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STABILITY OF PENDULUM-LIKE SYSTEMS WITH EXTERNAL DISTURBANCES

Vera B. Smirnova
Dept. of Mathematics
St. Petersburg State University
of Architecture and Civil Engineering,
St. Petersburg State University,
and Institute for Problems of Mechanical Engineering
of the Russian Academy of Sciences (IPME RAS),
Russia, root@al2189.spb.edu

Natalia V. Utina
Dept. of Mathematics
St. Petersburg State University
of Architecture and Civil Engineering,
Russia unv74@mail.ru

Anton V. Proskurnikov
Delft University of Technology, The Netherlands,
Institute for Problems of Mechanical Engineering
of the Russian Academy of Sciences (IPME RAS)
and ITMO University, Russia
anton.p.1982@ieee.org

Abstract
Systems with periodic nonlinearities, referred to as *pendulum–like systems* or systems with cylindric phase space, naturally arise in many applications. Considered in the Euclidean space, such systems are usually featured by an infinite sequence of equilibria, none of them being globally stable. Hence the system's "stability", understood as convergence of every solution to one of the equilibria points (gradient-like behavior, or phase locking), cannot be examined by standard tools of nonlinear control, ensuring global asymptotic stability of a single equilibrium. Nevertheless, it appears that a modification of absolute stability methods, originating from the works of V.M. Popov, allows to establish efficient criteria for gradient-like behavior of pendulum-like system, which also imply the system's robustness against a broad class of disturbances.

Key words
Stability, integral equation, periodic nonlinearity, robustness, pendulum-like system, phase-locked loop.

1 Introduction

Systems with periodic nonlinearities arise in abundance in nature and engineering applications and describe a broad class of phenomena, from the pendulum's swings to dynamics of vibrational units, electric machines and physical oscillators [Baker and Blackburn, 2005, Stoker, 1950, Leonov et al., 1996b, Blekhman, 2000]. An important example of system with periodic nonlinearity is a *phase locked loop* (PLL) and similar control circuits, providing synchronization between the internal controlled oscillator or clock and some exogenous signal [Gardner, 1966, Lindsey, 1972, Margaris, 2004, Best, 2003, Razavi, 2003]. PLL circuits are used for carrier recovery, frequency synthesis and time synchronization; periodic nonlinearities in their mathematical models naturally represent the nonlinear characteristics of phase detectors (comparators). Motivated by engineering applications [Stoker, 1950, Blekhman, 2000, Czolczynksi et al., 2012] the terms *pendulum–like system* or *synchronization system* has been coined to denote systems with periodic nonlinearities [Lindsey, 1972, Leonov, 2006].

Systems with periodic nonlinearities typically have infinite sequences of stable and unstable equilibria points, as exemplified by the simplest model of mathematical pendulum. In particular, such a system is *multistable*; it is often convenient to consider their state spaces as toric or cylindric manifolds [Kudrewicz and Wasowicz, 2007, Leonov et al., 1996a]. Many effects in such systems, e.g. oscillations, hidden attractors and "cycle slipping" [Chicone and Heitzman, 2013, Leonov et al., 2015b, Leonov et al., 2015a, Best et al., 2016, Dudkowski et al., 2016] are essentially *nonlocal* in the sense that they cannot be analyzed via linearization at equilibria points. One of the central problems, concerned with dynamics of synchronization systems, is the convergence of all solutions to equilibria points. This special type of stability is referred to as the *gradient-like* behavior [Leonov, 2006, Duan et al., 2007]; dealing with pendulum–like systems, it is also called *phase locking*. it excludes, in particular, the possibility of limit cycles and chaotic attractors.

Dynamical properties and control of pendulum–like systems have been extensively studied in the literature [Kudrewicz and Wasowicz, 2007, Seifullaev et al., 2016, Leonov et al., 1996a, Leonov et al., 1996b, Gelig et al., 2004, Leonov, 2006, Duan et al., 2007, Chicone and Heitzman, 2013, Best et al., 2016]. Most of the existing stability theorems here are based on Lyapunov methods, in particular, the Kalman-Yakubovich-Popov lemma [Popov, 1973, Gelig et al., 2004]. In this paper, we develop an alternative approach that stems from Popov's method of "a priori integral indices" [Popov, 1973, Rasvan, 2006, Yakubovich, 2002] or "integral quadratic constraints (IQC)" [Megretski and Rantzer, 1997]. Unlike Lyapunov method, this approach allows to examine not only ordinary differential equations with periodic nonlinearities, but also a broad class of distributed parameter synchronization systems, including e.g. delays [Wischert et al., 1992, Buckwalter and York, 2002] and non-rational filters [Tripathy et al., 2015]. The prerequisite for the applicability of Popov's method is the possibility to decompose the system as a feedback interconnection of a linear time-invariant block and a nonlinearity (such a decomposition is also known as the Lur'e form). Stability criteria reduce to a *frequency-domain* condition, involving the transfer function of the linear part, and some nonlinear algebraic conditions, restricting the periodic nonlinearity; in this sense the criteria are "frequency-algebraic".

Using Popov's method, we obtain novel criteria of phase-locking, extending the results from the previous works [Leonov et al., 1992, Leonov et al., 1996b, Perkin et al., 2009, Perkin et al., 2012, Smirnova et al., 2015, Smirnova et al., 2016, Smirnova and Proskurnikov, 2016] in two directions. First, we introduce frequency-domain conditions of a novel type, involving the values of the transfer function not on the imaginary axis but rather on an arbitrary vertical line $\{\alpha + \imath\omega : \omega \in \mathbb{R}\}$. Second, we consider synchronization systems in presence of uncertain *disturbances*. The mitigation of disturbances is an important problem in design of PLLs and other synchronization circuits, see e.g. [Hill and Cantoni, 2000, Cataliotti et al., 2007, Schilling et al., 2010], the existing mathematical results are however very limited and mainly deal with cancellation of harmonic disturbances [Schilling et al., 2010] and cycle slipping effects in presence of random excitations [Tausworthe, 1967, Ascheid and Meyr, 1982]. In this paper, we take an important step in analysis of pendulum–like systems' robustness against uncertain non-stochastic disturbances, addressing the case of disturbances with finite "energy".

