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Non-Darcian Flow in Fibre-Reinforced Biological Tissues

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3
4 **Abstract** Under suitable conditions, the motion of a
5 fluid in a porous medium can be studied by assum-
6 ing the validity of Darcy's law. Since many biological
7 tissues can be thought of as porous media, Darcy's
8 law is invoked in several biomechanical contexts, like
9 the transport of the chemical species needed for the
10 metabolism of tissue cells. Although Darcy's law sup-
11 plies physically sound results in many circumstances,
12 there may be cases in which the dynamic behaviour of
13 a biological fluid deviates from the Darcian one. The
14 scope of this work is to analyse some possible conse-
15 quences of such deviations, with emphasis on the fluid
16 velocity and pressure, which, in turn, influence the health
17 and correct functioning of the tissue cells. In particu-
18 lar, our study addresses the flow of an interstitial fluid
19 through a fibre-reinforced tissue, in which the fibres are
20 oriented statistically. We take articular cartilage as a
21 representative tissue of this type, and study the devi-
22 ation from Darcy's law known as "Forchheimer's cor-
23 rection". Moreover, we introduce two models of tissue

permeability, which lead to discrepant results when the
fluid velocity is described by Darcy's law. We show,
however, that the discrepancies in the description of
the flow can be reduced if Forchheimer's correction is
applied.

Keywords Biological tissue · Porous medium ·
Biphasic material · Fibre-reinforcement · Composite
materials · Transverse isotropy · Darcy's law ·
Forchheimer's correction

1 Introduction

Many biological tissues, such as articular cartilage [48,
49, 35, 34, 26], can be described as biphase systems com-
prising a fluid and a solid phase. In the course of its life,
a porous tissue responds to stimuli of various nature,
among which the mechanical ones contribute to vary its
shape and internal structure.

The fluid flowing through a tissue is affected by
the tissue's structural variations, and changes its veloc-
ity and pressure accordingly. Since these changes are
relevant for the tissue's health, it is important to un-
derstand the dynamics of biological fluids. In particu-
lar, the flow of the interstitial fluid in articular car-
tilage is related to the tissue's microstructure, which
changes in time because of the deformation of the non-
fibrous matrix (comprising mainly proteoglycans and
chondrocytes) and the reorientation of the collagen fi-
bres. Thus, the velocity of the fluid must be coupled
with the deformation, and should reflect the evolution
of the medium's anisotropy. At the tissue scale, the mo-
tion of the fluid is described by a velocity field, called
filtration velocity, obtained by eliminating the veloc-
ity fluctuations associated with the pore scale hetero-
geneities of the flow, and multiplying the result by the

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fluid volumetric fraction (see [21, 22] for the definition of mass average of the pore scale velocity). In many cases of biomechanical interest, the fluid filtration velocity is linearly related to the pressure gradient applied to the fluid. Such hypothesis is at the basis of Darcy’s law [20]. Apart from the description of the flow, Darcy’s law is also used to compute the advection velocity by which nutrients and other chemicals are conveyed to the cells [8].

We emphasise that the standard formulation of Darcy’s law stems from the assumption that the macroscopic inertial forces are negligible, the stress borne by the fluid is purely hydrostatic (no viscous stress is accounted for), and the dissipative forces exchanged among the tissue’s constituents are balanced by the pressure gradient, multiplied by the volumetric fraction of the fluid phase [20, 1]. The second-order tensor relating the pressure gradient to the filtration velocity is referred to as *hydraulic conductivity*, or *permeability*. More precisely, in the jargon of Porous Media, the “hydraulic conductivity” is the second-order tensor obtained by dividing the permeability tensor by the fluid viscosity [4], while, in Biomechanics, “(hydraulic) permeability” and “hydraulic conductivity” are usually regarded as synonyms, and are both expressed in $\text{mm}^4\text{N}^{-1}\text{s}^{-1}$ [1]. In the sequel, we follow the terminology used in Biomechanics.

Several anisotropic permeability models have been proposed to couple the fluid filtration velocity with the deformation of the matrix [42, 1]. Also, as deduced in the studies of Maroudas and Bulloch [31], collagen fibres contribute to the permeability of articular cartilage, and exert a resistance to the flow, which adds itself to that supplied by the matrix. This argument has been demonstrated under no or small deformations in [15, 14]. In these works, it is shown that the interstitial fluid flows more easily along the fibres than it does across them. The effect of the fibres on the permeability has also been studied under large deformations [12, 37, 38, 47].

Even though the models discussed so far are purely Darcian, there are cases in which Darcy’s law may cease to describe the flow adequately. A thorough discussion on this subject can be found in [4]. There exist, in fact, two typical situations in which Darcy’s law should be replaced by other descriptions of the flow. One of these occurs when the viscous stress tensor has to be considered in the momentum balance law of the fluid and, in this case, one speaks of Brinkman’s equation [4]. The other one, instead, leads to Forchheimer’s correction of Darcy’s law [4, 5], and arises when the relationship between the fluid filtration velocity and the pressure gradient acquires a quadratic term in the filtration velocity, with a coefficient depending on the mi-

crostructure of the porous medium [5]. We remark that Brinkman’s equation is necessary when boundary effects must be included in the description of the flow, and that Forchheimer’s correction is suggested when the flow is subjected to inertial effects [27]. It should be noticed, however, that these inertial effects are the microscopic ones, since Forchheimer’s correction is derived under the hypothesis of negligible macroscopic inertial forces in the momentum balance law (see the derivation in Section 3). In other words, Forchheimer’s correction is the representation at the tissue scale of microscopic inertial terms that contribute to the drag forces exchanged by the tissue’s constituents [5].

In this work we study Forchheimer’s correction within a nonlinear and anisotropic model of articular cartilage, which is regarded as a hyperelastic, fibre-reinforced tissue, undergoing finite deformations and in which the fibres are oriented statistically. We give ourselves this task for several reasons:

- (i) To improve the understanding of Forchheimer’s correction in the biomechanical context. Indeed, to the best of our knowledge, in Biomechanics Forchheimer’s correction has not been investigated until recently (one paper we are aware of is that of Khaled et al. [27]), whereas it is commonly employed in completely different contexts, like hydrogeology, for problems in which the deformability of the porous medium hosting the flow is usually disregarded.
- (ii) To enrich the description of the flow of biological fluids. Indeed, even though in Biomechanics it is usually believed that Darcy’s law is sufficient to model the flow, there can be situations (for example, in the benchmark tests performed to estimate the elastic and flow properties of articular cartilage) in which particularly severe loading conditions may trigger the microscopic inertial effects that call for Forchheimer’s correction.
- (iii) Since Forchheimer’s correction decreases the magnitude of the fluid filtration velocity and increases the fluid pressure, we use it to be more conservative in establishing the pressure threshold above which the tissue health may be compromised.
- (iv) Forchheimer’s correction introduces coefficients which may be tuned to fit experimental data. We show, indeed, that a partial agreement between two different permeability models, presented in [1] and [12], respectively, each with its own rationale, can be achieved by tuning a coefficient referred to as “trial friction factor” (see Section 5.1).

The work is organised as follows. In Section 2, we present the derivation of the model equations. In Section 3, we review Darcy’s law and Forchheimer’s correc-

tion. In Section 4, we present the constitutive framework. In Section 5, we describe the benchmark tests and the related numerical results. In Section 6, we summarise the main achievements of our work.

2 Biphasic Model of Fibre-Reinforced Hydrated Soft Tissues

Following [15,47], we assume that a Representative Elementary Volume (REV) exists, i.e., we admit that a region of space of constant size can be defined, whose characteristic length scale is sufficiently smaller than that of the tissue's coarse-scale heterogeneities, and sufficiently larger than that of the fine-scale ones [21]. The REV is partitioned into sub-regions, and each sub-region is occupied by one constituent of the tissue. The ratio between the measure of the sub-volume of the REV filled with the interstitial fluid and the measure of the REV is referred to as the fluid phase volumetric fraction, ϕ_f . Under the assumption of saturation, we denote by $\phi_s = 1 - \phi_f$ the volumetric fraction of the solid phase. The portion of REV filled with the solid phase is subdivided into two disjoint sub-regions, occupied by the matrix and the fibres with volumetric fractions ϕ_{0s} and ϕ_{1s} , respectively [47]. It holds that $\phi_{0s} + \phi_{1s} = \phi_s$, where ϕ_{0s} and ϕ_{1s} are expressed per unit volume of the REV.

2.1 Kinematics

As done in [47,17,9], we describe the kinematics of the considered biphasic system by adapting to our problem the theoretical framework developed for solid-fluid mixtures in [40,41]. Thus, two smooth material manifolds, \mathcal{M}_s and \mathcal{M}_f , are introduced, representing the solid and the fluid phase, respectively. The manifold \mathcal{M}_s is embedded into the three-dimensional Euclidean space \mathcal{S} , where it occupies the region $\mathcal{B} \subset \mathcal{S}$, called reference configuration.

Given the interval of time $\mathcal{J} \subset \mathbb{R}$ over which the system's evolution is observed, the motion χ of the solid phase maps \mathcal{B} into the *current configuration* $\chi(\mathcal{B}, t) \subset \mathcal{S}$. While in [40,41] the "points" of the manifolds \mathcal{M}_s are the particles of the solid constituent of a biphasic mixture, in the present framework each particle of \mathcal{M}_s includes both the matrix and the fibres, and both constituents are constrained to share the same motion χ . The motion of the fluid is represented by a one-parameter family of smooth embeddings \mathfrak{f} such that, at each $t \in \mathcal{J}$, the fluid particle $\mathcal{X}_f \in \mathcal{M}_f$ is embedded into the point $x \in \mathcal{S}$. The region $\mathcal{B}_t \subset \mathcal{S}$, at each point of which the solid and fluid particles coexist, is denoted

by $\mathcal{B}_t := \chi(\mathcal{B}, t) \cap \mathfrak{f}(\mathcal{M}_f, t)$. By definition of \mathcal{B}_t it holds that $x = \chi(X, t) = \mathfrak{f}(\mathcal{X}_f, t)$, for all $x \in \mathcal{B}_t$, with $X \in \mathcal{B}$ and $\mathcal{X}_f \in \mathcal{M}_f$.

For $x \in \mathcal{S}$, $T_x\mathcal{S}$ is the tangent space of \mathcal{S} attached at x , and $T\mathcal{S} = \sqcup_{x \in \mathcal{S}} T_x\mathcal{S}$ is the tangent bundle. Their duals are denoted by $T_x^*\mathcal{S}$ and $T^*\mathcal{S}$, respectively. Similarly, we define $T_X\mathcal{B}$ and $T_X^*\mathcal{B}$, with their bundles $T\mathcal{B}$ and $T^*\mathcal{B}$. Moreover, we define the tensor spaces of order $r + s$, where $r \geq 0$ and $s \geq 0$ are arbitrary positive integers [7,12], as

$$[T\mathcal{S}]^r_s = \underbrace{T\mathcal{S} \otimes \dots \otimes T\mathcal{S}}_{r \text{ times}} \otimes \underbrace{T^*\mathcal{S} \otimes \dots \otimes T^*\mathcal{S}}_{s \text{ times}}, \quad (1a)$$

$$[T\mathcal{B}]^r_s = \underbrace{T\mathcal{B} \otimes \dots \otimes T\mathcal{B}}_{r \text{ times}} \otimes \underbrace{T^*\mathcal{B} \otimes \dots \otimes T^*\mathcal{B}}_{s \text{ times}}. \quad (1b)$$

For $x \in \mathcal{B}_t$ and $t \in \mathcal{J}$, we introduce the velocity vectors $\mathbf{v}_s(x, t) \in T_x\mathcal{S}$ and $\mathbf{v}_f(x, t) \in T_x\mathcal{S}$, associated with the solid and fluid phase, respectively, the relative velocity $\mathbf{w} = \mathbf{v}_f - \mathbf{v}_s \in T\mathcal{S}$ and the filtration velocity $\mathbf{q} := \phi_f \mathbf{w} = \phi_f (\mathbf{v}_f - \mathbf{v}_s) \in T\mathcal{S}$ (note that, in several of our past works, we used \mathbf{w} to denote the filtration velocity instead of \mathbf{q}). Using the jargon of Marsden and Hughes [32], we also define the velocity vector fields covering $\chi(\cdot, t)$ and $\mathfrak{f}(\cdot, t)$, respectively, i.e., $\mathbf{u}_s(\cdot, t): \mathcal{B} \rightarrow T\mathcal{S}$ and $\mathbf{u}_f(\cdot, t): \mathcal{M}_f \rightarrow T\mathcal{S}$, which satisfy the equalities $\mathbf{v}_s(x, t) = \mathbf{u}_s(X, t) = \dot{\chi}(X, t)$ and $\mathbf{v}_f(x, t) = \mathbf{u}_f(\mathcal{X}_f, t) = \dot{\mathfrak{f}}(\mathcal{X}_f, t)$, with the superimposed "dot" standing for partial differentiation with respect to time. Finally, we introduce the deformation gradient tensor of the solid phase, \mathbf{F} , i.e., the tangent map $T\chi(\cdot, t) = \mathbf{F}(\cdot, t): T\mathcal{B} \rightarrow T\mathcal{S}$ of the solid phase motion $\chi(\cdot, t)$ [32]. For $X \in \mathcal{B}$ and $x = \chi(X, t)$, $\mathbf{F}(X, t): T_X\mathcal{B} \rightarrow T_x\mathcal{S}$ is a linear map transforming vectors of $T_X\mathcal{B}$ into vectors of $T_x\mathcal{S}$, and can be defined through the directional derivative of $\chi(\cdot, t)$ at $X \in \mathcal{B}$ along some vector $\mathbf{U} \in T_X\mathcal{B}$, i.e., $\partial_{\mathbf{U}}\chi(X, t) = \mathbf{F}(X, t)\mathbf{U} \in T_x\mathcal{S}$. The determinant of \mathbf{F} is denoted by $J = \det(\mathbf{F})$ and is required to be strictly positive in order for χ to be admissible. We emphasise that, since the matrix and the fibres are assumed to share the same motion, χ , they also share the same velocity, \mathbf{v}_s (or \mathbf{u}_s), and the same deformation gradient tensor \mathbf{F} . Along with \mathbf{F} , we also introduce the right and the left Cauchy-Green deformation tensors, denoted by $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{F}^T \mathbf{g} \mathbf{F}$ and $\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{F} \mathbf{G}^{-1} \mathbf{F}^T$, respectively, where $\mathbf{g} \in [T\mathcal{S}]_2^0$ and $\mathbf{G} \in [T\mathcal{B}]_2^0$ are the metric tensors associated with the spatial and material description of the system, respectively.

