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# Non-Darcian Flow in Fibre-Reinforced Biological Tissues

Alfio Grillo · Melania Carfagna · Salvatore Federico

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Abstract Under suitable conditions, the motion of a fluid in a porous medium can be studied by assum-5 ing the validity of Darcy's law. Since many biological 6 tissues can be thought of as porous media, Darcy's 7 law is invoked in several biomechanical contexts, like 8 the transport of the chemical species needed for the q metabolism of tissue cells. Although Darcy's law sup-10 plies physically sound results in many circumstances, 11 there may be cases in which the dynamic behaviour of 12 a biological fluid deviates from the Darcian one. The 13 scope of this work is to analyse some possible conse-14 quences of such deviations, with emphasis on the fluid 15 velocity and pressure, which, in turn, influence the health 16 and correct functioning of the tissue cells. In particu-17 lar, our study addresses the flow of an interstitial fluid 18 through a fibre-reinforced tissue, in which the fibres are 19 oriented statistically. We take articular cartilage as a 20 representative tissue of this type, and study the devi-21 ation from Darcy's law known as "Forchheimer's cor-22 rection". Moreover, we introduce two models of tissue 23

A. Grillo (Corresponding Author) · M. Carfagna Dept of Mathematical Sciences (DISMA) "G.L. Lagrange", Politecnico di Torino Corso Duca degli Abruzzi 24, 10129, Torino, Italy Tel.: +39-011-0907531 Fax: +39-011-0907599 E-mail: alfio.grillo@polito.it E-mail: melania.carfagna@polito.it

S. Federico

2500 University Drive NW, Calgary, AB, T2N 1N4, Canada E-mail: salvatore.federico@ucalgary.ca

permeability, which lead to discrepant results when the fluid velocity is described by Darcy's law. We show, however, that the discrepancies in the description of the flow can be reduced if Forchheimer's correction is applied. 28

KeywordsBiological tissue · Porous medium ·29Biphasic material · Fibre-reinforcement · Composite30materials · Transverse isotropy · Darcy's law ·31Forchheimer's correction32

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#### 1 Introduction

Many biological tissues, such as articular cartilage [48, 49, 35, 34, 26], can be described as biphasic systems comprising a fluid and a solid phase. In the course of its life, a porous tissue responds to stimuli of various nature, among which the mechanical ones contribute to vary its shape and internal structure.

The fluid flowing through a tissue is affected by 40 the tissue's structural variations, and changes its veloc-41 ity and pressure accordingly. Since these changes are 42 relevant for the tissue's health, it is important to un-43 derstand the dynamics of biological fluids. In partic-44 ular, the flow of the interstitial fluid in articular car-45 tilage is related to the tissue's microstructure, which 46 changes in time because of the deformation of the non-47 fibrous matrix (comprising mainly proteoglycans and 48 chondrocytes) and the reorientation of the collagen fi-49 bres. Thus, the velocity of the fluid must be coupled 50 with the deformation, and should reflect the evolution 51 of the medium's anisotropy. At the tissue scale, the mo-52 tion of the fluid is described by a velocity field, called 53 filtration velocity, obtained by eliminating the veloc-54 ity fluctuations associated with the pore scale hetero-55 geneities of the flow, and multiplying the result by the 56

Dept of Mechanical and Manufacturing Engineering, The University of Calgary

fluid volumetric fraction (see [21, 22] for the definition of 57 mass average of the pore scale velocity). In many cases 58 of biomechanical interest, the fluid filtration velocity is 59 linearly related to the pressure gradient applied to the 60 fluid. Such hypothesis is at the basis of Darcy's law [20]. 61 Apart from the description of the flow, Darcy's law is 62 also used to compute the advection velocity by which 63 nutrients and other chemicals are conveyed to the cells 64 [8]. 65

We emphasise that the standard formulation of Darcy's 66 law stems from the assumption that the macroscopic in-67 ertial forces are negligible, the stress borne by the fluid 68 is purely hydrostatic (no viscous stress is accounted 69 for), and the dissipative forces exchanged among the 70 tissue's constituents are balanced by the pressure gra-71 dient, multiplied by the volumetric fraction of the fluid 72 phase [20,1]. The second-order tensor relating the pres-73 sure gradient to the filtration velocity is referred to as 74 hydraulic conductivity, or permeability. More precisely, 75 in the jargon of Porous Media, the "hydraulic conduc-76 tivity" is the second-order tensor obtained by dividing 77 the permeability tensor by the fluid viscosity [4], while, 78 in Biomechanics, "(hydraulic) permeability" and "hy-79 draulic conductivity" are usually regarded as synonyms, 80 and are both expressed in  $mm^4N^{-1}s^{-1}$  [1]. In the se-81 quel, we follow the terminology used in Biomechanics. 82

Several anisotropic permeability models have been 83 proposed to couple the fluid filtration velocity with the 84 deformation of the matrix [42,1]. Also, as deduced in 85 the studies of Maroudas and Bullogh [31], collagen fi-86 87 bres contribute to the permeability of articular cartilage, and exert a resistance to the flow, which adds it-88 self to that supplied by the matrix. This argument has 89 been demonstrated under no or small deformations in 90 [15,14]. In these works, it is shown that the intersti-91 tial fluid flows more easily along the fibres than it does 92 across them. The effect of the fibres on the permeability 93 has also been studied under large deformations [12, 37,94 38,47]. 95

Even though the models discussed so far are purely 96 Darcian, there are cases in which Darcy's law may cease 97 to describe the flow adequately. A thorough discussion 98 on this subject can be found in [4]. There exist, in 99 fact, two typical situations in which Darcy's law should 100 be replaced by other descriptions of the flow. One of 101 these occurs when the viscous stress tensor has to be 102 considered in the momentum balance law of the fluid 103 and, in this case, one speaks of Brinkman's equation 104 [4]. The other one, instead, leads to Forchheimer's cor-105 rection of Darcy's law [4,5], and arises when the re-106 lationship between the fluid filtration velocity and the 107 108 pressure gradient acquires a quadratic term in the filtration velocity, with a coefficient depending on the mi-109

crostructure of the porous medium [5]. We remark that 110 Brinkman's equation is necessary when boundary ef-111 fects must be included in the description of the flow, 112 and that Forchheimer's correction is suggested when 113 the flow is subjected to inertial effects [27]. It should 114 be noticed, however, that these inertial effects are the 115 microscopic ones, since Forchheimer's correction is de-116 rived under the hypothesis of negligible macroscopic 117 inertial forces in the momentum balance law (see the 118 derivation in Section 3). In other words, Forchheimer's 119 correction is the representation at the tissue scale of 120 microscopic inertial terms that contribute to the drag 121 forces exchanged by the tissue's constituents [5]. 122

In this work we study Forchheimer's correction within <sup>123</sup> a nonlinear and anisotropic model of articular cartilage, <sup>124</sup> which is regarded as a hyperelastic, fibre-reinforced tissue, undergoing finite deformations and in which the <sup>125</sup> fibres are oriented statistically. We give ourselves this <sup>127</sup> task for several reasons: <sup>128</sup>

- (i) To improve the understanding of Forchheimer's 129 correction in the biomechanical context. Indeed, 130 to the best of our knowledge, in Biomechanics 131 Forchheimer's correction has not been investigated 132 until recently (one paper we are aware of is that 133 of Khaled et al. [27]), whereas it is commonly em-134 ployed in completely different contexts, like hydro-135 geology, for problems in which the deformability 136 of the porous medium hosting the flow is usually 137 disregarded. 138
- (ii) To enrich the description of the flow of biological 139 fluids. Indeed, even though in Biomechanics it is 140 usually believed that Darcy's law is sufficient to 141 model the flow, there can be situations (for exam-142 ple, in the benchmark tests performed to estimate 143 the elastic and flow properties of articular carti-144 lage) in which particularly severe loading condi-145 tions may trigger the microscopic inertial effects 146 that call for Forchheimer's correction. 147
- (iii) Since Forchheimer's correction decreases the magnitude of the fluid filtration velocity and increases
   the fluid pressure, we use it to be more conservative in establishing the pressure threshold above which the tissue health may be compromised.
- (iv) Forchheimer's correction introduces coefficients which<sup>53</sup> may be tuned to fit experimental data. We show, <sup>154</sup> indeed, that a partial agreement between two different permeability models, presented in [1] and <sup>156</sup> [12], respectively, each with its own rationale, can <sup>157</sup> be achieved by tuning a coefficient referred to as <sup>158</sup> "trial friction factor" (see Section 5.1). <sup>159</sup>

The work is organised as follows. In Section 2, we 160 present the derivation of the model equations. In Section 3, we review Darcy's law and Forchheimer's correction 162

tion. In Section 4, we present the constitutive framework. In Section 5, we describe the benchmark tests
and the related numerical results. In Section 6, we summarise the main achievements of our work.

# 2 Biphasic Model of Fibre-Reinforced Hydrated Soft Tissues

Following [15, 47], we assume that a Representative El-169 ementary Volume (REV) exists, i.e., we admit that a 170 region of space of constant size can be defined, whose 171 characteristic length scale is sufficiently smaller than 172 that of the tissue's coarse-scale heterogeneities, and suf-173 ficiently larger than that of the fine-scale ones [21]. 174 The REV is partitioned into sub-regions, and each sub-175 region is occupied by one constituent of the tissue. The 176 ratio between the measure of the sub-volume of the 177 REV filled with the interstitial fluid and the measure 178 of the REV is referred to as the fluid phase volumetric 179 fraction,  $\phi_{\rm f}$ . Under the assumption of saturation, we de-180 note by  $\phi_s = 1 - \phi_f$  the volumetric fraction of the solid 181 phase. The portion of REV filled with the solid phase 182 is subdivided into two disjoint sub-regions, occupied by 183 the matrix and the fibres with volumetric fractions  $\phi_{0s}$ 184 and  $\phi_{1s}$ , respectively [47]. It holds that  $\phi_{0s} + \phi_{1s} = \phi_s$ , 185 where  $\phi_{0s}$  and  $\phi_{1s}$  are expressed per unit volume of the 186 187 REV.

#### 188 2.1 Kinematics

As done in [47, 17, 9], we describe the kinematics of the 189 considered biphasic system by adapting to our problem 190 the theoretical framework developed for solid-fluid mix-191 tures in [40,41]. Thus, two smooth material manifolds, 192  $\mathcal{M}_{s}$  and  $\mathcal{M}_{f}$ , are introduced, representing the solid and 193 the fluid phase, respectively. The manifold  $\mathcal{M}_{s}$  is em-194 bedded into the three-dimensional Euclidean space S, 195 where it occupies the region  $\mathcal{B} \subset S$ , called reference 196 configuration. 197

Given the interval of time  $\mathcal{I} \subset \mathbb{R}$  over which the sys-198 tem's evolution is observed, the motion  $\chi$  of the solid 199 phase maps  $\mathcal{B}$  into the current configuration  $\chi(\mathcal{B},t) \subset$ 200 S. While in [40,41] the "points" of the manifolds  $\mathcal{M}_{s}$ 201 are the particles of the solid constituent of a bipha-202 sic mixture, in the present framework each particle of 203  $\mathcal{M}_s$  includes both the matrix and the fibres, and both 204 constituents are constrained to share the same motion 205  $\chi$ . The motion of the fluid is represented by a one-206 parameter family of smooth embeddings f such that, 207 at each  $t \in \mathcal{I}$ , the fluid particle  $\mathfrak{X}_{\mathbf{f}} \in \mathcal{M}_{\mathbf{f}}$  is embedded 208 into the point  $x \in S$ . The region  $\mathcal{B}_t \subset S$ , at each point 209 of which the solid and fluid particles coexist, is denoted 210

by  $\mathcal{B}_t := \chi(\mathcal{B}, t) \cap \mathfrak{f}(\mathcal{M}_{\mathfrak{f}}, t)$ . By definition of  $\mathcal{B}_t$  it holds that  $x = \chi(X, t) = \mathfrak{f}(\mathcal{X}_{\mathfrak{f}}, t)$ , for all  $x \in \mathcal{B}_t$ , with  $X \in \mathcal{B}$ and  $\mathcal{X}_{\mathfrak{f}} \in \mathcal{M}_{\mathfrak{f}}$ .