In this article we generalize the results of the paper presented at the 8th International Scientific Conference on Physics and Control (PhysCon 2017) [Smirnova et al., 2017] .

2 Problem Setup. Preliminaries

Consider a control system described by integro–differential equations

$$\dot{\sigma}(t) = b(t) + R(\psi(\sigma(t-h)) + f(t-h)) - \\ - \int_0^t \gamma(t-\tau)(\psi(\sigma(\tau)) + f(\tau))\,d\tau \quad (t > 0). \tag{1}$$

Here $\sigma(t) = (\sigma_1(t), \ldots, \sigma_l(t))^\top$, $\psi : \mathbb{R}^l \to \mathbb{R}^l$ and $\psi(\sigma) = (\psi_1(\sigma_1), \ldots, \psi_l(\sigma_l))^\top$, $f : [-h, +\infty) \to \mathbb{R}^l$, $b : [0, +\infty) \to \mathbb{R}^l$, $\gamma : [0, +\infty) \to \mathbb{R}^{l\times l}$, $R \in \mathbb{R}^{l\times l}$, $h \geq 0$. The solution of (1) is defined by initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{2}$$

Assume that the following conditions are true:
1) the function $b(\cdot)$ is continuous and $b(t) \to 0$ as $t \to \infty$; the function $\gamma(\cdot)$ is piece–wise continuous with N breaks;
2)

$$|b(t)|e^{rt}, |\gamma(t)|e^{rt} \in L_2[0, +\infty) \tag{3}$$

for certain $r > 0$;
3)

$$\lim_{t\to+\infty} f(t) = L, \tag{4}$$

where $L = (L_1, \ldots, L_l)^\top$, $L_j \in \mathbb{R}$;
4) the function $\sigma^0(\cdot)$ is continuous and $\sigma(0 + 0) = \sigma^0(0)$;
5) each map ψ_j is Δ_j-periodic ($\psi_j(\sigma_j + \Delta_j) = \psi_j(\sigma_j)$); it is C^1-smooth with

$$\alpha_{1j} := \inf_{\zeta\in[0,\Delta_j)} \psi_j'(\zeta); \\ \alpha_{2j} := \sup_{\zeta\in[0,\Delta_j)} \psi_j'(\zeta) \tag{5}$$

(it is clear that $\alpha_{1j}\alpha_{2j} < 0$);
6) the functions

$$\varphi_j(\zeta) \triangleq \psi_j(\zeta) + L_j \tag{6}$$

have simple isolated roots.

The goal of the paper is to establish the conditions for convergence of the solutions of (1). We extend here the frequency–algebraic criteria for gradient-like behavior of pendulum-like systems without external disturbance ($f(t) \equiv 0$) [Perkin et al., 2012].

The frequency–algebraic criteria we are going to prove contain a frequency–domain inequality involving the transfer matrix of the linear part together with varying parameters and nonlinear algebraic restrictions on the varying parameters. So we need to introduce the transfer matrix of the linear part of (1):

$$K(p) := -Re^{-ph} + \int\limits_0^\infty \gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}). \tag{7}$$

Let

$$m_{1j} \le \alpha_{1j}, \quad m_{2j} \ge \alpha_{2j}. \tag{8}$$

Notice that m_{ij} $(i = 1, 2; j = 1, \dots, l)$ may be either a certain number or ∞. In the latter case we put $m_{ij}^{-1} = 0$. Let $M_i = diag\,\{m_{i1}^{-1}, \dots, m_{il}^{-1}\}$ $(i = 1, 2)$. Introduce diagonal matrices $\varkappa = diag\,\{\varkappa_1, \dots, \varkappa_l\}$, $\varepsilon = diag\,\{\varepsilon_1, \dots, \varepsilon_l\}$, $\tau = diag\,\{\tau_1, \dots, \tau_l\}$ and $\delta = diag\,\{\delta_1, \dots, \delta_l\}$ and determine the frequency–domain inequality

$$\begin{aligned}\Pi_\lambda(\omega) &\triangleq \Re e\{\varkappa K(\imath\omega - \lambda) - (K(\imath\omega - \lambda) + \\ &+ (\imath\omega - \lambda)M_1)^*\tau(K(\imath\omega - \lambda) + (\imath\omega - \lambda)M_2) - \\ &- K(\imath\omega - \lambda)^*\varepsilon K(\imath\omega - \lambda)\} - \delta \ge 0.\end{aligned} \tag{9}$$

Here $\imath^2 = -1$, the symbol $(*)$ means Hermitian conjugation and

$$\Re e H \triangleq \frac{1}{2}(H + H^*), \quad H \in \mathbb{R}^{l\times l}. \tag{10}$$

In next two sections we present a number of theorems which guarantee the convergence of any solution of (1). In section 3 the case of arbitrary parameters $M_i\,(i = 1, 2)$ and τ is considered. In this case the inequality (9) with $\lambda = 0$ only can be used, which means that the values of the transfer matrix ought to be calculated on imaginary axis. The section 4 is devoted to particular case of $\tau M_1 M_2 = 0$ and the values of the transfer matrix can be calculated on arbitrary line parallel to the imaginary axis.

By (4) and (6) one can rewrite the system (1) in the form

$$\begin{aligned}\dot\sigma(t) &= b(t) + R(\varphi(\sigma(t - h)) + g(t - h)) - \\ &- \int\limits_0^t \gamma(t - \tau)(\varphi(\sigma(\tau)) + g(\tau))\,d\tau \quad (t > 0),\end{aligned} \tag{11}$$

where
$\varphi(\sigma) = (\varphi_1(\sigma_1), \dots, \varphi_l(\sigma_l))^\top$,
$g(t) = f(t) - L$.
Throughout the paper we shall use the functions

$$\Phi_j(\zeta) \triangleq \sqrt{\left(1 - m_{1j}^{-1}\varphi_j'(\zeta)\right)\left(1 - m_{2j}^{-1}\varphi_j'(\zeta)\right)} \tag{12}$$

and

$$P_j(\zeta; \alpha, \beta) \triangleq \sqrt{1 + \frac{\alpha}{\beta}\Phi_j^2(\zeta)} \tag{13}$$

where $\alpha > 0$ and $\beta > 0$ are parameters.