2.2 Balance Laws and Dissipation Inequality

We introduce the mass balance laws for the three constituents considered in the present model of articular

257 cartilage, i.e., matrix, fibres and fluid phase, with the
258 corresponding mass densities ϱ_{0s} , ϱ_{1s} , and ϱ_f , which we
259 regard here as constant (cf. e.g. [47]). In material for-
260 malism, these balance laws read

$$\dot{\Phi}_{\alpha s} = 0, \quad \alpha \in \{0, 1\}, \quad (2a)$$

$$\dot{J} + \text{Div } \mathbf{Q} = 0. \quad (2b)$$

261 where $\Phi_{\alpha s} := J\phi_{\alpha s}$ is the constant volumetric fraction
262 of the α th solid constituent (i.e., matrix or fibres) in the
263 reference configuration, and $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$ is the *material*
264 *filtration velocity*, i.e., the backward Piola transform of
265 the filtration velocity \mathbf{q} . Note that the material form of
266 the volumetric fraction of the solid phase, $\Phi_s = \Phi_{0s} +$
267 Φ_{1s} , is constant in time too, whereas the material form
268 of the fluid phase volumetric fraction is given by $\Phi_f =$
269 $J - \Phi_s$.

270 Next, we introduce the momentum balance laws,
271 under the hypothesis that inertial and external body
272 forces are negligible, i.e.,

$$\text{div } \boldsymbol{\sigma}_s + \boldsymbol{\pi}_s = \mathbf{0}, \quad (3a)$$

$$\text{div } \boldsymbol{\sigma}_f + \boldsymbol{\pi}_f = \mathbf{0}, \quad (3b)$$

273 where $\boldsymbol{\pi}_s$ and $\boldsymbol{\pi}_f$ represent the force densities due to the
274 momentum exchange between the solid and the fluid
275 constituent, and

$$\boldsymbol{\sigma}_s = -\phi_s p \mathbf{g}^{-1} + \boldsymbol{\sigma}_{sc}, \quad (4a)$$

$$\boldsymbol{\sigma}_f = -\phi_f p \mathbf{g}^{-1} = -(1 - \phi_s)p \mathbf{g}^{-1} \quad (4b)$$

276 are the Cauchy stress tensors associated with the solid
277 and the fluid phase, respectively. In (4a) and (4b), p is a
278 hydrostatic pressure called *pore pressure*, and $\boldsymbol{\sigma}_{sc}$ is the
279 *constitutive part* of $\boldsymbol{\sigma}_s$. Since the system under study is
280 closed with respect to momentum, the condition $\boldsymbol{\pi}_s +$
281 $\boldsymbol{\pi}_f = \mathbf{0}$ has to apply. Hence, by adding together (3a)
282 and (3b), one obtains

$$\text{div } \boldsymbol{\sigma} \equiv \text{div}(-p \mathbf{g}^{-1} + \boldsymbol{\sigma}_{sc}) = \mathbf{0}, \quad (5a)$$

$$-\mathbf{g}^{-1} \text{grad}(\phi_f p) + \boldsymbol{\pi}_f = \mathbf{0}, \quad (5b)$$

283 where $\boldsymbol{\sigma} = \boldsymbol{\sigma}_s + \boldsymbol{\sigma}_f$ is the total Cauchy stress tensor.
284 For the system under study, the local dissipation, \mathfrak{D} ,
285 computed per unit volume of \mathcal{B}_t , is given by

$$\mathfrak{D} = -\boldsymbol{\pi}_{fd} \cdot \frac{\mathbf{q}}{\phi_f} \geq 0, \quad (6)$$

286 where the term $\boldsymbol{\pi}_{fd} := \boldsymbol{\pi}_f - p \mathbf{g}^{-1} \text{grad} \phi_f$, i.e., the dis-
287 sipative force density [20], permits to reformulate (5b)
288 as (see e.g. [20, 6])

$$\boldsymbol{\pi}_{fd} = \phi_f \mathbf{g}^{-1} \text{grad} p. \quad (7)$$

3 Darcy's Law and Forchheimer's Correction 289

290 In its classical formulation, Darcy's law is obtained un-
291 der the hypothesis that $\boldsymbol{\pi}_{fd}$ is determined through a
292 constitutive function, $\hat{\boldsymbol{\pi}}_{fd}$, of the deformation gradient
293 tensor, \mathbf{F} , and the filtration velocity, \mathbf{q} , with linear de-
294 pendence on \mathbf{q} , i.e.,

$$\boldsymbol{\pi}_{fd} = \hat{\boldsymbol{\pi}}_{fd}(\mathbf{F}, \mathbf{q}) = -\mathbf{g}^{-1} \hat{\mathbf{r}}(\mathbf{F}) \mathbf{q}, \quad (8)$$

295 where $\mathbf{r} = \hat{\mathbf{r}}(\mathbf{F}) \in [TS]_2^0$ is the *resistivity tensor*. We
296 assume that \mathbf{r} is symmetric and positive-definite and,
297 thus, also invertible. Hence, by substituting (8) into (7),
298 and solving for \mathbf{q} , we obtain Darcy's law

$$\mathbf{q} \equiv \mathbf{q}_D = -\phi_f \mathbf{r}^{-1} \text{grad} p = -\mathbf{k} \text{grad} p, \quad (9)$$

299 where $\mathbf{k} = \phi_f \mathbf{r}^{-1} \in [TS]_0^2$ is the *permeability tensor*. In
300 this work, we study the case in which $\hat{\boldsymbol{\pi}}_{fd}$ is a quadratic
301 function of the filtration velocity (i.e., (8) no longer
302 applies), but the simplified momentum balance law (7)
303 is still valid. When these conditions apply, one speaks of
304 Forchheimer's correction to Darcy's law [5]. Following
305 [53], we can express the relation between \mathbf{q}_D and the
306 "corrected" filtration velocity, \mathbf{q} , as (with our notation)

$$(\mathbf{i} + \mathcal{F})\mathbf{q} = \mathbf{q}_D, \quad (10)$$

307 where \mathbf{i} is the identity tensor, and the tensor $\mathcal{F} \in [TS]_1^1$
308 is the "*Forchheimer's correction tensor*" [53]. To ob-
309 tain (10), Whitaker studied a porous medium subjected
310 to no deformation, and applied the volume-averaging
311 method to the Navier-Stokes equation modelling the
312 pore scale dynamics of the fluid [53]. In his work, the
313 correction tensor \mathcal{F} was determined by solving auxil-
314 iary "*closure problems*" under the assumption that, at
315 a sufficiently fine scale, the porous medium enjoys the
316 discrete symmetry of spatial periodicity. Moreover, \mathcal{F}
317 was proven to depend linearly on the norm of the filtra-
318 tion velocity, in the limit of sufficiently small Reynolds
319 numbers [53].

320 By adapting the theoretical framework of [5] to our
321 problem, we show that Forchheimer's correction (10)
322 can be deduced from the dissipation inequality (6). To
323 accomplish our task, we suppose that the theoretical
324 framework deduced in [53] for non deformable porous
325 media can describe also those tissues undergoing (finite)
326 deformations, even though such deformations can com-
327 promise the periodicity of the internal structure. Thus,
328 by relaxing the hypothesis of periodic internal struc-
329 ture, we postulate that the dissipative force $\boldsymbol{\pi}_{fd}$ takes
330 on the form

$$\boldsymbol{\pi}_{fd} = -[\mathbf{i} + \|\mathbf{q}\| \mathcal{A}] \mathbf{g}^{-1} \mathbf{r} \mathbf{q}, \quad (11)$$

where we have set $\mathcal{F} := \|\mathbf{q}\|\mathcal{A}$, and refer to $\mathcal{A} \in [TS]_1^1$ as to the *tensorial Forchheimer coefficient*. By substituting (11) into (7), multiplying both sides of the resulting expression by the inverse of the resistivity tensor, \mathbf{r}^{-1} , and invoking the definition of the permeability tensor, $\mathbf{k} = \phi_f \mathbf{r}^{-1}$, we obtain

$$[\mathbf{i} + \|\mathbf{q}\|\mathbf{k}\mathfrak{A}\mathbf{k}^{-1}]\mathbf{q} = \mathbf{q}_D, \quad (12)$$

where $\mathfrak{A} := \mathbf{g}\mathcal{A}\mathbf{g}^{-1}$ is the counterpart of \mathcal{A} in the tensor space $[TS]_1^1$. Our result (12) is consistent with similar results found in the literature (cf. e.g. [52], in which the case of an anisotropic porous medium is considered).

To express the Forchheimer coefficient \mathfrak{A} , we introduce the ‘‘associated’’ permeability tensor $\boldsymbol{\kappa} = \mathbf{g}\mathbf{k} \in [TS]_1^1$ and we assume $\mathfrak{A} := \varrho_f \boldsymbol{\kappa} \boldsymbol{\beta}$, where $\boldsymbol{\beta} \in [TS]_1^1$ is called *non-Darcy coefficient tensor* [52], and is defined according to the empirical law $\boldsymbol{\beta} = c_0 \phi_f^{c_1} \mu^{c_2} \boldsymbol{\kappa}^{c_2}$ adapted from [46], in which μ is the viscosity of the fluid, and $c_0 \geq 0$, c_1 , and c_2 are real constants. In the jargon of Thauvin and Mohanty [46], formulae of this type are said to be ‘‘correlations’’, since they express the non-Darcy coefficient in terms of other relevant parameters pertaining to the flow as well as the structure of the considered porous medium. Since c_2 is a real number, the power law $\boldsymbol{\kappa}^{c_2}$ is conveniently written in spectral form as

$$\boldsymbol{\kappa}^{c_2} = \sum_{\alpha=1}^3 (k_\alpha)^{c_2} \mathbf{n}^\alpha \otimes \mathbf{n}_\alpha, \quad (13)$$

where k_α is the α th eigenvalue of the permeability tensor, $\mathbf{n}^\alpha \in T^*\mathcal{S} = [TS]_1^0$ is its corresponding eigenvector (determined by $[\boldsymbol{\kappa} - k_\alpha \mathbf{i}^T]\mathbf{n}^\alpha = \mathbf{0}$), and $\mathbf{n}_\alpha = \mathbf{g}^{-1}\mathbf{n}^\alpha$ is the associated eigenvector. By employing (13), the Forchheimer coefficient \mathfrak{A} acquires the expression

$$\begin{aligned} \mathfrak{A} &:= \varrho_f \boldsymbol{\kappa} \boldsymbol{\beta} = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \boldsymbol{\kappa} \boldsymbol{\kappa}^{c_2} \\ &= c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \sum_{\alpha=1}^3 (k_\alpha)^{1+c_2} \mathbf{n}^\alpha \otimes \mathbf{n}_\alpha. \end{aligned} \quad (14)$$

According to (14), the tensors $\boldsymbol{\kappa}$ and $\boldsymbol{\beta}$ are coaxial, and thus commute, i.e., it holds that $\mathfrak{A} = \varrho_f \boldsymbol{\kappa} \boldsymbol{\beta} = \varrho_f \boldsymbol{\beta} \boldsymbol{\kappa}$. This implies

$$\begin{aligned} \mathbf{k}\mathfrak{A}\mathbf{k}^{-1} &= \mathbf{k}(\varrho_f \boldsymbol{\kappa} \boldsymbol{\beta})\mathbf{k}^{-1} = \mathbf{k}(\varrho_f \boldsymbol{\beta} \boldsymbol{\kappa})\mathbf{k}^{-1} \\ &= \mathbf{g}^{-1}(\varrho_f \boldsymbol{\kappa} \boldsymbol{\beta})\mathbf{g} = \mathbf{g}^{-1}\mathfrak{A}\mathbf{g} = \mathcal{A}, \end{aligned} \quad (15)$$

and, consequently, the relation (12) becomes

$$[\mathbf{i} + \|\mathbf{q}\|\mathcal{A}]\mathbf{q} = \mathbf{q}_D. \quad (16)$$

Finally, with the aid of (14), the identity $\mathcal{A} = \mathbf{g}^{-1}\mathfrak{A}\mathbf{g}$ leads to

$$\mathcal{A} = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \sum_{\alpha=1}^3 (k_\alpha)^{1+c_2} \mathbf{n}_\alpha \otimes \mathbf{n}^\alpha. \quad (17)$$

We remark that introducing the Forchheimer coefficient into (11) is equivalent to defining an effective resistivity tensor, $\mathbf{r}_F := \mathbf{r} + \|\mathbf{q}\|\mathfrak{A}\mathbf{r}$. Hence, $\boldsymbol{\pi}_{fd}$ admits the expression $\boldsymbol{\pi}_{fd} = -\mathbf{g}^{-1}\mathbf{r}_F\mathbf{q}$, which is formally similar to (8), but accounts for Forchheimer’s correction. Moreover, computing explicitly \mathbf{r}_F , with $\mathbf{r} = \phi_f \mathbf{k}^{-1}$ and \mathfrak{A} given in (14), yields

$$\begin{aligned} \mathbf{r}_F &= \phi_f \mathbf{k}^{-1} \\ &+ c_0 \|\mathbf{q}\| \varrho_f \phi_f^{1+c_1} \mu^{c_2} \left(\sum_{\alpha=1}^3 (k_\alpha)^{c_2} \mathbf{n}^\alpha \otimes \mathbf{n}^\alpha \right). \end{aligned} \quad (18)$$

Since the hypothesis of positive-definiteness of \mathbf{k} implies that \mathbf{r}_F is positive-definite too, the dissipation inequality is respected, and can be written in compact form as $\mathcal{D} = \phi_f^{-1} \mathbf{r}_F : (\mathbf{q} \otimes \mathbf{q}) \geq 0$.