For  $x \in S$ ,  $T_xS$  is the tangent space of S attached at x, and  $TS = \bigsqcup_{x \in S} T_xS$  is the tangent bundle. Their duals are denoted by  $T_x^*S$  and  $T^*S$ , respectively. Similarly, we define  $T_X\mathcal{B}$  and  $T_X^*\mathcal{B}$ , with their bundles  $T\mathcal{B}$ and  $T^*\mathcal{B}$ . Moreover, we define the tensor spaces of order r + s, where  $r \ge 0$  and  $s \ge 0$  are arbitrary positive integers [7,12], as

$$[TS]^r{}_s = \underbrace{TS \otimes \ldots \otimes TS}_{r \text{ times}} \otimes \underbrace{T^*S \otimes \ldots \otimes T^*S}_{s \text{ times}}, \tag{1a}$$

$$[T\mathcal{B}]^{r}{}_{s} = \underbrace{T\mathcal{B} \otimes \ldots \otimes T\mathcal{B}}_{r \text{ times}} \otimes \underbrace{T^{*}\mathcal{B} \otimes \ldots \otimes T^{*}\mathcal{B}}_{s \text{ times}}.$$
 (1b)

For  $x \in \mathcal{B}_t$  and  $t \in \mathcal{I}$ , we introduce the velocity 221 vectors  $\boldsymbol{v}_{s}(x,t) \in T_{x} S$  and  $\boldsymbol{v}_{f}(x,t) \in T_{x} S$ , associated 222 with the solid and fluid phase, respectively, the rela-223 tive velocity  $m{w} = m{v}_{
m f} - m{v}_{
m s} \in T \mathbb{S}$  and the filtration ve-224 locity  $\boldsymbol{q} := \phi_{\rm f} \boldsymbol{w} = \phi_{\rm f} (\boldsymbol{v}_{\rm f} - \boldsymbol{v}_{\rm s}) \in TS$  (note that, in 225 several of our past works, we used  $\boldsymbol{w}$  to denote the 226 filtration velocity instead of q). Using the jargon of 227 Marsden and Hughes [32], we also define the velocity 228 vector fields covering  $\chi(\cdot, t)$  and  $\mathfrak{f}(\cdot, t)$ , respectively, 229 i.e.,  $\boldsymbol{u}_{s}(\cdot,t): \mathcal{B} \to TS$  and  $\boldsymbol{u}_{f}(\cdot,t): \mathcal{M}_{f} \to TS$ , which 230 satisfy the equalities  $\boldsymbol{v}_{s}(x,t) = \boldsymbol{u}_{s}(X,t) = \dot{\chi}(X,t)$  and 231  $\boldsymbol{v}_{\mathrm{f}}(x,t) = \boldsymbol{u}_{\mathrm{f}}(\mathfrak{X}_{\mathrm{f}},t) = \mathfrak{f}(\mathfrak{X}_{\mathrm{f}},t),$  with the superimposed 232 "dot" standing for partial differentiation with respect 233 to time. Finally, we introduce the deformation gradi-234 ent tensor of the solid phase, F, i.e., the tangent map 235  $T\chi(\cdot,t) = \mathbf{F}(\cdot,t): T\mathcal{B} \to T\mathcal{S}$  of the solid phase mo-236 tion  $\chi(\cdot, t)$  [32]. For  $X \in \mathcal{B}$  and  $x = \chi(X, t), F(X, t)$ : 237  $T_X \mathcal{B} \to T_x \mathcal{S}$  is a linear map transforming vectors of 238  $T_X \mathcal{B}$  into vectors of  $T_x \mathcal{S},$  and can be defined through the 239 directional derivative of  $\chi(\cdot, t)$  at  $X \in \mathcal{B}$  along some 240 vector  $\boldsymbol{U} \in T_X \mathcal{B}$ , i.e.,  $\partial_{\boldsymbol{U}} \chi(\boldsymbol{X}, t) = \boldsymbol{F}(\boldsymbol{X}, t) \boldsymbol{U} \in T_x \mathcal{S}$ . 241 The determinant of F is denoted by  $J = \det(F)$  and is 242 required to be strictly positive in order for  $\chi$  to be ad-243 missible. We emphasise that, since the matrix and the 244 fibres are assumed to share the same motion,  $\chi$ , they 245 also share the same velocity,  $\boldsymbol{v}_{s}$  (or  $\boldsymbol{u}_{s}$ ), and the same 246 deformation gradient tensor F. Along with F, we also 247 introduce the right and the left Cauchy-Green defor-248 mation tensors, denoted by  $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{g} \boldsymbol{F}$  and 249  $\boldsymbol{b} = \boldsymbol{F}.\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}\boldsymbol{G}^{-1}\boldsymbol{F}^{\mathrm{T}}$ , respectively, where  $\boldsymbol{g} \in [TS]_{2}^{0}$ 250 and  $\boldsymbol{G} \in [T\mathcal{B}]_2^0$  are the metric tensors associated with 251 the spatial and material description of the system, re-252 spectively. 253

#### 2.2 Balance Laws and Dissipation Inequality

We introduce the mass balance laws for the three constituents considered in the present model of articular 256

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cartilage, i.e., matrix, fibres and fluid phase, with the corresponding mass densities  $\rho_{0s}$ ,  $\rho_{1s}$ , and  $\rho_{f}$ , which we regard here as constant (cf. e.g. [47]). In material formalism, these balance laws read

$$\dot{\Phi}_{\alpha s} = 0, \qquad \qquad \alpha \in \{0, 1\}, \qquad (2a)$$

$$J + \operatorname{Div} \boldsymbol{Q} = 0. \tag{2b}$$

where  $\Phi_{\alpha s} := J \phi_{\alpha s}$  is the constant volumetric fraction 261 of the  $\alpha$ th solid constituent (i.e., matrix or fibres) in the 262 reference configuration, and  $\boldsymbol{Q} = J \boldsymbol{F}^{-1} \boldsymbol{q}$  is the material 263 filtration velocity, i.e., the backward Piola transform of 264 the filtration velocity  $\boldsymbol{q}$ . Note that the material form of 265 the volumetric fraction of the solid phase,  $\Phi_{\rm s} = \Phi_{\rm 0s} +$ 266  $\Phi_{1s}$ , is constant in time too, whereas the material form 267 of the fluid phase volumetric fraction is given by  $\Phi_{\rm f} =$ 268  $J - \Phi_s$ . 269

Next, we introduce the momentum balance laws, under the hypothesis that inertial and external body forces are negligible, i.e.,

$$\operatorname{div} \boldsymbol{\sigma}_{s} + \boldsymbol{\pi}_{s} = \mathbf{0}, \tag{3a}$$

$$\operatorname{div} \boldsymbol{\sigma}_{\mathrm{f}} + \boldsymbol{\pi}_{\mathrm{f}} = \mathbf{0}, \tag{3b}$$

where  $\pi_{\rm s}$  and  $\pi_{\rm f}$  represent the force densities due to the momentum exchange between the solid and the fluid constituent, and

$$\boldsymbol{\sigma}_{\rm s} = -\phi_{\rm s} \, p \, \boldsymbol{g}^{-1} + \boldsymbol{\sigma}_{\rm sc}, \tag{4a}$$

$$\boldsymbol{\sigma}_{\rm f} = -\phi_{\rm f} \, p \, \boldsymbol{g}^{-1} = -(1 - \phi_{\rm s}) p \, \boldsymbol{g}^{-1} \tag{4b}$$

are the Cauchy stress tensors associated with the solid and the fluid phase, respectively. In (4a) and (4b), p is a hydrostatic pressure called *pore pressure*, and  $\sigma_{sc}$  is the *constitutive part* of  $\sigma_s$ . Since the system under study is closed with respect to momentum, the condition  $\pi_s +$  $\pi_f = 0$  has to apply. Hence, by adding together (3a) and (3b), one obtains

$$\operatorname{div} \boldsymbol{\sigma} \equiv \operatorname{div}(-p \, \boldsymbol{g}^{-1} + \boldsymbol{\sigma}_{\mathrm{sc}}) = \boldsymbol{0}, \tag{5a}$$

$$-\boldsymbol{g}^{-1}\operatorname{grad}(\phi_{\rm f}\,p) + \boldsymbol{\pi}_{\rm f} = \boldsymbol{0},\tag{5b}$$

where  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{s} + \boldsymbol{\sigma}_{f}$  is the total Cauchy stress tensor. For the system under study, the local dissipation,  $\mathfrak{D}$ , computed per unit volume of  $\mathcal{B}_{t}$ , is given by

$$\mathfrak{D} = -\pi_{\mathrm{fd}} \cdot \frac{\boldsymbol{q}}{\phi_{\mathrm{f}}} \ge 0, \tag{6}$$

where the term  $\pi_{\rm fd} := \pi_{\rm f} - p \, g^{-1} \operatorname{grad} \phi_{\rm f}$ , i.e., the dissipative force density [20], permits to reformulate (5b) as (see e.g. [20,6])

$$\boldsymbol{\pi}_{\rm fd} = \phi_{\rm f} \boldsymbol{g}^{-1} \operatorname{grad} \boldsymbol{p}. \tag{7}$$

### 3 Darcy's Law and Forchheimer's Correction

In its classical formulation, Darcy's law is obtained under the hypothesis that  $\pi_{\rm fd}$  is determined through a constitutive function,  $\hat{\pi}_{\rm fd}$ , of the deformation gradient tensor, F, and the filtration velocity, q, with linear dependence on q, i.e., 294

$$\boldsymbol{\pi}_{\rm fd} = \hat{\boldsymbol{\pi}}_{\rm fd}(\boldsymbol{F}, \boldsymbol{q}) = -\boldsymbol{g}^{-1} \hat{\boldsymbol{r}}(\boldsymbol{F}) \boldsymbol{q}, \qquad (8)$$

where  $\mathbf{r} = \hat{\mathbf{r}}(\mathbf{F}) \in [TS]_2^0$  is the resistivity tensor. We 295 assume that  $\mathbf{r}$  is symmetric and positive-definite and, 296 thus, also invertible. Hence, by substituting (8) into (7), 297 and solving for  $\mathbf{q}$ , we obtain Darcy's law 298

$$\boldsymbol{q} \equiv \boldsymbol{q}_{\mathrm{D}} = -\phi_{\mathrm{f}} \boldsymbol{r}^{-1} \operatorname{grad} \boldsymbol{p} = -\boldsymbol{k} \operatorname{grad} \boldsymbol{p}, \qquad (9)$$

where  $\mathbf{k} = \phi_{\rm f} \mathbf{r}^{-1} \in [TS]_0^2$  is the permeability tensor. In 299 this work, we study the case in which  $\hat{\pi}_{\rm fd}$  is a quadratic 300 function of the filtration velocity (i.e., (8) no longer 301 applies), but the simplified momentum balance law (7)302 is still valid. When these conditions apply, one speaks of 303 Forchheimer's correction to Darcy's law [5]. Following 304 [53], we can express the relation between  $q_{\rm D}$  and the 305 "corrected" filtration velocity,  $\boldsymbol{q}$ , as (with our notation) 306

$$(\boldsymbol{i} + \boldsymbol{\mathcal{F}})\boldsymbol{q} = \boldsymbol{q}_{\mathrm{D}},\tag{10}$$

where i is the identity tensor, and the tensor  $\mathcal{F} \in [TS]^{1}_{1}$ 307 is the "Forchheimer's correction tensor" [53]. To ob-308 tain (10). Whitaker studied a porous medium subjected 309 to no deformation, and applied the volume-averaging 310 method to the Navier-Stokes equation modelling the 311 pore scale dynamics of the fluid [53]. In his work, the 312 correction tensor  $\mathcal{F}$  was determined by solving auxil-313 iary "closure problems" under the assumption that, at 314 a sufficiently fine scale, the porous medium enjoys the 315 discrete symmetry of spatial periodicity. Moreover,  $\mathcal{F}$ 316 was proven to depend linearly on the norm of the filtra-317 tion velocity, in the limit of sufficiently small Reynolds 318 numbers [53]. 319

By adapting the theoretical framework of [5] to our 320 problem, we show that Forchheimer's correction (10)321 can be deduced from the dissipation inequality (6). To 322 accomplish our task, we suppose that the theoretical 323 framework deduced in [53] for non deformable porous 324 media can describe also those tissues undergoing (finite) 325 deformations, even though such deformations can com-326 promise the periodicity of the internal structure. Thus, 327 by relaxing the hypothesis of periodic internal struc-328 ture, we postulate that the dissipative force  $\pi_{\rm fd}$  takes 329 on the form 330

$$\boldsymbol{\pi}_{\mathrm{fd}} = -\left[\boldsymbol{i} + \|\boldsymbol{q}\|\boldsymbol{\mathcal{A}}\right]\boldsymbol{g}^{-1}\boldsymbol{r}\boldsymbol{q},\tag{11}$$

where we have set  $\mathcal{F} := \|\boldsymbol{q}\| \mathcal{A}$ , and refer to  $\mathcal{A} \in [TS]^{1}_{1}$ as to the *tensorial Forchheimer coefficient*. By substituting (11) into (7), multiplying both sides of the resulting expression by the inverse of the resistivity tensor,  $\boldsymbol{r}^{-1}$ , and invoking the definition of the permeability tensor,  $\boldsymbol{k} = \phi_{\rm f} \boldsymbol{r}^{-1}$ , we obtain

$$[\boldsymbol{i} + \|\boldsymbol{q}\|\boldsymbol{k}\boldsymbol{\mathfrak{A}}\boldsymbol{k}^{-1}]\boldsymbol{q} = \boldsymbol{q}_{\mathrm{D}},\tag{12}$$

where  $\mathfrak{A} := g\mathcal{A}g^{-1}$  is the counterpart of  $\mathcal{A}$  in the tensor space  $[TS]_1^1$ . Our result (12) is consistent with similar results found in the literature (cf. e.g. [52], in which the case of an anisotropic porous medium is considered).

To express the Forchheimer coefficient  $\mathfrak{A}$ , we intro-341 duce the "associated" permeability tensor  $\kappa = gk \in$ 342  $[TS]_1^1$  and we assume  $\mathfrak{A} := \rho_f \kappa \beta$ , where  $\beta \in [TS]_1^1$ 343 is called non-Darcy coefficient tensor [52], and is de-344 fined according to the empirical law  $\boldsymbol{\beta} = c_0 \phi_{\rm f}^{c_1} \mu^{c_2} \boldsymbol{\kappa}^{c_2}$ 345 adapted from [46], in which  $\mu$  is the viscosity of the 346 fluid, and  $c_0 \ge 0$ ,  $c_1$ , and  $c_2$  are real constants. In the 347 jargon of Thauvin and Mohanty [46], formulae of this 348 type are said to be "correlations", since they express 349 the non-Darcy coefficient in terms of other relevant pa-350 rameters pertaining to the flow as well as the structure 351 of the considered porous medium. Since  $c_2$  is a real 352 number, the power law  $\kappa^{c_2}$  is conveniently written in 353 spectral form as 354

$$\boldsymbol{\kappa}^{c_2} = \sum_{\alpha=1}^3 (k_\alpha)^{c_2} \boldsymbol{n}^\alpha \otimes \boldsymbol{n}_\alpha, \qquad (13)$$

where  $k_{\alpha}$  is the  $\alpha$ th eigenvalue of the permeability tensor,  $\mathbf{n}^{\alpha} \in T^* S = [TS]_1^0$  is its corresponding eigencovector (determined by  $[\mathbf{\kappa} - k_{\alpha} \mathbf{i}^{\mathrm{T}}] \mathbf{n}^{\alpha} = \mathbf{0}$ ), and  $\mathbf{n}_{\alpha} =$  $g^{-1} \mathbf{n}^{\alpha}$  is the associated eigenvector. By employing (13), the Forchheimer coefficient  $\mathfrak{A}$  acquires the expression

$$\mathbf{\mathcal{A}} := \varrho_{\mathrm{f}} \boldsymbol{\kappa} \boldsymbol{\beta} = c_0 \varrho_{\mathrm{f}} \phi_{\mathrm{f}}^{c_1} \mu^{c_2} \boldsymbol{\kappa} \boldsymbol{\kappa}^{c_2}$$
$$= c_0 \varrho_{\mathrm{f}} \phi_{\mathrm{f}}^{c_1} \mu^{c_2} \sum_{\alpha=1}^{3} (k_\alpha)^{1+c_2} \boldsymbol{n}^{\alpha} \otimes \boldsymbol{n}_{\alpha}.$$
(14)

