</td>
<td>240.62</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>240.75</td>
<td>240.76</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>241.45</td>
<td>241.46</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>243.94</td>
<td>243.96</td>
<td>0.01</td>
</tr>
<tr>
<td>0.100</td>
<td>253.62</td>
<td>253.72</td>
<td>0.04</td>
</tr>
<tr>
<td>0.200</td>
<td>289.02</td>
<td>289.37</td>
<td>0.12</td>
</tr>
<tr>
<td>0.400</td>
<td>399.25</td>
<td>400.24</td>
<td>0.25</td>
</tr>
<tr>
<td>0.500</td>
<td>464.00</td>
<td>465.28</td>
<td>0.28</td>
</tr>
<tr>
<td>1.000</td>
<td>805.64</td>
<td>807.78</td>
<td>0.27</td>
</tr>
<tr>
<td>$a/h=10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>469.46</td>
<td>469.45</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>469.52</td>
<td>469.52</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>469.86</td>
<td>469.87</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>471.04</td>
<td>471.09</td>
<td>0.01</td>
</tr>
<tr>
<td>0.100</td>
<td>475.75</td>
<td>475.95</td>
<td>0.04</td>
</tr>
<tr>
<td>0.200</td>
<td>494.09</td>
<td>494.83</td>
<td>0.15</td>
</tr>
<tr>
<td>0.400</td>
<td>560.73</td>
<td>563.22</td>
<td>0.44</td>
</tr>
<tr>
<td>0.500</td>
<td>605.19</td>
<td>608.67</td>
<td>0.57</td>
</tr>
<tr>
<td>1.000</td>
<td>877.57</td>
<td>884.46</td>
<td>0.78</td>
</tr>
<tr>
<td>$a/h=5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>861.99</td>
<td>861.99</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>862.02</td>
<td>862.02</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>862.16</td>
<td>862.18</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>862.65</td>
<td>862.72</td>
<td>0.01</td>
</tr>
<tr>
<td>0.100</td>
<td>864.61</td>
<td>864.89</td>
<td>0.03</td>
</tr>
<tr>
<td>0.200</td>
<td>872.38</td>
<td>873.47</td>
<td>0.12</td>
</tr>
<tr>
<td>0.400</td>
<td>902.54</td>
<td>906.54</td>
<td>0.44</td>
</tr>
<tr>
<td>0.500</td>
<td>924.23</td>
<td>930.08</td>
<td>0.63</td>
</tr>
<tr>
<td>1.000</td>
<td>1080.3</td>
<td>1094.2</td>
<td>1.29</td>
</tr>
</tbody>
</table>

Table 19: Simply supported isotropic one-layered spherical shell panel. First mode for $m=1$ and $n=1$, frequency $f$ in Hz. Effects of curvature $a/R_\alpha$ and thickness $a/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{n,\beta}=1) - 3D}{3D} \times 100$. 