Before going further, we emphasise that the tensorial Forchheimer coefficient \mathcal{A} written in (17) stems from the empirical laws expressing \mathfrak{A} and $\boldsymbol{\beta}$, in which the coefficients c_0 , c_1 , and c_2 are to be determined experimentally. Thauvin and Mohanty [46] studied non-deforming isotropic porous media, for which it holds that $\mathbf{k} = k_0 \mathbf{g}^{-1}$, and the non-Darcy coefficient tensor is represented by the scalar quantity $\beta = c_0 \phi_f^{c_1} \mu^{c_2} k_0^{c_2}$. Moreover, they found several expressions for β , each corresponding to a set of scalars $\{c_0, c_1, c_2\}$, obtained for different pore structures and system sizes. Some of the correlations considered in [46] were assumed to depend also on the (scalar) tortuosity of the porous medium. On the contrary, since we are not aware of expressions of $\boldsymbol{\beta}$ explicitly determined for articular cartilage, in the present work we consider $\boldsymbol{\beta}$, by choosing c_0 , c_1 , and c_2 from the literature, with a certain amount of freedom ascribable to the lack of experimental data. The tortuosity is not taken into account in the realisations of the non-Darcy coefficient tensor $\boldsymbol{\beta}$ considered here, since, to the best of our knowledge, there is no experimental evidence of such parameter in articular cartilage.

4 Materials Reinforced by Statistically Oriented Fibres

Following the line of thought and notation in [12], the porous fibre-reinforced composite material studied in this work is assumed to have a statistical distribution of fibres, described by the probability distribution function $\Psi : \mathbb{S}^2\mathcal{B} \rightarrow \mathbb{R}_0^+$, where $\mathbb{S}^2\mathcal{B}$ is the collection of all vectors $\mathbf{M}_X \in T_X\mathcal{B}$, with X varying in \mathcal{B} , such that $\|\mathbf{M}_X\| = 1$. The value $\Psi(\mathbf{M}_X)$ represents the probability density that, at a point X , a fibre is locally aligned along \mathbf{M}_X . The probability density satisfies the normalisation condition $\int_{\mathbb{S}^2\mathcal{B}} \Psi(\mathbf{M}) = 1$ and, since in

412 this work we restrict our attention to phenomena that
 413 involve only the direction of the fibres, but not their
 414 sense, we require Ψ to fulfil also the symmetry condi-
 415 tion $\Psi(-\mathbf{M}) = \Psi(\mathbf{M})$. We also introduce the notation

$$\langle\langle \mathfrak{F} \rangle\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \Psi(\mathbf{M}) \mathfrak{F}(\mathbf{M}), \quad (19)$$

416 denoting the *directional average* of the quantity \mathfrak{F} with
 417 respect to the probability density Ψ .

418 The composite is assumed to exhibit hyperelastic
 419 behaviour from the reference configuration \mathcal{B} , and its
 420 elastic potential is constructed by superimposing the
 421 elastic contribution of the matrix to that of the fibres,
 422 i.e.,

$$\hat{W}(\mathbf{C}) = \Phi_s \hat{U}(J) + \Phi_{0s} \hat{W}_0(\mathbf{C}) + \Phi_{1s} \hat{W}_e(\mathbf{C}), \quad (20)$$

423 with $J \equiv J(\mathbf{C}) = \sqrt{\det(\mathbf{C})}$

$$\hat{U}(J) = \mathcal{H}(J_{\text{cr}} - J)(J - J_{\text{cr}})^{2q}(J - \Phi_s)^{-r}, \quad (21a)$$

$$\hat{W}_0(\mathbf{C}) = \alpha_0 \frac{\exp(\alpha_1[I_1 - 3] + \alpha_2[I_2 - 3])}{[I_3]^{\alpha_3}}, \quad (21b)$$

$$\hat{W}_e(\mathbf{C}) = \hat{W}_{1i}(\mathbf{C}) + \langle\langle \hat{W}_{1a}(\mathbf{C}, \mathbf{A}) \rangle\rangle, \quad (21c)$$

$$\hat{W}_{1a}(\mathbf{C}, \mathbf{A}) = \mathcal{H}(I_4 - 1)^{\frac{1}{2}} c [I_4 - 1]^2, \quad (21d)$$

424 where, for brevity, we used I_1, I_2, I_3 for the three prin-
 425 cipal invariants $I_1(\mathbf{C}) = \text{tr}(\mathbf{C})$, $I_2(\mathbf{C}) = \frac{1}{2}\{\text{tr}(\mathbf{C})^2 -$
 426 $\text{tr}(\mathbf{C}^2)\}$, $I_3(\mathbf{C}) = \det(\mathbf{C})$ of the right Cauchy-Green
 427 deformation tensor \mathbf{C} , and I_4 for the fourth invariant
 428 $I_4(\mathbf{C}, \mathbf{A}) = \mathbf{C} : \mathbf{A}$ of \mathbf{C} [44], in which $\mathbf{A} = \mathbf{M} \otimes \mathbf{M}$
 429 is the structure tensor field.

430 The contribution $\hat{U}(J)$ is a penalty term depend-
 431 ing solely on J , and accounting for the fact that, after
 432 the fluid has flown away and the pores are closed, the
 433 tissue behaves as an incompressible material. In (21a),
 434 \mathcal{H} is the Heaviside function, $J_{\text{cr}} \in]\Phi_s, 1]$, $q \geq 2$ and
 435 $r \in]0, 1]$. The quantity J_{cr} specifies a ‘‘critical value’’
 436 of the volume ratio J , below which the penalty term is
 437 active. When this occurs, $\hat{U}(J)$ diverges for $J \rightarrow \Phi_s^+$,
 438 thereby preventing the violation of the unilateral con-
 439 straint $J \geq \Phi_s$. A representation of $\hat{U}(J)$ is in Fig.
 440 1. The elastic potential $\hat{W}_0(\mathbf{C})$ in (21b) describes the
 441 hyperelastic response of the matrix alone, which is as-
 442 sumed to be isotropic. The constitutive expression of
 443 $\hat{W}_0(\mathbf{C})$ is taken from [26], where the coefficients $\alpha_0, \alpha_1,$
 444 α_2 , and α_3 are model parameters. In (21c) and (21d),
 445 $\hat{W}_{1i}(\mathbf{C})$ denotes the isotropic part of the fibre elastic po-
 446 tential, $\hat{W}_{1a}(\mathbf{C}, \mathbf{A})$ denotes the anisotropic elastic po-
 447 tential depending on the local direction of fibre align-
 448 ment, c is a material parameter, and the Heaviside step
 449 function is introduced to eliminate the contribution of
 450 the fibres that are not stretched (i.e., those for which
 451 $I_4 \leq 1$).

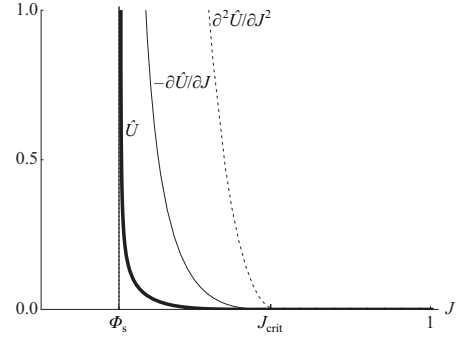


Fig. 1 Graphical representation of the penalty term $\hat{U}(J)$. The vertical asymptote at $J = \Phi_s$ expresses that $\hat{U}(J)$ diverges at compaction, i.e., when the lower bound $J = \Phi_s$ of the admissible values of J is approached. The penalty term is active only for $J < J_{\text{crit}}$, and is zero otherwise. Graphics adapted from [12].

The anisotropic part of $\hat{W}(\mathbf{C})$ generates the anisotropic
 contribution to σ_{sc} given by

$$\hat{\sigma}_a(\mathbf{F}) = \frac{2\Phi_{1s}}{J} \langle\langle \mathcal{H}(I_4 - 1)c[I_4 - 1]\mathbf{F}\mathbf{A}\mathbf{F}^T \rangle\rangle, \quad (22)$$

452 where again, for brevity, we used I_4 for $I_4(\mathbf{C}, \mathbf{A})$. To
 453 study the material symmetries satisfied by the consti-
 454 tutive tensor function $\hat{\sigma}_a(\mathbf{F})$, we choose arbitrarily $X \in$
 455 \mathcal{B} , and consider the group of all proper rotations about
 456 a material unit vector \mathbf{M} attached at X , i.e.,

$$\mathcal{G}_X(\mathbf{M}) := \{\mathbf{H} \in \text{Orth}^+ : \mathbf{H}\mathbf{M} = \pm\mathbf{M}\}. \quad (23)$$

457 Hence, we notice that the integrand of (22) is a trans-
 458 versely isotropic tensor function with respect to \mathbf{M} be-
 459 cause, for all $\mathbf{H} \in \mathcal{G}_X(\mathbf{M})$, the structure tensor fulfils
 460 the equality $\mathbf{A} = \mathbf{H}\mathbf{A}\mathbf{H}^T$, I_4 is (by definition) invari-
 461 ant under the transformation $\mathbf{F} \mapsto \tilde{\mathbf{F}} = \mathbf{F}\mathbf{H}$, and so is
 462 also the tensor $\mathbf{F}\mathbf{A}\mathbf{F}^T$, i.e.,

$$\mathbf{F}\mathbf{A}\mathbf{F}^T \mapsto \tilde{\mathbf{F}}\tilde{\mathbf{A}}\tilde{\mathbf{F}}^T = \mathbf{F}\mathbf{H}\mathbf{A}\mathbf{H}^T\mathbf{F}^T = \mathbf{F}\mathbf{A}\mathbf{F}^T. \quad (24)$$

463 If there exists a polar axis \mathbf{M}_0 , such that the probabil-
 464 ity density Ψ is restricted by the symmetry condition

$$\Psi(\mathbf{H}\mathbf{M}) = \Psi(\mathbf{M}), \quad (25)$$

465 for every $\mathbf{M} \in \mathbb{S}^2 \mathcal{B}$ and for every $\mathbf{H} \in \text{Orth}^+$ such
 466 that $\mathbf{H}\mathbf{M}_0 = \pm\mathbf{M}_0$, then Ψ is said to be transversely
 467 isotropic with respect to \mathbf{M}_0 and, because of the in-
 468 tegration over all possible directions performed in (22),
 469 $\hat{\sigma}_a(\mathbf{F})$ is invariant under arbitrary rotations about \mathbf{M}_0 .

470 Permeability is the material property describing the
 471 ability of a fluid to flow through the pore space of a
 472 porous medium. In this work, we focus on two per-
 473 meability models that have been recently conceived
 474 to study the coupling between fluid flow and deformation
 475 in anisotropic porous media undergoing finite deforma-
 476 tions. These models were presented in [1] and [12], and

are hereafter referred to as the ‘‘AW-model’’ and ‘‘FG-model’’, respectively.

The FG-model [12] extends the results obtained in [15] to the framework of finite deformations, and was employed in [47] to investigate the influence of the fibres’ orientation on the permeability of articular cartilage. In the FG-model, the permeability \mathbf{k} is the result of an upscaling method. More precisely, in [15], at each spatial point $x \in \mathcal{B}_t \subset \mathcal{S}$, a (rectified) fibre is assumed to be locally aligned along the unit vector $\mathbf{m} \in T_x\mathcal{S}$. Then, a REV is attached at x and its size is chosen in such a way that it comprehends only the fibre passing from x and the portion of matrix in which the fibre is embedded. Hence, the permeability of the REV, \mathbf{k}_{REV} , is determined by enforcing a self-consistent method [39] (see [15] for details) under the hypothesis of validity of Darcy’s law at the REV scale and in the limit of vanishing fibre permeability and small fibre volumetric fraction. The result obtained within this approach is then generalised to the case of arbitrary fibre volumetric fractions by adopting differential schemes for composite materials [33,36], and supposing that the permeability of the matrix has a spherical representation. Thus, the REV permeability determined in [15] reads

$$\mathbf{k}_{\text{REV}} = k_0[1 - \phi_{1s}]^2 \mathbf{g}^{-1} + k_0[1 - \phi_{1s}]\phi_{1s}\mathbf{a}, \quad (26)$$

where

$$\begin{aligned} \mathbf{a} &= \mathbf{m} \otimes \mathbf{m} = \frac{\mathbf{FM}}{\|\mathbf{FM}\|} \otimes \frac{\mathbf{FM}}{\|\mathbf{FM}\|} \\ &= \frac{1}{I_4(\mathbf{C}, \mathbf{A})} \mathbf{FAF}^T, \end{aligned} \quad (27)$$

and $\mathbf{m} = \mathbf{FM}/\|\mathbf{FM}\|$. We remark that the contributions of the fibre to the permeability of the REV manifest themselves exclusively through the spatial structure tensor \mathbf{a} and the fibre volumetric fraction ϕ_{1s} . Following the constitutive framework of Holmes and Mow [26], the scalar permeability k_0 is expressed as a constitutive function of the deformation through the volume ratio J , i.e.,

$$\begin{aligned} k_0 &:= \hat{k}_0(J) \\ &= k_{0R} \left[\frac{J - \Phi_s}{1 - \Phi_s} \right]^{\kappa_0} \exp\left(\frac{1}{2}m_0[J^2 - 1]\right), \end{aligned} \quad (28)$$

where κ_0 and m_0 are material parameters, k_{0R} is the scalar permeability in the undeformed configuration, and the condition $\lim_{J \rightarrow \Phi_s} \hat{k}(J) = 0$ is respected, since the permeability has to vanish at compaction. By computing \mathbf{k}_{REV} at $X \in \mathcal{B}$, and considering the group $\mathcal{G}_X(\mathcal{M})$ defined in (23), it holds that $\hat{\mathbf{k}}_{\text{REV}}(\mathbf{FH}, \mathbf{A}) = \hat{\mathbf{k}}_{\text{REV}}(\mathbf{F}, \mathbf{A})$, for all $\mathbf{H} \in \mathcal{G}_X(\mathcal{M})$. Thus, the REV permeability is transversely isotropic with respect to \mathcal{M} .