We shall also need the constants

$$\nu_j := \frac{\int\limits_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int\limits_0^{\Delta_j} |\varphi_j(\zeta)|d\zeta}, \quad \nu_{0j} := \frac{\int\limits_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int\limits_0^{\Delta_j} \Phi_j(\zeta)|\varphi_j(\zeta)|d\zeta}. \tag{14}$$

$$\nu_{1j}(\alpha, \beta) := \frac{\int\limits_0^{\Delta_j} \varphi_j(\zeta)\,d\zeta}{\int\limits_0^{\Delta_j} |\varphi_j(\zeta)|P_j(\zeta; \alpha, \beta)\,d\zeta}. \tag{15}$$

3 Frequency–algebraic Stability Conditions

In this section we assume in addition that function $g(t)$ is differentiable and the inclusions

$$|g(t)|, \dot g(t)| \in L_2[0, +\infty) \tag{16}$$

are satisfied.

Theorem 1. *Suppose there exist positive definite matrices $\varkappa$, δ, τ, ε, matrices M_1 and M_2, and numbers $a_j \in [0, 1]$ $(j = 1, \dots, l)$ such that the following conditions are fulfilled:*

1) for $\lambda = 0$ and all $\omega \ge 0$ the frequency–domain inequality (9) is true, i.e. for all $\omega \ge 0$

$$\Pi_0(\omega) \ge 0; \tag{17}$$

2) the quadratic forms

$$\begin{aligned}Q_j(x, y, z) &:= \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \varkappa_j a_j \nu_j xy + \\ &+ \varkappa_j(1 - a_j)\nu_{0j} yz \quad (j = 1, \dots, l)\end{aligned} \tag{18}$$

are positive definite.
Then

$$\dot{\sigma}(t) \to 0 \quad as\ t \to +\infty \tag{19}$$

and

$$\sigma_j(t) \to q_j \quad as\ t \to +\infty, \tag{20}$$

where $\psi_j(q_j) = -L_j \quad (j = 1, \dots, l)$.

Proof. The proof is based on Popov's method of a priori integral indices [Popov, 1961, Popov, 1973, Rasvan, 2006, Yakubovich, 2002] which is traditionally used for stability investigation of integral and integro–differential equations. A prerequisite for Popov's method applicability is the existence of an integral quadratic constraint, satisfied by the nonlinearities.

Let $\eta(t) = \varphi(\sigma(t))$ and $\xi(t) = \eta(t) + g(t)$. Introduce the auxiliary function

$$\mu(t) \triangleq \begin{cases} 0, & t < 0, \\ t, & 0 \le t \le 1, \\ 1. & t > 1. \end{cases} \tag{21}$$

For $T \ge 1$ and a given solution of (1) consider the functions

$$\xi_T(t) \triangleq \begin{cases} \mu(t)\xi(t), & t < T, \\ \xi(T)e^{c(T-t)} & t \ge T \quad (c > 0); \end{cases} \tag{22}$$

$$\begin{gathered} \sigma_0(t) = b(t) + R\xi(t-h)(1-\mu(t-h)) - \\ - \int_0^t (1-\mu(\tau))\gamma(t-\tau)\xi(\tau)\,d\tau. \end{gathered} \tag{23}$$

It is easy to see that

$$|\sigma_0(t)|e^{rt} \in L_2[0, +\infty). \tag{24}$$

Introduce the function

$$\sigma_T(t) = R\xi_T(t-h) - \int_0^t \gamma(t-\tau)\xi_T(\tau)\,d\tau. \tag{25}$$

It is clear that

$$\dot{\sigma}(t) = \sigma_0(t) + \sigma_T(t) \quad \text{for } t \in [0, T]. \tag{26}$$

Consider a set of functionals $(T \ge 1)$

$$\begin{gathered} J_T \triangleq \int_0^\infty \{\sigma_T^* \varkappa \xi_T + \xi_T^* \delta \xi_T + \\ + \sigma_T^* \varepsilon \sigma_T + (\sigma_T - M_1\dot{\xi}_T)^* \tau (\sigma_T - M_2\dot{\xi}_T)\}\,dt. \end{gathered} \tag{27}$$

Due to Plancherel theorem we have

$$\begin{gathered} J_T = \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \{\mathfrak{F}(\sigma_T)^* \varkappa \mathfrak{F}(\xi_T) + \mathfrak{F}(\xi_T)^* \delta \mathfrak{F}(\xi_T) + \\ + \mathfrak{F}(\sigma_T)^* \varepsilon \mathfrak{F}(\sigma_T) + (\mathfrak{F}(\sigma_T) - M_1\mathfrak{F}(\dot{\xi}_T))^* \tau (\mathfrak{F}(\sigma_T) - \\ - M_2\mathfrak{F}(\dot{\xi}_T))\}\,d\omega, \end{gathered} \tag{28}$$

where $\mathfrak{F}$ stands for the Fourier transform. Notice that

$$\mathfrak{F}(\sigma_T)(\imath\omega) = -K(\imath\omega)\mathfrak{F}(\xi_T)(\imath\omega) \tag{29}$$

and

$$\mathfrak{F}(\dot{\xi}_T)(\imath\omega) = \imath\omega\mathfrak{F}(\xi_T)(\imath\omega). \tag{30}$$

So

$$J_T = -\tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \mathfrak{F}^*(\xi_T)(\imath\omega)\Pi_0(\omega)\mathfrak{F}(\xi_T)(\imath\omega)\,d\omega \tag{31}$$

and by virtue of (17)

$$J_T \le 0. \tag{32}$$

On the other hand the following equality is true

$$J_T = \rho_T + J_0 + J_{1T} + J_{2T} + J_{3T}, \tag{33}$$

where

$$\begin{gathered} \rho_T \triangleq \int_0^T \{\dot{\sigma}^* \varkappa \xi + \xi^* \delta \xi + \dot{\sigma}^* \varepsilon \dot{\sigma} + \\ + (\dot{\sigma} - M_1\dot{\xi})^* \tau (\dot{\sigma} - M_2\dot{\xi})\}\,dt; \end{gathered} \tag{34}$$

