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Material Description of Fluxes in Terms of Differential Forms

Dedicated to Prof. David Steigmann in Recognition of His Contributions

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Abstract The flux of a certain extensive physical quantity across a surface is often represented by the integral over the surface of the component of a pseudo-vector normal to the surface. A pseudo-vector is in fact a possible representation of a second-order differential form, i.e., a skew-symmetric second-order covariant tensor, which follows the regular transformation laws of tensors. However, because of the skew-symmetry of differential forms, the associated pseudo-vector follows a transformation law that is different from that of proper vectors, and is named after the Italian mathematical physicist Gabrio Piola (1794-1850). In this work, we employ the methods of Differential Geometry and the representation in terms of differential forms to demonstrate how the flux of an extensive quantity transforms from the spatial to the material point of view. After an introduction to the theory of differential forms, their transformation laws, and their role in Integration Theory, we apply them to the case of first-order transport laws such as Darcy's law and Ohm's law.

Keywords Differential Geometry · Differential Form · Flux · Material · Spatial · Darcy's Law · Ohm's Law

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1 Introduction

In the elementary treatment of Classical Physics and Continuum Mechanics, several quantities are referred to as pseudo-vectors, because the transformation laws they obey are different from those of vectors. Indeed, under any transformation other than a proper orthogonal transformation (proper means that the determinant is equal to 1), pseudo-vectors take a multiplicative factor equal to the determinant of the transformation, contrary to what vectors do. For instance, under reflection of one of the axes (a particular case of improper orthogonal transformation, for which the determinant is -1), pseudo-vectors are not only reflected, but also see their sense reversed, unlike vectors, which are just reflected. The first examples that come in the study of elementary Mechanics are the moment of a force and the angular velocity. Pseudo-vectors also come into play in the calculation of the flux of an extensive physical quantity across a surface, which is the integral over the surface of the component of the pseudo-vector normal to the surface itself. Examples are the mass and charge density currents in Fluid Mechanics and Electromagnetism. Continuum Mechanics also presents us with objects that can be righteously called pseudo-scalars because, unlike scalars, which are invariant under any transformation, they transform with a multiplicative factor equal to the determinant of the transformation and, for instance, suffer a change in sign under an improper orthogonal transformation. Examples are the so-called "volume element" of the theory of integration, and the scalar product of a vector and a pseudo-vector.

In the Continuum Mechanics of solids, it is most often necessary to transform the various physical quantities from the spatial picture of Mechanics, in which the equations are naturally written, to the material picture, in which it is usually most convenient to build a constitutive framework. The original ideas on the transformation from spatial to material picture dates back to the Italian mathematical physicist Gabrio Piola (see [4], as well as [20]) and modern interpretations have been very often proposed in recent works in Continuum Mechanics (among many others, see, e.g., [3,2]). In the modern language of Continuum Mechanics, the transformation from the spatial to the material picture of Mechanics is called a pull-back, a terminology that is shared by Differential Geometry.

In Continuum Mechanics one always works with a non-orthogonal transformation, i.e., the deformation gradient. Therefore, the pull-back laws of pseudo-vectors and pseudo-scalars differ from those of vectors and scalars (the latter being invariant), respectively, by a multiplicative factor equal to the determinant of the deformation gradient. In fact, in the customary three-dimensional case, a pseudo-vector represents a second-order differential form (or two-form) and a pseudo-scalar represents a (non-vanishing) third-order differential form (or non-vanishing three-form, or volume form) [7,24]. Differential forms are skew-symmetric covariant tensors, and follow the regular transformation rules of tensors. The form of that these rules take in the representation in terms of pseudo-vectors and pseudo-scalars is that of a Piola transformation. In the case of pseudo-vectors and pseudo-scalars, the Piola transformation preserves the invariance of fluxes across deforming oriented surfaces and of the extent of physical quantities over volumes, respectively, when passing from the current configuration to the body manifold (or the reference configuration).

In this work, we first briefly recall the tensor algebra notation, show the definitions of r-forms, the particular cases of n-forms and $(n-1)$-forms and the associated pseudo-vectors and pseudo-scalars (Section 2). Then, after introducing the basic definitions of Continuum Kinematics (Section 3), we introduce differential forms on general manifolds, their relationship to the Theory of Integration, including how fluxes are calculated as integrals of $(n-1)$-forms, and demonstrate how the pull-backs of pseudo-vectors and pseudo-scalars are obtained from those of the corresponding $(n-1)$- and n-forms (Section 4). Finally, we apply this method to first-order transport laws such as Darcy's law in the Theory of Porous Media and Ohm's law in Electromagnetism, and show how the Piola transformation on the flux quantity induces another Piola transformation on the first leg of the tensor providing the constitutive relation between the flux and the generalised force density, e.g., on the first leg of the permeability tensor, which constitutively relates filtration velocity with the gradient of the pore pressure in a porous medium (Section 5).

The purpose of this work is to move a step toward a unified formalism, which might help to account for phenomena characterised by a seemingly different Physics, which nevertheless is described by the very same Mathematics.

Remark. Throughout this work, for the sake of generality, both the body $\mathcal{B}$ and the space $\mathcal{S}$ are treated as differentiable manifolds or, when the metric structure is required, as Riemannian manifolds (differentiable manifolds are treated exhaustively, e.g., in the treatise by Epstein [7]). However, if one prefers, the physical space $\mathcal{S}$ can be regarded as a three-dimensional affine space, and the body $\mathcal{B}$, or any arbitrary reference configuration $\mathcal{B}_R$, as an open subset of $\mathcal{S}$. A very exhaustive introduction to affine spaces is out of the scope of this work (again, see [7]) but, roughly speaking, an affine space consists of a set $\mathcal{A}$, called the point space, and a function that maps a pair (x,y) of points x,y of $\mathcal{A}$ into an element $\boldsymbol{u} = y-x$ of a vector space $\mathcal{V}$, called the supporting or modelling space. In the affine space $\mathcal{A}$ one can thus attach a vector $\boldsymbol{u} = y-x$ at every point x. The vector space of all vectors emanating from a point x is called the tangent space, $T_x\mathcal{A}$, at x. The dual space of $T_x\mathcal{A}$, i.e., the set of al linear maps from $T_x\mathcal{A}$ to the real numbers $\mathbb{R}$, is called the cotangent space, $T_x^\star\mathcal{A}$, at x. The the disjoint unions of all tangent spaces and of all cotangent spaces for all points $x \in \mathcal{A}$ are the tangent bundle $T\mathcal{A}$ and the cotangent bundle $T^\star\mathcal{A}$, respectively. When the point space $\mathcal{A}$ and the supporting space $\mathcal{V}$ are both $\mathbb{R}^3$, one obtains the familiar affine space $\mathbb{E}^3$ of Classical Mechanics.