By exploiting (26), the FG-model obtains the spatial permeability, \mathbf{k} , by integrating \mathbf{k}_{REV} over all possible directions, which results in a constitutive function of the deformation gradient alone [12,47], i.e.,

$$\begin{aligned} \mathbf{k}_{\text{FG}} &= \hat{\mathbf{k}}_{\text{FG}}(\mathbf{F}) = \langle\langle \hat{\mathbf{k}}_{\text{REV}}(\mathbf{F}, \mathbf{A}) \rangle\rangle \\ &= J^{-2} \hat{k}_0(J) [J - \Phi_{1s}]^2 \mathbf{g}^{-1} \\ &\quad + J^{-2} \hat{k}_0(J) [J - \Phi_{1s}] \Phi_{1s} \mathbf{F} \hat{\mathbf{Z}}(\mathbf{C}(\mathbf{F})) \mathbf{F}^T, \end{aligned} \quad (29)$$

where \mathbf{C} in (29) is understood as a function of \mathbf{F} , and we set $\mathbf{Z} = \langle\langle \frac{\mathbf{A}}{I_4(\mathbf{C}, \mathbf{A})} \rangle\rangle$.

The backward Piola transformation of (29), i.e., $\mathbf{K}_{\text{FG}} = J\mathbf{F}^{-1}\mathbf{k}_{\text{FG}}\mathbf{F}^{-T}$, produces the material permeability of the FG-model:

$$\begin{aligned} \mathbf{K}_{\text{FG}} &= \hat{\mathbf{K}}_{\text{FG}}(\mathbf{C}) = \frac{\hat{k}_0(J) [J - \Phi_{1s}]^2}{J} \mathbf{C}^{-1} \\ &\quad + \frac{\hat{k}_0(J) [J - \Phi_{1s}] \Phi_{1s}}{J} \hat{\mathbf{Z}}(\mathbf{C}). \end{aligned} \quad (30)$$

The AW-model considers several classes of material symmetries and it supplies for each of those the corresponding permeability tensor. To this purpose, it employs the Representation Theorems for functions valued in the space of symmetric second-order tensors [43, 29]. In the case of transverse isotropy with respect to a direction $\mathbf{M} \in T\mathcal{B}$, the AW-model defines the spatial permeability tensor as

$$\begin{aligned} \mathbf{k}_{\text{AW}} &= k_{0i} \mathbf{g}^{-1} + k_{1t} \mathbf{b} + 2k_{2t} \mathbf{b}^2 \\ &\quad + [k_{1a} - k_{1t}] \mathbf{a} + 2[k_{2a} - k_{2t}] \text{sym}(\mathbf{a}, \mathbf{b}), \end{aligned} \quad (31)$$

where the coefficients k_{0i} , k_{1a} , k_{1t} , k_{2a} , and k_{2t} are, in general, functions of the invariants I_1 , I_2 , I_3 , I_4 , and $I_5 = \mathbf{C}^2 : \mathbf{A}$ [44]. We remark that, while in this work \mathbf{a} is defined by (27), the spatial structure tensor used in [1] is given by \mathbf{FAF}^T and is, thus, *not* normalised. Therefore, the coefficients k_{1a} , k_{1t} , k_{2a} , and k_{2t} appearing in (31) must be adjusted accordingly in order for (31) to be consistent with the expression provided in [1].

By comparing (26) and (31), one can see that \mathbf{k}_{REV} is retrieved from \mathbf{k}_{AW} in the limit of vanishing k_{1t} , k_{2t} and k_{2a} , and provided that the identifications

$$k_{0i} := \hat{k}_{0i}(J) \equiv J^{-2} \hat{k}_0(J) [J - \Phi_{1s}]^2, \quad (32a)$$

$$k_{1a} := \hat{k}_{1a}(J) \equiv J^{-2} \hat{k}_0(J) [J - \Phi_{1s}] \Phi_{1s} \quad (32b)$$

are made. In fact, whereas neglecting k_{1t} and k_{2t} can be physically motivated by the observation that the permeability along the fibres is much higher than that across the fibres [47], the absence of a coefficient of the type k_{2a} in the expression of \mathbf{k}_{REV} descends from the chosen upscaling criterion. We regard this feature as a weak point of the FG-model.

From here on, we consider only a “reduced” and slightly modified version of the AW-model, obtained by setting k_{1t} , k_{2t} , and k_{2a} equal to zero, and choosing $k_{0i} = \hat{k}_{0i}(J) \equiv \hat{k}_0(J)$, and $k_{1a} := \hat{k}_{1a}(J) \equiv J^{-2}\hat{k}_0(J)$ (cf. the form of k_{1a} with Equation (39) of [1]). Then, we write the statistical average of the material permeability of the AW-model, $\mathbf{K}_{\text{AW}} = J\mathbf{F}^{-1}\mathbf{k}_{\text{AW}}\mathbf{F}^{-\text{T}}$, as

$$\langle\langle \mathbf{K}_{\text{AW}} \rangle\rangle = J\hat{k}_0(J)\mathbf{C}^{-1} + J^{-1}\hat{k}_0(J)\hat{\mathbf{Z}}(\mathbf{C}). \quad (33)$$

The difference between (33) and the permeability that would be obtained by adopting the original model by Ateshian and Weiss [1] is due to the division by I_4 in the definition of the spatial structure tensor \mathbf{a} (cf. (27)). Indeed, if the spatial structure tensor were not normalised, as is the case in [1], the second term on the right-hand-side of (33) would read $J^{-1}\hat{k}_0(J)\langle\langle \mathbf{A} \rangle\rangle$.

Notice that, while the FG-model predicts that \mathbf{k}_{REV} depends explicitly both on the fibre’s volumetric fraction and on the fibre’s orientation, \mathbf{k}_{AW} depends on the direction of transverse isotropy, but may be independent on ϕ_{1s} . The dependence of \mathbf{k}_{AW} on ϕ_{1s} , however, can be accounted for by extending the constitutive framework.

5 Benchmark Tests

The model described in the previous sections requires to determine the unknowns $\mathcal{U} = \{\chi, p, \mathbf{q} \text{ or } \mathbf{Q}\}$ through the solution of the equations

$$\text{Div} \left(-Jp\mathbf{g}^{-1}\mathbf{F}^{-\text{T}} + \mathbf{P}_{\text{sc}} \right) = \mathbf{0}, \quad (34a)$$

$$\dot{J} + \text{Div} \mathbf{Q} = 0, \quad (34b)$$

$$(\mathbf{I} + \|\mathbf{q}\|\mathbf{F}^{-1}\mathbf{A}\mathbf{F})\mathbf{Q} = \mathbf{Q}_{\text{D}}, \quad (34c)$$

in which (34c) is the material form of Forchheimer’s correction. In (34a)–(34c), \mathbf{I} is the identity tensor in $[\mathcal{T}\mathcal{B}]^1_1$, $\mathbf{Q}_{\text{D}} = -\mathbf{K} \text{Grad} p$ is the material form of Darcy’s law, where \mathbf{K} is given either by (30) or by (33), depending on whether the FG- or the AW-model is used.

Equations (34a)–(34c) must be equipped with the initial and boundary conditions specifying the type of benchmark problem that has to be solved. To this end, we assume that \mathcal{B} coincides with the configuration of a cylindrical sample at time $t_0 = 0$, regarded as undeformed and unloaded, and we partition the boundary of \mathcal{B} as $\partial\mathcal{B} = \Gamma_{\text{L}} \cup \Gamma_{\text{U}} \cup \Gamma_{\text{B}}$, where Γ_{L} , Γ_{U} , and Γ_{B} represent the lower, upper, and lateral surfaces of $\partial\mathcal{B}$, respectively. As benchmark problems, we consider two unconfined compression tests. In both tests, a cylindrical sample of height $L = 1$ mm and circular cross-section of diameter $D = 3$ mm is inserted between two parallel, impermeable and rigid plates, and compressed along

the direction \mathbf{M}_0 of its geometrical axis. For this purpose, a loading history is imposed to the upper plate, while the lower plate is kept fixed. The two plates remain parallel to each other over the entire duration of the tests. In the following, we consider the Cartesian orthonormal vector bases $\{\mathcal{E}_I\}_{I=1}^3$ and $\{\varepsilon_i\}_{i=1}^3$, associated with \mathcal{B} and \mathcal{S} , respectively. We assume $\{\mathcal{E}_I\}_{I=1}^3$ and $\{\varepsilon_i\}_{i=1}^3$ to be collinear and choose \mathcal{E}_3 coincident with \mathbf{M}_0 .

In the first test, Γ_{L} is clamped at the lower plate. Thus, the original cylindrical shape of the sample is lost during deformation, although each cross section maintains the polar symmetry with respect to the axis $\mathbf{M}_0 \equiv \mathcal{E}_3$. Accordingly, for all times $t \in]t_0, T_{\text{end}}]$, the boundary conditions read

$$\text{On } \Gamma_{\text{U}}, \quad \begin{cases} \chi^3 = \mathbf{g}, \\ (-\mathbf{K} \text{Grad} p) \cdot \mathbf{N} = 0, \end{cases} \quad (35a)$$

$$\text{On } \Gamma_{\text{B}}, \quad \begin{cases} (-Jp\mathbf{g}^{-1}\mathbf{F}^{-\text{T}} + \mathbf{P}_{\text{sc}}) \cdot \mathbf{N} = \mathbf{0}, \\ p = 0, \end{cases} \quad (35b)$$

$$\text{On } \Gamma_{\text{L}}, \quad \begin{cases} \chi(X, t) - \chi(X, 0) = \mathbf{0}, \\ (-\mathbf{K} \text{Grad} p) \cdot \mathbf{N} = 0, \end{cases} \quad (35c)$$

where \mathbf{N} is the unit vector normal to the surface of the sample, and \mathbf{g} is the loading history

$$\mathbf{g}(t) = \begin{cases} L - \frac{t}{T_{\text{ramp}}}u_{\text{T}}, & \text{for } t \in [0, T_{\text{ramp}}], \\ L - u_{\text{T}}, & \text{for } t \in]T_{\text{ramp}}, T_{\text{end}}]. \end{cases} \quad (36)$$

Here, $u_{\text{T}} = 0.2$ mm is the target displacement and $T_{\text{ramp}} = 20$ s is the final instant of time of the loading ramp. The load (36) is kept up to $T_{\text{end}} = 50$ s.

In the second test, which we call “cylindrical unconfined compression test”, we assume that the cylindrical shape of the sample is preserved by requiring that Γ_{U} and Γ_{L} glide frictionlessly on the plates’ surfaces in axial-symmetric way and that Γ_{B} is a free boundary, although the sample is inhomogeneous [cf. (38a)–(40)]. In fact, the inhomogeneity of the sample in the direction $\mathbf{M}_0 \equiv \mathcal{E}_3$ causes the axial strain and radial displacement to be non-constant with the space variable X^3 . Still, even when we consider an inhomogeneous sample [see Eqs. (38a)–(40)], in our simulations the deformed configurations of the sample deviates only slightly from the cylindrical shape (data not shown). Thus, for the purposes of this work, and in particular for the results reported in section 5.2, we approximate the sample’s deformation with a deformation preserving the cylindrical shape. In this case, the conditions (35a) and (35b) as well as the no-flux condition on Γ_{L} still apply, while the null displacement condition on Γ_{L} has to be replaced by the condition $\chi^3(X, t) = 0$, for all $X \in \Gamma_{\text{L}}$ and for all $t \in]t_0, T_{\text{end}}]$.

640 With respect to the orthonormal vector basis $\{\mathbf{E}_I\}_{I=1}^3$,
 641 \mathbf{M} is written as

$$\begin{aligned} \mathbf{M} &= \hat{\mathbf{M}}(\Theta, \Phi) \\ &= \sin \Theta \cos \Phi \mathbf{E}_1 + \sin \Theta \sin \Phi \mathbf{E}_2 + \cos \Theta \mathbf{E}_3, \end{aligned} \quad (37)$$

642 where $\Theta \in [0, \pi]$ is the co-latitude and $\Phi \in [0, 2\pi]$ is the
 643 longitude, and the transverse isotropy of the probability
 644 density Ψ means that there exists a function $\wp : [0, \pi] \rightarrow$
 645 \mathbb{R}_0^+ such that the conditions $\Psi(\mathbf{M}) = \Psi(\hat{\mathbf{M}}(\Theta, \Phi)) =$
 646 $\wp(\Theta)$ is verified for all $\Phi \in [0, 2\pi]$. The function \wp must
 647 comply with the normalisation condition and with the
 648 symmetry condition $\wp(\Theta) = \wp(\pi - \Theta)$, for all $\Theta \in$
 649 $[0, \pi]$, which corresponds to $\Psi(\mathbf{M}) = \Psi(-\mathbf{M})$. More-
 650 over, since in this work we compute the statistical aver-
 651 ages of functions that depend on direction only through
 652 the structure tensor, we are allowed to restrict the aver-
 653 aging integrals to one hemisphere only (e.g. the “north-
 654 ern” hemisphere $\mathbb{S}^{2+}\mathcal{B}$). Hence, we introduce the prob-
 655 ability density $\bar{\wp} : [0, \pi/2] \rightarrow \mathbb{R}_0^+$ such that the normal-
 656 isation reads $2\pi \int_0^{\pi/2} \bar{\wp}(\Theta) \sin(\Theta) d\Theta = 1$. In this work,
 657 we use the pseudo-Gaussian distribution [13]

$$\bar{\wp}(\Theta, \xi) = \frac{\mathbf{p}(\Theta, \xi)}{2\pi \int_0^{\pi/2} \mathbf{p}(\Theta', \xi) \sin(\Theta') d\Theta'}, \quad (38a)$$

$$\mathbf{p}(\Theta, \xi) = \exp\left(-\frac{[\Theta - Q(\xi)]^2}{2[\omega(\xi)]^2}\right), \quad (38b)$$

658 where both the mean angle $Q(\xi)$ and the standard devi-
 659 ation $\omega(\xi)$ depend on the normalised axial coordinate
 660 $\xi = X^3/L$, and are given by [13]

$$Q(\xi) = \frac{\pi}{2} \left[1 - \cos\left(\left(-\frac{2}{3}\xi^2 + \frac{5}{3}\xi\right)\frac{\pi}{2}\right) \right], \quad (39a)$$

$$\omega(\xi) = 10^3[(1 - \xi)\xi]^4 + 0.03. \quad (39b)$$

661 A graphical representation of the functions $\mathbf{p}(\Theta, \xi)$,
 662 $Q(\xi)$, and $\omega(\xi)$, defined in (38b), (39a), and (39b), re-
 663 spectively, is reported in Fig. 2. The angle $Q(\xi)$ ranges
 664 continuously from $Q(0) = 0$ rad at the lower boundary
 665 (cartilage-bone interface) to $Q(1) = \pi/2$ rad at the up-
 666 per boundary (articular surface). The variance, in turn,
 667 is greater in the middle zone, since in that zone the fi-
 668 bres are almost randomly oriented, and thus the tissue
 669 could be thought of as isotropic. Hence, the probability
 670 density tends to be peaked around 0 rad for ξ approach-
 671 ing zero, and around $\pi/2$ rad, for ξ approaching unity.