$$\begin{gathered} J_{1T} \triangleq -\int_0^T \{\sigma_0^* \varkappa \xi_T + 2\sigma_0^* \varepsilon \dot{\sigma} - \\ - \sigma_0^*(\varepsilon + \tau)\sigma_0 + \sigma_0^* \tau (\dot{\sigma} - M_2\dot{\xi}_T) + \\ + (\dot{\sigma} - M_1\dot{\xi}_T)^* \tau \sigma_0\}\,dt; \end{gathered} \tag{35}$$

$$J_0 \triangleq \int_0^1 \{(\mu-1)\dot\sigma^*\varkappa\xi + (\mu^2-1)\xi^*\delta\xi - \\ -(\mu\dot{\hat\xi} - \dot\xi)^* M_1\tau\dot\sigma - \dot\sigma^*\tau M_2(\mu\dot{\hat\xi} - \dot\xi) + \\ +\mu\dot{\hat\xi}^* M_1\tau M_2\mu\dot{\hat\xi} - \dot\xi^* M_1\tau M_2\xi\}\, dt; \tag{36}$$

$$J_{2T} \triangleq \int_T^\infty \{\sigma_T^*\varkappa\xi(T)e^{c(T-t)} + \xi(T)^*\delta\xi(T)e^{2c(T-t)} + \\ +c^2\xi(T)^*M_1\tau M_2\xi(T)e^{2c(T-t)} - \\ -ce^{c(T-t)}\xi(T)^*(M_1+M_2)\tau\sigma_T\}\, dt; \tag{37}$$

$$J_{3T} \triangleq \int_T^\infty \sigma_T^*(\varepsilon+\tau)\sigma_T\, dt. \tag{38}$$

Notice that from the assumptions 1)–5) it follows that $\eta(t)$, $\dot\sigma(t)$, $\dot\eta(t)$ are bounded on $\mathbb{R}_+$. Hence and from the estimates (24) and (16) it follows that

$$|J_{1T}| \le C_2, \tag{39}$$

where C_2 does not depend on T. It is easy to see that

$$|J_{2T}| < C_3, \tag{40}$$

where C_3 does not depend on T. Then inequalities (32), (39), (40) together with (33) imply that

$$\rho_T < C_0, \quad \forall T > 1, \tag{41}$$

where C_0 does not depend on T.

Let us evaluate the functional ρ_T from bellow. We have

$$\rho_T \ge \int_0^T G_1(\dot\sigma,\eta,\dot\eta)\, dt + \int_0^T G_2(\dot\sigma,\eta,\dot\eta,g,\dot g)\, dt. \tag{42}$$

where

$$G_1(\dot\sigma,\eta,\dot\eta) \triangleq \dot\sigma^*\varkappa\eta + \eta^*\delta\eta + \dot\sigma^*\varepsilon\dot\sigma + \\ +(\dot\sigma - M_1\dot\eta)^*\tau(\dot\sigma - M_2\dot\eta); \tag{43}$$

$$G_2(\dot\sigma,\eta,\dot\eta,g,\dot g) \triangleq \dot\sigma^*\varkappa g + 2g^*\delta\eta - \dot g^*M_1\tau(\dot\sigma - M_2\dot\eta) - \\ -(\dot\sigma - M_1\dot\eta)^*\tau M_2\dot g + \dot g^*M_1\tau M_2\dot g. \tag{44}$$

It is obvious that

$$\rho_T \ge \int_0^T G_1(\dot\sigma,\eta,\dot\eta)\, dt - \int_0^T |G_2(\dot\sigma,\eta,\dot\eta,g,\dot g)|\, dt. \tag{45}$$

Notice that

$$\int_0^T |\dot g^* M_1\tau M_2\dot g|\, dt \le C_4, \quad \forall T > 0. \tag{46}$$

On the other hand for any $\varepsilon_1 > 0$ the following inequalities are valid:

$$|\dot\sigma^*\varkappa g| \le \varepsilon_1|\dot\sigma|^2 + \tfrac{1}{4\varepsilon_1}(\max_{j=1,\dots,l}\varkappa_j^2)|g|^2; \tag{47}$$

$$|\eta^*\delta g| \le \varepsilon_1|\eta|^2 + \tfrac{1}{4\varepsilon_1}(\max_{j=1,\dots,l}\delta_j^2)|g|^2; \tag{48}$$

$$|(\dot\sigma - M_1\dot\eta)^*\tau M_2\dot g| \le \varepsilon_1|\dot\sigma - M_1\dot\eta|^2 + \\ +\tfrac{1}{4\varepsilon_1}(\max_{j=1,\dots,l}(\tau_j m_{2j}^{-1})^2)|\dot g|^2; \tag{49}$$

$$|g^*M_1\tau(\dot\sigma - M_2\dot\eta)| \le \varepsilon_1|\dot\sigma - M_2\dot\eta|^2 + \\ +\tfrac{1}{4\varepsilon_1}(\max_{j=1,\dots,l}(\tau_j m_{1j}^{-1})^2)|\dot g|^2. \tag{50}$$

The identities

$$\dot\sigma_j - m_{ij}^{-1}\dot\eta_j = \dot\sigma_j(1 - m_{ij}^{-1}\tfrac{d\varphi_j(\zeta)}{d\zeta}) \\ (i=1,2;\ j=1,\dots,l) \tag{51}$$

imply that

$$|\dot\sigma - M_i\dot\eta|^2 \le C_5|\dot\sigma|^2. \tag{52}$$

Introduce the functional

$$\rho_T^0 \triangleq \int_0^T \{\dot\sigma^*\varkappa\eta + \eta\delta_0\eta + \dot\sigma^*\varepsilon_0\dot\sigma + \\ +(\dot\sigma - M_1\dot\eta)^*\tau(\dot\sigma - M_2\dot\eta)\}\, dt \tag{53}$$

with diagonal matrices $\varepsilon_0 = diag\,\{\varepsilon_{01},\dots,\varepsilon_{0l}\}$, $\delta_0 = diag\,\{\delta_{01},\dots,\delta_{0l}\}$ defined by the formulas

$$\delta_0 = \delta - 2\varepsilon_1 E_l;\ \varepsilon_0 = \varepsilon - (1+2C_5)\varepsilon_1 E_l. \tag{54}$$

From estimate (45) and inequalities (46)– (52) and assumption (16) it follows that

$$\rho_T \geq \rho_T^0 - C_6, \tag{55}$$

where a positive constant C_6 depends on ε_1 but does not depend on T.