2 Tensor Algebra

In this section, we briefly illustrate the tensor algebra notation [13,10,11] employed in this work, then introduce r-forms in an n-dimensional vector space, and subsequently present the important cases of n-forms and $(n-1)$-forms. Note that, for the sake of a succinct, yet reasonably self-contained presentation of these concepts, we avoid introducing the wedge product $\wedge$, skew-symmetrisation of the tensor product $\otimes$, and instead rely on use of the Ricci/Levi-Civita symbol (Equation (6)). Exhaustive introductions to r-forms and spaces of r-forms can be found, e.g., in the works by Epstein [7] and Segev [24].

2.1 Tensors on a Vector Space

Given a vector space $\mathcal{V}$ on the real numbers $\mathbb{R}$, its dual space, i.e., the space of all linear forms $\boldsymbol{\pi} : \mathcal{V} \to \mathbb{R}$, is denoted $\mathcal{V}^\star$. The space of all multilinear forms

$$\mathbb{A} : \underbrace{\mathcal{V}^\star \times \ldots \times \mathcal{V}^\star}_{r\text{ times}} \times \underbrace{\mathcal{V} \times \ldots \times \mathcal{V}}_{s\text{ times}} \to \mathbb{R}, \tag{1}$$

i.e., all tensors of order $r+s$ with the first r legs being vectorial and the last s legs being covectorial, is denoted

$$\mathcal{V}^r{}_s = \underbrace{\mathcal{V} \otimes \ldots \otimes \mathcal{V}}_{r\text{ times}} \otimes \underbrace{\mathcal{V}^\star \otimes \ldots \otimes \mathcal{V}^\star}_{s\text{ times}}. \tag{2}$$

Note that the identifications $\mathcal{V}^0_0 \equiv \mathbb{R}$, $\mathcal{V}^1_0 \equiv \mathcal{V}$, $\mathcal{V}^0_1 \equiv \mathcal{V}^\star$ hold, and that, if $\mathcal{V}$ has dimension n, the dimension of $\mathcal{V}^r{}_s$ is n^{r+s}.

If $\mathcal{V}$ has dimension n, considering a basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of $\mathcal{V}$, together with the associated basis $\{\boldsymbol{e}^i\}_{i=1}^n$ of $\mathcal{V}^\star$, the components of a tensor $\mathbb{A} \in \mathcal{V}^r{}_s$ with respect to the given bases are $\mathrm{A}^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$, with the first r indices being contravariant and the last s indices being covariant, i.e.,

$$\mathbb{A} = \mathrm{A}^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \boldsymbol{e}_{i_1} \otimes \ldots \otimes \boldsymbol{e}_{i_r} \otimes \boldsymbol{e}^{j_1} \otimes \ldots \otimes \boldsymbol{e}^{j_s}, \tag{3}$$

where the components $\mathrm{A}^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ are, by definition,

$$\mathrm{A}^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = \mathbb{A}(\boldsymbol{e}^{i_1}, \ldots, \boldsymbol{e}^{i_r}, \boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_s}). \tag{4}$$

2.2 r-Forms on a Vector Space

Given a vector space $\mathcal{V}$ of dimension n, and $r \leq n$, an r-form (or form of order r, or multi-covector of order r) is a tensor $\boldsymbol{\beta} \in \mathcal{V}^0_r$ that is skew-symmetric, i.e., it is invariant for even permutations of the arguments and changes sign for odd permutations of the arguments. Therefore, for every set of r vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset \mathcal{V}$,

$$\boldsymbol{\beta}(\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_r}) = \varepsilon_{i_1 \ldots i_r} \boldsymbol{\beta}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r), \tag{5}$$

where $\varepsilon_{i_1 \ldots i_r}$ is the Ricci/Levi-Civita permutation symbol, defined as

$$\varepsilon_{i_1 \ldots i_r} = \begin{cases} +1 \text{ for } \{i_1, \ldots, i_r\} \text{ even w.r.t. } \{1, \ldots, r\}, \\ -1 \text{ for } \{i_1, \ldots, i_r\} \text{ odd w.r.t. } \{1, \ldots, r\}. \end{cases} \tag{6}$$

All r-forms in $\mathcal{V}^0_r$ constitute a subspace denoted $\Lambda_r(\mathcal{V})$, whose dimension can be shown to be equal to the binomial coefficient $\binom{n}{r} = \frac{n!}{(n-r)!r!}$, where $n = \dim\mathcal{V}$ [7,24]. With the help of the Tartaglia-Pascal triangle, the scheme below reports the dimension of $\Lambda_r(\mathcal{V})$ for every order r and for every dimension n of the "mother" space $\mathcal{V}$.

$$\begin{array}{rcccccccccccl}
 & & & & & & & & & & & & \dim\mathcal{V}: \\
 & & & & & & 1 & & & & & & 0 \\
 & & & & & 1 & & 1 & & & & & 1 \\
 & & & & 1 & & 2 & & 1 & & & & 2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & 4 \\
 & \ldots & \ldots & & & & \ldots & & & \ldots & \ldots & & \ldots \\
\dim\Lambda_r(\mathcal{V}): & 1 & n & & & & \binom{n}{r} & & & n & 1 & & n \\
 & & & & & & & & & & & & \\
\text{order } r: & 0 & 1 & & & & r & & & n-1 & n & &
\end{array}$$

For a given n, the spaces $\Lambda_r(\mathcal{V})$ of r-forms constitute a "fusiform" structure [7], with $r = n$ being the maximum possible order for an r-form, and with the pairs of spaces of the forms of order r and $n - r$ having the same dimension. It is immediate to make the conventional identifications $\Lambda_0(\mathcal{V}) \equiv \mathbb{R}$ between zero-forms and scalars, and $\Lambda_1(\mathcal{V}) \equiv \mathcal{V}^\star$ between one-forms and covectors. The spaces $\Lambda_n(\mathcal{V})$ and $\Lambda_{n-1}(\mathcal{V})$ "look a lot" like $\mathbb{R}$ and $\mathcal{V}$, respectively, but they do not quite coincide with these. Indeed, we shall show that n-forms and $(n-1)$-forms obey transformation laws that are *different* from those which scalars (which are invariant) and vectors obey. For this reason, they are often referred to as pseudo-scalars and pseudo-vectors.