672 The model parameters used for the numerical sim-
 673 ulations of the considered benchmark tests are taken
 674 from [47]. By employing experimental data available
 675 in the literature [24, 10, 2], we provide polynomial ex-
 676 pressions for the volumetric fractions, i.e., $\Phi_{0s}(\xi) =$
 677 $-0.062\xi^2 + 0.038\xi + 0.046$, $\Phi_{1s}(\xi) = 0.062\xi^2 - 0.138\xi +$
 678 0.204 , and $1 - \Phi_s(\xi) = 0.100\xi^2 + 0.750$. To determine

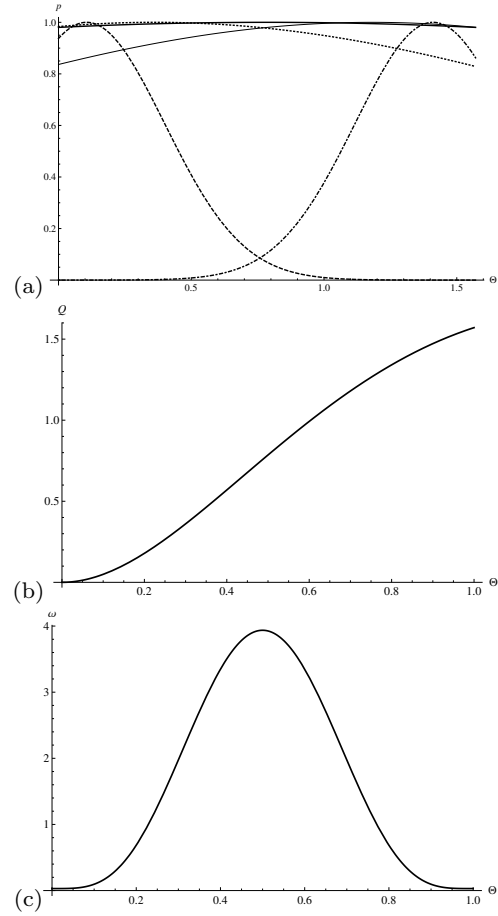


Fig. 2 Graphical representation of (a) the probability density distribution in (38b) as a function of Θ parameterised by ξ (dashed line: $\xi = 0.15$; dotted line: $\xi = 0.30$; thick line: $\xi = 0.50$; thin line: $\xi = 0.70$; dashed-dotted line: $\xi = 0.85$); (b) the histological profile of the mean angle $Q(\xi)$ given in (39a); and (c) the standard deviation $\omega(\xi)$ given in (39b).

\mathbf{K}_{FG} and $\langle\langle \mathbf{K}_{\text{AW}} \rangle\rangle$ (cf. (30) and (33), respectively), it is
 679 necessary to specify κ_0 , m_0 , and $\hat{k}_{0\text{R}}$. As done in [47],
 680 we take here $\kappa_0 = 0.0848$ and $m_0 = 4.638$ [25], and we
 681 prescribe $k_{0\text{R}}$ to be a function of the normalised axial
 682 coordinate [54], i.e.,
 683

$$k_{0\text{R}} \equiv k_{0\text{R}}(\xi) = k_{0\text{R}}^{(0)} \left[\frac{e_{\text{R}}(\xi)}{e_{\text{R}}^{(0)}} \right]^{\kappa_0} \exp\left(\frac{1}{2} m_0 \left[\left(\frac{1+e_{\text{R}}(\xi)}{1+e_{\text{R}}^{(0)}} \right)^2 - 1 \right]\right), \quad (40)$$

684 where $k_{0\text{R}}^{(0)} = 0.003 \text{ mm}^4 \text{ N}^{-1} \text{ s}^{-1}$ is a referential value
 685 of the scalar permeability (taken of the same order
 686 of magnitude as that reported in [3]), while $e_{\text{R}}(\xi) =$
 687 $(1 - \Phi_s(\xi))/\Phi_s(\xi)$ is the *void ratio* associated with the
 688 undeformed configuration (i.e., the ratio between the
 689 fluid and the solid volumetric fractions in the unde-
 690 formed configuration), and $e_{\text{R}}^{(0)} = 4.0$ [34] is a referen-
 691 tial value for $e_{\text{R}}(\xi)$. Moreover, we assume $\hat{W}_{1i} = \hat{W}_0$

and, as done in [47], we compute the parameters α_0 , α_1 , and α_2 featuring in (21b) by imposing the condition that the elastic coefficients obtained by linearising $\hat{W}_0(\mathbf{C})$ are identical to those experimentally determined in [3], and fitted to the biphasic indentation model presented in [30]. Since the samples of articular cartilage used for the experiments reported in [3] were intact and comprised both the matrix of proteoglycans and the chondrocytes, we conclude that α_0 , α_1 , and α_2 refer to the mixture of proteoglycans and cartilage cells. For this purpose, we adopt the formulae provided in [26], in which α_0 , α_1 , and α_2 are written as functions of the Lamé's constants. This calculation leads to $\alpha_0 = 0.1250$ MPa, $\alpha_1 = 0.7778$, and $\alpha_2 = 0.1111$. Finally, we choose $c = 7.5$ MPa [47].

Equations (34a)–(34c) are solved numerically by means of Finite Element methods. This necessitates to introduce the weak forms of (34a)–(34c), which are obtained by multiplying each of these equations by the corresponding test function, integrating the resulting expressions over \mathcal{B} , and applying Gauss Theorem, where required, along with the boundary conditions characterising the chosen benchmark tests. Thus, one obtains:

$$\int_{\mathcal{B}} \{\tilde{p}j - (\text{Grad } \tilde{p})\mathbf{Q}\} = 0, \quad (41a)$$

$$\int_{\mathcal{B}} \mathbf{P} : \mathbf{g} \text{Grad } \tilde{\mathbf{u}} = 0, \quad (41b)$$

$$\int_{\mathcal{B}} \{[(\mathbf{I} + \|\mathbf{q}\| \mathbf{F}^{-1} \mathcal{A} \mathbf{F})\mathbf{Q}] \cdot \tilde{\mathbf{Q}} - \mathbf{Q}_D \cdot \tilde{\mathbf{Q}}\} = 0, \quad (41c)$$

where \tilde{p} , $\tilde{\mathbf{u}}$, and $\tilde{\mathbf{Q}}$ are test functions, taken in suitable functional spaces, and referred to as virtual pressure, virtual velocity, and virtual (material) filtration velocity, respectively. For the finite element discretisation of the problem, piecewise quadratic Lagrange interpolation functions are used for all the unknowns and the corresponding test functions. Equation (41c) is an algebraic auxiliary equation that has been introduced to compute \mathbf{Q} numerically when its analytical determination is cumbersome (for example, when the Forchheimer coefficient tensor, \mathcal{A} , is not spherical).

To compute the required statistical averages, we employ the Spherical Design Algorithm [19,11,9]. Finally, we notice that in (34c) and (41c) $\|\mathbf{q}\|$ can be written as $\|\mathbf{q}\| = J^{-1} \sqrt{\mathbf{C} : (\mathbf{Q} \otimes \mathbf{Q})}$, where the term under square root supplies a further coupling between deformation and the flow direction [16].

5.1 The “Equivalent” Scalar Forchheimer Coefficient

Although the permeabilities predicted by both the FG- and the AW-model are not represented by spherical ten-

sors, we start by defining the *equivalent scalar permeability* [16],

$$k_{\text{eq}} := \sqrt{\frac{1}{3} \text{tr}[\mathbf{k} \cdot \mathbf{k}^T]}, \quad (42)$$

which we employ to construct the equivalent non-Darcy coefficient, $\beta_{\text{eq}} = c_0 \phi_f^{c_1} \mu^{c_2} k_{\text{eq}}^{c_2}$, and the “equivalent” scalar Forchheimer’s correction, \mathcal{A}_{eq} , i.e.,

$$\mathcal{A}_{\text{eq}} := \varrho_f k_{\text{eq}} \beta_{\text{eq}} = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} k_{\text{eq}}^{1+c_2}, \quad (43)$$

with $c_0 \geq 0$. The factor β_{eq} may depend on c_0 , c_1 , and c_2 in several ways. A review on the subject can be found, for example, in [46,16].

The equivalent scalar Forchheimer coefficient \mathcal{A}_{eq} is determined to invert (34c) analytically, and to study the simplest case of interaction between the anisotropy of the medium and the nonlinearity of the flow. Indeed, if $\mathcal{A}_{\text{eq}} \mathbf{i}$ is used instead of \mathcal{A} in (16), Forchheimer’s correction becomes

$$[1 + \|\mathbf{q}\| \mathcal{A}_{\text{eq}}] \mathbf{q} = \mathbf{q}_D, \quad (44)$$

which is remnant of the result obtained in [23]. An advantage of working with (44) is that it can be readily solved for \mathbf{q} in spite of the nonlinearity of the product $\|\mathbf{q}\| \mathbf{q}$. Indeed, taking the norm of both sides of (44), and rearranging all terms, the equality (44) can be turned into a quadratic equation in $\|\mathbf{q}\|$ [18,16] whose only admissible solution is given by

$$\|\mathbf{q}\| = \frac{-1 + \sqrt{1 + 4 \mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|}}{2 \mathcal{A}_{\text{eq}}}. \quad (45)$$

Since (45) expresses $\|\mathbf{q}\|$ as a function of $\|\mathbf{q}_D\|$, we can solve (44) for \mathbf{q} , i.e.,

$$\mathbf{q} = f \mathbf{q}_D, \quad (46a)$$

$$f := \frac{2}{1 + \sqrt{1 + 4 \mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|}}, \quad (46b)$$

where f is referred to as *friction factor*. As shown in [16], f can be understood as a function of the product $\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|$, and, with a slight abuse of notation, we set $f = f(\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|)$. In particular, f is such that $f(0) = 1$, $f(\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|) \sim 1 - \mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|$ for $\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\| \rightarrow 0$, and $f(\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|) \sim (\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|)^{-1/2}$ for $\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\| \rightarrow +\infty$. We remark that the definition of f given in (46b) looks much like a result obtained in [23] (cf. Equation (20) in [23]).

If the exponents c_1 and c_2 in (43) are chosen as $c_1 = -11/2$ and $c_2 = -1/2$, $\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\|$ can be expressed as a function of the Reynolds number Re_D [4], i.e.,

$$\mathcal{A}_{\text{eq}} \|\mathbf{q}_D\| = c_0 \phi_f^{-5} \text{Re}_D, \quad (47a)$$

$$\text{Re}_D = \varrho_f \sqrt{\frac{k_{\text{eq}} / \phi_f}{\mu}} \|\mathbf{q}_D\|. \quad (47b)$$

Therefore, by substituting (47a) into (46b), f can be expressed as a function of Re_D and ϕ_f , with $c_0 \geq 0$ being the only tuneable parameter, i.e.,

$$f = \frac{2}{1 + \sqrt{1 + 4c_0\phi_f^{-5}\text{Re}_D}}. \quad (48)$$

We emphasise that, while c_0 is assumed to be constant in this work, f varies in space and time, since so do also ϕ_f and Re_D . Moreover, since the filtration velocity is given by $\mathbf{q} = f\mathbf{q}_D$, and f is determined either by (46b) or by (48), the equations necessary to close the model reduce to (34a) and (34b). Finally, we remark that the definition (47b) of Re_D is slightly different from the one given in [4], in which the characteristic value of $\|\mathbf{q}_D\|$ is divided by the characteristic volumetric fraction of the fluid phase.

Equation (48) implies that the strength of Forchheimer's correction is influenced by c_0 . Indeed, the magnitude of the filtration velocity converges to that predicted by Darcy's law in the limit $c_0 \rightarrow 0$, and tends towards zero for increasing c_0 . This description can be formalised by recognising that, for every $\phi_f^{-5}\text{Re}_D$, f can be written as $f = \hat{f}(c_0)$, and can be thus identified with the value taken at c_0 by the strictly monotonically decreasing function $\hat{f} : [0, +\infty[\rightarrow]0, 1]$. This function is defined by the right-hand-side of (48), and satisfies the conditions $\hat{f}(0) = 1$ and $\lim_{c_0 \rightarrow +\infty} \hat{f}(c_0) = 0$. We notice that, since \hat{f} is continuous and strictly monotonically decreasing over $[0, +\infty[$, it is invertible and its inverse $\hat{f}^{-1} :]0, 1] \rightarrow [0, +\infty[$ is continuous. Since we do not have experimental data for c_0 , we use the invertibility of \hat{f} to determine a prescribed value of c_0 such that f stays within a certain acceptable range. More precisely, in a preliminary test, for which $f = 1$, we calculate $\mathcal{R}_0 = \phi_f^{-5}\text{Re}_D$ at a given point and instant of time, and then, by selecting a *trial* friction factor $f_{\text{trial}} \in]0, 1]$, we obtain the corresponding value of c_0 as $c_0 = \hat{f}^{-1}(f_{\text{trial}})$. By substituting this result into (48), f can be related to f_{trial} :

$$f = \frac{2}{1 + \sqrt{1 + \frac{4-4f_{\text{trial}}}{f_{\text{trial}}^2} \phi_f^{-5}\text{Re}_D \mathcal{R}_0}}. \quad (49)$$

In (49), \mathcal{R}_0 is the value of $\phi_f^{-5}\text{Re}_D$ at a point X_U of the boundary line of Γ_U and at time T_{ramp} . Consistently with the behaviour outlined above, f tends to unity in the limit $f_{\text{trial}} \rightarrow 1$, thereby meaning that the filtration velocity tends to converge to Darcy's solution. Since we expect that Forchheimer's correction is moderate in articular cartilage, we regard only small deviations of the flow from the predictions of Darcy's law as physically admissible. Although this suggests to restrict \hat{f}^{-1} to

values of f_{trial} sufficiently close to unity, for the sake of completeness we consider f_{trial} ranging from 0.1 to 0.9.