According to (14), the tensors  $\boldsymbol{\kappa}$  and  $\boldsymbol{\beta}$  are coaxial, and thus commute, i.e., it holds that  $\mathfrak{A} = \varrho_{\mathrm{f}}\boldsymbol{\kappa}\boldsymbol{\beta} = \varrho_{\mathrm{f}}\boldsymbol{\beta}\boldsymbol{\kappa}$ . This implies

$$k\mathfrak{A}k^{-1} = k(\varrho_{\mathrm{f}}\kappa\beta)k^{-1} = k(\varrho_{\mathrm{f}}\beta\kappa)k^{-1}$$
$$= g^{-1}(\varrho_{\mathrm{f}}\kappa\beta)g = g^{-1}\mathfrak{A}g = \mathcal{A}, \qquad (15)$$

and, consequently, the relation (12) becomes

$$[\mathbf{i} + \|\mathbf{q}\|\mathcal{A}]\mathbf{q} = \mathbf{q}_{\mathrm{D}}.\tag{16}$$

Finally, with the aid of (14), the identity  $\mathcal{A} = g^{-1}\mathfrak{A}g$ leads to

$$\mathcal{A} = c_0 \varrho_{\rm f} \phi_{\rm f}^{c_1} \mu^{c_2} \sum_{\alpha=1}^3 (k_\alpha)^{1+c_2} \boldsymbol{n}_\alpha \otimes \boldsymbol{n}^\alpha.$$
(17)

We remark that introducing the Forchheimer coefficient into (11) is equivalent to defining an effective resistivity tensor,  $\mathbf{r}_F := \mathbf{r} + \|\mathbf{q}\| \mathfrak{A} \mathbf{r}$ . Hence,  $\pi_{\rm fd}$  admits the sepression  $\pi_{\rm fd} = -\mathbf{g}^{-1} \mathbf{r}_F \mathbf{q}$ , which is formally similar to (8), but accounts for Forchheimer's correction. Moreover, computing explicitly  $\mathbf{r}_F$ , with  $\mathbf{r} = \phi_{\rm f} \mathbf{k}^{-1}$  and  $\mathfrak{A}$  given in (14), yields 372

$$\boldsymbol{r}_{F} = \phi_{\mathrm{f}} \boldsymbol{k}^{-1} + c_{0} \|\boldsymbol{q}\| \varrho_{\mathrm{f}} \phi_{\mathrm{f}}^{1+c_{1}} \mu^{c_{2}} \left( \sum_{\alpha=1}^{3} (k_{\alpha})^{c_{2}} \boldsymbol{n}^{\alpha} \otimes \boldsymbol{n}^{\alpha} \right).$$
(18)

Since the hypothesis of positive-definiteness of  $\boldsymbol{k}$  implies that  $\boldsymbol{r}_F$  is positive-definite too, the dissipation inequality is respected, and can be written in compact form as  $\mathfrak{D} = \phi_{\mathrm{f}}^{-1} \boldsymbol{r}_F : (\boldsymbol{q} \otimes \boldsymbol{q}) \geq 0.$  376

Before going further, we emphasise that the ten-377 sorial Forchheimer coefficient  $\mathcal{A}$  written in (17) stems 378 from the empirical laws expressing  $\mathfrak{A}$  and  $\beta$ , in which 379 the coefficients  $c_0$ ,  $c_1$ , and  $c_2$  are to be determined ex-380 perimentally. Thauvin and Mohanty [46] studied non-381 deforming isotropic porous media, for which it holds 382 that  $\mathbf{k} = k_0 \mathbf{g}^{-1}$ , and the non-Darcy coefficient tensor 383 is represented by the scalar quantity  $\beta = c_0 \phi_{\rm f}^{c_1} \mu^{c_2} k_0^{c_2}$ . 384 Moreover, they found several expressions for  $\beta$ , each 385 corresponding to a set of scalars  $\{c_0, c_1, c_2\}$ , obtained 386 for different pore structures and system sizes. Some 387 of the correlations considered in [46] were assumed to 388 depend also on the (scalar) tortuosity of the porous 389 medium. On the contrary, since we are not aware of 390 expressions of  $\beta$  explicitly determined for articular car-391 tilage, in the present work we consider  $\beta$ , by choosing 392  $c_0, c_1, and c_2$  from the literature, with a certain amount 393 of freedom ascribable to the lack of experimental data. 394 The tortuosity is not taken into account in the realisa-395 tions of the non-Darcy coefficient tensor  $\beta$  considered 396 here, since, to the best of our knowledge, there is no 397 experimental evidence of such parameter in articular 398 cartilage. 399

### 4 Materials Reinforced by Statistically Oriented Fibres

Following the line of thought and notation in [12], the 402 porous fibre-reinforced composite material studied in 403 this work is assumed to have a statistical distribution 404 of fibres, described by the probability distribution func-405 tion  $\Psi: \mathbb{S}^2\mathcal{B} \to \mathbb{R}^+_0$ , where  $\mathbb{S}^2\mathcal{B}$  is the collection of all 406 vectors  $M_X \in T_X \mathcal{B}$ , with X varying in  $\mathcal{B}$ , such that 407  $\|\boldsymbol{M}_X\| = 1$ . The value  $\Psi(\boldsymbol{M}_X)$  represents the prob-408 ability density that, at a point X, a fibre is locally 409 aligned along  $M_X$ . The probability density satisfies the 410 normalisation condition  $\int_{\mathbb{S}^{2}\mathcal{B}} \Psi(\mathbf{M}) = 1$  and, since in 411

400

this work we restrict our attention to phenomena that involve only the direction of the fibres, but not their sense, we require  $\Psi$  to fulfil also the symmetry condition  $\Psi(-M) = \Psi(M)$ . We also introduce the notation

$$\langle\!\langle \mathfrak{F} \rangle\!\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \Psi(\boldsymbol{M}) \,\mathfrak{F}(\boldsymbol{M}),$$
 (19)

denoting the *directional average* of the quantity  $\mathfrak{F}$  with respect to the probability density  $\Psi$ .

The composite is assumed to exhibit hyperelastic behaviour from the reference configuration  $\mathcal{B}$ , and its elastic potential is constructed by superimposing the elastic contribution of the matrix to that of the fibres, i.e.,

$$\hat{W}(C) = \Phi_{\rm s} \hat{U}(J) + \Phi_{\rm 0s} \hat{W}_0(C) + \Phi_{\rm 1s} \hat{W}_{\rm e}(C), \qquad (20)$$

with  $J \equiv J(\boldsymbol{C}) = \sqrt{\det(\boldsymbol{C})}$ 

$$\hat{U}(J) = \mathcal{H}(J_{\rm cr} - J)(J - J_{\rm cr})^{2q}(J - \Phi_{\rm s})^{-r},$$
(21a)

$$\hat{W}_0(\boldsymbol{C}) = \alpha_0 \frac{\exp\left(\alpha_1 [I_1 - 3] + \alpha_2 [I_2 - 3]\right)}{[I_3]^{\alpha_3}},$$
(21b)

$$\hat{W}_{e}(\boldsymbol{C}) = \hat{W}_{1i}(\boldsymbol{C}) + \langle\!\langle \hat{W}_{1a}(\boldsymbol{C}, \boldsymbol{A}) \rangle\!\rangle, \qquad (21c)$$

$$\hat{W}_{1a}(\boldsymbol{C}, \boldsymbol{A}) = \mathcal{H}(I_4 - 1) \frac{1}{2} c [I_4 - 1]^2,$$
 (21d)

where, for brevity, we used  $I_1$ ,  $I_2$ ,  $I_3$  for the three principal invariants  $I_1(\mathbf{C}) = \operatorname{tr}(\mathbf{C})$ ,  $I_2(\mathbf{C}) = \frac{1}{2} \{ [\operatorname{tr}(\mathbf{C})]^2 - \operatorname{tr}(\mathbf{C}^2) \}$ ,  $I_3(\mathbf{C}) = \operatorname{det}(\mathbf{C})$  of the right Cauchy-Green deformation tensor  $\mathbf{C}$ , and  $I_4$  for the fourth invariant  $I_4(\mathbf{C}, \mathbf{A}) = \mathbf{C} : \mathbf{A}$  of  $\mathbf{C}$  [44], in which  $\mathbf{A} = \mathbf{M} \otimes \mathbf{M}$  is the structure tensor field.

The contribution  $\hat{U}(J)$  is a penalty term depend-430 ing solely on J, and accounting for the fact that, after 431 the fluid has flown away and the pores are closed, the 432 tissue behaves as an incompressible material. In (21a), 433  $\mathcal{H}$  is the Heaviside function,  $J_{\rm cr} \in ]\Phi_{\rm s}, 1], q \geq 2$  and 434  $r \in ]0,1]$ . The quantity  $J_{cr}$  specifies a "critical value" 435 of the volume ratio J, below which the penalty term is 436 active. When this occurs,  $\hat{U}(J)$  diverges for  $J \to \Phi_s^+$ , 437 thereby preventing the violation of the unilateral con-438 straint  $J \geq \Phi_{\rm s}$ . A representation of U(J) is in Fig. 439 1. The elastic potential  $\hat{W}_0(\mathbf{C})$  in (21b) describes the 440 hyperelastic response of the matrix alone, which is as-441 sumed to be isotropic. The constitutive expression of 442  $\hat{W}_0(\mathbf{C})$  is taken from [26], where the coefficients  $\alpha_0, \alpha_1$ , 443  $\alpha_2$ , and  $\alpha_3$  are model parameters. In (21c) and (21d), 444  $\hat{W}_{1i}(\boldsymbol{C})$  denotes the isotropic part of the fibre elastic po-445 tential,  $\hat{W}_{1a}(\boldsymbol{C}, \boldsymbol{A})$  denotes the anisotropic elastic po-446 tential depending on the local direction of fibre align-447 ment, c is a material parameter, and the Heaviside step 448 function is introduced to eliminate the contribution of 449 450 the fibres that are not stretched (i.e., those for which  $I_4 \le 1$ ). 451



Fig. 1 Graphical representation of the penalty term  $\hat{U}(J)$ . The vertical asymptote at  $J = \Phi_s$  expresses that  $\hat{U}(J)$  diverges at compaction, i.e., when the lower bound  $J = \Phi_s$  of the admissible values of J is approached. The penalty term is active only for  $J < J_{crit}$ , and is zero otherwise. Graphics adapted from [12].

The anisotropic part of W(C) generates the anisotropic<sub>452</sub> contribution to  $\sigma_{\rm sc}$  given by 453

$$\hat{\boldsymbol{\sigma}}_{\mathrm{a}}(\boldsymbol{F}) = \frac{2\Phi_{\mathrm{1s}}}{J} \langle\!\langle \mathcal{H}(I_4 - 1)c[I_4 - 1]\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{\mathrm{T}}\rangle\!\rangle, \qquad (22)$$

where again, for brevity, we used  $I_4$  for  $I_4(\boldsymbol{C}, \boldsymbol{A})$ . To study the material symmetries safisfied by the constitutive tensor function  $\hat{\boldsymbol{\sigma}}_{a}(\boldsymbol{F})$ , we choose arbitrarily  $X \in$  $\mathcal{B}$ , and consider the group of all proper rotations about a material unit vector  $\boldsymbol{M}$  attached at X, i.e., 458

$$\mathcal{G}_X(\boldsymbol{M}) := \{ \boldsymbol{H} \in \operatorname{Orth}^+ \colon \boldsymbol{H}\boldsymbol{M} = \pm \boldsymbol{M} \}.$$
(23)

Hence, we notice that the integrand of (22) is a transversely isotropic tensor function with respect to  $\boldsymbol{M}$  because, for all  $\boldsymbol{H} \in \mathcal{G}_X(\boldsymbol{M})$ , the structure tensor fulfils the equality  $\boldsymbol{A} = \boldsymbol{H}\boldsymbol{A}\boldsymbol{H}^{\mathrm{T}}$ ,  $I_4$  is (by definition) invariant under the transformation  $\boldsymbol{F} \mapsto \tilde{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{H}$ , and so is also the tensor  $\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{\mathrm{T}}$ , i.e., 464

$$\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{\mathrm{T}}\mapsto\tilde{\boldsymbol{F}}\boldsymbol{A}\tilde{\boldsymbol{F}}^{\mathrm{T}}=\boldsymbol{F}\boldsymbol{H}\boldsymbol{A}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}=\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{\mathrm{T}}.$$
 (24)

If there exists a polar axis  $M_0$ , such that the probability density  $\Psi$  is restricted by the symmetry condition 465

$$\Psi(\boldsymbol{H}\boldsymbol{M}) = \Psi(\boldsymbol{M}),\tag{25}$$

for every  $\boldsymbol{M} \in \mathbb{S}^2 \mathcal{B}$  and for every  $\boldsymbol{H} \in \text{Orth}^+$  such 467 that  $\boldsymbol{H}\boldsymbol{M}_0 = \pm \boldsymbol{M}_0$ , then  $\boldsymbol{\Psi}$  is said to be transversely 468 isotropic with respect to  $\boldsymbol{M}_0$  and, because of the integration over all possible directions performed in (22), 470  $\hat{\boldsymbol{\sigma}}_{a}(\boldsymbol{F})$  is invariant under arbitrary rotations about  $\boldsymbol{M}_0$ . 471

Permeability is the material property describing the ability of a fluid to flow through the pore space of a porous medium. In this work, we focus on two permeability models that have been recently conceived to study the coupling between fluid flow and deformation in anisotropic porous media undergoing finite deformations. These models were presented in [1] and [12], and 478 are hereafter referred to as the "AW-model" and "FG-model", respectively.