Let us choose $\varepsilon_1 > 0$ so small that the quadratic forms

$$\begin{array}{l} Q_{0j}(x,y,z) := \varepsilon_{0j}x^2 + \delta_{0j}y^2 + \tau_j z^2 + \\ +\varkappa_j\nu_j a_j xy + \varkappa_j \nu_{0j}(1-a_j)yz \quad (j=1,\dots,l) \end{array} \tag{56}$$

are positive definite. Once the number ε_1 is defined, the estimates (41) and (55) allow to affirm that

$$\rho_T^0 \leq C_7, \tag{57}$$

where C_7 does not depend on T.

Now we can apply special technique destined for stability investigation of control systems with periodic nonlinearities. Its main idea is to separate from periodic functions in ρ_T^0 the functions with null average on the period. We repeat here the argument of paper [Perkin et al., 2012]. New periodic functions are determined:

$$\begin{array}{l} F_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_j\, |\varphi_j(\zeta)|\,, \\ \Psi_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_{0j}\, |\varphi_j(\zeta)|\, \Phi_j(\zeta). \end{array} \tag{58}$$

It is evident that

$$\int_0^{\Delta_j} F_j(\zeta)d\zeta = \int_0^{\Delta_j} \Psi_j(\zeta)d\zeta = 0. \tag{59}$$

The functional ρ_T^0 is decomposed into two summands:

$$\begin{array}{l} \rho_T^0 = \displaystyle\int_0^T \sum_{j=1}^l Q_{0j}(\dot\sigma_j, |\eta_j|, \Phi_j(\sigma_j)\dot\sigma_j)\,dt + \\ + \displaystyle\sum_{j=1}^l \left(\varkappa_j \int_{\sigma_j(0)}^{\sigma_j(T)} (a_j F_j(\zeta) + (1-a_j)\Psi_j(\zeta))d\zeta \right). \end{array} \tag{60}$$

It follows from (57) and (59) that

$$\int_0^T \sum_{j=1}^l Q_{0j}(\dot\sigma_j, |\eta_j|, \Phi(\sigma_j)\dot\sigma_j)\,dt < C_9, \tag{61}$$

where C_9 does not depend on T. Since all the quadratic forms Q_{0j} are positive definite the estimate (61) implies that

$$\begin{array}{l} \dot\sigma_j(t) \in L_2[0,+\infty), \\ \varphi_j(\sigma(t)) \in L_2[0,+\infty) \quad (j=1,\dots,l). \end{array} \tag{62}$$

Any $\varphi_j(\sigma(t))$ is uniformly continuous on $[0,+\infty)$. So according to Barbalat lemma [Popov, 1973] it tends to zero as $t \to +\infty$. Then $\sigma_j(t)$ tends to a zero of $\varphi_j(\zeta)$ as $t \to +\infty$. Since the functions $\dot\sigma_j(t)$ are uniformly continuous on $[0,+\infty)$, they tend to zero as $t \to +\infty$. Theorem 1 is proved. □

Next we present a modification of Theorem 1. The following assertion contains the same frequency–domain condition but other algebraic requirements on variable parameters.

Theorem 2. *Suppose there exist positive definite matrices $\varkappa$, δ, τ, ε, matrices M_1 and M_2 such that for all $\omega \geq 0$ frequency–domain inequality* (17) *holds and algebraic inequalities*

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{1j}(\tau_j,\varepsilon_j)|\varkappa_j \quad (j=1,\dots,l) \tag{63}$$

are true.

Then the conclusion of Theorem 1 is valid.

Proof. The proof of this theorem is based on the proof of Theorem 1. We repeat all the argument of Theorem 1 up to the formula (55) and choose $\varepsilon_1 > 0$ so small that the inequalities

$$2\sqrt{\varepsilon_{0j}\delta_{0j}} > |\nu_{1j}(\tau_j,\varepsilon_{0j})|\varkappa_j \quad (j=1,\dots,l) \tag{64}$$

are true. The inequality (55) implies the inequality (57). Then we determine new periodic functions

$$Y_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_{1j}(\tau_j,\varepsilon_{0j})|\varphi_j(\zeta)|P_j(\zeta;\tau_j,\varepsilon_{0j}). \tag{65}$$

Notice that

$$\int_0^{\Delta_j} Y_j(\zeta)\,d\zeta = 0. \tag{66}$$

With the help of Y_j and the quadratic forms

$$W_j(x,y) = \varepsilon_{0j}x^2 + \delta_{0j}y^2 + \varkappa_j\nu_{1j}(\tau_j,\varepsilon_{0j})xy \tag{67}$$

the functional ρ_T^0 can be decomposed as follows

$$\begin{array}{l} \rho_T^0 = \displaystyle\int_0^T \sum_{j=1}^l W_j(\dot\sigma_j P_j(\sigma_j;\tau_j,\varepsilon_{0j}), |\eta_j|)\,dt + \\ + \displaystyle\sum_{j=1}^l \int_{\sigma_j(0)}^{\sigma_j(T)} \varkappa_j Y_j(\zeta)\,d\zeta. \end{array} \tag{68}$$

It follows from (57) and (66) that

$$\int_0^T \sum_{j=1}^l W_j(\dot\sigma_j P_j(\sigma_j;\tau_j,\varepsilon_{0j}), \eta_j)\,dt < C_{10}, \tag{69}$$

where C_{10} does not depend on T. In virtue of (64) each W_j is positive definite. Then (69) implies (62). Thus Theorem 2 is proved. □