We close this section with an important definition that will be employed later. Given a vector $\boldsymbol{u} \in \mathcal{V}$ and an r-form $\boldsymbol{\beta} \in \Lambda_r(\mathcal{V})$, their interior product $\boldsymbol{\iota_u \beta} \in \Lambda_{r-1}(\mathcal{V})$ is the $(r-1)$-form given by the contraction of $\boldsymbol{u}$ with the first leg of $\boldsymbol{\beta}$, i.e., for every system of $r-1$ vectors $\{\boldsymbol{v}_2, ..., \boldsymbol{v}_r\} \subset \mathcal{V}$,

$$(\boldsymbol{\iota_u \beta})(\boldsymbol{v}_2, ..., \boldsymbol{v}_r) = \boldsymbol{\beta}(\boldsymbol{u}, \boldsymbol{v}_2, ..., \boldsymbol{v}_r). \tag{7}$$

2.3 n-Forms and Their Transformation Laws

In a vector space $\mathcal{V}$ of dimension n, the space $\Lambda_n(\mathcal{V})$ has dimension one, and thus an n-form has only one independent component with respect to a given basis $\{\boldsymbol{e}_i\}_{i=1}^n$, as it can be seen by using the definition of skew-symmetry, which implies

$$\boldsymbol{\mu}(\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_n}) = \varepsilon_{i_1 \dots i_n} \boldsymbol{\mu}(\boldsymbol{e}_1, ..., \boldsymbol{e}_n) = h\, \varepsilon_{i_1 \dots i_n}, \tag{8}$$

where the well-defined scalar $h = \boldsymbol{\mu}(\boldsymbol{e}_1, ..., \boldsymbol{e}_n)$ is the only independent component of $\boldsymbol{\mu}$. The n-form $\boldsymbol{\mu}$ can therefore be expressed, in components, as

$$\boldsymbol{\mu} = h\, \varepsilon_{i_1 \dots i_n} \boldsymbol{e}^{i_1} \otimes \dots \otimes \boldsymbol{e}^{i_n}. \tag{9}$$

Given the basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of $\mathcal{V}$, the unique n-form with independent component equal to one is called determinant with respect to $\{\boldsymbol{e}_i\}_{i=1}^n$ and is denoted by det, i.e.,

$$\det(\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_n}) = \varepsilon_{i_1 \dots i_n} \;\Rightarrow\; \det(\boldsymbol{e}_1, ..., \boldsymbol{e}_n) = 1, \tag{10}$$

and, via the definitions of multi linearity and skew-symmetry, defines the determinant of the system of vectors $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ with respect to the basis $\{\boldsymbol{e}_i\}_{i=1}^n$ as the scalar

$$\begin{aligned} \det(\boldsymbol{v}_1, ..., \boldsymbol{v}_n) &= \det(v_1^{i_1} \boldsymbol{e}_{i_1}, ..., v_n^{i_n} \boldsymbol{e}_{i_n}) \\ &= v_1^{i_1} ... v_n^{i_n} \det(\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_n}) \\ &= \varepsilon_{i_1 \dots i_n} v_1^{i_1} ... v_n^{i_n}. \end{aligned} \tag{11}$$

The determinant of a matrix $[\![a^i{}_j]\!]$ is defined as the determinant of the system of vectors $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ such that $[\![a^i{}_j]\!] = [\![v^i_j]\!]$. A discussion on the definition of determinant for the case of second-order tensors belonging to $\mathcal{V}^1{}_1$ ("mixed" tensors), $\mathcal{V}^2_0$ ("contravariant" tensors) and $\mathcal{V}^0_2$ ("covariant" tensors) is given in [12].

As mentioned before, both spaces $\Lambda_0(\mathcal{V}) \equiv \mathbb{R}$ and $\Lambda_n(\mathcal{V})$ have dimension one. However, whereas a scalar of $\mathbb{R}$ is invariant under a change of basis in $\mathcal{V}$, an n-form is not. Indeed, if $\{\boldsymbol{e}_i\}_{i=1}^n$ and $\{\boldsymbol{e}'_j\}_{j=1}^n$ are two bases of $\mathcal{V}$ related by

$$\boldsymbol{e}'_j = a^i{}_j \boldsymbol{e}_i, \quad \boldsymbol{e}_i = b^j{}_i \boldsymbol{e}'_j, \tag{12}$$

where the matrices $[\![a^i{}_j]\!]$ and $[\![b^j{}_i]\!]$ are one the inverse of the other, then the component of an n-form $\boldsymbol{\mu}$ transforms according to

$$\begin{aligned} h' = \boldsymbol{\mu}(\boldsymbol{e}'_1, ..., \boldsymbol{e}'_n) &= \boldsymbol{\mu}(a^{i_1}{}_1 \boldsymbol{e}_{i_1}, ..., a^{i_n}{}_n \boldsymbol{e}_{i_n}) \\ &= a^{i_1}{}_1 ... a^{i_1}{}_n \boldsymbol{\mu}(\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_n}) \\ &= a^{i_1}{}_1 ... a^{i_n}{}_n \varepsilon_{i_1 \dots i_n} \boldsymbol{\mu}(\boldsymbol{e}_1, ..., \boldsymbol{e}_n) = \det[\![a^i{}_j]\!]\, h. \end{aligned} \tag{13}$$

Therefore, the component of an n-form transforms with a coefficient equal to the determinant of the change of basis, and this is why it is often called a pseudo-scalar. Note that, under a change of basis for which the determinant is equal to one (e.g. a proper orthogonal change of basis), an n-form remains invariant, i.e., the difference between the component of an n-form and the corresponding scalar is unnoticeable.

2.4 $(n-1)$-Forms and Axial Vectors

In a vector space $\mathcal{V}$ of dimension n, the space $\Lambda_{n-1}(\mathcal{V})$ has dimension n, like $\mathcal{V}^\star$ and $\mathcal{V}$ itself. Because of this, with every $(n-1)$-form $\boldsymbol{\omega} \in \Lambda_{n-1}(\mathcal{V})$, it is possible to univocally associate a vector $\boldsymbol{u} \in \mathcal{V}$ with respect to a non-vanishing n-form $\boldsymbol{\mu} \in \Lambda_n(\mathcal{V})$, via the interior product

$$\iota_{\boldsymbol{u}}\boldsymbol{\mu} = \boldsymbol{\omega}. \tag{14}$$

The vector $\boldsymbol{u}$ is called the axial vector of $\boldsymbol{\omega}$ with respect to $\boldsymbol{\mu}$ and, in elementary Algebra and Mechanics, is called a pseudo-vector as it follows a transformation law that is different from that of vectors, and it is obtained via the transformation followed by the corresponding $(n-1)$-form $\boldsymbol{\omega}$, which will be shown later in Section 4.

3 Kinematics of the Deformation

Here we report some definitions about the kinematics of deformation in Continuum Mechanics. The notation follows generally that of the treatise by Marsden and Hughes [18] and that used in some previous works [12,11]. The presentation is fairly standard, except for the fact that, in order to illustrate the transformations in a more general case, we keep the dimension n rather than going to the customary dimension 3.