The FG-model [12] extends the results obtained in 481 [15] to the framework of finite deformations, and was 482 employed in [47] to investigate the influence of the fi-483 bres' orientation on the permeability of articular carti-484 lage. In the FG-model, the permeability  $\boldsymbol{k}$  is the result 485 of an upscaling method. More precisely, in [15], at each 486 spatial point  $x \in \mathcal{B}_t \subset \mathcal{S}$ , a (rectified) fibre is assumed 487 to be locally aligned along the unit vector  $\boldsymbol{m} \in T_x S$ . 488 Then, a REV is attached at x and its size is chosen in 489 such a way that it comprehends only the fibre passing 490 from x and the portion of matrix in which the fibre is 491 embedded. Hence, the permeability of the REV,  $k_{\text{REV}}$ , 492 is determined by enforcing a self-consistent method [39] 493 (see [15] for details) under the hypothesis of validity 494 of Darcy's law at the REV scale and in the limit of 495 vanishing fibre permeability and small fibre volumetric 496 fraction. The result obtained within this approach is 497 then generalised to the case of arbitrary fibre volumetric 498 fractions by adopting differential schemes for composite 499 materials [33,36], and supposing that the permeability 500 of the matrix has a spherical representation. Thus, the 501 REV permeability determined in [15] reads 502

$$\boldsymbol{k}_{\text{REV}} = k_0 [1 - \phi_{1\text{s}}]^2 \boldsymbol{g}^{-1} + k_0 [1 - \phi_{1\text{s}}] \phi_{1\text{s}} \boldsymbol{a}, \qquad (26)$$

503 where

$$a = m \otimes m = \frac{FM}{\|FM\|} \otimes \frac{FM}{\|FM\|}$$
$$= \frac{1}{I_4(C, A)} FAF^{\mathrm{T}}, \qquad (27)$$

and m = FM/||FM||. We remark that the contribu-504 tions of the fibre to the permeability of the REV man-505 ifest themselves exclusively through the spatial struc-506 ture tensor  $\boldsymbol{a}$  and the fibre volumetric fraction  $\phi_{1s}$ . Fol-507 lowing the constitutive framework of Holmes and Mow 508 [26], the scalar permeability  $k_0$  is expressed as a consti-509 tutive function of the deformation through the volume 510 ratio J, i.e., 511

$$k_{0} := \hat{k}_{0}(J)$$
  
=  $k_{0\mathrm{R}} \left[ \frac{J - \Phi_{\mathrm{s}}}{1 - \Phi_{\mathrm{s}}} \right]^{\kappa_{0}} \exp\left(\frac{1}{2}m_{0}[J^{2} - 1]\right),$  (28)

where  $\kappa_0$  and  $m_0$  are material parameters,  $k_{0R}$  is the 512 scalar permeability in the undeformed configuration, 513 and the condition  $\lim_{J\to\Phi_s} \hat{k}(J) = 0$  is respected, since 514 the permeability has to vanish at compaction. By com-515 puting  $\mathbf{k}_{\text{REV}}$  at  $X \in \mathcal{B}$ , and considering the group 516  $\mathcal{G}_X(\boldsymbol{M})$  defined in (23), it holds that  $\boldsymbol{k}_{\text{REV}}(\boldsymbol{FH},\boldsymbol{A}) =$ 517  $\hat{k}_{\text{REV}}(F, A)$ , for all  $H \in \mathcal{G}_X(M)$ . Thus, the REV per-518 meability is transversely isotropic with respect to M. 519

By exploiting (26), the FG-model obtains the spatial permeability,  $\mathbf{k}$ , by integrating  $\mathbf{k}_{\text{REV}}$  over all possible directions, which results in a constitutive function of the deformation gradient alone [12,47], i.e., 523

$$\boldsymbol{k}_{\mathrm{FG}} = \boldsymbol{k}_{\mathrm{FG}}(\boldsymbol{F}) = \langle\!\langle \boldsymbol{k}_{\mathrm{REV}}(\boldsymbol{F}, \boldsymbol{A}) \rangle\!\rangle$$
  
$$= J^{-2} \hat{k}_0(J) [J - \Phi_{1\mathrm{s}}]^2 \boldsymbol{g}^{-1}$$
  
$$+ J^{-2} \hat{k}_0(J) [J - \Phi_{1\mathrm{s}}] \Phi_{1\mathrm{s}} \boldsymbol{F} \hat{\boldsymbol{Z}}(\boldsymbol{C}(\boldsymbol{F})) \boldsymbol{F}^{\mathrm{T}}, \qquad (29)$$

where C in (29) is understood as a function of F, and we set  $Z = \langle \frac{A}{I_4(C,A)} \rangle$ .

The backward Piola transformation of (29), i.e.,  $K_{\text{FG}} = 26$  $JF^{-1}k_{\text{FG}}F^{-\text{T}}$ , produces the material permeability of 527 the FG-model: 528

$$\boldsymbol{K}_{\rm FG} = \hat{\boldsymbol{K}}_{\rm FG}(\boldsymbol{C}) = \frac{k_0(J)[J - \Phi_{\rm 1s}]^2}{J} \boldsymbol{C}^{-1} + \frac{\hat{k}_0(J)[J - \Phi_{\rm 1s}]\Phi_{\rm 1s}}{J} \hat{\boldsymbol{Z}}(\boldsymbol{C}).$$
(30)

The AW-model considers several classes of mate-529 rial symmetries and it supplies for each of those the 530 corresponding permeability tensor. To this purpose, it 531 employs the Representation Theorems for functions val-532 ued in the space of symmetric second-order tensors [43, 533 29]. In the case of transverse isotropy with respect to a 534 direction  $M \in T\mathcal{B}$ , the AW-model defines the spatial 535 permeability tensor as 536

$$k_{AW} = k_{0i}g^{-1} + k_{1t}b + 2k_{2t}b^{2} + [k_{1a} - k_{1t}]a + 2[k_{2a} - k_{2t}]sym(a.b), \qquad (31)$$

where the coefficients  $k_{0i}$ ,  $k_{1a}$ ,  $k_{1t}$ ,  $k_{2a}$ , and  $k_{2t}$  are, in 537 general, functions of the invariants  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ , and 538  $I_5 = \mathbf{C}^2 : \mathbf{A}$  [44]. We remark that, while in this work 539 a is defined by (27), the spatial structure tensor used 540 in [1] is given by  $\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{\mathrm{T}}$  and is, thus, *not* normalised. 541 Therefore, the coefficients  $k_{1a}$ ,  $k_{1t}$ ,  $k_{2a}$ , and  $k_{2t}$  appear-542 ing in (31) must be adjusted accordingly in order for 543 (31) to be consistent with the expression provided in 544 [1].545

By comparing (26) and (31), one can see that  $\mathbf{k}_{\text{REV}}$  546 is retrieved from  $\mathbf{k}_{\text{AW}}$  in the limit of vanishing  $k_{1t}$ ,  $k_{2t}$  547 and  $k_{2a}$ , and provided that the identifications 548

$$k_{0i} := \hat{k}_{0i}(J) \equiv J^{-2} \hat{k}_0(J) [J - \Phi_{1s}]^2, \qquad (32a)$$

$$k_{1a} := \hat{k}_{1a}(J) \equiv J^{-2} \hat{k}_0(J) [J - \Phi_{1s}] \Phi_{1s}$$
(32b)

are made. In fact, whereas neglecting  $k_{1t}$  and  $k_{2t}$  can be physically motivated by the observation that the permeability along the fibres is much higher than that across the fibres [47], the absence of a coefficient of the type  $k_{2a}$  in the expression of  $k_{REV}$  descends from the chosen upscaling criterion. We regard this feature as a weak point of the FG-model. From here on, we consider only a "reduced" and slightly modified version of the AW-model, obtained by setting  $k_{1t}$ ,  $k_{2t}$ , and  $k_{2a}$  equal to zero, and choosing  $k_{0i} = \hat{k}_{0i}(J) \equiv \hat{k}_0(J)$ , and  $k_{1a} := \hat{k}_{1a}(J) \equiv J^{-2}\hat{k}_0(J)$ (cf. the form of  $k_{1a}$  with Equation (39) of [1]). Then, we write the statistical average of the material permeability of the AW-model,  $\mathbf{K}_{AW} = J \mathbf{F}^{-1} \mathbf{k}_{AW} \mathbf{F}^{-T}$ , as

$$\langle\!\langle \boldsymbol{K}_{\text{AW}} \rangle\!\rangle = J \hat{k}_0(J) \boldsymbol{C}^{-1} + J^{-1} \hat{k}_0(J) \hat{\boldsymbol{Z}}(\boldsymbol{C}).$$
 (33)

The difference between (33) and the permeability that would be obtained by adopting the original model by Ateshian and Weiss [1] is due to the division by  $I_4$ in the definition of the spatial structure tensor  $\boldsymbol{a}$  (cf. (27)). Indeed, if the spatial structure tensor were not normalised, as is the case in [1], the second term on the right-hand-side of (33) would read  $J^{-1}\hat{k}_0(J)\langle\langle \boldsymbol{A}\rangle\rangle$ .

Notice that, while the FG-model predicts that  $k_{\text{REV}}$ depends explicitly both on the fibre's volumetric fraction and on the fibre's orientation,  $k_{\text{AW}}$  depends on the direction of transverse isotropy, but may be independent on  $\phi_{1s}$ . The dependence of  $k_{\text{AW}}$  on  $\phi_{1s}$ , however, can be accounted for by extending the constitutive framework.

#### 577 5 Benchmark Tests

The model described in the previous sections requires to determine the unknowns  $\mathcal{U} = \{\chi, p, q \text{ or } Q\}$  through the solution of the equations

$$\operatorname{Div}\left(-Jp\,\boldsymbol{g}^{-1}\boldsymbol{F}^{-\mathrm{T}}+\boldsymbol{P}_{\mathrm{sc}}\right)=\boldsymbol{0},\tag{34a}$$

$$\dot{J} + \operatorname{Div} \boldsymbol{Q} = 0, \tag{34b}$$

$$(\boldsymbol{I} + \|\boldsymbol{q}\|\boldsymbol{F}^{-1}\mathcal{A}\boldsymbol{F})\boldsymbol{Q} = \boldsymbol{Q}_{\mathrm{D}},\tag{34c}$$

<sup>581</sup> in which (34c) is the material form of Forchheimer's <sup>582</sup> correction. In (34a)-(34c),  $\boldsymbol{I}$  is the identity tensor in <sup>583</sup>  $[T\mathcal{B}]^{1}_{1}, \boldsymbol{Q}_{\mathrm{D}} = -\boldsymbol{K} \operatorname{Grad} p$  is the material form of Darcy's <sup>584</sup> law, where  $\boldsymbol{K}$  is given either by (30) or by (33), depend-<sup>585</sup> ing on whether the FG- or the AW-model is used.

Equations (34a)-(34c) must be equipped with the 586 initial and boundary conditions specifying the type of 587 benchmark problem that has to be solved. To this end, 588 we assume that  $\mathcal{B}$  coincides with the configuration of 589 a cylindrical sample at time  $t_0 = 0$ , regarded as unde-590 formed and unloaded, and we partition the boundary of 591  $\mathcal{B}$  as  $\partial \mathcal{B} = \Gamma_{\rm L} \cup \Gamma_{\rm U} \cup \Gamma_{\rm B}$ , where  $\Gamma_{\rm L}$ ,  $\Gamma_{\rm U}$ , and  $\Gamma_{\rm B}$  represent 592 the lower, upper, and lateral surfaces of  $\partial \mathcal{B}$ , respec-593 tively. As benchmark problems, we consider two un-594 confined compression tests. In both tests, a cylindrical 595 sample of height  $L = 1 \,\mathrm{mm}$  and circular cross-section 596 of diameter  $D = 3 \,\mathrm{mm}$  is inserted between two paral-597 lel, impermeable and rigid plates, and compressed along 598

the direction  $M_0$  of its geometrical axis. For this pur-599 pose, a loading history is imposed to the upper plate, 600 while the lower plate is kept fixed. The two plates re-601 main parallel to each other over the entire duration of 602 the tests. In the following, we consider the Cartesian 603 orthonormal vector bases  $\{\mathbf{\mathcal{E}}_I\}_{I=1}^3$  and  $\{\boldsymbol{\varepsilon}_i\}_{i=1}^3$ , associ-604 ated with  $\mathcal{B}$  and  $\mathcal{S}$ , respectively. We assume  $\{\mathcal{E}_I\}_{I=1}^3$ 605 and  $\{\boldsymbol{\varepsilon}_i\}_{i=1}^3$  to be collinear and choose  $\boldsymbol{\mathcal{E}}_3$  coincident 606 with  $M_0$ . 607

In the first test,  $\Gamma_{\rm L}$  is clamped at the lower plate. <sup>608</sup> Thus, the original cylindrical shape of the sample is <sup>609</sup> lost during deformation, although each cross section <sup>610</sup> maintains the polar symmetry with respect to the axis <sup>611</sup>  $M_0 \equiv \mathcal{E}_3$ . Accordingly, for all times  $t \in [t_0, T_{\rm end}]$ , the <sup>612</sup> boundary conditions read <sup>613</sup>

On 
$$\Gamma_{\rm U}$$
,  $\begin{cases} \chi^3 = \mathfrak{g}, \\ (-\boldsymbol{K} \operatorname{Grad} p) \cdot \boldsymbol{N} = 0, \end{cases}$  (35a)

On 
$$\Gamma_{\rm B}$$
,  $\begin{cases} (-Jp\boldsymbol{g}^{-1}\boldsymbol{F}^{-{\rm T}} + \boldsymbol{P}_{\rm sc}).\boldsymbol{N} = \boldsymbol{0}, \\ p = 0, \end{cases}$  (35b)

On 
$$\Gamma_{\rm L}$$
, 
$$\begin{cases} \chi(X,t) - \chi(X,0) = \mathbf{0}, \\ (-\mathbf{K} \operatorname{Grad} p) \cdot \mathbf{N} = 0, \end{cases}$$
 (35c)

where N is the unit vector normal to the surface of the sample, and  $\mathfrak{g}$  is the loading history <sup>615</sup>

$$\mathfrak{g}(t) = \begin{cases} L - \frac{t}{T_{\mathrm{ramp}}} u_{\mathrm{T}}, & \text{for } t \in [0, T_{\mathrm{ramp}}], \\ L - u_{\mathrm{T}}, & \text{for } t \in ]T_{\mathrm{ramp}}, T_{\mathrm{end}}]. \end{cases}$$
(36)

Here,  $u_{\rm T} = 0.2 \,\mathrm{mm}$  is the target displacement and  $T_{\rm ramp} = {}_{616} 20 \,\mathrm{s}$  is the final instant of time of the loading ramp. The  ${}_{617} \log (36)$  is kept up to  $T_{\rm end} = 50 \,\mathrm{s}$ .