4 Generalized Frequency–algebraic Criteria

In this section we suppose that R is diagonal and $h = 0$. We assume also that $b(t)$ is differentiable on $[0, +\infty)$, $\gamma(t)$ is differentiable for $t \in [0, +\infty)$ with the exception of discontinuity points and

$$|\dot{b}(t)|e^{rt}, |\dot{\gamma}(t)|e^{rt} \in L_2[0, +\infty). \tag{70}$$

As to $g(t)$ it is sufficient to presuppose that it has a bounded derivative and

$$g(t) \in L_1[0, +\infty). \tag{71}$$

We also suppose that

$$\varphi_j(\zeta) \in C^2(\mathbb{R}). \tag{72}$$

Theorem 3. *Suppose there exist positive definite matrices $\varkappa$, δ, ε, matrices M_i $(i = 1, 2)$, matrix τ with $\tau_j \geq 0$, and numbers $a_j \in [0, 1]$ such that the following conditions are fulfilled:*
1) for a certain $\lambda \in [0; \frac{r}{2})$ and all $\omega \geq 0$ the frequency–domain inequality (9) *is true;*
2)

$$\tau_j m_{1j}^{-1} m_{2j}^{-1} = 0, \ \forall j = 1, 2, \ldots, l; \tag{73}$$

3)

$$R^* \tau M_2 + M_1 \tau R \leq 0; \tag{74}$$

4) the quadratic forms

$$\begin{aligned}\bar{Q}_j(x, y, z) &:= \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \bar{\varkappa}_j a_j \nu_j xy + \\ &+ \bar{\varkappa}_j (1 - a_j) \nu_{0j} yz \quad (j = 1, \ldots, l),\end{aligned} \tag{75}$$

where $\bar{\varkappa}_j = \varkappa_j + 2\lambda\tau_j(m_{1j}^{-1} + m_{2j}^{-1})$ and $a_j = 1$ if $\tau_j = 0$, are positive definite.
Then the conclusion of Theorem 1 is true.

Proof. Throughout the proof we use the notation

$$[f]^s(t) \triangleq f(t)e^{st}. \tag{76}$$

We shall also use the functions defined in the previous section. The whole scheme of the proof is alike that of Theorem 1, though the "cut–function" $\xi_T(t)$ is changed here for the "cut–function"

$$\xi_T^1(t) = \begin{cases} \mu(t)\xi(t), & t \leq T, \\ 0, & t > T. \end{cases} \tag{77}$$

Let

$$\sigma_T^1(t) = R\xi_T^1(t) - \int_0^t \gamma(t - \tau)\xi_T^1(\tau)\, d\tau. \tag{78}$$

Consider the functions $[\xi_T^1]^\lambda(t)$ and $[\sigma_T^1]^\lambda(t)$. We have

$$[\sigma_T^1]^\lambda(t) = R[\xi_T^1]^\lambda(t) - \int_0^t [\gamma]^\lambda(t - \tau)[\xi_T^1]^\lambda(\tau)\, d\tau. \tag{79}$$

It is obvious that

$$\dot{\sigma}(t) = \sigma_0(t) + \sigma_T^1(t) \quad \text{for}\, t \in [0, T]. \tag{80}$$

It is also clear that $\forall T > 1$

$$|[\xi_T^1]^\lambda|, |[\sigma_T^1]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty), \tag{81}$$

and

$$|[\dot{\sigma}_T^1]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty). \tag{82}$$

It is easy to demonstrate that

$$|[\dot{\sigma}_0]^\lambda(t)|, |[\sigma_0]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty). \tag{83}$$

Notice that

$$\mathfrak{F}([\sigma_T^1]^\lambda) = -K(\imath\omega - \lambda)\mathfrak{F}([\xi_T^1]^\lambda) \tag{84}$$

Consider the functional

$$\begin{aligned} J_{\lambda T}^1 &\triangleq \int_0^\infty \{([\sigma_T^1]^\lambda)^*(\varkappa + \lambda M_1\tau + \\ &+ \lambda\tau M_2)[\xi_T^1]^\lambda + ([\sigma_T^1]^\lambda)^*(\varepsilon + \tau)[\sigma_T^1]^\lambda + \\ &+ (\tfrac{d}{dt}([\sigma_T^1]^\lambda - R[\xi_T^1]^\lambda))^*\tau M_2[\xi_T^1]^\lambda + \\ &+ ([\xi_T^1]^\lambda)^* M_1\tau\tfrac{d}{dt}([\sigma_T^1]^\lambda - R[\xi_T^1]^\lambda) + \\ &+ ([\xi_T^1]^\lambda)^*\delta([\xi_T^1]^\lambda)\}\, dt. \end{aligned} \tag{85}$$

Due to Plancherel theorem

$$J_{\lambda T}^1 = -\frac{1}{2\pi}\int_0^\infty \{\mathfrak{F}^*([\xi_T^1]^\lambda)\Pi_\lambda(\omega)\mathfrak{F}([\xi_T^1]^\lambda)\}\, d\omega. \tag{86}$$

So it follows from condition 1) that

$$J^1_{\lambda T} \leq 0. \tag{87}$$

On the other hand we have

$$\begin{aligned} &J^1_{\lambda T} = \rho^1_{\lambda T} + J^0_{\lambda T} + J^+_{\lambda T} - \\ &- \int\limits_0^T ([\xi^1_T]^\lambda)^* (R^* \tau M_2 + M_1 \tau R)\, d([\xi^1_T]^\lambda) + \\ &+ \int\limits_T^\infty ([\sigma^1_T]^\lambda)^* (\varepsilon + \tau) [\sigma^1_T]^\lambda \, dt \end{aligned} \tag{88}$$

where

$$\begin{aligned} &\rho^1_{\lambda T} \triangleq \int\limits_0^T \big\{ ([\dot\sigma]^\lambda)^* (\varkappa + \lambda M_1 \tau + \\ &+ \lambda \tau M_2) [\xi]^\lambda + ([\dot\sigma]^\lambda)^* (\varepsilon + \tau) [\dot\sigma]^\lambda + \\ &+ (\tfrac{d}{dt}([\dot\sigma]^\lambda))^* \tau M_2) [\xi]^\lambda + ([\xi]^\lambda)^* M_1 \tau \tfrac{d}{dt}([\dot\sigma]^\lambda) + \\ &+ ([\xi]^\lambda)^* \delta([\xi]^\lambda) \big\} \, dt, \end{aligned} \tag{89}$$

$$\begin{aligned} &J^0_{\lambda T} \triangleq - \int\limits_0^T \{ ([\sigma_0]^\lambda)^* (\varkappa + \lambda M_1 \tau + \\ &+ \lambda \tau M_2) [\xi^1_T]^\lambda - ([\sigma_0]^\lambda)^* (\varepsilon + \tau) [\sigma_0]^\lambda + \\ &+ 2([\sigma_0]^\lambda)^* (\varepsilon + \tau) [\dot\sigma^1_T]^\lambda + \\ &+ (\tfrac{d}{dt}([\sigma_0]^\lambda))^* \tau M_2 [\xi^1_T]^\lambda + ([\xi^1_T]^\lambda)^* M_1 \tau \tfrac{d}{dt}([\sigma_0]^\lambda) \} \, dt \end{aligned} \tag{90}$$

and $J^+_{\lambda T}$ can be obtained from J_0 by replacing in (36) $\varkappa$ by $(\varkappa + \lambda M_1 \tau + \lambda \tau M_2)$, ξ by $[\xi]^\lambda$ and $\dot\sigma$ by $[\dot\sigma]^\lambda$. In virtue of (83)

$$|J^0_{\lambda T}| \leq C_{11}, \tag{91}$$

where C_{11} does not depend on T.