3.1 Deformation and Configuration Map

We work on two n-dimensional manifolds: the body $\mathcal{B}$ and the physical space $\mathcal{S}$. The configuration map describing the deformation is, for the moment, assumed time-independent, and is defined as an embedding (i.e., a differentiable map on whose image the inverse map is defined and differentiable; with a subtle abuse of terminology, the configuration map is often said to be a diffeomorphism)

$$\chi : \mathcal{B} \to \mathcal{S} : X \mapsto x = \chi(X). \tag{15}$$

Note that, if one wishes, it is possible to choose a reference configuration $\chi_R : \mathcal{B} \to \mathcal{B}_R \subset \mathcal{S}$ and then refer to the map $\chi \circ \chi_R^{-1} : \mathcal{B}_R \to \mathcal{S}$, which differs from (15) by a mere change of coordinates [8]. Here, we prefer to follow the modern approach to Continuum Mechanics, in which no particular reference configuration is chosen.

3.2 Physical Quantities

Material and spatial physical quantities are tensor fields of the type

$$\mathbb{A} : \mathcal{B} \quad\;\; \to [T\mathcal{B}]^r{}_s : X \mapsto \mathbb{A}(X), \tag{16}$$
$$\mathbb{a} : \chi(\mathcal{B}) \to [T\mathcal{S}]^r{}_s : x \mapsto \mathbb{a}(x), \tag{17}$$

respectively, where $T\mathcal{B}$ and $T\mathcal{S}$ are the tangent bundles of $\mathcal{B}$ and $\mathcal{S}$, disjoint union of all tangent spaces $T_X\mathcal{B}$ and $T_x\mathcal{S}$ at all points $X \in \mathcal{B}$ and $x \in \mathcal{S}$, respectively. We also recall that the dual spaces of the tangent spaces $T_X\mathcal{B}$ and $T_x\mathcal{S}$ are the cotangent spaces $T_X^\star\mathcal{B}$ and $T_x^\star\mathcal{S}$, the disjoint unions of which are the cotangent bundles $T^\star\mathcal{B}$ and $T^\star\mathcal{S}$.

3.3 Deformation Gradient, Push-Forward, Pull-Back

The deformation gradient $\boldsymbol{F}$ is defined as the tangent map of χ,

$$T\chi = \boldsymbol{F} : T\mathcal{B} \to T\mathcal{S}, \tag{18}$$

such that, at each point $X \in \mathcal{B}$, the two-point tensor

$$(T\chi)(X) = \boldsymbol{F}(X) : T_X\mathcal{B} \to T_x\mathcal{S}, \tag{19}$$

is the the Frechét differential of χ at X. Thus, in the coordinate charts $\{\hat{X}^A\}$ in $\mathcal{B}$ and $\{\hat{x}^a\}$ in $\mathcal{S}$, we have

$$F^a{}_A(X) = \chi^a{}_{,A}(X). \tag{20}$$

Note that, as two-point tensor fields, the deformation gradient, its inverse, its transpose and its inverse transpose are defined as

$$\boldsymbol{F} \quad : \mathcal{B} \qquad \to T\mathcal{S} \otimes T^\star\mathcal{B}, \tag{21}$$
$$\boldsymbol{F}^{-1} : \chi(\mathcal{B}) \to T\mathcal{B} \otimes T^\star\mathcal{S}, \tag{22}$$
$$\boldsymbol{F}^{T} \ : \chi(\mathcal{B}) \to T^\star\mathcal{B} \otimes T\mathcal{S}, \tag{23}$$
$$\boldsymbol{F}^{-T} : \mathcal{B} \qquad \to T^\star\mathcal{S} \otimes T\mathcal{B}, \tag{24}$$

respectively.

Given a material vector field $\boldsymbol{U} : \mathcal{B} \to T\mathcal{B}$, its push-forward is the spatial vector field $\boldsymbol{u} = \chi_* \boldsymbol{U} : \chi(\mathcal{B}) \to T\mathcal{S}$, defined as

$$\boldsymbol{u} = \chi_* \boldsymbol{U} = (\boldsymbol{F}\boldsymbol{U}) \circ \chi^{-1}. \tag{25}$$

The inverse operation is called pull-back:

$$\boldsymbol{U} = \chi^* \boldsymbol{u} = (\boldsymbol{F}^{-1}\boldsymbol{u}) \circ \chi. \tag{26}$$

The push-forward of a material covector field (i.e., a one-form) $\boldsymbol{\Pi} : \mathcal{B} \to T^\star\mathcal{B}$ is the spatial covector field $\boldsymbol{\pi} = \chi_* \boldsymbol{\Pi} : \chi(\mathcal{B}) \to T^\star\mathcal{S}$, defined via the pull-back of vector fields of Equation (26),

$$\begin{aligned} (\chi_* \boldsymbol{\Pi})\boldsymbol{u} = [\boldsymbol{\Pi}\,(\chi^* \boldsymbol{u})] \circ \chi^{-1} \Rightarrow \\ (\boldsymbol{\Pi} \circ \chi^{-1})(\boldsymbol{F}^{-1}\boldsymbol{u}) = [(\boldsymbol{F}^{-T}\boldsymbol{\Pi}) \circ \chi^{-1}]\,\boldsymbol{u}, \end{aligned} \tag{27}$$

from which the push-forward rule is

$$\boldsymbol{\pi} = \chi_* \boldsymbol{\Pi} = (\boldsymbol{F}^{-T}\boldsymbol{\Pi}) \circ \chi^{-1}, \tag{28}$$

and therefore the pull-back rule is

$$\boldsymbol{\Pi} = \chi^* \boldsymbol{\pi} = (\boldsymbol{F}^{T}\boldsymbol{\pi}) \circ \chi. \tag{29}$$

The push-forward of tensor fields valued in $[T\mathcal{B}]^r{}_s$ and the pull-back of tensors in $[T\mathcal{S}]^r{}_s$ are obtained by performing pull-backs and push-forwards of each vector or covector leg of the tensor. The next section specialises the pull-back transformation laws to differential forms, i.e., fields valued in spaces of r-forms and, in particular, to the cases of differential n-forms and $(n-1)$-forms in the context of the Theory of Integration.

4 Differential Forms on Manifolds

A spatial differential form of order $r \leq n$ in the n-dimensional manifold $\mathcal{S}$ is an r-form-valued field on $\mathcal{S}$, i.e., a mapping

$$\boldsymbol{\beta} : \mathcal{S} \to \Lambda_r(T\mathcal{S}) : x \mapsto \boldsymbol{\beta}(x). \tag{30}$$

The definition of material forms (on $\mathcal{B}$) is analogous. An r-differential form can be called, with a slight abuse of terminology, an r-form, whenever there is no danger of confusion between the field $\boldsymbol{\beta}$ and its value $\boldsymbol{\beta}(x)$.

In this section we describe the role of r-forms in the Theory of Integration on manifolds and enunciate the theorem of the change of variables in integrals, before going to the crucial point of this work: the transformation rules of integrals of n-forms and $(n-1)$-forms.