In the second test, which we call "cylindrical un-619 confined compression test", we assume that the cylin-620 drical shape of the sample is preserved by requiring that 621  $\Gamma_{\rm U}$  and  $\Gamma_{\rm L}$  glide frictionlessly on the plates' surfaces in 622 axial-symmetric way and that  $\Gamma_{\rm B}$  is a free boundary, al-623 though the sample is inhomogeneous [cf. (38a)-(40)]. In 624 fact, the inhomogeneity of the sample in the direction 625  $M_0 \equiv \mathcal{E}_3$  causes the axial strain and radial displace-626 ment to be non-constant with the space variable  $X^3$ . 627 Still, even when we consider an inhomogeneous sample 628 [see Eqs. (38a)-(40)], in our simulations the deformed 629 configurations of the sample deviates only slightly from 630 the cylindrical shape (data not shown). Thus, for the 631 purposes of this work, and in particular for the results 632 reported in section 5.2, we approximate the sample's de-633 formation with a deformation preserving the cylindrical 634 shape. In this case, the conditions (35a) and (35b) as 635 well as the no-flux condition on  $\Gamma_{\rm L}$  still apply, while the 636 null displacement condition on  $\Gamma_{\rm L}$  has to be replaced by 637 the condition  $\chi^3(X,t) = 0$ , for all  $X \in \Gamma_L$  and for all 638  $t \in [t_0, T_{\text{end}}].$ 639 640 With respect to the orthonormal vector basis  $\{\mathcal{E}_I\}_{I=1}^3$ , 641 M is written as

$$M = \hat{M}(\Theta, \Phi)$$
  
= sin  $\Theta$  cos  $\Phi \mathcal{E}_1$  + sin  $\Theta$  sin  $\Phi \mathcal{E}_2$  + cos  $\Theta \mathcal{E}_3$ , (37)

where  $\Theta \in [0, \pi]$  is the co-latitude and  $\Phi \in [0, 2\pi]$  is the 642 longitude, and the transverse isotropy of the probability 643 density  $\Psi$  means that there exists a function  $\wp: [0,\pi] \to$ 644  $\mathbb{R}^+_0$  such that the conditions  $\Psi(\mathbf{M}) = \Psi(\mathbf{M}(\Theta, \Phi)) =$ 645  $\wp(\Theta)$  is verified for all  $\Phi \in [0, 2\pi]$ . The function  $\wp$  must 646 comply with the normalisation condition and with the 647 symmetry condition  $\wp(\Theta) = \wp(\pi - \Theta)$ , for all  $\Theta \in$ 648  $[0,\pi]$ , which corresponds to  $\Psi(\mathbf{M}) = \Psi(-\mathbf{M})$ . More-649 over, since in this work we compute the statistical aver-650 ages of functions that depend on direction only through 651 the structure tensor, we are allowed to restrict the aver-652 aging integrals to one hemisphere only (e.g. the "north-653 ern" hemisphere  $\mathbb{S}^{2+}\mathcal{B}$ ). Hence, we introduce the prob-654 ability density  $\bar{\wp}: [0, \pi/2] \to \mathbb{R}_0^+$  such that the normal-655 isation reads  $2\pi \int_0^{\pi/2} \bar{\wp}(\Theta) \sin(\Theta) d\Theta = 1$ . In this work, 656 we use the pseudo-Gaussian distribution [13] 657

$$\bar{\wp}(\Theta,\xi) = \frac{\mathfrak{p}(\Theta,\xi)}{2\pi \int_0^{\pi/2} \mathfrak{p}(\Theta',\xi) \sin(\Theta') \mathrm{d}\Theta'},\tag{38a}$$

$$\mathfrak{p}(\Theta,\xi) = \exp\left(-\frac{[\Theta - Q(\xi)]^2}{2[\omega(\xi)]^2}\right),\tag{38b}$$

where both the mean angle  $Q(\xi)$  and the standard deviation  $\omega(\xi)$  depend on the normalised axial coordinate  $\xi = X^3/L$ , and are given by [13]

$$Q(\xi) = \frac{\pi}{2} \left[ 1 - \cos\left( \left( -\frac{2}{3}\xi^2 + \frac{5}{3}\xi \right) \frac{\pi}{2} \right) \right],$$
(39a)

$$\omega(\xi) = 10^3 [(1 - \xi)\xi]^4 + 0.03.$$
(39b)

A graphical representation of the functions  $\mathfrak{p}(\Theta, \xi)$ , 661  $Q(\xi)$ , and  $\omega(\xi)$ , defined in (38b), (39a), and (39b), re-662 spectively, is reported in Fig. 2. The angle  $Q(\xi)$  ranges 663 continuously from Q(0) = 0 rad at the lower boundary 664 (cartilage-bone interface) to  $Q(1) = \pi/2$  rad at the up-665 per boundary (articular surface). The variance, in turn, 666 is greater in the middle zone, since in that zone the fi-667 bres are almost randomly oriented, and thus the tissue 668 could be thought of as isotropic. Hence, the probability 669 density tends to be peaked around 0 rad for  $\xi$  approach-670 ing zero, and around  $\pi/2$  rad, for  $\xi$  approaching unity. 671

The model parameters used for the numerical simulations of the considered benchmark tests are taken from [47]. By employing experimental data available in the literature [24,10,2], we provide polynomial expressions for the volumetric fractions, i.e.,  $\Phi_{0s}(\xi) =$  $-0.062 \xi^2 + 0.038 \xi + 0.046$ ,  $\Phi_{1s}(\xi) = 0.062 \xi^2 - 0.138 \xi +$ 0.204, and  $1 - \Phi_s(\xi) = 0.100 \xi^2 + 0.750$ . To determine



Fig. 2 Graphical representation of (a) the probability density distribution in (38b) as a function of  $\Theta$  parameterised by  $\xi$  (dashed line:  $\xi = 0.15$ ; dotted line:  $\xi = 0.30$ ; thick line:  $\xi = 0.50$ ; thin line:  $\xi = 0.70$ ; dashed-dotted line:  $\xi = 0.85$ ); (b) the histological profile of the mean angle  $Q(\xi)$  given in (39a); and (c) the standard deviation  $\omega(\xi)$  given in (39b).

 $\boldsymbol{K}_{\text{FG}}$  and  $\langle\!\langle \boldsymbol{K}_{\text{AW}} \rangle\!\rangle$  (cf. (30) and (33), respectively), it is necessary to specify  $\kappa_0$ ,  $m_0$ , and  $\hat{k}_{0\text{R}}$ . As done in [47], we take here  $\kappa_0 = 0.0848$  and  $m_0 = 4.638$  [25], and we prescribe  $k_{0\text{R}}$  to be a function of the normalised axial coordinate [54], i.e., 683

$$k_{0R} \equiv k_{0R}(\xi) = k_{0R}^{(0)} \left[ \frac{e_{R}(\xi)}{e_{R}^{(0)}} \right]^{\kappa_{0}} \exp\left(\frac{1}{2}m_{0}\left[ \left(\frac{1+e_{R}(\xi)}{1+e_{R}^{(0)}}\right)^{2} - 1 \right] \right), \quad (40)$$

where  $k_{0R}^{(0)} = 0.003 \,\mathrm{mm^4 N^{-1} s^{-1}}$  is a referential value of the scalar permeability (taken of the same order of magnitude as that reported in [3]), while  $e_{\mathrm{R}}(\xi) =$  $(1 - \Phi_{\mathrm{s}}(\xi))/\Phi_{\mathrm{s}}(\xi)$  is the *void ratio* associated with the undeformed configuration (i.e., the ratio between the fluid and the solid volumetric fractions in the undeformed configuration), and  $e_{\mathrm{R}}^{(0)} = 4.0$  [34] is a referential value for  $e_{\mathrm{R}}(\xi)$ . Moreover, we assume  $\hat{W}_{\mathrm{1i}} = \hat{W}_{0}$ 

and, as done in [47], we compute the parameters  $\alpha_0$ , 692  $\alpha_1$ , and  $\alpha_2$  featuring in (21b) by imposing the condi-693 tion that the elastic coefficients obtained by linearis-694 ing  $W_0(\mathbf{C})$  are identical to those experimentally de-695 termined in [3], and fitted to the biphasic indentation 696 model presented in [30]. Since the samples of articular 697 cartilage used for the experiments reported in [3] were 698 intact and comprised both the matrix of proteoglycans 699 and the chondrocytes, we conclude that  $\alpha_0$ ,  $\alpha_1$ , and 700  $\alpha_2$  refer to the mixture of proteoglycans and cartilage 701 cells. For this purpose, we adopt the formulae provided 702 in [26], in which  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  are written as func-703 tions of the Lamé's constants. This calculation leads 704 to  $\alpha_0 = 0.1250 \text{ MPa}$ ,  $\alpha_1 = 0.7778$ , and  $\alpha_2 = 0.1111$ . 705 Finally, we choose c = 7.5 MPa [47]. 706

Equations (34a)–(34c) are solved numerically by means 707 of Finite Element methods. This necessitates to intro-708 duce the weak forms of (34a)–(34c), which are obtained 709 by multiplying each of these equations by the corre-710 sponding test function, integrating the resulting expres-711 sions over  $\mathcal{B}$ , and applying Gauss Theorem, where re-712 quired, along with the boundary conditions character-713 ising the chosen benchmark tests. Thus, one obtains: 714

$$\int_{\mathcal{B}} \{ \tilde{p}\dot{J} - (\operatorname{Grad}\tilde{p})\boldsymbol{Q} \} = 0, \qquad (41a)$$

$$\int_{\mathcal{B}} \boldsymbol{P} : \boldsymbol{g} \operatorname{Grad} \tilde{\boldsymbol{u}} = 0, \tag{41b}$$

$$\int_{\mathcal{B}} \{ [(\boldsymbol{I} + \|\boldsymbol{q}\|\boldsymbol{F}^{-1}\mathcal{A}\boldsymbol{F})\boldsymbol{Q}].\tilde{\boldsymbol{Q}} - \boldsymbol{Q}_{\mathrm{D}}.\tilde{\boldsymbol{Q}} \} = 0, \qquad (41c)$$

where  $\tilde{p}$ ,  $\tilde{u}$ , and  $\tilde{Q}$  are test functions, taken in suitable 715 functional spaces, and referred to as virtual pressure, 716 virtual velocity, and virtual (material) filtration veloc-717 ity, respectively. For the finite element discretisation of 718 the problem, piecewise quadratic Lagrange interpola-719 tion functions are used for all the unknowns and the 720 corresponding test functions. Equation (41c) is an al-721 gebraic auxiliary equation that has been introduced to 722 compute Q numerically when its analytical determina-723 tion is cumbersome (for example, when the Forchheimer 724 coefficient tensor,  $\mathcal{A}$ , is not spherical). 725

To compute the required statistical averages, we em-726 ploy the Spherical Design Algorithm [19, 11, 9]. Finally, 727 we notice that in (34c) and (41c)  $\|\boldsymbol{q}\|$  can be written as 728  $\|\boldsymbol{q}\| = J^{-1}\sqrt{\boldsymbol{C}:(\boldsymbol{Q}\otimes\boldsymbol{Q})},$  where the term under square 729 root supplies a further coupling between deformation 730 and the flow direction [16]. 731

5.1 The "Equivalent" Scalar Forchheimer Coefficient 732

Although the permeabilities predicted by both the FG-733 and the AW-model are not represented by spherical ten-734

sors, we start by defining the equivalent scalar perme-735 ability [16], 736

$$k_{\rm eq} := \sqrt{\frac{1}{3} {\rm tr}[\boldsymbol{k}.\boldsymbol{k}^{\rm T}]},\tag{42}$$

which we employ to construct the equivalent non-Darcy 737 coefficient,  $\beta_{eq} = c_0 \phi_f^{c_1} \mu^{c_2} k_{eq}^{c_2}$ , and the "equivalent" 738 scalar Forchheimer's correction,  $\mathcal{A}_{eq}$ , i.e., 739

$$\mathcal{A}_{\rm eq} := \varrho_{\rm f} k_{\rm eq} \beta_{\rm eq} = c_0 \varrho_{\rm f} \phi_{\rm f}^{c_1} \mu^{c_2} k_{\rm eq}^{1+c_2}, \qquad (43)$$

with  $c_0 \geq 0$ . The factor  $\beta_{eq}$  may depend on  $c_0, c_1,$ 740 and  $c_2$  in several ways. A review on the subject can be 741 found, for example, in [46, 16]. 742

The equivalent scalar Forchheimer coefficient  $\mathcal{A}_{eq}$  is 743 determined to invert (34c) analytically, and to study 744 the simplest case of interaction between the anisotropy 745 of the medium and the nonlinearity of the flow. Indeed, 746 if  $\mathcal{A}_{eq}i$  is used instead of  $\mathcal{A}$  in (16), Forchheimer's cor-747 rection becomes 748

$$[1 + \|\boldsymbol{q}\|\mathcal{A}_{eq}]\boldsymbol{q} = \boldsymbol{q}_{D}, \tag{44}$$

which is remnant of the result obtained in [23]. An ad-749 vantage of working with (44) is that it can be readily 750 solved for q in spite of the nonlinearity of the product 751  $\|\boldsymbol{q}\|\boldsymbol{q}$ . Indeed, taking the norm of both sides of (44), and 752 rearranging all terms, the equality (44) can be turned 753 into a quadratic equation in  $\|\boldsymbol{q}\|$  [18,16] whose only ad-754 missible solution is given by 755

$$\|\boldsymbol{q}\| = \frac{-1 + \sqrt{1 + 4\mathcal{A}_{eq}} \|\boldsymbol{q}_{D}\|}{2\mathcal{A}_{eq}}.$$
(45)