In the simulations performed in this work, the Reynolds number associated with Darcy's law ranges between 10^{-8} and 10^{-7} , thereby corresponding to a maximum velocity magnitude of about 10^{-5} m/s (see Fig. 4). This range is often distinctive of a purely Darcian regime [4]. A plausible range of variation for f_{trial} and c_0 could be obtained by means of the comparison between the FG- and the AW-model, as done in the present work. Notice that, for a porous medium with $\phi_f \approx 0.75$, the coefficient c_0 would have approximatively the same order of magnitude as Re_D^{-1} , and decreases with f_{trial} .

It should be noticed that, even though f_{trial} is constant, the friction factor reported in (49) depends on space and time, and may also deviate appreciably from f_{trial} . Furthermore, if evaluated at (X_U, T_{ramp}) , it does not return f_{trial} . Indeed, computing $\phi_f^{-5}\text{Re}_D$ in (X_U, T_{ramp}) by accounting for Forchheimer's correction yields a value \mathcal{R}_1 different from \mathcal{R}_0 , which is instead computed by using Darcy's law only. In fact, as we will see in the following, Forchheimer's correction leads to higher pressures and lower magnitudes of the velocity field in the domain, thereby leading to usually lower Reynolds number. In this respect, the friction factor (49) is "inconsistent". This discrepancy, however, can be reduced by iterating the determination of the friction factor as shown in Algorithm 1.

Algorithm 1 Procedure for determining the friction factor f

```

1: Choose a tolerance  $\text{TOL} > 0$  and  $f_{\text{trial}} \in ]0, 1]$ ;
2: Compute  $\mathcal{R}_0$  by using Darcy's law;
3: Compute  $f_0 = \frac{2}{1 + \sqrt{1 + \frac{4-4f_{\text{trial}}}{f_{\text{trial}}^2} \phi_f^{-5}\text{Re}_D \mathcal{R}_0}}$  [cf. Eq. (49)];
4: if  $|f_0(X_U, T_{\text{ramp}}) - f_{\text{trial}}| < \text{TOL}$  then
5:    $f = f_0$ ;
6: else
7:    $k = 0$ ;
8:   Compute  $\mathcal{R}_{k+1}$  by using  $f_k$ ;
9:   Compute  $f_{k+1} = \frac{2}{1 + \sqrt{1 + \frac{4-4f_{\text{trial}}}{f_{\text{trial}}^2} \phi_f^{-5}\text{Re}_D \mathcal{R}_{k+1}}}$ ;
10:  if  $|f_{k+1}(X_U, T_{\text{ramp}}) - f_{\text{trial}}| < \text{TOL}$  then
11:     $f = f_{k+1}$ 
12:  else
13:     $k = k + 1$ ;
14:  Go to 8;

```

To see how the friction factor varies in space, and to highlight how its spatial distribution is influenced by the fluid filtration velocity, which, in turn, depends through the permeability tensor on the anisotropy and inhomogeneity of the tissue, we run two simulations of the first benchmark problem (i.e., an unconfined compression test in which the lower boundary of the sample is clamped). For the first simulation, we consider the

851 transversely isotropic and inhomogeneous model dis-
 852 cussed in the previous sections, while for the second
 853 simulation we study a simplified framework in which
 854 the anisotropic contribution of the fibres is not taken
 855 into account, and all material parameters are constant
 856 with the depth of the sample. In particular, we set
 857 $\bar{\Phi}_s = \bar{\Phi}_{0s} = 0.15$. Hence, the scalar permeability in (40)
 858 becomes constant through the depth of the sample, and
 859 equal to $k_{0R} = 0.0188 \text{ mm}^4 \text{ N}^{-1} \text{ s}^{-1}$.

860 By comparing Fig. 3 with Fig. 4, which represent
 861 f and \mathbf{q}_D at $t = T_{\text{ramp}}$, respectively, we notice that, as
 862 expected, the friction factor is higher in the zones of the
 863 sample in which the Reynolds number is lower, i.e., in
 864 the central zone of the sample, and it approaches f_{trial}
 865 in the external zone, for both the inhomogeneous (Fig.
 866 3(a) and Fig. 4(a)) and the homogeneous case (Fig.
 867 3(b) and Fig. 4(b)). As experimentally observed, both
 868 the porosity and the permeability of articular cartilage
 869 experience strong variations along the tissue's depth. If
 870 the inhomogeneity of these physical quantities is mod-
 871 elled, the pathways of the fluid inside the tissue vary
 872 sensibly with respect to the homogeneous case. On the
 873 contrary, as shown in Fig. 3(b) and Fig. 4(b), when $\bar{\Phi}_s$
 874 and k_{0R} are assumed to be constant, the variation of
 875 both the filtration velocity and the friction factor along
 876 the sample depth is less pronounced than it is in the
 877 inhomogeneous (and transversely isotropic) case.

878 We report in Fig. 5 the patterns of \mathbf{q} at $t = T_{\text{ramp}}$,
 879 as obtained by employing Forchheimer's correction with
 880 $f_{\text{trial}} = 0.1$, both in the inhomogeneous and anisotropic
 881 case (Fig. 5a) and in the homogeneous and isotropic
 882 case (Fig. 5b). We observe that, when Forchheimer's
 883 correction is introduced, the filtration velocity tends
 884 to become more spatially uniform than that predicted
 885 by Darcy's law, and a small distortion of the stream-
 886 lines occurs at the bottom of the sample, where zero-
 887 displacement boundary conditions are imposed.

888 In Fig. 6 we show the influence of Forchheimer's cor-
 889 rection on the magnitude of the filtration velocity and
 890 pressure for different values of f_{trial} . At each time, the
 891 values on the vertical axis refer to the maxima attained
 892 by the magnitude of the filtration velocity, $\|\mathbf{q}\|$, and
 893 pressure, p , within the sample. In particular, $\|\mathbf{q}\|$ is eval-
 894 uated at the point X_U defined above, while p is taken at
 895 the point $X_L = (0, 0, 0)$ (centre of the lower boundary
 896 of the sample). First, we report the results of two sim-
 897 ulations, performed by using Darcy's law, in which the
 898 permeability is given once by the FG-model and once by
 899 AW-model. Looking at Fig. 6, we notice that the results
 900 predicted by the Darcy-based FG- and AW-model are
 901 in remarkable disagreement with each other, although
 902 they both seem to be physically plausible. In particu-
 903 lar, the AW-model returns a higher filtration velocity

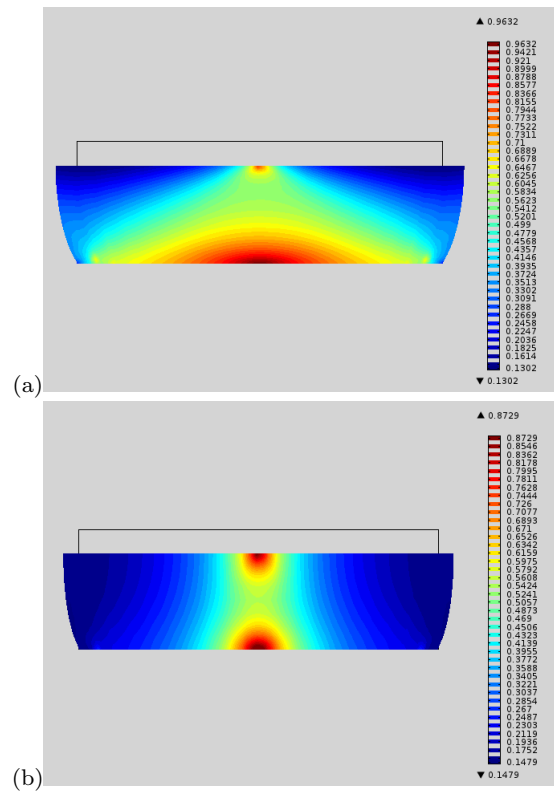


Fig. 3 The friction factor f varies according to the variation of both the filtration velocity and the permeability (computed here with the AW-model). (a): Transversely isotropic and inhomogeneous model with $f_{\text{trial}} = 0.3$. (b): Homogeneous and isotropic case with $f_{\text{trial}} = 0.3$. Close to the outer wall of the sample, where the filtration velocity is higher, the values of f are smaller (thereby yielding a stronger Forchheimer's correction) than those at the centre of the sample. The plots are evaluated at $t = T_{\text{ramp}}$. (Colour figure online)

904 and a lower pore pressure (black dotted curve) than
 905 those obtained by using the FG-model (black curves).
 906 By tuning f_{trial} , a partial agreement between the two
 907 models can be achieved. Indeed, as we can see from
 908 Fig. 6, Forchheimer's correction contributes to lower
 909 the magnitude of the filtration velocity and to raise the
 910 pressure, thereby reducing the mean distance between
 911 the results of the AW-model and those of the FG-model.
 912 Moreover, an *optimal* value of f_{trial} can be obtained by
 913 means of an optimisation procedure that minimises the
 914 distance between the magnitude of Darcy's velocity ob-
 915 tained by means of the FG-model, and the magnitude
 916 of the filtration velocity obtained with the AW-model
 917 modified by Forchheimer's correction. Here, we set

$$f_{\text{trial}} = f_{\text{opt}} = \tilde{f}_{\text{trial}}(\xi) = 2.38 \xi^3 - 3.51 \xi^2 + 1.69 \xi + 0.07. \quad (50)$$

918 Such optimal value varies in space, due to the spatial
 919 variations of the computed f (see Fig. 3).

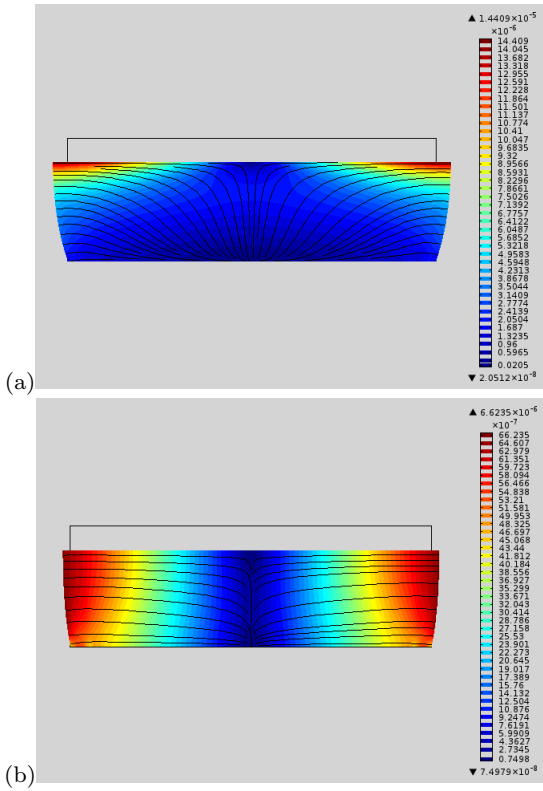


Fig. 4 Patterns of Darcy's filtration velocity \mathbf{q}_D at $t = T_{\text{ramp}}$ as predicted by the AW-model. (a): Transversely isotropic and inhomogeneous model. (b): Isotropic and homogeneous model. The black curves represent the streamlines. The zones of higher velocity correspond to the zones of lower friction factor in Fig. 3. (Colour figure online)

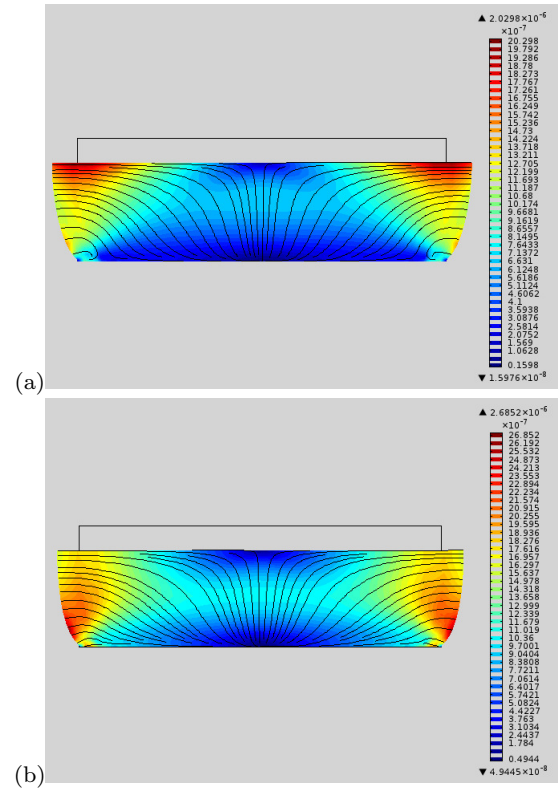


Fig. 5 Patterns of filtration velocity \mathbf{q} at $t = T_{\text{ramp}}$ for $f_{\text{trial}} = 0.1$. (a): Transversely isotropic and inhomogeneous model. (b): Isotropic and homogeneous case. The black curves represent the streamlines. Both simulations are obtained for the AW-model. The filtration velocity is more uniform in the domain, and lower than that obtained in the Darcian case (cf. Fig. 4). (Colour figure online)

920 It is important to notice that, when Forchheimer's
 921 correction is introduced, both the magnitude of the fil-
 922 tration velocity and the pressure relax towards the sta-
 923 tionary states more slowly than in the Darcian case.