Notice also that the condition 3) of the theorem provides that the forth summand in right part of (88) is positive. So it follows from (87), (88) and (91) that

$$\rho^1_{\lambda T} \leq C_{12}, \tag{92}$$

where C_{12} does not depend on T.

Consider now the functional

$$\begin{aligned} &\rho_{\lambda T} \triangleq \int\limits_0^T \{ \dot\sigma^* (\varkappa + 2\lambda(M_1 \tau + \tau M_2)) \xi + \xi^* \delta \xi + \\ &+ \dot\sigma^* \varepsilon \dot\sigma + (\dot\sigma - M_1 \dot\xi)^* \tau (\dot\sigma - M_2 \dot\xi) \} \, dt \end{aligned} \tag{93}$$

We have by virtue of (73)

$$\begin{aligned} &\rho_{\lambda T} = \int\limits_0^T \{ \dot\sigma^* (\varkappa + 2\lambda(M_1 \tau + \tau M_2)) \xi + \xi^* \delta \xi + \\ &+ \dot\sigma^* (\varepsilon + \tau) \dot\sigma - \dot\sigma^* \tau M_2 \dot\xi - \dot\xi^* M_1 \tau \dot\sigma \} \, dt. \end{aligned} \tag{94}$$

With the help of equalities

$$\begin{aligned} &\int\limits_0^T \dot\sigma^* \tau M_2 \dot\xi \, dt = \dot\sigma^* \tau M_2 \xi \big|_0^T - \\ &- \int\limits_0^T \ddot\sigma^* \tau M_2 \xi \, dt, \end{aligned} \tag{95}$$

$$\begin{aligned} &\int\limits_0^T \dot\xi^* M_1 \tau \dot\sigma \, dt = \xi^* M_1 \tau \dot\sigma \big|_0^T - \\ &- \int\limits_0^T \xi^* M_1 \tau \ddot\sigma \, dt \end{aligned} \tag{96}$$

we get

$$\rho_{\lambda T} \leq I_{\lambda T} + C_{13}, \tag{97}$$

where

$$\begin{aligned} &I_{\lambda T} = \int\limits_0^T \{ \dot\sigma^* (\varkappa + 2\lambda(M_1 \tau + \tau M_2)) \xi + \xi^* \delta \xi + \\ &+ \dot\sigma^* (\varepsilon + \tau) \dot\sigma + \ddot\sigma^* \tau M_2 \xi + \xi^* M_1 \tau \ddot\sigma \} \, dt. \end{aligned} \tag{98}$$

and C_{13} does not depend on T. Further

$$\begin{aligned} &I_{\lambda T} = \int\limits_0^T e^{-2\lambda t} \{ ([\dot\sigma]^\lambda)^* (\varkappa + 2\lambda(M_1 \tau + \tau M_2)) [\xi]^\lambda + \\ &+ ([\xi]^\lambda)^* \delta [\xi]^\lambda + ([\dot\sigma]^\lambda)^* (\varepsilon + \tau) [\dot\sigma]^\lambda + \\ &+ (\tfrac{d}{dt}([\dot\sigma]^\lambda) - \lambda([\dot\sigma]^\lambda))^* \tau M_2 [\xi]^\lambda + \\ &+ ([\xi]^\lambda)^* M_1 \tau (\tfrac{d}{dt}([\dot\sigma]^\lambda) - \lambda([\dot\sigma]^\lambda)) \} \, dt = \rho^1_{\lambda T'}, \end{aligned} \tag{99}$$

where $T' \in [0, T]$. From (92), (97) and (99) it follows that

$$\rho_{\lambda T} \leq C_{14}, \tag{100}$$

where C_{14} does not depend on T.

The formula (100) is alike the formula (41). So from this we can exploit the argument of Theorem 1. Let us evaluate the functional $\rho_{\lambda T}$:

$$\rho_{\lambda T} \geq \int_0^T \bar{G}_1(\dot{\sigma}, \eta, \dot{\eta})\, dt - |\int_0^T \bar{G}_2(\dot{\sigma}, \eta, \dot{\eta}, g, \dot{g})\, dt|, \tag{101}$$

where

$$\begin{aligned}\bar{G}_1(\dot{\sigma}, \eta, \dot{\eta}) &\triangleq \dot{\sigma}^* \bar{\varkappa}\eta + \eta^* \delta\eta + \dot{\sigma}^* \varepsilon\dot{\sigma} + \\ &+ (\dot{\sigma} - M_1\dot{\eta})^* \tau(\dot{\sigma} - M_2\dot{\eta});\end{aligned} \tag{102}$$

$$\begin{aligned}\bar{G}_2(\dot{\sigma}, \eta, \dot{\eta}, g, \dot{g}) &\triangleq \dot{\sigma}^* \bar{\varkappa} g + 2g^* \delta\eta - \\ &- \dot{g}^* M_1\tau(\dot{\sigma} - M_2\dot{\eta}) - (\dot{\sigma} - M_1\dot{\eta})^* \tau M_2\dot{g}.\end{aligned} \tag{103}$$