Since (45) expresses  $\|\boldsymbol{q}\|$  as a function of  $\|\boldsymbol{q}_{\mathrm{D}}\|$ , we can 756 solve (44) for  $\boldsymbol{q}$ , i.e., 757

$$\boldsymbol{q} = f \boldsymbol{q}_{\mathrm{D}},\tag{46a}$$

$$f := \frac{2}{1 + \sqrt{1 + 4\mathcal{A}_{\text{eq}} \|\boldsymbol{q}_{\text{D}}\|}},\tag{46b}$$

where f is referred to as *friction factor*. As shown in 758 [16], f can be understood as a function of the product 759  $\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|$ , and, with a slight abuse of notation, we set 760  $f = f(\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|)$ . In particular, f is such that f(0) =761 1,  $f(\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|) \sim 1 - \mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|$  for  $\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \| \to 0$ , and  $f(\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|) \sim (\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|)^{-1/2}$  for  $\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \| \to +\infty$ . We 762 763 remark that the definition of f given in (46b) looks 764 much like a result obtained in [23] (cf. Equation (20) in 765 [23]).766

If the exponents  $c_1$  and  $c_2$  in (43) are chosen as 767  $c_1 = -11/2$  and  $c_2 = -1/2$ ,  $\mathcal{A}_{eq} \| \boldsymbol{q}_{D} \|$  can be expressed 768 as a function of the Reynolds number  $\text{Re}_{D}$  [4], i.e.,

$$\mathcal{A}_{\mathrm{eq}} \| \boldsymbol{q}_{\mathrm{D}} \| = c_0 \phi_{\mathrm{f}}^{-5} \mathrm{Re}_{\mathrm{D}}, \qquad (47a)$$

$$\operatorname{Re}_{\mathrm{D}} = \varrho_{\mathrm{f}} \sqrt{\frac{k_{\mathrm{eq}}/\phi_{\mathrm{f}}}{\mu}} \|\boldsymbol{q}_{\mathrm{D}}\|.$$
(47b)

Therefore, by substituting (47a) into (46b), f can be expressed as a function of Re<sub>D</sub> and  $\phi_{\rm f}$ , with  $c_0 \geq 0$ being the only tuneable parameter, i.e.,

$$f = \frac{2}{1 + \sqrt{1 + 4c_0 \phi_{\rm f}^{-5} {\rm Re}_{\rm D}}}.$$
(48)

We emphasise that, while  $c_0$  is assumed to be constant 773 in this work, f varies in space and time, since so do also 774  $\phi_{\rm f}$  and Re<sub>D</sub>. Moreover, since the filtration velocity is 775 given by  $\boldsymbol{q} = f \boldsymbol{q}_{\mathrm{D}}$ , and f is determined either by (46b) 776 or by (48), the equations necessary to close the model 777 reduce to (34a) and (34b). Finally, we remark that the 778 definition (47b) of  $Re_D$  is slightly different from the one 779 given in [4], in which the characteristic value of  $\|\boldsymbol{q}_{\mathrm{D}}\|$  is 780 divided by the characteristic volumetric fraction of the 781 fluid phase. 782

Equation (48) implies that the strength of Forch-783 heimer's correction is influenced by  $c_0$ . Indeed, the mag-784 nitude of the filtration velocity converges to that pre-785 dicted by Darcy's law in the limit  $c_0 \rightarrow 0$ , and tends 786 towards zero for increasing  $c_0$ . This description can be 787 formalised by recognising that, for every  $\phi_{\rm f}^{-5} {\rm Re}_{\rm D}$ , f can 788 be written as  $f = \hat{f}(c_0)$ , and can be thus identified with 789 the value taken at  $c_0$  by the strictly monotonically de-790 creasing function  $\hat{f}: [0, +\infty[ \rightarrow ]0, 1]$ . This function is 791 defined by the right-hand-side of (48), and satisfies the 792 conditions  $\hat{f}(0) = 1$  and  $\lim_{c_0 \to +\infty} \hat{f}(c_0) = 0$ . We notice 793 that, since  $\hat{f}$  is continuous and strictly monotonically 794 decreasing over  $[0, +\infty)$ , it is invertible and its inverse 795  $\hat{f}^{-1}: [0,1] \to [0,+\infty]$  is continuous. Since we do not 796 have experimental data for  $c_0$ , we use the invertibility 797 of  $\hat{f}$  to determine a prescribed value of  $c_0$  such that f798 stays within a certain acceptable range. More precisely, 799 in a preliminary test, for which f = 1, we calculate 800  $\mathcal{R}_0 = \phi_f^{-5} \text{Re}_D$  at a given point and instant of time, and 801 then, by selecting a *trial* friction factor  $f_{\text{trial}} \in [0, 1]$ , we 802 obtain the corresponding value of  $c_0$  as  $c_0 = \hat{f}^{-1}(f_{\text{trial}})$ . 803 By substituting this result into (48), f can be related 804 to  $f_{\text{trial}}$ : 805

$$f = \frac{2}{1 + \sqrt{1 + \frac{4 - 4f_{\text{trial}}}{f_{\text{trial}}^2} \frac{\phi_f^{-5} \text{Re}_D}{\mathcal{R}_0}}}.$$
 (49)

In (49),  $\mathcal{R}_0$  is the value of  $\phi_f^{-5} \text{Re}_D$  at a point  $X_U$  of the 806 boundary line of  $\Gamma_{\rm U}$  and at time  $T_{\rm ramp}$ . Consistently 807 with the behaviour outlined above, f tends to unity in 808 the limit  $f_{\text{trial}} \rightarrow 1$ , thereby meaning that the filtration 809 velocity tends to converge to Darcy's solution. Since we 810 expect that Forchheimer's correction is moderate in ar-811 ticular cartilage, we regard only small deviations of the 812 flow from the predictions of Darcy's law as physically 813 admissible. Although this suggests to restrict  $\hat{f}^{-1}$  to 814

values of  $f_{\rm trial}$  sufficiently close to unity, for the sake of completeness we consider  $f_{\rm trial}$  ranging from 0.1 to 0.9.

In the simulations performed in this work, the Reynoldse17 number associated with Darcy's law ranges between 818  $10^{-8}$  and  $10^{-7}$ , thereby corresponding to a maximum 819 velocity magnitude of about  $10^{-5}$  m/s (see Fig. 4). This 820 range is often distinctive of a purely Darcian regime [4]. 821 A plausible range of variation for  $f_{\text{trial}}$  and  $c_0$  could be 822 obtained by means of the comparison between the FG-823 and the AW-model, as done in the present work. Notice 824 that, for a porous medium with  $\phi_{\rm f} \approx 0.75$ , the coeffi-825 cient  $c_0$  would have approximatively the same order of 826 magnitude as  $\operatorname{Re}_{D}^{-1}$ , and decreases with  $f_{\text{trial}}$ . 827

It should be noticed that, even though  $f_{\text{trial}}$  is con-828 stant, the friction factor reported in (49) depends on 829 space and time, and may also deviate appreciably from 830  $f_{\text{trial}}$ . Furthermore, if evaluated at  $(X_{\text{U}}, T_{\text{ramp}})$ , it does 831 not return  $f_{\text{trial}}$ . Indeed, computing  $\phi_{\text{f}}^{-5} \text{Re}_{\text{D}}$  in  $(X_{\text{U}}, T_{\text{ramp}})_{32}$ by accounting for Forchheimer's correction yields a value 833  $\mathcal{R}_1$  different from  $\mathcal{R}_0$ , which is instead computed by us-834 ing Darcy's law only. In fact, as we will see in the follow-835 ing, Forchheimer's correction leads to higher pressures 836 and lower magnitudes of the velocity field in the do-837 main, thereby leading to usually lower Reynolds num-838 ber. In this respect, the friction factor (49) is "inconsis-839 tent". This discrepancy, however, can be reduced by it-840 erating the determination of the friction factor as shown 841 in Algorithm 1. 842

# **Algorithm 1** Procedure for determining the friction factor f

Choose a tolerance TOL > 0 and  $f_{\text{trial}} \in [0, 1];$ 1: Compute  $\mathcal{R}_0$  by using Darcy's law; 3: Compute  $f_0$ [cf. Eq. (49)]; -5 ReD  $1 + \sqrt{1 + \frac{4 - 4f_{\text{trial}}}{f_{\text{trial}}^2} \frac{\phi_{\text{f}}^{-5} \text{Re}}{\mathcal{R}_0}}$ 4: if  $|f_0(X_U, T_{\text{ramp}}) - f_{\text{trial}}| < \text{TOL then}$ 5: $f = f_0;$ else6: 7: Compute  $\mathcal{R}_{k+1}$  by using  $f_k$ ; 8: <u>9</u>: Compute  $f_{k+1}$  $\int_{k+1}^{\infty} \frac{1}{1+\sqrt{1+\frac{4-4f_{\text{trial}}}{f_{\text{trial}}^2}}} \frac{\phi_{\text{f}}^{-5}\text{Rep}}{\frac{3}{R+1}},$ if  $|f_{k+1}(X_{\text{U}}, T_{\text{ramp}}) - f_{\text{trial}}| < \text{TOL then}$ 10: 11: 12: $= f_{k+1}$ else 13:k = k + 1;14:Go to 8:

To see how the friction factor varies in space, and 843 to highlight how its spatial distribution is influenced 844 by the fluid filtration velocity, which, in turn, depends 845 through the permeability tensor on the anisotropy and 846 inhomogeneity of the tissue, we run two simulations of 847 the first benchmark problem (i.e., an unconfined com-848 pression test in which the lower boundary of the sample 849 is clamped). For the first simulation, we consider the 850

transversely isotropic and inhomogeneous model dis-851 cussed in the previous sections, while for the second 852 simulation we study a simplified framework in which 853 the anisotropic contribution of the fibres is not taken 854 into account, and all material parameters are constant 855 with the depth of the sample. In particular, we set 856  $\Phi_{\rm s} = \Phi_{0\rm s} = 0.15$ . Hence, the scalar permeability in (40) 857 becomes constant through the depth of the sample, and 858 equal to  $k_{0R} = 0.0188 \,\mathrm{mm^4 N^{-1} s^{-1}}.$ 859

By comparing Fig. 3 with Fig. 4, which represent 860 f and  $q_{\rm D}$  at  $t = T_{\rm ramp}$ , respectively, we notice that, as 861 expected, the friction factor is higher in the zones of the 862 sample in which the Reynolds number is lower, i.e., in 863 the central zone of the sample, and it approaches  $f_{\rm trial}$ 864 in the external zone, for both the inhomogeneous (Fig. 865 3(a) and Fig. 4(a)) and the homogeneous case (Fig. 866 3(b) and Fig. 4(b)). As experimentally observed, both 867 the porosity and the permeability of articular cartilage 868 experience strong variations along the tissue's depth. If 869 the inhomogeneity of these physical quantities is mod-870 elled, the pathways of the fluid inside the tissue vary 871 sensibly with respect to the homogeneous case. On the 872 contrary, as shown in Fig. 3(b) and Fig. 4(b), when  $\Phi_{\rm s}$ 873 and  $k_{0R}$  are assumed to be constant, the variation of 874 both the filtration velocity and the friction factor along 875 the sample depth is less pronounced than it is in the 876 inhomogeneous (and transversely isotropic) case. 877

We report in Fig. 5 the patterns of  $\boldsymbol{q}$  at  $t = T_{\text{ramp}}$ , 878 as obtained by employing Forchheimer's correction with 879  $f_{\text{trial}} = 0.1$ , both in the inhomogeneous and anisotropic 880 case (Fig. 5a) and in the homogeneous and isotropic 881 case (Fig. 5b). We observe that, when Forchheimer's 882 correction is introduced, the filtration velocity tends 883 to become more spatially uniform than that predicted 884 by Darcy's law, and a small distortion of the stream-885 lines occurs at the bottom of the sample, where zero-886 displacement boundary conditions are imposed. 887

In Fig. 6 we show the influence of Forchheimer's cor-888 rection on the magnitude of the filtration velocity and 889 pressure for different values of  $f_{\text{trial}}$ . At each time, the 890 values on the vertical axis refer to the maxima attained 891 by the magnitude of the filtration velocity,  $\|\boldsymbol{q}\|$ , and 892 pressure, p, within the sample. In particular,  $\|q\|$  is eval-893 uated at the point  $X_{\rm U}$  defined above, while p is taken at 894 the point  $X_{\rm L} = (0, 0, 0)$  (centre of the lower boundary 895 of the sample). First, we report the results of two sim-896 ulations, performed by using Darcy's law, in which the 897 permeability is given once by the FG-model and once by 898 AW-model. Looking at Fig. 6, we notice that the results 899 predicted by the Darcy-based FG- and AW-model are 900 in remarkable disagreement with each other, although 901 they both seem to be physically plausible. In particu-902 lar, the AW-model returns a higher filtration velocity 903





Fig. 3 The friction factor f varies according to the variation of both the filtration velocity and the permeability (computed here with the AW-model). (a): Transversely isotropic and inhomogeneous model with  $f_{\text{trial}} = 0.3$ . (b): Homogeneous and isotropic case with  $f_{\text{trial}} = 0.3$ . Close to the outer wall of the sample, where the filtration velocity is higher, the values of f are smaller (thereby yielding a stronger Forchheimer's correction) than those at the centre of the sample. The plots are evaluated at  $t = T_{\text{ramp}}$ . (Colour figure online)

and a lower pore pressure (black dotted curve) than 904 those obtained by using the FG-model (black curves). 905 By tuning  $f_{\text{trial}}$ , a partial agreement between the two 906 models can be achieved. Indeed, as we can see from 907 Fig. 6, Forchheimer's correction contributes to lower 908 the magnitude of the filtration velocity and to raise the 909 pressure, thereby reducing the mean distance between 910 the results of the AW-model and those of the FG-model. 911 Moreover, an *optimal* value of  $f_{\text{trial}}$  can be obtained by 912 means of an optimisation procedure that minimises the 913 distance between the magnitude of Darcy's velocity ob-914 tained by means of the FG-model, and the magnitude 915 of the filtration velocity obtained with the AW-model 916 modified by Forchheimer's correction. Here, we set 917

$$f_{\text{trial}} = f_{\text{opt}} = \tilde{f}_{\text{trial}}(\xi)$$
  
= 2.38  $\xi^3 - 3.51 \xi^2 + 1.69 \xi + 0.07.$  (50)

Such optimal value varies in space, due to the spatial  $_{918}$  variations of the computed f (see Fig. 3).  $_{919}$ 



**Fig. 4** Patterns of Darcy's filtration velocity  $q_{\rm D}$  at  $t = T_{\rm ramp}$  as predicted by the AW-model. (a): Transversely isotropic and inhomogeneous model. (b): Isotropic and homogeneous model. The black curves represent the streamlines. The zones of higher velocity correspond to the zones of lower friction factor in Fig. 3. (Colour figure online)

It is important to notice that, when Forchheimer's correction is introduced, both the magnitude of the filtration velocity and the pressure relax towards the stationary states more slowly than in the Darcian case.