4.1 Differential Forms and Integration

Differential r-forms on an n-dimensional manifold $\mathcal{S}$ are intimately connected with the theory of integration in that they induce a measure on sub-manifolds (i.e., subsets of $\mathcal{S}$ possessing the structure of manifold by themselves) of the same order r. Here we are interested in the case of n-forms and $(n-1)$-forms on an n-dimensional manifold.

A non-vanishing n-form $\boldsymbol{\theta} : \mathcal{S} \to \Lambda_n(T\mathcal{S})$, also called a volume form, is a volume integrand on the manifold itself and represents the density of a certain extensive quantity q. Therefore, the integral of $\boldsymbol{\theta}$ is the extent of q over an n-dimensional submanifold $\mathcal{C} \subset \mathcal{S}$:

$$\mathrm{Extent}(q, \mathcal{C}) = \int_{\mathcal{C}} \boldsymbol{\theta}. \tag{31}$$

In a given chart, an n-form is uniquely determined by its single scalar component. Therefore, often, one takes a suitable volume form $\boldsymbol{\mu}$ to calculate the physical volume of $\mathcal{C}$, and then derives any other volume form via multiplication by a non-vanishing function ρ, exactly like in measure theory (see, e.g., [22]), so that

$$\mathrm{Volume}(\mathcal{C}) = \int_{\mathcal{C}} \boldsymbol{\mu}, \quad \mathrm{Extent}(q, \mathcal{C}) = \int_{\mathcal{C}} \rho\, \boldsymbol{\mu}, \tag{32}$$

which would read $\int_{\mathcal{C}} \mathrm{d}\upsilon$ and $\int_{\mathcal{C}} \rho\, \mathrm{d}\upsilon$ in the traditional formalism.

Similarly, an $(n-1)$-form $\boldsymbol{\omega} : s \to \Lambda_{n-1}(T\mathcal{S})$ is an integrand on a hypersurface $s \subset \mathcal{S}$ and its integral represents the flux of an extensive quantity q across the hypersurface s:

$$\mathrm{Flux}(q, s) = \int_s \boldsymbol{\omega}. \tag{33}$$

When a metric tensor $\boldsymbol{g}$ (i.e., a symmetric and positive-definite tensor field valued in $[T\mathcal{S}]^0_2$) is available in $\mathcal{S}$, the integral of an $(n-1)$-form $\boldsymbol{\omega}$ on a surface s can be expressed in terms of the axial vector field $\boldsymbol{w}$ of $\boldsymbol{\omega}$ with respect to the volume form $\boldsymbol{\mu}$, i.e., $\boldsymbol{w}$ is such that $\boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu} = \boldsymbol{\omega}$. Indeed, the metric $\boldsymbol{g}$ allows for the definition of the normal covector $\boldsymbol{n}$ to s (such that its squared norm is $\|\boldsymbol{n}\|^2 = \boldsymbol{n}.\boldsymbol{n} = \langle \boldsymbol{n}, \boldsymbol{n} \rangle = n_a\, g^{ab}\, n_b = 1$) and of the associated normal vector $\boldsymbol{n}^\sharp = \boldsymbol{g}^{-1}\boldsymbol{n}$ (with components $n^a = g^{ab} n_b$). Exploiting the identity $\boldsymbol{i}^T = \boldsymbol{n}\boldsymbol{n}^\sharp$ (in components, $n_a n^b = \delta_a{}^b$, which are the components of the transpose of the spatial identity tensor $\boldsymbol{i}$), the $(n-1)$-form $\boldsymbol{\omega}$ can be written

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\iota}_{\boldsymbol{w}}\, \boldsymbol{\mu} = \boldsymbol{\iota}_{[\boldsymbol{w}\boldsymbol{i}^T]}\, \boldsymbol{\mu} = \boldsymbol{\iota}_{[\boldsymbol{w}\boldsymbol{n}\boldsymbol{n}^\sharp]}\, \boldsymbol{\mu} \\ &= (\boldsymbol{w}\boldsymbol{n})\, \boldsymbol{\iota}_{\boldsymbol{n}^\sharp}\, \boldsymbol{\mu} = (\boldsymbol{w}\boldsymbol{n})\, \boldsymbol{\alpha} \end{aligned} \tag{34}$$

where $\boldsymbol{w}\boldsymbol{n} \equiv \langle \boldsymbol{w}|\boldsymbol{n}\rangle = w^a n_a$ is the contraction of the vector $\boldsymbol{w}$ and the covector $\boldsymbol{n}$, and

$$\boldsymbol{\alpha} = \boldsymbol{\iota}_{\boldsymbol{n}^\sharp}\,\boldsymbol{\mu} \tag{35}$$

is the $(n-1)$-form induced by the volume form $\boldsymbol{\mu}$ and the metric $\boldsymbol{g}$ on the hypersurface s. With this definition, the flux of an extensive quantity q across s can be expressed in the alternative notation

$$\mathrm{Flux}(q,s) = \int_s (\boldsymbol{w}\boldsymbol{n})\,\boldsymbol{\alpha} \equiv \int_s \boldsymbol{w}\boldsymbol{n}. \tag{36}$$

In the traditional notation, the flux reads $\int_s \boldsymbol{w}\boldsymbol{n}\,\mathrm{d}a$ or $\int_s \boldsymbol{w}\,\mathrm{d}\boldsymbol{a}$, where $\mathrm{d}\boldsymbol{a}$ is the "element of area" inclusive of the normal $\boldsymbol{n}$.

4.2 Theorem of the Change of Variables

In the context of Continuum Mechanics, the theorem of the change of variables in integrals is used to transform integrals from the spatial to the material picture. If $\chi : \mathcal{B} \to \mathcal{S}$ is a configuration, $\mathcal{D}$ is an r-dimensional sub-manifold of the n-dimensional body manifold $\mathcal{B}$, and $\chi(\mathcal{D})$ is its image through χ, the spatial integral of the r-form $\boldsymbol{\beta}$ transforms according to

$$\int_{\chi(\mathcal{D})} \boldsymbol{\beta} = \int_{\mathcal{D}} \chi^*\boldsymbol{\beta}, \tag{37}$$

where the "change of variables" is precisely the configuration χ. Therefore, the need arises to calculate the pull-backs of differential forms. In the jargon of Continuum Mechanics, the pull-backs of volume forms and $(n-1)$-forms are called Piola transformations, a terminology that actually refers to the pseudo-scalar and the pseudo-vector quantities they are associated with, respectively.