Let us consider the second summand in right part of (101). In virtue of (3), (4), (71) we have

$$\int_0^T |\dot{\sigma}^* \bar{\varkappa} g|\, dt < C_{15}, \tag{104}$$

$$\int_0^T |g^* \delta\eta|\, dt < C_{16}, \tag{105}$$

where C_{15} and C_{16} do not depend on T. On the other hand

$$\begin{aligned}&|\int_0^T (\dot{\sigma} - M_1\dot{\eta})^* \tau M_2\dot{g}\, dt| \leq \\ &\leq |\int_0^T (\ddot{\sigma} - M_1\ddot{\eta})^* \tau M_2 g\, dt| + \\ &+ |(\dot{\sigma} - M_1\dot{\eta})^* \tau M_2 g\Big|_0^T|\end{aligned} \tag{106}$$

and

$$\begin{aligned}&|\int_0^T (\dot{\sigma} - M_2\dot{\eta})^* M_1\tau\dot{g}\, dt| \leq \\ &\leq |\int_0^T (\ddot{\sigma} - M_2\ddot{\eta})^* M_1\tau g\, dt| + \\ &+ |(\dot{\sigma} - M_2\dot{\eta})^* M_1\tau g\Big|_0^T|.\end{aligned} \tag{107}$$

The assumptions (3), (70), (71) and (72) imply that the right parts of equalities (106) and (107) are bounded by constants that do not depend on T. So it follows from (100) and (104)– (107) that

$$\rho^0_{\lambda T} \triangleq \int_0^T \bar{G}_1(\dot{\sigma}, \eta, \dot{\eta})\, dt \tag{108}$$

is bounded by a constant which does not depend on T:

$$\rho^0_{\lambda T} \leq C_{17}. \tag{109}$$

No we can use functions (58) and repeat the proof of Theorem 1 replacing ρ^0_T by $\rho^0_{\lambda T}$ and Q_{0j} by $\bar{Q}_j$.
□

Theorem 4. *Suppose there exist positive definite matrices $\varkappa$, δ, ε, matrices M_1 and M_2, and nonnegative matrix τ such that following conditions are fulfilled:*
1) for a certain $\lambda \in [0; \frac{r}{2})$ and all $\omega \geq 0$ the frequency–domain inequality (9) *is true;*
2) $\tau_j m_{1j}^{-1} m_{2j}^{-1} = 0,\ \forall j = 1, 2, \ldots, l$;
3)

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{1j}(\tau_j, \varepsilon_j)|\bar{\varkappa}_j \quad (j = 1, \ldots, l), \tag{110}$$

where $\bar{\varkappa}_j = \varkappa_j + 2\lambda\tau_j(m_{1j}^{-1} + m_{2j}^{-1})$.
Then the conclusion of Theorem 1 is valid.

The proof of Theorem 4 follows from that of Theorem 3 and of Theorem 2.

5 Systems with Lumped Parameters

Let now a pendulum–like system be described by ordinary differential equations:

$$\begin{aligned}&\dot{z}(t) = Az(t) + B(\psi(\sigma(t)) + f(t)),\\ &\dot{\sigma}(t) = C^* z(t) + R(\psi(\sigma(t)) + f(t))\\ &(t > 0).\end{aligned} \tag{111}$$

Here $R \in \mathbb{R}^{l\times l}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$,$C \in \mathbb{R}^{n\times l}$; $z : [0, +\infty) \to \mathbb{R}^n$; functions ψ and f are described in section 2. The solution of (111) is defined by initial conditions

$$z(0) = z^0, \tag{112}$$

$$\sigma(0) = \sigma^0. \tag{113}$$

We assume that A is a Hurwitz matrix and the conditions 3), 5) and 6) of section 2 remain valid.

System (111), (112) can be easily reduced to the system of Volterra equations

$$z(t) = e^{At}z^0 + \int_0^t e^{A(t-\tau)}B(\psi(\sigma(\tau)) + f(\tau))\,d\tau \quad (t > 0), \tag{114}$$

$$\dot\sigma(t) = C^* e^{At}z^0 + R(\psi(\sigma(t)) + f(t)) + \int_0^t C^* e^{A(t-\tau)}B(\psi(\sigma(\tau)) + f(\tau))\,d\tau \quad (t > 0). \tag{115}$$

The equation (115) coincides with the equation (1) with

$$h = 0,\ b(t) = C^* e^{At}z^0,\ \gamma(t) = -C^* e^{At}B. \tag{116}$$

Since A is a Hurwitz matrix the assumptions 1), 2) of section 2 as well as the inclusions (70) are true for $b(t)$ and $\gamma(t)$. So all the four theorems proved in the previous sections can be applied to the equation (114). The frequency–algebraic conditions of either of the theorems guarantee that

$$|\psi(\sigma(t)) + f(t)| \in L_2[0, +\infty), \tag{117}$$

$$|\dot\sigma(t)| \to 0 \text{ as } t \to +\infty. \tag{118}$$

$$\sigma_j \to q_j \text{ as } t \to +\infty \quad (j = 1, 2, \ldots, l), \tag{119}$$

where $\psi_j(q_j) = -L_j$. As $|e^{A(t)}B| \in L_2[0, +\infty)$ it follows from (117) [Gelig, 1966] that

$$|z(t)| \to 0 \text{ as } t \to +\infty. \tag{120}$$

Example. Theorem 1 was applied to the phase–locked loop (PLL) with integrating filter and sine–shaped characteristic of phase detector. For the system $l = 1$, $n = 1$,

$$K(p) = \frac{T}{Tp + 1} \quad (T > 0), \tag{121}$$

$$\psi_1(\zeta) = \sin\zeta - \beta \quad (\beta \in (0, 1)). \tag{122}$$

The mathematical description of this PLL coincides with that of mathematical pendulum

$$\ddot\sigma + a\dot\sigma + (\sin\sigma - \beta) = 0 \quad (a = \frac{1}{T}). \tag{123}$$

By computer simulation, for various coefficients T the values β_T such that the system (121), (122) with $\beta \le \beta_T$ is stable were established. The latter were compared with frontier values of stability domain on the plane $\{T^2, \beta\}$ computed by qualitatively–numerical methods [Belyustina et al., 1970]. It turned out that Theorem 1 guarantees no less than 80% of genuine stability domain.

6 Conclusion

In this paper we consider the infinite–dimensional synchronization system with uncertain disturbances. The system has a denumerable set of stable and unstable equilibria. We offer a number of novel frequency–domain criteria which guarantee that any solution of the system converges to one of equilibria. The techniques used stem from Popov's method of "a priori integral indices". New types of Popov functionals destined for systems with periodic nonlinearities are exploited.

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