#### <sup>924</sup> 5.2 Diagonal Forchheimer Coefficient Tensor

In this section, we consider the benchmark test of the 925 second kind, in which the original shape of the sample 926 is approximately maintained by the deformation. We 927 say "approximately" because, in spite of tissue's inho-928 mogeneity, the deformed shape of the sample deviates 929 only slightly from the original, cylindrical one. We de-930 note by  $\{E_I(X)\}_{I=1}^3 \in T_X \mathcal{B}$  and  $\{e_i(x)\}_{i=1}^3 \in T_x \mathcal{S}$  the 931 collinear, orthonormal vector bases attached at  $X \in \mathcal{B}$ 932 and  $x = \chi(X,t) \in S$ , respectively, with  $E_1(X)$  and 933  $e_1(x)$  oriented radially,  $E_2(X)$  and  $e_2(x)$  circumferen-934 tially, and  $E_3(X)$  and  $e_3(x)$  axially. Forchheimer's cor-935 rection tensor  $\mathcal{A}$  is diagonal with respect to the basis 936



**Fig. 5** Patterns of filtration velocity  $\boldsymbol{q}$  at  $t = T_{\text{ramp}}$  for  $f_{\text{trial}} = 0.1$ . (a): Transversely isotropic and inhomogeneous model. (b): Isotropic and homogeneous case. The black curves represent the streamlines. Both simulations are obtained for the AW-model. The filtration velocity is more uniform in the domain, and lower than that obtained in the Darcian case (cf. Fig. 4). (Colour figure online)

$$\{e_i(x)\}_{i=1}^3 \in T_x \mathcal{S}$$
, and can be written as

$$\mathcal{A} = \sum_{i=1}^{3} \mathcal{A}^{i}{}_{i} \boldsymbol{e}_{i} \otimes \boldsymbol{e}^{i} = \sum_{i=1}^{3} \mathcal{A}_{i} \boldsymbol{e}_{i} \otimes \boldsymbol{e}^{i}, \qquad (51)$$

where  $\{e^{i}(x)\}_{i=1}^{3} \in T_{x}^{*}S$  is the covector basis dual of  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ , and  $\mathcal{A}_{i} \equiv \mathcal{A}^{i}_{i} > 0, i \in \{1, 2, 3\}$  (no  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$ ) sum with respect to i), are the transversal and axial  $\{e_{i}(x)\}_{i=1}^{3} \in T_{x}S$  (no  $e_{i}(x)$ ) sum with respect to i), are the transversal and axial  $e_{i}(x)$  (no  $e_{i}(x)$ ) such that the transversal is the transversal and axial  $e_{i}(x)$  (no  $e_{i}(x)$ ) such that the transversal is the transversal is the transversal axial  $e_{i}(x)$  (no  $e_{i}(x)$ ) such that the transversal is the

$$\left[\sum_{i=1}^{3} (1 + \mathcal{A}_i \| \boldsymbol{q} \|) \boldsymbol{e}_i \otimes \boldsymbol{e}^i\right] \boldsymbol{q} = \boldsymbol{q}_{\mathrm{D}}.$$
(52)

Let  $\mathcal{A}_{\mathrm{M}} = \max_{i \in \{1,2,3\}} \{\mathcal{A}_i\}$  and  $\mathcal{A}_{\mathrm{m}} = \min_{i \in \{1,2,3\}} \{\mathcal{A}_i\}$  943 be the maximum and the minimum eigenvalue of  $\mathcal{A}$ , 944 respectively. Hence, the inequality  $\mathcal{A}_{\mathrm{M}} \geq \mathcal{A}_{\mathrm{m}}$ , with the 945 equality sign being satisfied in the isotropic case, implies the estimates 947

$$(1 + \mathcal{A}_{\rm m} \|\boldsymbol{q}\|) \|\boldsymbol{q}\| \le \|\boldsymbol{q}_{\rm D}\| \le (1 + \mathcal{A}_{\rm M} \|\boldsymbol{q}\|) \|\boldsymbol{q}\|.$$
(53)



**Fig. 6** Magnitude of the filtration velocity ||q|| (a) and pressure p (b), for different values of  $f_{\text{trial}}$ , computed at the point of the sample in which each of these quantities attains its maximum. The black solid curves with and without markers represent the output of the AW and the FG model, respectively, in a purely Darcian regime. The coloured curves are obtained by means of the scalar Forchheimer's correction applied to the AW permeability model. (Colour figure online)

<sup>948</sup> By equating  $\|\boldsymbol{q}_{\mathrm{D}}\|$  with its lower and upper bound, <sup>949</sup> Eq. (53) allows to deduce two admissible extremal so-<sup>950</sup> lutions for  $\|\boldsymbol{q}\|$ , i.e.,

$$\gamma(\mathcal{A}_{\mathrm{M}}) := \frac{-1 + \sqrt{1 + 4\mathcal{A}_{\mathrm{M}} \|\boldsymbol{q}_{\mathrm{D}}\|}}{2\mathcal{A}_{\mathrm{M}}},\tag{54a}$$

$$\gamma(\mathcal{A}_{\mathrm{m}}) := \frac{-1 + \sqrt{1 + 4\mathcal{A}_{\mathrm{m}} \|\boldsymbol{q}_{\mathrm{D}}\|}}{2\mathcal{A}_{\mathrm{m}}}.$$
(54b)

<sup>951</sup> It can be proven that the inequality  $\gamma(\mathcal{A}_{\rm M}) \leq \gamma(\mathcal{A}_{\rm m})$ <sup>952</sup> holds true. Consistently, the magnitude of the filtration <sup>953</sup> velocity is said to be admissible if it complies with the <sup>954</sup> chain of inequalities

$$\gamma(\mathcal{A}_{\mathrm{M}}) \le \|\boldsymbol{q}\| \le \gamma(\mathcal{A}_{\mathrm{m}}). \tag{55}$$

We remark that  $\gamma(\mathcal{A}_{\mathrm{M}})$  and  $\gamma(\mathcal{A}_{\mathrm{m}})$  depend on  $\|\boldsymbol{q}_{\mathrm{D}}\|$ , which, in turn, depends on the permeability and pressure gradient. The lower and the upper bounds of  $\|\boldsymbol{q}\|$  are obtained by evaluating the same function,  $\gamma$ , once in the maximum and once in the minimum eigenvalue of  $\mathcal{A}$ . Thus, if we set  $||\mathbf{q}|| = \gamma(\mathcal{A}_j)$ , with  $j \in \{1, 2, 3\}$ , and substitute the result into (52), we obtain

$$+ \mathcal{A}_{i} \|\boldsymbol{q}\| = 1 + \mathcal{A}_{i} \gamma(\mathcal{A}_{j})$$

$$= \frac{(2 - \zeta_{ij}) + \zeta_{ij} \sqrt{1 + 4\mathcal{A}_{j}} \|\boldsymbol{q}_{\mathrm{D}}\|}{2}$$

$$=: \frac{1}{f_{ij}},$$

$$(56)$$

1

where  $\zeta_{ij} := \mathcal{A}_i / \mathcal{A}_j$  is referred to as anisotropy ratio, and  $f_{ij}$  is said to be the corresponding friction factor. Note that  $\zeta_{ji} = 1/\zeta_{ij}$ , and (52) becomes

$$\left[\sum_{i=1}^{3} \frac{1}{f_{ij}} \boldsymbol{e}_i \otimes \boldsymbol{e}^i\right] \boldsymbol{q} = \boldsymbol{q}_{\mathrm{D}}, \quad j \in \{1, 2, 3\},$$
(57)

which can be inverted to express the filtration velocity as 965

$$\boldsymbol{q}_{(j)} = \left[\sum_{i=1}^{3} f_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}^i\right] \boldsymbol{q}_{\mathrm{D}}, \quad j \in \{1, 2, 3\}.$$
(58)

where  $q_{(i)}$  is the value of the filtration velocity whose 967 norm is  $\gamma(\mathcal{A}_i)$ , with  $j \in \{1, 2, 3\}$ . To complete the 968 description, we provide an explicit expression for the 969 anisotropy factors,  $\zeta_{ij}$ , with  $i, j \in \{1, 2, 3\}$ . In the AW-970 model, the spatial permeability tensor is given by k =971  $J^{-1}F\langle\!\!\langle K_{\rm AW}\rangle\!\!\rangle F^{\rm T}$ , and depends on the tensor Z. Due 972 to the particular form of the deformation, Z possesses 973 transverse isotropy with respect to  $E_3$ , which is paral-974 lel to  $M_0$ . Since  $A_0 := E_3 \otimes E_3$  and  $T_0 = E_1 \otimes E_1 + C_1$ 975  $E_2 \otimes E_2$  span the space of all symmetric second-order 976 tensors of the type  $[T\mathcal{B}]_0^2$  exhibiting transverse isotropy 977 with respect to  $M_0 \equiv E_3$ , we can write 978

$$\boldsymbol{Z} = \left\langle\!\!\left\langle \frac{\boldsymbol{A}}{I_4(\boldsymbol{C},\boldsymbol{A})} \right\rangle\!\!\right\rangle = Z_{\rm t} \boldsymbol{T}_0 + Z_{\rm a} \boldsymbol{A}_0, \tag{59}$$

with  $Z_t$  and  $Z_a$  being the transverse and axial components of Z, respectively. Since, for the considered benchmark test, F is assumed to admit in cylindrical coordinates the representation  $F = \lambda_1 e_1 \otimes E^1 + \lambda_2 e_2 \otimes$  $E^2 + \lambda_3 e_3 \otimes E^3$ , the tensor  $z = FZF^T$  is given by

$$\begin{aligned} \boldsymbol{z} &= Z_{t}\boldsymbol{b} + (Z_{a} - Z_{t})\boldsymbol{F}\boldsymbol{A}_{0}\boldsymbol{F}^{\mathrm{T}} \\ &= Z_{t}\boldsymbol{b} + (Z_{a} - Z_{t})I_{40}\boldsymbol{a}_{0}, \end{aligned}$$
(60)

with  $\boldsymbol{b} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{e}_i \otimes \boldsymbol{e}_i$  being the left Cauchy-Green get deformation tensor,  $I_{40} = \boldsymbol{C} : \boldsymbol{A}_0 = \lambda_3^2$ , and  $\boldsymbol{a}_0 = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ . We remark that  $\boldsymbol{Z}$  features the nonzero transversal component  $Z_t$  even though  $I_4^{-1}\boldsymbol{A}$  does not have any get transverse component along the local transverse projection tensor  $T = G^{-1} - A$  [50, 51, 47]. Consequently, the spatial permeability is given by

$$\begin{aligned} \boldsymbol{k} &= \hat{k}_0(J)\boldsymbol{g}^{-1} + J^{-2}\hat{k}_0(J)Z_{\mathbf{t}}\boldsymbol{b} \\ &+ J^{-2}\hat{k}_0(J)(Z_{\mathbf{a}} - Z_{\mathbf{t}})I_{40}\boldsymbol{a}_0, \end{aligned}$$
(61)

and, more explicitly, it can be written as  $\mathbf{k} = \sum_{i=1}^{3} k_i \, \mathbf{e}_i \otimes \mathbf{e}_i$ , where the scalar permeabilities are defined by

$$k_1 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_1^2, \tag{62a}$$

$$k_2 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_t\lambda_2^2, \tag{62b}$$

$$k_3 = \hat{k}_0(J) + J^{-2}\hat{k}_0(J)Z_2\lambda_2^2.$$
(62c)

Accordingly, the eigenvalues of the Forchheimer coefficient tensor read

$$\mathcal{A}_1 = c_0 \varrho_{\rm f} \phi_{\rm f}^{c_1} \mu^{c_2} \left( \hat{k}_0(J) + J^{-2} \hat{k}_0(J) Z_{\rm t} \lambda_1^2 \right)^{1+c_2}, \quad (63a)$$

$$\mathcal{A}_{2} = c_{0} \varrho_{\rm f} \phi_{\rm f}^{c_{1}} \mu^{c_{2}} \left( \hat{k}_{0}(J) + J^{-2} \hat{k}_{0}(J) Z_{\rm t} \lambda_{2}^{2} \right)^{1+c_{2}}, \quad (63b)$$

$$\mathcal{A}_{3} = c_{0}\varrho_{\rm f}\phi_{\rm f}^{c_{1}}\mu^{c_{2}}\left(\hat{k}_{0}(J) + J^{-2}\hat{k}_{0}(J)Z_{\rm a}\lambda_{3}^{2}\right)^{1+c_{2}}.$$
 (63c)

<sup>995</sup> Therefore, if we choose the anisotropy factors  $\zeta_{31}$  and  $\zeta_{32}$ , we obtain

$$\zeta_{3j} = \left[\frac{J^2 + Z_{\rm a}\lambda_3^2}{J^2 + Z_{\rm t}\lambda_j^2}\right]^{1+c_2}, \quad j = \{1, 2\}.$$
(64)

<sup>997</sup> In the undeformed configuration, it holds that  $Z_{\rm a}$  + <sup>998</sup>  $2Z_{\rm t} = 1$ , which yields

$$\zeta_{31} = \zeta_{32} = \zeta = \left[\frac{1+Z_{\rm a}}{1+Z_{\rm t}}\right]^{1+c_2} = \left[\frac{2-2Z_{\rm t}}{1+Z_{\rm t}}\right]^{1+c_2}.$$
 (65)

<sup>999</sup> Note that, in the undeformed configuration, it holds that  $\zeta_{12} = \zeta_{21} = 1$ . The correlations used in this work to express  $\beta_{eq}$  were taken from [28], and are referred to as Coles&Hartman correlation,  $\beta_{eq} = c_0 \phi_f^{0.449} \mu^{-1.88} k_{eq}^{-1.88}$ and Geertsma correlation,  $\beta_{eq} = c_0 \phi_f^{-5.5} \mu^{-0.5} k_{eq}^{-0.5}$ . For the computations, the Coles&Hartman correlation has been approximated by setting  $c_2 = -2$ .