4.3 Change of Variables: Volume Forms

Let $\boldsymbol{\mu} : \mathcal{S} \to \Lambda_n(T\mathcal{S})$ and $\boldsymbol{\mathcal{M}} : \mathcal{B} \to \Lambda_n(T\mathcal{B})$ be a spatial and a material volume form. If $\{\boldsymbol{e}_a\}_{a=1}^n$ and $\{\boldsymbol{E}_A\}_{A=1}^n$ are the bases induced by the coordinate charts $\{\hat{x}^a\}$ and $\{\hat{X}^a\}$, respectively, the component forms of $\boldsymbol{\mu}$ and $\boldsymbol{\mathcal{M}}$ read

$$\boldsymbol{\mu} = h\,\varepsilon_{a_1\ldots a_n}\boldsymbol{e}^{a_1}\otimes\ldots\otimes\boldsymbol{e}^{a_n}, \tag{38}$$

$$\boldsymbol{\mathcal{M}} = H\,\varepsilon_{A_1\ldots A_n}\boldsymbol{E}^{A_1}\otimes\ldots\otimes\boldsymbol{E}^{A_n}. \tag{39}$$

The pull-back $\chi^*\boldsymbol{\mu}$ can be calculated explicitly, as

$$\begin{aligned}\chi^*\boldsymbol{\mu} &= (h\circ\chi)\,\varepsilon_{a_1\ldots a_n}(\boldsymbol{F}^T\boldsymbol{e}^{a_1})\circ\chi\otimes\ldots\otimes(\boldsymbol{F}^T\boldsymbol{e}^{a_n})\circ\chi \\ &= (h\circ\chi)\,\varepsilon_{a_1\ldots a_n}F^{a_1}{}_{A_1}\ldots F^{a_n}{}_{A_n}\boldsymbol{E}^{A_1}\otimes\ldots\otimes\boldsymbol{E}^{A_n} \\ &= (h\circ\chi)\det[\![F^a{}_A]\!]\,\varepsilon_{A_1\ldots A_n}\boldsymbol{E}^{A_1}\otimes\ldots\otimes\boldsymbol{E}^{A_n},\end{aligned} \tag{40}$$

i.e., $\chi^*\boldsymbol{\mu}$ is the volume form on $\mathcal{B}$ with independent component $(h\circ\chi)\det[\![F^a{}_A]\!]$. We remark that the determinant $\det[\![F^a{}_A]\!]$ is *not* a scalar invariant of the deformation gradient $\boldsymbol{F}$ [18], as it depends on the choice of the coordinate charts $\{\hat{x}^a\}$ and $\{\hat{X}^a\}$. Therefore, it is convenient to express the pull-back $\chi^*\boldsymbol{\mu}$ in terms of the material volume form $\mathcal{M}$ [12], as

$$\begin{aligned}\chi^*\boldsymbol{\mu} &= (h\circ\chi)\,\det[\![F^a{}_A]\!]\,\frac{1}{H}\,H\,\varepsilon_{A_1\dots A_n}\boldsymbol{E}^{A_1}\otimes\dots\otimes\boldsymbol{E}^{A_n}\\ &= (h\circ\chi)\,\det[\![F^a{}_A]\!]\,\frac{1}{H}\,\mathcal{M}\\ &= J\,\mathcal{M},\end{aligned} \tag{41}$$

where

$$J = \det\boldsymbol{F} \equiv (h\circ\chi)\,\det[\![F^a{}_A]\!]\,\frac{1}{H} \tag{42}$$

is indeed a scalar invariant, which is *defined* as the determinant of the two-point tensor $\boldsymbol{F}$ with respect to the volume form $\boldsymbol{\mu}$ on $\mathcal{S}$ and the volume form $\mathcal{M}$ on $\mathcal{B}$ (see [12], which also reports an expression of Equation (42) for the case in which metric tensors are available and the corresponding induced volume forms are employed). In practice, it can be fairly easily shown that, since h and H transform according to Equation (13), the factor $(h\circ\chi)/H$ makes J an invariant. Therefore, for the case of volume forms, the theorem of the change of variables can be expressed as

$$\int_{\chi(\mathcal{B})}\boldsymbol{\mu} = \int_{\mathcal{B}}\chi^*\boldsymbol{\mu} = \int_{\mathcal{B}} J\,\mathcal{M}, \tag{43}$$

which in the traditional notation reads $d\upsilon = J\,dV$.

4.4 Change of Variables: $(n-1)$-Forms

Let $\boldsymbol{\mu}$ and $\mathcal{M}$ be a spatial and a material volume form as above, S a hypersurface in $\mathcal{B}$, $s=\chi(S)$ its image in $\mathcal{S}$ through the configuration χ, and $\boldsymbol{\omega}: s\to\Lambda_{n-1}(T\mathcal{S})$ a spatial $(n-1)$-form, with axial vector $\boldsymbol{w}$ with respect to the volume form $\boldsymbol{\mu}$, i.e., $\boldsymbol{w}$ is such that $\boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu}=\boldsymbol{\omega}$. The pull-back of $\boldsymbol{\omega}$ is obtained in terms of its axial vector $\boldsymbol{w}$, by exploiting the distributivity of the pull-back operation and the fact that the interior product of a vector and an r-form is merely the contraction of the vector with the first leg of the r-form. Indeed,

$$\begin{aligned}\chi^*\boldsymbol{\omega} &= \chi^*[\boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu}] = \boldsymbol{\iota}_{[\chi^*\boldsymbol{w}]}\,\chi^*\boldsymbol{\mu} = \boldsymbol{\iota}_{[(\boldsymbol{F}^{-1}\boldsymbol{w})\circ\chi]}\,J\mathcal{M}\\ &= \boldsymbol{\iota}_{[J\,(\boldsymbol{w}\circ\chi)\boldsymbol{F}^{-T}]}\,\mathcal{M} = \boldsymbol{\iota}_{\boldsymbol{W}}\,\mathcal{M} = \boldsymbol{\Omega},\end{aligned} \tag{44}$$

where

$$\boldsymbol{W} = J\,(\boldsymbol{w}\circ\chi)\,\boldsymbol{F}^{-T} = J\,(\boldsymbol{F}^{-1}\boldsymbol{w})\circ\chi \tag{45}$$

is called the Piola transform of $\boldsymbol{w}$, and is the axial vector of the pulled-back $(n-1)$-form $\boldsymbol{\Omega}=\chi^*\boldsymbol{\omega}: S\to\Lambda_{n-1}(T\mathcal{B})$ with respect to the material volume form $\mathcal{M}$.