924 5.2 Diagonal Forchheimer Coefficient Tensor

925 In this section, we consider the benchmark test of the
 926 second kind, in which the original shape of the sample
 927 is approximately maintained by the deformation. We
 928 say "approximately" because, in spite of tissue's inho-
 929 mogeneity, the deformed shape of the sample deviates
 930 only slightly from the original, cylindrical one. We de-
 931 note by $\{\mathbf{E}_I(X)\}_{I=1}^3 \in T_X\mathcal{B}$ and $\{\mathbf{e}_i(x)\}_{i=1}^3 \in T_x\mathcal{S}$ the
 932 collinear, orthonormal vector bases attached at $X \in \mathcal{B}$
 933 and $x = \chi(X, t) \in \mathcal{S}$, respectively, with $\mathbf{E}_1(X)$ and
 934 $\mathbf{e}_1(x)$ oriented radially, $\mathbf{E}_2(X)$ and $\mathbf{e}_2(x)$ circumfer-
 935 entially, and $\mathbf{E}_3(X)$ and $\mathbf{e}_3(x)$ axially. Forchheimer's cor-
 936 rection tensor \mathcal{A} is diagonal with respect to the basis

$\{\mathbf{e}_i(x)\}_{i=1}^3 \in T_x\mathcal{S}$, and can be written as 937

$$\mathcal{A} = \sum_{i=1}^3 \mathcal{A}^i \mathbf{e}_i \otimes \mathbf{e}^i = \sum_{i=1}^3 \mathcal{A}_i \mathbf{e}_i \otimes \mathbf{e}^i, \quad (51)$$

938 where $\{\mathbf{e}^i(x)\}_{i=1}^3 \in T_x^*\mathcal{S}$ is the covector basis dual of
 939 $\{\mathbf{e}_i(x)\}_{i=1}^3 \in T_x\mathcal{S}$, and $\mathcal{A}_i \equiv \mathcal{A}^i > 0$, $i \in \{1, 2, 3\}$ (no
 940 sum with respect to i), are the transversal and axial
 941 components of \mathcal{A} , respectively. Equation (51) permits
 942 to rewrite (16) as

$$\left[\sum_{i=1}^3 (1 + \mathcal{A}_i \|\mathbf{q}\|) \mathbf{e}_i \otimes \mathbf{e}^i \right] \mathbf{q} = \mathbf{q}_D. \quad (52)$$

Let $\mathcal{A}_M = \max_{i \in \{1, 2, 3\}} \{\mathcal{A}_i\}$ and $\mathcal{A}_m = \min_{i \in \{1, 2, 3\}} \{\mathcal{A}_i\}$
 943 be the maximum and the minimum eigenvalue of \mathcal{A} ,
 944 respectively. Hence, the inequality $\mathcal{A}_M \geq \mathcal{A}_m$, with the
 945 equality sign being satisfied in the isotropic case, im-
 946 plies the estimates 947

$$(1 + \mathcal{A}_m \|\mathbf{q}\|) \|\mathbf{q}\| \leq \|\mathbf{q}_D\| \leq (1 + \mathcal{A}_M \|\mathbf{q}\|) \|\mathbf{q}\|. \quad (53)$$

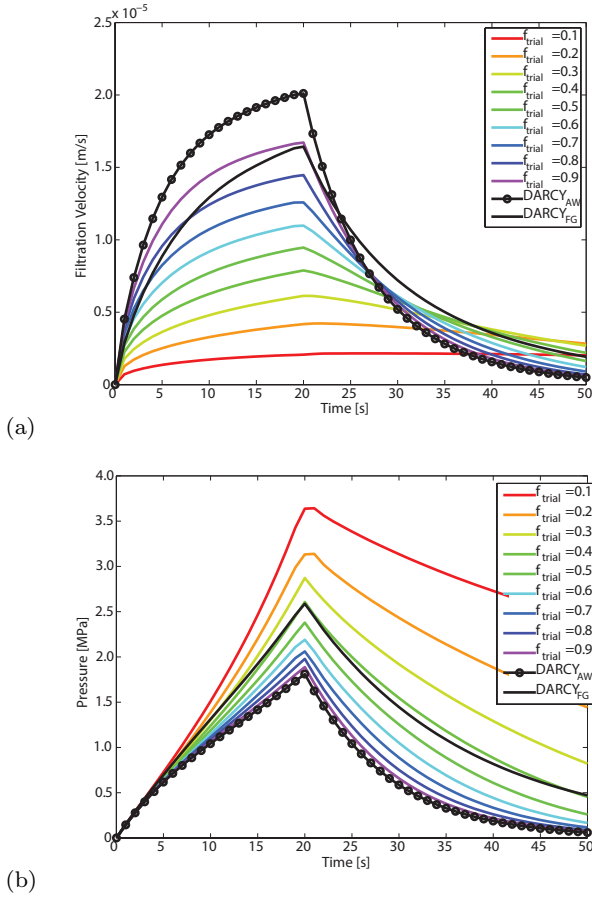


Fig. 6 Magnitude of the filtration velocity $\|\mathbf{q}\|$ (a) and pressure p (b), for different values of f_{trial} , computed at the point of the sample in which each of these quantities attains its maximum. The black solid curves with and without markers represent the output of the AW and the FG model, respectively, in a purely Darcian regime. The coloured curves are obtained by means of the scalar Forchheimer's correction applied to the AW permeability model. (Colour figure online)

By equating $\|\mathbf{q}_D\|$ with its lower and upper bound, Eq. (53) allows to deduce two admissible extremal solutions for $\|\mathbf{q}\|$, i.e.,

$$\gamma(\mathcal{A}_M) := \frac{-1 + \sqrt{1 + 4\mathcal{A}_M\|\mathbf{q}_D\|}}{2\mathcal{A}_M}, \quad (54a)$$

$$\gamma(\mathcal{A}_m) := \frac{-1 + \sqrt{1 + 4\mathcal{A}_m\|\mathbf{q}_D\|}}{2\mathcal{A}_m}. \quad (54b)$$

It can be proven that the inequality $\gamma(\mathcal{A}_M) \leq \gamma(\mathcal{A}_m)$ holds true. Consistently, the magnitude of the filtration velocity is said to be admissible if it complies with the chain of inequalities

$$\gamma(\mathcal{A}_M) \leq \|\mathbf{q}\| \leq \gamma(\mathcal{A}_m). \quad (55)$$

We remark that $\gamma(\mathcal{A}_M)$ and $\gamma(\mathcal{A}_m)$ depend on $\|\mathbf{q}_D\|$, which, in turn, depends on the permeability and pressure gradient. The lower and the upper bounds of $\|\mathbf{q}\|$

are obtained by evaluating the same function, γ , once in the maximum and once in the minimum eigenvalue of \mathcal{A} . Thus, if we set $\|\mathbf{q}\| = \gamma(\mathcal{A}_j)$, with $j \in \{1, 2, 3\}$, and substitute the result into (52), we obtain

$$\begin{aligned} 1 + \mathcal{A}_i\|\mathbf{q}\| &= 1 + \mathcal{A}_i\gamma(\mathcal{A}_j) \\ &= \frac{(2 - \zeta_{ij}) + \zeta_{ij}\sqrt{1 + 4\mathcal{A}_j\|\mathbf{q}_D\|}}{2} \\ &=: \frac{1}{f_{ij}}, \end{aligned} \quad (56)$$

where $\zeta_{ij} := \mathcal{A}_i/\mathcal{A}_j$ is referred to as *anisotropy ratio*, and f_{ij} is said to be the corresponding *friction factor*. Note that $\zeta_{ji} = 1/\zeta_{ij}$, and (52) becomes

$$\left[\sum_{i=1}^3 \frac{1}{f_{ij}} \mathbf{e}_i \otimes \mathbf{e}^i \right] \mathbf{q} = \mathbf{q}_D, \quad j \in \{1, 2, 3\}, \quad (57)$$

which can be inverted to express the filtration velocity as

$$\mathbf{q}_{(j)} = \left[\sum_{i=1}^3 f_{ij} \mathbf{e}_i \otimes \mathbf{e}^i \right] \mathbf{q}_D, \quad j \in \{1, 2, 3\}. \quad (58)$$

where $\mathbf{q}_{(j)}$ is the value of the filtration velocity whose norm is $\gamma(\mathcal{A}_j)$, with $j \in \{1, 2, 3\}$. To complete the description, we provide an explicit expression for the anisotropy factors, ζ_{ij} , with $i, j \in \{1, 2, 3\}$. In the AW-model, the spatial permeability tensor is given by $\mathbf{k} = J^{-1}\mathbf{F}\langle\langle\mathbf{K}_{AW}\rangle\rangle\mathbf{F}^T$, and depends on the tensor \mathbf{Z} . Due to the particular form of the deformation, \mathbf{Z} possesses transverse isotropy with respect to \mathbf{E}_3 , which is parallel to \mathbf{M}_0 . Since $\mathbf{A}_0 := \mathbf{E}_3 \otimes \mathbf{E}_3$ and $\mathbf{T}_0 = \mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2$ span the space of all symmetric second-order tensors of the type $[\mathcal{TB}]_0^2$ exhibiting transverse isotropy with respect to $\mathbf{M}_0 \equiv \mathbf{E}_3$, we can write

$$\mathbf{Z} = \left\langle\left\langle \frac{\mathbf{A}}{I_4(\mathbf{C}, \mathbf{A})} \right\rangle\right\rangle = Z_t\mathbf{T}_0 + Z_a\mathbf{A}_0, \quad (59)$$

with Z_t and Z_a being the transverse and axial components of \mathbf{Z} , respectively. Since, for the considered benchmark test, \mathbf{F} is assumed to admit in cylindrical coordinates the representation $\mathbf{F} = \lambda_1\mathbf{e}_1 \otimes \mathbf{E}^1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{E}^2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{E}^3$, the tensor $\mathbf{z} = \mathbf{F}\mathbf{Z}\mathbf{F}^T$ is given by

$$\begin{aligned} \mathbf{z} &= Z_t\mathbf{b} + (Z_a - Z_t)\mathbf{F}\mathbf{A}_0\mathbf{F}^T \\ &= Z_t\mathbf{b} + (Z_a - Z_t)I_{40}\mathbf{a}_0, \end{aligned} \quad (60)$$

with $\mathbf{b} = \sum_{i=1}^3 \lambda_i^2 \mathbf{e}_i \otimes \mathbf{e}_i$ being the left Cauchy-Green deformation tensor, $I_{40} = \mathbf{C} : \mathbf{A}_0 = \lambda_3^2$, and $\mathbf{a}_0 = \mathbf{e}_3 \otimes \mathbf{e}_3$. We remark that \mathbf{Z} features the nonzero transversal component Z_t even though $I_4^{-1}\mathbf{A}$ does not have any

transverse component along the local transverse projection tensor $\mathbf{T} = \mathbf{G}^{-1} - \mathbf{A}$ [50, 51, 47]. Consequently, the spatial permeability is given by

$$\mathbf{k} = \hat{k}_0(J)\mathbf{g}^{-1} + J^{-2}\hat{k}_0(J)Z_t\mathbf{b} + J^{-2}\hat{k}_0(J)(Z_a - Z_t)I_{40}\mathbf{a}_0, \quad (61)$$

and, more explicitly, it can be written as $\mathbf{k} = \sum_{i=1}^3 k_i \mathbf{e}_i \otimes \mathbf{e}_i$, where the scalar permeabilities are defined by

$$k_1 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_1^2, \quad (62a)$$

$$k_2 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_2^2, \quad (62b)$$

$$k_3 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_a\lambda_3^2. \quad (62c)$$

Accordingly, the eigenvalues of the Forchheimer coefficient tensor read

$$\mathcal{A}_1 = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \left(\hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_1^2 \right)^{1+c_2}, \quad (63a)$$

$$\mathcal{A}_2 = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \left(\hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_2^2 \right)^{1+c_2}, \quad (63b)$$

$$\mathcal{A}_3 = c_0 \varrho_f \phi_f^{c_1} \mu^{c_2} \left(\hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_a\lambda_3^2 \right)^{1+c_2}. \quad (63c)$$

Therefore, if we choose the anisotropy factors ζ_{31} and ζ_{32} , we obtain

$$\zeta_{3j} = \left[\frac{J^2 + Z_a\lambda_j^2}{J^2 + Z_t\lambda_j^2} \right]^{1+c_2}, \quad j = \{1, 2\}. \quad (64)$$

In the undeformed configuration, it holds that $Z_a + 2Z_t = 1$, which yields

$$\zeta_{31} = \zeta_{32} = \zeta = \left[\frac{1 + Z_a}{1 + Z_t} \right]^{1+c_2} = \left[\frac{2 - 2Z_t}{1 + Z_t} \right]^{1+c_2}. \quad (65)$$

Note that, in the undeformed configuration, it holds that $\zeta_{12} = \zeta_{21} = 1$. The correlations used in this work to express β_{eq} were taken from [28], and are referred to as Coles&Hartman correlation, $\beta_{\text{eq}} = c_0 \phi_f^{0.449} \mu^{-1.88} k_{\text{eq}}^{-1.88}$, and Geertsma correlation, $\beta_{\text{eq}} = c_0 \phi_f^{-5.5} \mu^{-0.5} k_{\text{eq}}^{-0.5}$. For the computations, the Coles&Hartman correlation has been approximated by setting $c_2 = -2$.