<table>
<thead>
<tr>
<th>$a/R_\alpha$</th>
<th>3D</th>
<th>3D($H_{\alpha,\beta}$=1)</th>
<th>$\Delta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a/h=100</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>46.893</td>
<td>46.893</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>47.219</td>
<td>47.219</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>48.893</td>
<td>48.893</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>54.451</td>
<td>54.451</td>
<td>0.00</td>
</tr>
<tr>
<td>0.100</td>
<td>72.520</td>
<td>72.523</td>
<td>0.00</td>
</tr>
<tr>
<td>0.200</td>
<td>119.94</td>
<td>119.94</td>
<td>0.00</td>
</tr>
<tr>
<td>0.400</td>
<td>223.80</td>
<td>223.81</td>
<td>0.00</td>
</tr>
<tr>
<td>0.500</td>
<td>275.76</td>
<td>275.77</td>
<td>0.00</td>
</tr>
<tr>
<td>1.000</td>
<td>517.77</td>
<td>517.78</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>a/h=50</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>93.380</td>
<td>93.380</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>93.543</td>
<td>93.543</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>94.392</td>
<td>94.393</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>97.364</td>
<td>97.365</td>
<td>0.00</td>
</tr>
<tr>
<td>0.100</td>
<td>108.42</td>
<td>108.43</td>
<td>0.01</td>
</tr>
<tr>
<td>0.200</td>
<td>144.25</td>
<td>144.27</td>
<td>0.01</td>
</tr>
<tr>
<td>0.400</td>
<td>237.12</td>
<td>237.16</td>
<td>0.02</td>
</tr>
<tr>
<td>0.500</td>
<td>286.32</td>
<td>286.37</td>
<td>0.02</td>
</tr>
<tr>
<td>1.000</td>
<td>522.08</td>
<td>522.13</td>
<td>0.01</td>
</tr>
<tr>
<td><strong>a/h=20</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>226.75</td>
<td>226.75</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>226.81</td>
<td>226.81</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>227.14</td>
<td>227.14</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>228.33</td>
<td>228.33</td>
<td>0.00</td>
</tr>
<tr>
<td>0.100</td>
<td>233.00</td>
<td>233.01</td>
<td>0.00</td>
</tr>
<tr>
<td>0.200</td>
<td>250.73</td>
<td>250.78</td>
<td>0.02</td>
</tr>
<tr>
<td>0.400</td>
<td>310.54</td>
<td>310.69</td>
<td>0.05</td>
</tr>
<tr>
<td>0.500</td>
<td>347.64</td>
<td>347.83</td>
<td>0.05</td>
</tr>
<tr>
<td>1.000</td>
<td>549.53</td>
<td>549.70</td>
<td>0.03</td>
</tr>
<tr>
<td><strong>a/h=10</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>414.54</td>
<td>414.54</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>414.57</td>
<td>414.57</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>414.72</td>
<td>414.72</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>415.27</td>
<td>415.27</td>
<td>0.00</td>
</tr>
<tr>
<td>0.100</td>
<td>417.45</td>
<td>417.47</td>
<td>0.00</td>
</tr>
<tr>
<td>0.200</td>
<td>426.04</td>
<td>426.08</td>
<td>0.01</td>
</tr>
<tr>
<td>0.400</td>
<td>458.13</td>
<td>458.27</td>
<td>0.03</td>
</tr>
<tr>
<td>0.500</td>
<td>480.17</td>
<td>480.34</td>
<td>0.03</td>
</tr>
<tr>
<td>1.000</td>
<td>620.53</td>
<td>620.37</td>
<td>-0.03</td>
</tr>
<tr>
<td><strong>a/h=5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>653.81</td>
<td>653.81</td>
<td>0.00</td>
</tr>
<tr>
<td>0.010</td>
<td>653.82</td>
<td>653.82</td>
<td>0.00</td>
</tr>
<tr>
<td>0.025</td>
<td>653.88</td>
<td>653.87</td>
<td>0.00</td>
</tr>
<tr>
<td>0.050</td>
<td>654.09</td>
<td>654.08</td>
<td>0.00</td>
</tr>
<tr>
<td>0.100</td>
<td>654.95</td>
<td>654.91</td>
<td>-0.01</td>
</tr>
<tr>
<td>0.200</td>
<td>658.35</td>
<td>658.25</td>
<td>-0.01</td>
</tr>
<tr>
<td>0.400</td>
<td>671.57</td>
<td>671.01</td>
<td>-0.08</td>
</tr>
<tr>
<td>0.500</td>
<td>681.09</td>
<td>680.21</td>
<td>-0.13</td>
</tr>
<tr>
<td>1.000</td>
<td>750.14</td>
<td>746.48</td>
<td>-0.49</td>
</tr>
</tbody>
</table>

Table 20: Simply supported composite 90°/0°/90° spherical shell panel. First mode for $m=1$ and $n=1$, frequency $f$ in Hz. Effects of curvature $a/R_\alpha$ and thickness $a/h$. Error in percentage calculated as $\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1)-3D}{3D} \times 100$. 
Table 21: Simply supported sandwich spherical shell panel. First mode for several combinations of half-wave numbers \((m,n)\), frequency \(f\) in Hz. Effects of curvature \(a/R\) and thickness \(a/h\). Error in percentage calculated as \(\Delta(\%) = \frac{3D(H_{\alpha,\beta}=1) - 3D}{3D} \times 100\).
Figure 1: Shell classifications: shallow and deep shells, thin and thick shells.

Figure 2: Reference system, geometrical parameters and notations for shells.

Figure 3: Reference systems and thickness coordinates $z$ and $\tilde{z}$ for shells.