If metric tensors $\boldsymbol{g}$ and $\boldsymbol{G}$ are available in $\mathcal{S}$ and $\mathcal{B}$, the normal covectors $\boldsymbol{n}$ and $\boldsymbol{N}$ and the associated normal vectors $\boldsymbol{n}^\sharp$ and $\boldsymbol{N}^\sharp$ can be defined on s and S. Therefore, following Equation (35), it is possible to define the $(n-1)$-forms $\boldsymbol{\alpha}=\boldsymbol{\iota}_{\boldsymbol{n}^\sharp}\boldsymbol{\mu}$ and $\boldsymbol{\mathcal{A}}=\boldsymbol{\iota}_{\boldsymbol{N}^\sharp}\mathcal{M}$ induced on s by $\boldsymbol{\mu}$ and on S by $\mathcal{M}$. Therefore, using the pull-back rules for $(n-1)$-forms in Equation

(44) and for their axial vectors in Equation (45), the theorem of the change of variables (37) takes the form

$$\int_{\chi(S)} \boldsymbol{\omega} = \int_{\chi(S)} \boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu} = \int_{\chi(S)} (\boldsymbol{w}\boldsymbol{n})\,\boldsymbol{\alpha} = \int_S (\boldsymbol{W}\boldsymbol{N})\boldsymbol{\mathcal{A}}$$
$$= \int_S [J\,(\boldsymbol{w}\circ\chi)\boldsymbol{F}^{-T}\boldsymbol{N}]\boldsymbol{\mathcal{A}} = \int_S \boldsymbol{\iota}_{\boldsymbol{W}}\boldsymbol{\mathcal{M}} = \int_S \boldsymbol{\Omega} = \int_S \chi^*\boldsymbol{\omega} \tag{46}$$

from which, omitting the $(n-1)$-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\mathcal{A}}$, as in Equation (36),

$$\int_{\chi(S)} \boldsymbol{w}\boldsymbol{n} = \int_S J\,(\boldsymbol{w}\circ\chi)\,\boldsymbol{F}^{-T}\boldsymbol{N} = \int_S \boldsymbol{W}\boldsymbol{N}. \tag{47}$$

In the traditional notation, this becomes Nanson's formula, which can be alternatively written $\boldsymbol{n}\,\mathrm{d}a = J\boldsymbol{F}^{-T}\boldsymbol{N}\,\mathrm{d}A$ or, by including the normal corresponding to each "area element", $\mathrm{d}\boldsymbol{a} = J\boldsymbol{F}^{-T}\,\mathrm{d}\boldsymbol{A}$.

5 Application to First-Order Transport Laws

A first-order transport law is most often used in Physics and Mechanics to formalise a constitutive relation between a flux density and a generalised force density. Naturally, higher-order laws are always possible in principle, and sometimes necessary to represent certain phenomena. However, a first-order law is sufficient for a vast range of phenomena, and has the objective advantage of a simpler mathematical structure. The general structure of a first-order linear transport law is usually written

$$\boldsymbol{w} = \boldsymbol{k}\boldsymbol{h}, \tag{48}$$

where the spatial vector field $\boldsymbol{w}$, valued in $T\mathcal{S}$, is the flux density or current density of a certain extensive quantity q, the second-order tensor field $\boldsymbol{k}$, valued in $[T\mathcal{S}]^2_0$ (i.e., a "contravariant" tensor) is a permittivity, and the covector field $\boldsymbol{h}$, valued in $T^\star\mathcal{S}$, is a generalised force density. The flux density $\boldsymbol{w}$ may, or may not, be given by the product of a (pseudo-scalar) density and a vector field.

As seen in Section 4.1, in the three-dimensional space $\mathcal{S}$, the flux density $\boldsymbol{w}$ is nothing but the axial vector of a two-form $\boldsymbol{\omega}$ with respect to a volume form $\boldsymbol{\mu}$, via the interior product $\boldsymbol{\omega} = \boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu}$. Therefore, in terms of forms, Equation (48) reads

$$\boldsymbol{\omega} = \mathsf{k}\boldsymbol{h}, \tag{49}$$

where the $[T\mathcal{S}]^2_0$-valued tensor field $\boldsymbol{k}$ has been replaced by the tensor field k, valued in $\Lambda_2(T\mathcal{S})\otimes T\mathcal{S}$, i.e., the first vector leg of $\boldsymbol{k}$ has been replaced by a two-form in the definition of k. If we define the third-order tensor field $\boldsymbol{\iota}_{\boldsymbol{\mu}}$, valued in $\Lambda_2(T\mathcal{S})\otimes T^\star\mathcal{S}$, via the isomorphism $\boldsymbol{\iota}_{\boldsymbol{\mu}}\boldsymbol{w} = \boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu}$, then k is given by contracting the last leg (the covector leg) of $\boldsymbol{\iota}_{\boldsymbol{\mu}}$ with the first leg of $\boldsymbol{k}$, i.e.,

$$\mathsf{k} = \boldsymbol{\iota}_{\boldsymbol{\mu}}\boldsymbol{k}. \tag{50}$$

In components, if $\boldsymbol{k} = k^{ab}\boldsymbol{e}_a\otimes\boldsymbol{e}_b$ and $\boldsymbol{h} = h_b\boldsymbol{e}^b$,

$$\boldsymbol{\omega} = \boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu} = \boldsymbol{\iota}_{[\boldsymbol{k}\boldsymbol{h}]}\boldsymbol{\mu} = \boldsymbol{\iota}_{[k^{ab}h_b\boldsymbol{e}_a]}\boldsymbol{\mu} = k^{ab}\,(\boldsymbol{\iota}_{\boldsymbol{e}_a}\boldsymbol{\mu})\,h_b = k^{ab}\,(\boldsymbol{\iota}_{\boldsymbol{e}_a}\boldsymbol{\mu})\,\boldsymbol{e}_b(\boldsymbol{h}) = [k^{ab}\,(\boldsymbol{\iota}_{\boldsymbol{e}_a}\boldsymbol{\mu})\otimes\boldsymbol{e}_b]\boldsymbol{h}, \tag{51}$$

from which

$$\mathsf{k} = k^{ab}\,(\boldsymbol{\iota}_{\boldsymbol{e}_a}\boldsymbol{\mu})\otimes\boldsymbol{e}_b. \tag{52}$$

Therefore, not only $\boldsymbol{\omega}$, but also the first leg of the tensor $\mathbf{k}$ transforms like a two-form. In the corresponding vectorial equation (48), the Piola transformation reads

$$J\,(\boldsymbol{w}\circ\chi)\,\boldsymbol{F}^{-T} = [J\,(\boldsymbol{F}^{-1}\boldsymbol{k})\circ\chi]\,\boldsymbol{F}^{-T}\,(\boldsymbol{F}^{T}\boldsymbol{h})\circ\chi, \tag{53}$$

where

$$\boldsymbol{W} = J\,(\boldsymbol{w}\circ\chi)\,\boldsymbol{F}^{-T}, \quad \boldsymbol{K} = [J\,(\boldsymbol{F}^{-1}\boldsymbol{k})\circ\chi]\,\boldsymbol{F}^{-T}, \quad \boldsymbol{H} = (\boldsymbol{F}^{T}\boldsymbol{h})\circ\chi, \tag{54}$$

are the material flux density (Piola transform of the spatial flux density), the material permittivity (Piola transform on the first leg and pull-back on the second leg of the spatial permittivity) and the material generalised force density (pull-back of the spatial generalised force density). The material equations (54) have been shown for the case of Darcy's law [23, 17, 3, 15, 1, 16, 25, 11], as well as for the analogous case of the polarisation of a dielectric [9, 27, 26, 5, 19, 6], in the traditional manner, without the use of differential forms.