The curves in Fig. 7a refer to two different sets of computations of the magnitude of the filtration velocity: once by employing the equivalent scalar Forchheimer's coefficient \mathcal{A}_{eq} defined in (43) and the optimised friction factor f_{opt} , and once in the case of diagonal \mathcal{A} . The continuous curve is obtained for $\|\mathbf{q}\| = \gamma(\mathcal{A}_1)$, and the marked curve is obtained for $\|\mathbf{q}\| = \gamma(\mathcal{A}_3)$. From Fig. 7a, we can see that the curve obtained by expressing the norm $\|\mathbf{q}\|$ of the filtration velocity as a function of \mathcal{A}_1 is quite compatible with the one obtained as a result of the equivalent scalar case. The greatest distance between the two curves, i.e., the one obtained for $\|\mathbf{q}\| \equiv \gamma(\mathcal{A}_1)$ and the one obtained

for $\|\mathbf{q}\| \equiv \gamma(\mathcal{A}_3)$, can be registered in the neighbourhood of $t = T_{\text{ramp}}$, i.e., when Forchheimer's correction is more significant due to the higher values of the filtration velocity in the sample. In Fig. 7b, the friction factors f_{ij} , with $i, j = 1, 3$, are compared with f_{opt} . As a consequence of the inhomogeneity of the permeability through the depth of the sample, \mathcal{A}_1 and \mathcal{A}_3 acquire the role of maximum or minimum eigenvalue of \mathcal{A} , respectively. In particular, the axial friction factors f_{13} and f_{33} are higher than the longitudinal ones in the deep zone of the sample. Due to the randomness of the distribution of the fibres in the middle zone, also the material parameters are such that, in this zone, an isotropic behaviour can be observed. In this zone, indeed, all the friction factors merge, whereas at the top of the sample, the transversal friction factors f_{11} and f_{31} have a greater value than the axial ones. Moreover, at the top of the sample, the friction factors related to the transversal eigenvalue \mathcal{A}_1 , i.e., f_{11} and f_{13} , are both higher than the ones related to the axial eigenvalue \mathcal{A}_3 , i.e., f_{31} and f_{33} , whereas the latter two are higher than the transversal ones at the bottom.

5.3 Fully Tensorial Case

In this section, we simulate the second benchmark test, in which the original cylindrical shape of the sample is disrupted by the deformation, and we consider a not necessarily diagonal Forchheimer coefficient \mathcal{A} . In this case, which we call "fully tensorial case", we prefer to invert the relation (34c) numerically. For determining \mathcal{A} , we employ the non-Darcy coefficient tensor β , with the exponents c_0 , c_1 , and c_2 predicted by the previously introduced approximation of the Coles&Hartman correlation. For comparison, we consider also the benchmark test of the first type (which approximately maintains the sample's cylindrical shape). In Fig. 8, we show the time variation of the magnitude of the filtration velocity, $\|\mathbf{q}\|$, for the fully tensorial case, and for the extremal values $\|\mathbf{q}\| = \gamma(\mathcal{A}_1)$ and $\|\mathbf{q}\| = \gamma(\mathcal{A}_3)$, obtained in the case of diagonal Forchheimer coefficient tensor. From Fig. 8, we see that the Coles&Hartman correlation induces a greater difference between the extremal curves, with respect to those plotted in Fig. 7, which were obtained for c_1 and c_2 taken from the Geertsma correlation, and $c_0 = \hat{f}^{-1}(f_{\text{opt}})$, with $f_{\text{opt}} = \hat{f}_{\text{trial}}(\xi)$ as in (50).

Figure 8 represents also a validation of the results obtained by solving numerically (34c). Indeed, the numerical results of the two extremal cases, $\|\mathbf{q}\| = \gamma(\mathcal{A}_1)$ and $\|\mathbf{q}\| = \gamma(\mathcal{A}_3)$, act as an upper and a lower bound for the results of the fully tensorial case, depending on which eigenvalue attains the maximum and minimum

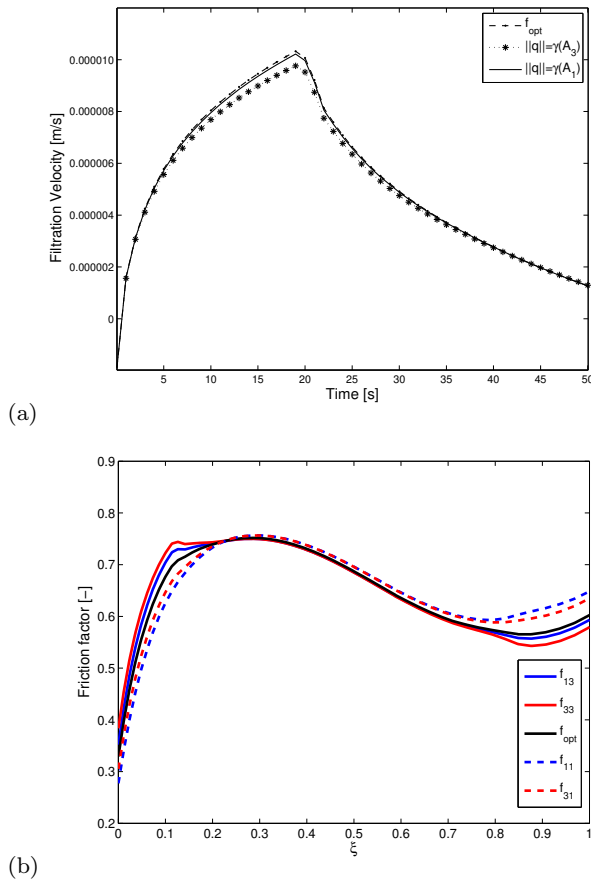


Fig. 7 (a): Filtration velocity evaluated at the upper external point of the sample. (b): Friction factors related to each of them for $t = T_{\text{ramp}}$ vs the normalised axial coordinate ξ . The vertical line along which the variation of the friction factor is observed intersects the lower boundary at $X^1 = X^2 = 0.5$ mm. The blue curves are obtained by means of the formula (56), with $i = 1$ and $j = 1$, for the dashed curve, and $j = 3$, for the solid curve. Analogously, the red curves are obtained by choosing $i = 3$. (Colour figure online)

1070 value, respectively. Thus, in Fig. 7a, we see that the
 1071 magnitude of the filtration velocity obtained as an out-
 1072 come of the fully tensorial case lies in between the two
 1073 extremal solutions, and it is quite compatible with the
 1074 result obtained with the equivalent scalar Forchheimer's
 1075 correction.

1076 In Fig. 9a, the results obtained with the scalar Forch-
 1077 heimer's correction, which corresponds here to the opti-
 1078 mised friction factor, f_{opt} , are compared with those ob-
 1079 tained with the fully tensorial correction for the case of
 1080 clamped lower boundary of the sample (this boundary
 1081 condition is closer to the system's phenomenology, since
 1082 it simulates the attachment of articular cartilage to the
 1083 subchondral bone). For completeness, we report also
 1084 the magnitude of the filtration velocity as predicted by
 1085 the FG- and the AW-model within the Darcian regime.

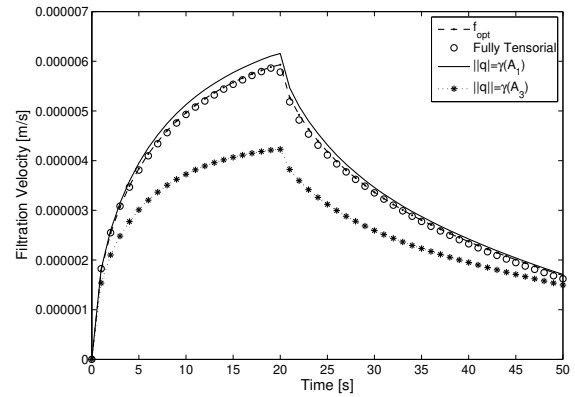


Fig. 8 Magnitude of the filtration velocity for the fully tensorial case (plain rings), for the diagonal case and extremal values $\|q\| = \gamma(A_1)$ and $\|q\| = \gamma(A_3)$ (continuous line and asterisks, respectively), and for the equivalent scalar case with optimised friction factor f_{opt} . All velocities are evaluated at a point X_U of the boundary line of I_U by employing the Coles&Hartman correlation, with $c_0 = 5 \cdot 10^{-18}$.

As seen in Fig. 7a, also with a different boundary condition, the equivalent scalar Forchheimer coefficient and the fully tensorial Forchheimer coefficient return, in our work, a quite compatible numerical result. Indeed, the curves representing the fully tensorial case (plain circles in Fig. 9a) are almost overlapped to the dashed ones, which represent the scalar equivalent case. Finally, Fig. 9b shows the time variation of the pressure at the centre of the lower boundary of the sample. We remark that, in contrast to what happens to the magnitude of the filtration velocity, the pressure predicted by the fully tensorial model is lower than that obtained by the equivalent scalar model. Moreover, the curves obtained within the Darcian regime by employing the FG- and the AW-model predict sufficiently smaller values of pressure and, in particular, the lowest pressures are those predicted by the AW-model. Finally, we notice that, also in the tensorial case, Forchheimer's correction implies that the magnitude of the filtration velocity and pressure relax towards the stationary states more slowly than in the Darcian case.

6 Discussion and Conclusions

In this work, we studied some consequences of Forchheimer's correction to Darcy's law in the study of the fluid flow in a hydrated biological tissue such as articular cartilage. To imitate the internal structure of the examined target tissue, its reinforcing fibres were assumed to be oriented statistically, as predicted by a probability density compatible with the tissue's histology. Also the volumetric fractions of matrix and fibres

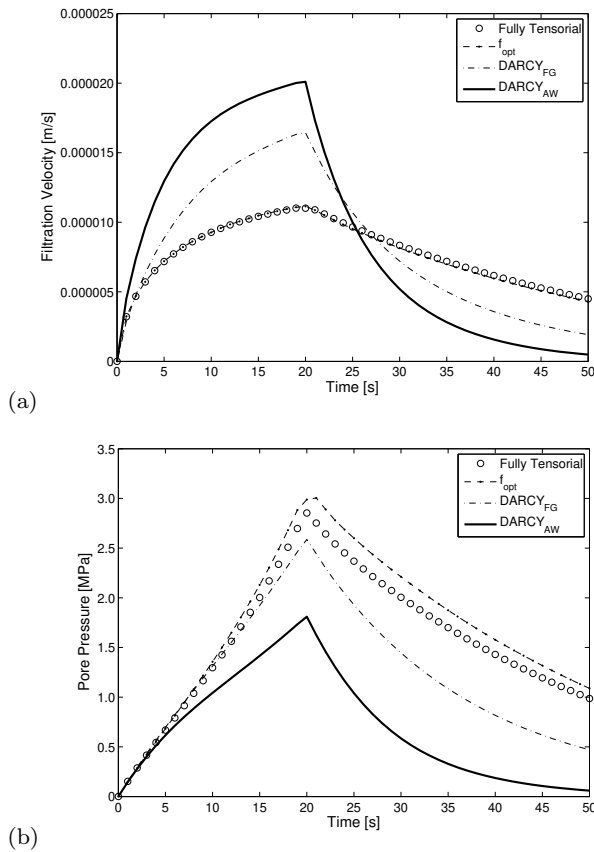


Fig. 9 (a): Magnitude of the filtration velocity for the fully tensorial case (plain rings), for the equivalent scalar case with optimised friction factor f_{opt} (asterisks), for the Darcian regime and FG-model (dash-dotted line), and for the Darcian regime and AW-model (continuous line). All velocities are evaluated at a point X_U of the boundary line of Γ_U , for the Coles&Hartman correlation, with $c_0 = 5 \cdot 10^{-18}$. (b): Pore pressure versus time (curves as in point (a)). Pressures are evaluated at $X_L = (0, 0, 0)$ for the Coles&Hartman correlation, with $c_0 = 5 \cdot 10^{-18}$.

were deduced from experimental data taken from the literature. The mechanical response of the solid matrix of the sample was hypothesised to be hyperelastic, and characterised by the elastic potential defined in (20). Moreover, to study the flow of the interstitial fluid, the FG-model [12] and the AW-model [1] of permeability were compared.

We developed the theory of Forchheimer’s correction for the case of a tensorial Forchheimer’s coefficient. However, in order to adapt our study to well-established derivations of Forchheimer’s correction available in the literature [46], we first introduced an “equivalent” scalar coefficient, \mathcal{A}_{eq} , and the friction factor, f . We observed that the inhomogeneity and anisotropy of the sample yield patterns of f and \mathbf{q}_D that are different from those obtained in the isotropic and homogeneous case, and produce an increase of the maximum value of both f

and $\|\mathbf{q}_D\|$ (see Figs. 3 and 4). The increase of $\|\mathbf{q}_D\|$ might be ascribable to microstructural effects. By introducing Forchheimer’s correction, we obtained a reduction of the magnitude of the filtration velocity (see Fig. 5) with respect to the Darcian description. Moreover, a redistribution of the flow pattern, which tends to become spatially uniform, can be observed. By comparing Figs. 4 and 5, we can also observe that, by applying the same trial friction factor $f_{trial} = 0.1$ to both the inhomogeneous and anisotropic tissue and to the isotropic and homogeneous one, Forchheimer’s correction produces, in the former case, a maximum difference between the magnitudes of the filtration velocity $\|\mathbf{q}\|$ and $\|\mathbf{q}_D\|$ of about 85%, and of about 60% in the latter. Thus, we may conclude that the more inhomogeneous and complex the microstructure is, the more Forchheimer’s correction could be significant in studying the flow. Indeed, it is possible that also this result is due to the microstructure as well as to a better resolution of the interplay between deformation and flow.

To test the FG-model of permeability, which takes the sample’s microstructure explicitly into account, we compared it with the AW-model. From the results of this comparison (see Fig. 6), we observed that the two models are discrepant in the Darcian case, but that the discrepancies can be partially smoothed over by modulating the AW-model with the aid of Forchheimer’s correction and, thus, of the friction factor. We believe that this behaviour could be due to the fact that Forchheimer’s correction introduces new parameters into the flow model, which can thus be employed to better fit experimental results. We emphasise that, by modulating the AW-model, we by no means intended to correct it. Rather, we chose to modulate the AW-model because, contrary to the FG-model, it is not restricted by the use of Darcy’s law at the REV scale. An important conclusion is that Forchheimer’s correction implies an increase of the fluid pressure and a dilation of the relaxation times for both the filtration velocity and pressure. This behaviour can be observed both in the equivalent scalar case and in the fully tensorial one (see Fig. 9).

In the future, we would like to study the combined effect of Forchheimer’s correction and the Brinkman equation to study the boundary effects on the fluid behaviour. These, indeed, may lead to a more precise description of the flow in complex benchmark tests, such as the indentation test.

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1190 In Memoriam

1191 In memory of our master Prof. Gaetano Giaquinta (1945–
 1192 2016).

1193 Compliance with Ethical Standards

1194 The authors declare that they have no conflict of inter-
 1195 est.

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