The curves in Fig. 7a refer to two different sets 1006 of computations of the magnitude of the filtration ve-1007 locity: once by employing the equivalent scalar Forch-1008 heimer's coefficient  $\mathcal{A}_{eq}$  defined in (43) and the opti-1009 mised friction factor  $f_{opt}$ , and once in the case of di-1010 agonal  $\mathcal{A}$ . The continuous curve is obtained for  $\|q\| =$ 1011  $\gamma(\mathcal{A}_1)$ , and the marked curve is obtained for  $\|\boldsymbol{q}\| =$ 1012  $\gamma(\mathcal{A}_3)$ . From Fig. 7a, we can see that the curve ob-1013 tained by expressing the norm  $\|\boldsymbol{q}\|$  of the filtration ve-1014 locity as a function of  $\mathcal{A}_1$  is quite compatible with the 1015 one obtained as a result of the equivalent scalar case. 1016 The greatest distance between the two curves, i.e., the 1017 one obtained for  $\|\boldsymbol{q}\| \equiv \gamma(\mathcal{A}_1)$  and the one obtained 1018

for  $\|\boldsymbol{q}\| \equiv \gamma(\mathcal{A}_3)$ , can be registered in the neighbour-1019 hood of  $t = T_{\text{ramp}}$ , i.e., when Forchheimer's correction 1020 is more significant due to the higher values of the fil-1021 tration velocity in the sample. In Fig. 7b, the friction 1022 factors  $f_{ij}$ , with i, j = 1, 3, are compared with  $f_{opt}$ . 1023 As a consequence of the inhomogeneity of the perme-1024 ability through the depth of the sample,  $\mathcal{A}_1$  and  $\mathcal{A}_3$ 1025 acquire the role of maximum or minimum eigenvalue 1026 of  $\mathcal{A}$ , respectively. In particular, the axial friction fac-1027 tors  $f_{13}$  and  $f_{33}$  are higher than the longitudinal ones 1028 in the deep zone of the sample. Due to the randomness 1029 of the distribution of the fibres in the middle zone, also 1030 the material parameters are such that, in this zone, an 1031 isotropic behaviour can be observed. In this zone, in-1032 deed, all the friction factors merge, whereas at the top 1033 of the sample, the transversal friction factors  $f_{11}$  and 1034  $f_{31}$  have a greater value than the axial ones. Moreover, 1035 at the top of the sample, the friction factors related to 1036 the transversal eigenvalue  $\mathcal{A}_1$ , i.e.,  $f_{11}$  and  $f_{13}$ , are both 1037 higher than the ones related to the axial eigenvalue  $\mathcal{A}_3$ , 1038 i.e.,  $f_{31}$  and  $f_{33}$ , whereas the latter two are higher than 1039 the transversal ones at the bottom. 1040

#### 5.3 Fully Tensorial Case

In this section, we simulate the second benchmark test, 1042 in which the original cylindrical shape of the sample is 1043 disrupted by the deformation, and we consider a not 1044 necessarily diagonal Forchheimer coefficient  $\mathcal{A}$ . In this 1045 case, which we call "fully tensorial case", we prefer to 1046 invert the relation (34c) numerically. For determining 1047  $\mathcal{A}$ , we employ the non-Darcy coefficient tensor  $\mathcal{B}$ , with 1048 the exponents  $c_0$ ,  $c_1$ , and  $c_2$  predicted by the previously 1049 introduced approximation of the Coles&Hartman corre-1050 lation. For comparison, we consider also the benchmark 1051 test of the first type (which approximately maintains 1052 the sample's cylindrical shape). In Fig. 8, we show the 1053 time variation of the magnitude of the filtration veloc-1054 ity,  $\|\boldsymbol{q}\|$ , for the fully tensorial case, and for the ex-1055 tremal values  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_1)$  and  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_3)$ , obtained 1056 in the case of diagonal Forchheimer coefficient tensor. 1057 From Fig. 8, we see that the Coles&Hartman correla-1058 tion induces a greater difference between the extremal 1059 curves, with respect to those plotted in Fig. 7, which 1060 were obtained for  $c_1$  and  $c_2$  taken from the Geertsma 1061 correlation, and  $c_0 = \hat{f}^{-1}(f_{\text{opt}})$ , with  $f_{\text{opt}} = \tilde{f}_{\text{trial}}(\xi)$  as 1062 in (50). 1063

Figure 8 represents also a validation of the results 1064 obtained by solving numerically (34c). Indeed, the numerical results of the two extremal cases,  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_1)$  1066 and  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_3)$ , act as an upper and a lower bound 1067 for the results of the fully tensorial case, depending on 1068 which eigenvalue attains the maximum and minimum 1069

0.0000

[m/s]

ation Velocity

(a)

(b)

0.

0.

0.2L

0.2

0.3 0.4 0.5 0.6 0.7

0.1

Friction factor [-]

of the formula (56), with i = 1 and j = 1, for the dashed curve, and j = 3, for the solid curve. Analogously, the red curves are obtained by choosing i = 3. (Colour figure online) value, respectively. Thus, in Fig. 7a, we see that the magnitude of the filtration velocity obtained as an out-

Fig. 7 (a): Filtration velocity evaluated at the upper ex-

ternal point of the sample. (b): Friction factors related to

each of them for  $t = T_{ramp}$  vs the normalised axial coor-

dinate  $\xi$ . The vertical line along which the variation of the

friction factor is observed intersects the lower boundary at

 $X^1 = X^2 = 0.5$  mm. The blue curves are obtained by means

value, respectively. Thus, in Fig. 7a, we see that the
magnitude of the filtration velocity obtained as an outcome of the fully tensorial case lies in between the two
extremal solutions, and it is quite compatible with the
result obtained with the equivalent scalar Forchheimer's
correction.

In Fig. 9a, the results obtained with the scalar Forch-1076 heimer's correction, which corresponds here to the opti-1077 mised friction factor,  $f_{opt}$ , are compared with those ob-1078 tained with the fully tensorial correction for the case of 1079 clamped lower boundary of the sample (this boundary 1080 condition is closer to the system's phenomenology, since 1081 it simulates the attachment of articular cartilage to the 1082 subchondral bone). For completeness, we report also 1083 the magnitude of the filtration velocity as predicted by 1084 the FG- and the AW-model within the Darcian regime. 1085

**Fig. 8** Magnitude of the filtration velocity for the fully tensorial case (plain rings), for the diagonal case and extremal values  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_1)$  and  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_3)$  (continuous line and asterisks respectively), and for the casivelant case are written as the result of the result.

**Fig. 6** Magnitude of the nitration velocity for the fully tensorial case (plain rings), for the diagonal case and extremal values  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_1)$  and  $\|\boldsymbol{q}\| = \gamma(\mathcal{A}_3)$  (continuous line and asterisks, respectively), and for the equivalent scalar case with optimised friction factor  $f_{\text{opt}}$ . All velocities are evaluated at a point  $X_{\text{U}}$  of the boundary line of  $\Gamma_{\text{U}}$  by employing the Coles&Hartman correlation, with  $c_0 = 5 \cdot 10^{-18}$ .

As seen in Fig. 7a, also with a different boundary condi-1086 tion, the equivalent scalar Forchheimer coefficient and 1087 the fully tensorial Forchheimer coefficient return, in our 1088 work, a quite compatible numerical result. Indeed, the 1089 curves representing the fully tensorial case (plain circles 1090 in Fig. 9a) are almost overlapped to the dashed ones, 1091 which represent the scalar equivalent case. Finally, Fig. 1092 9b shows the time variation of the pressure at the cen-1093 tre of the lower boundary of the sample. We remark 1094 that, in contrast to what happens to the magnitude 1095 of the filtration velocity, the pressure predicted by the 1096 fully tensorial model is lower than that obtained by 1097 the equivalent scalar model. Moreover, the curves ob-1098 tained within the Darcian regime by employing the FG-1099 and the AW-model predict sufficiently smaller values 1100 of pressure and, in particular, the lowest pressures are 1101 those predicted by the AW-model. Finally, we notice 1102 that, also in the tensorial case, Forchheimer's correc-1103 tion implies that the magnitude of the filtration velocity 1104 and pressure relax towards the stationary states more 1105 slowly than in the Darcian case. 1106

## 6 Discussion and Conclusions

In this work, we studied some consequences of Forch-1108 heimer's correction to Darcy's law in the study of the 1109 fluid flow in a hydrated biological tissue such as ar-1110 ticular cartilage. To imitate the internal structure of 1111 the examined target tissue, its reinforcing fibres were 1112 assumed to be oriented statistically, as predicted by a 1113 probability density compatible with the tissue's histol-1114 ogy. Also the volumetric fractions of matrix and fibres 1115



0.8

- '<sub>opt</sub> ||q||=γ(A<sub>3</sub>)

||q||=γ(A





Fig. 9 (a): Magnitude of the filtration velocity for the fully tensorial case (plain rings), for the equivalent scalar case with optimised friction factor  $f_{\rm opt}$  (asterisks), for the Darcian regime and FG-model (dash-dotted line), and for the Darcian regime and AW-model (continuous line). All velocities are evaluated at a point  $X_{\rm U}$  of the boundary line of  $\Gamma_{\rm U}$ , for the Coles&Hartman correlation, with  $c_0 = 5 \cdot 10^{-18}$ . (b): Pore pressure versus time (curves as in point (a)). Pressures are evaluated at  $X_{\rm L} = (0,0,0)$  for the Coles&Hartman correlation, with  $c_0 = 5 \cdot 10^{-18}$ .

were deduced from experimental data taken from the literature. The mechanical response of the solid matrix of the sample was hypothesised to be hyperelastic, and characterised by the elastic potential defined in (20). Moreover, to study the flow of the interstitial fluid, the FG-model [12] and the AW-model [1] of permeability were compared.

We developed the theory of Forchheimer's correc-1123 tion for the case of a tensorial Forchheimer's coefficient. 1124 However, in order to adapt our study to well-established 1125 derivations of Forchheimer's correction available in the 1126 literature [46], we first introduced an "equivalent" scalar 1127 coefficient,  $\mathcal{A}_{eq}$ , and the friction factor, f. We observed 1128 that the inhomogeneity and anisotropy of the sample 1129 yield patterns of f and  $q_{\rm D}$  that are different from those 1130 obtained in the isotropic and homogeneous case, and 1131 produce an increase of the maximum value of both f1132

and  $\|\boldsymbol{q}_{\mathrm{D}}\|$  (see Figs. 3 and 4). The increase of  $\|\boldsymbol{q}_{\mathrm{D}}\|$ 1133 might be ascribable to microstructural effects. By in-1134 troducing Forchheimer's correction, we obtained a re-1135 duction of the magnitude of the filtration velocity (see 1136 Fig. 5) with respect to the Darcian description. More-1137 over, a redistribution of the flow pattern, which tends 1138 to become spatially uniform, can be observed. By com-1139 paring Figs. 4 and 5, we can also observe that, by ap-1140 plying the same trial friction factor  $f_{\text{trial}} = 0.1$  to both 1141 the inhomogeneous and anisotropic tissue and to the 1142 isotropic and homogeneous one, Forchheimer's correc-1143 tion produces, in the former case, a maximum differ-1144 ence between the magnitudes of the filtration velocity 1145  $\|\boldsymbol{q}\|$  and  $\|\boldsymbol{q}_{\mathrm{D}}\|$  of about 85%, and of about 60% in the 1146 latter. Thus, we may conclude that the more inhomo-1147 geneous and complex the microstructure is, the more 1148 Forchheimer's correction could be significant in study-1149 ing the flow. Indeed, it is possible that also this result 1150 is due to the microstructure as well as to a better res-1151 olution of the interplay between deformation and flow. 1152

To test the FG-model of permeability, which takes 1153 the sample's microstructure explicitly into account, we 1154 compared it with the AW-model. From the results of 1155 this comparison (see Fig. 6), we observed that the two 1156 models are discrepant in the Darcian case, but that the 1157 discrepancies can be partially smoothed over by mod-1158 ulating the AW-model with the aid of Forchheimer's 1159 correction and, thus, of the friction factor. We believe 1160 that this behaviour could be due to the fact that Forch-1161 heimer's correction introduces new parameters into the 1162 flow model, which can thus be employed to better fit 1163 experimental results. We emphasise that, by modulat-1164 ing the AW-model, we by no means intended to correct 1165 it. Rather, we chose to modulate the AW-model be-1166 cause, contrary to the FG-model, it is not restricted by 1167 the use of Darcy's law at the REV scale. An important 1168 conclusion is that Forchheimer's correction implies an 1169 increase of the fluid pressure and a dilation of the relax-1170 ation times for both the filtration velocity and pressure. 1171 This behaviour can be observed both in the equivalent 1172 scalar case and in the fully tensorial one (see Fig. 9). 1173

In the future, we would like to study the combined 1174 effect of Forchheimer's correction and the Brinkman 1175 equation to study the boundary effects on the fluid behaviour. These, indeed, may lead to a more precise description of the flow in complex benchmark tests, such 1178 as the indentation test. 1179

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#### 1190 In Memoriam

In memory of our master Prof. Gaetano Giaquinta (1945-1191 2016). 1192

#### **Compliance with Ethical Standards** 1193

The authors declare that they have no conflict of inter-1194 est. 1195

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