In some cases, the flux density pseudo-vector $\boldsymbol{w}$ may be given as the product of a pseudo-scalar density ρ times a proper vector field $\boldsymbol{v}$, i.e.,

$$\boldsymbol{w} = \rho\,\boldsymbol{v}, \tag{55}$$

which, in terms of the associated two- and three-forms, reads

$$\boldsymbol{\omega} = \boldsymbol{\iota}_{\boldsymbol{w}}\boldsymbol{\mu} = \boldsymbol{\iota}_{[\rho\,\boldsymbol{v}]}\boldsymbol{\mu} = \boldsymbol{\iota}_{\boldsymbol{v}}(\rho\,\boldsymbol{\mu}). \tag{56}$$

Therefore, if the volume form $\boldsymbol{\mu}$ is thought to be associated with a measure of physical volume, the new three-form $\rho\boldsymbol{\mu}$ is the density of a certain extensive quantity q. The extent of the quantity q in the body $\mathcal{B}$ is, thus,

$$\mathrm{Extent}(q,\chi(\mathcal{B})) = \int_{\chi(\mathcal{B})} \rho\,\boldsymbol{\mu} = \int_{\mathcal{B}} \chi^{*}[\rho\,\boldsymbol{\mu}] = \int_{\mathcal{B}} (\rho\circ\chi)\,(J\boldsymbol{\mathcal{M}}) = \int_{\mathcal{B}} J\,(\rho\circ\chi)\,\boldsymbol{\mathcal{M}}, \tag{57}$$

which, in the traditional notation, reads $\int_{\chi(\mathcal{B})}\rho\,d\upsilon = \int_{\mathcal{B}} J\,(\rho\circ\chi)\,dV$. Furthermore, the flux of the extensive quantity q across a material surface S, with image $s=\chi(S)$, can be written in terms of the two-form $\boldsymbol{\omega}$, as in the standard case described by Equation (33), with the pull-back shown in Equation (46),

$$\mathrm{Flux}(q,\chi(S)) = \int_{\chi(S)} \boldsymbol{\omega} = \int_{S} \chi^{*}\boldsymbol{\omega} = \int_{S} \boldsymbol{\Omega}, \tag{58}$$

or, whenever metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$ are available in $\mathcal{B}$ and $\mathcal{S}$, and the normal covectors to S and $s=\chi(S)$ can be defined, the flux can be written in terms of pseudo-vectors as in Equation (46), i.e.,

$$\mathrm{Flux}(q,\chi(S)) = \int_{\chi(S)} \boldsymbol{w}\boldsymbol{n} = \int_{S} J\,(\boldsymbol{w}\circ\chi)\,\boldsymbol{F}^{-T}\boldsymbol{N} = \int_{S} \boldsymbol{W}\boldsymbol{N}, \tag{59}$$

where we have omitted writing the two-forms $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\alpha}$, induced by $\boldsymbol{G}$ and $\boldsymbol{g}$ on S and $s=\chi(S)$.

The two cases that we report as an example are Darcy's law for fluid filtration in a porous medium and Ohm's law for the conduction of charges in an electrical conductor. We chose these two cases because their flux densities both have pseudo-vectors expressible as the product of a pseudo-scalar density and a proper vector. Because of this analogy, and of the fact that it is well-established to study electromagnetism in terms of forms (see, e.g., [14, 18, 24]), it is interesting to report a treatment of Darcy's law too in this geometric formalism.

In Darcy's law,

$$\boldsymbol{w} = \phi_f\,(\boldsymbol{v}_f - \boldsymbol{v}_s) = \boldsymbol{k}\boldsymbol{h} = -\boldsymbol{k}\,(\operatorname{grad} p - \rho_{fT}\,\boldsymbol{f}), \tag{60}$$

the flux density $\boldsymbol{w} = \phi_f\,(\boldsymbol{v}_f - \boldsymbol{v}_s)$ is the filtration velocity, obtained as the product of the fluid volumetric fraction ϕ_f (pseudo-scalar density) and the velocity $\boldsymbol{v}_f - \boldsymbol{v}_s$ of the fluid relative to the solid (proper vector), $\boldsymbol{k}$ is the permeability tensor (function of fluid viscosity and fluid volumetric fraction), and the generalised force density $\boldsymbol{h} = -(\operatorname{grad} p - \rho_{fT}\,\boldsymbol{f})$ is comprised of the negative of the gradient of the pore pressure p and the body force term (usually gravity).

In Ohm's law,

$$\boldsymbol{j} = \rho\,\boldsymbol{v} = \boldsymbol{\kappa}\boldsymbol{e} = -\boldsymbol{\kappa}\operatorname{grad}\varphi, \tag{61}$$

the flux density $\boldsymbol{w} \equiv \boldsymbol{j} = \rho\,\boldsymbol{v}$ is the current density, obtained as the product of the charge density ρ (pseudo-scalar density) and the velocity $\boldsymbol{v}$ of the charges, $\boldsymbol{k} \equiv \boldsymbol{\kappa}$ is the conductivity tensor, and the generalised force density $\boldsymbol{h} \equiv \boldsymbol{e}$ is the electric field, given by the negative of the gradient of the scalar potential φ.

Remark. We observe that, in the work of Noll (e.g., [21]), a body is viewed as a differentiable manifold, the physical space is modelled as a Euclidean space (i.e., an affine space with a metric), and configurations are viewed as charts of the body manifold. Thus, there is no preferred reference configuration for the body in space. In this context, the flux field is represented by a single $(n-1)$-form on the body manifold independently of any configuration, and the various spatial fields associated with that form are simply different representations of a single geometric object.

6 Summary

In this work, we gave an overview of the tools of Differential Geometry needed for the description of those quantities that, in elementary Physics and Mechanics, are called pseudo-vectors and pseudo-scalars. The nature of these objects and the transformation laws they obey is completely unveiled if they are described as two-forms and three-forms, respectively, in the three-dimensional space of Classical Mechanics. In particular, we studied the case in which the integration of a two-form over a surface represents the flux of a certain extensive physical quantity across that surface. As an example of application to first-order transport laws, we reported Darcy's law for fluid filtration in a porous medium and Ohm's law for the conduction of a current in a conductor. This work contributes to the path towards a unified geometrical formalism in Continuum Mechanics, within which it is possible to represent different physical phenomena sharing the same mathematical structure, and to rigorously describe the transformation laws which the various physical quantities at play must obey.

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