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# An adaptive $h p$-DG-FE Method for Elliptic Problems. Convergence and Optimality in the 1D Case 

Dedicated to the memory of Professor Ben-yu Guo

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#### Abstract

We propose and analyze an $h p$-adaptive DG-FEM algorithm, termed hp-ADFEM, and a realization of it in one space dimension which is convergent, instance optimal, and $h$ - and p-robust. The procedure consists of iterating two routines: one hinges on Binev's algorithm for the adaptive $h p$-approximation of a given function, and finds a near-best $h p$-approximation of the current discrete solution and data to a desired accuracy; the other one improves the discrete solution to a finer but comparable accuracy, by iteratively applying Dörfler marking and $h$-refinement.


## 1 Introduction

The design and analysis of adaptive $h p$-type finite element methods for elliptic problems is significantly more challenging than for $h$-type methods. Indeed, as demonstrated e.g. by some examples given in [6, Sect.1], one should include in the adaptive procedure the possibility of stepping back from an early choice between $h$-refinement and $p$-enrichment: while the true structure of the solution reveals itself along the iterations, one should be able to re-distribute the allocated degrees of freedom between $h$ - and $p$-resolution. The existence of (rather) pathological situations has not prevented the development of practical $h p$-adaptive algorithms that work (see e.g. [9] and the references therein), but in most cases these procedures are not supported by a sound mathematical theory, which assesses the optimality, and even the convergence, of the method (unless a-priori assumptions on the structure of the solution are made).

The crucial issue is an approximation problem: how can we build an $h p$-finite element space of minimal dimension in which a given function can be approximated with a prescribed accuracy? A constructive answer to this question has been given by P. Binev in the past few years (see [5]), who designed a greedy $h p$-algorithm, which is incremental with respect to the dimension and has instance optimality properties (see Sect. 2.3).

With a good answer to such an approximation problem, one may think of recursively applying the $h p$-adaptive algorithm to a sequence of Galerkin discrete solutions of the elliptic problem, built in a way to get closer and closer to the exact solution. This idea

[^0]has been implemented in [6], where a general framework for adaptive $h p$-discretizations has been devised, and an adaptive algorithm termed hp-AFEM has been proposed, which guarantees convergence and instance optimality of the sequence of generated Galerkin solutions. The algorithm is both $h$ - and $p$-optimal in one space dimension, whereas in higher dimensions $p$-robustness is lost, partly due to the need of going from the nonconforming meshes produced by Binev's algorithm to the conforming ones needed in a continuous Galerkin method, and partly due to the use of a residual-based error estimator (the latter obstruction may be removed by resorting to equilibrated flux estimators, as done in [7]).

Since Binev's algorithm produces non-conforming meshes and discontinuous approximations, it is quite natural to associate to it a Discontinuous, rather than a Continuous, Galerkin discretization of the elliptic problem. The purpose of this paper is to take a step forward in this direction. In particular, hereafter we propose an $h p$-adaptive DG-FEM algorithm, termed hp-ADFEM, and a realization of it in one space dimension which is convergent, instance optimal, and $h$ - and $p$-robust. No restriction on the relative size of neighboring elements, nor on the polynomial degrees used on them, is required. In building a discrete solution that matches a prescribed accuracy, we extend to the $h p$-case the approach developed in [4] for $h$-type DG methods, using in the analysis several results on $h p$-type a posteriori error estimators (see e.g. [8] and the references therein). The multi-dimensional case is currently under investigation [1]; while our general convergence theorem holds in any dimension, proving $p$-robustness seems to require a grading property in the distribution of polynomial degrees over the partition, which is not guaranteed by the algorithm proposed in [5].

The paper is organized as follows. In Sect. 2, we introduce our general framework for the $h p$-approximation of a given function, and we present Binev algorithm. Sect. 3 describes the $h p$-DG discretizations that we consider, and collects some of their properties. Sect. 4 contains the general convergence and instance optimality result, based on the concatenation of Binev's algorithm and a procedure to compute DG-solutions with polynomial data, matching a prescribed tolerance. Finally, in Sect. 5 we illustrate a possible realization of this procedure, which is based on the classical SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE paradigm.

The following notation will be used thoughout the paper. By $A \lesssim B$ we will mean that $A$ can be bounded by a multiple of $B$, independently of parameters which $A$ and $B$ may depend on. Likewise, $A \simeq B$ is defined as both $A \lesssim B$ and $B \lesssim A$.
C.C. wishes to remember the long-lasting friendship and mutual esteem with Professor Ben-yu Guo, a person of great humanity and a devoted scientist.

## $2 h p$-partitions and $h p$-approximations

Let $\Omega$ be a bounded open interval of the real line. In view of the $h p$-adaptive discretization of a boundary-value problem therein, we introduce some notation concerning partitions in $\Omega$ and function spaces built on them.

### 2.1 Partitions of the domain

We assume that we are given an essentially disjoint initial partition $\mathcal{K}_{0}$ of $\bar{\Omega}$ into finitely many closed subintervals, which will be the initial geometric elements; the initial subdivision may depend upon the data of the problem at hand. Then, we apply subsequent dyadic subdivisions, by halving each element $K$ that we encounter into two closed subintervals $K^{\prime}$ and $K^{\prime \prime}$ of equal size, the 'children' of $K$, such that $K=K^{\prime} \cup K^{\prime \prime}$ and $\left|K^{\prime} \cap K^{\prime \prime}\right|=0$.

The set $\mathfrak{K}$ of all these geometric elements forms an infinite binary 'master tree', having as its roots the elements of the initial partition of $\bar{\Omega}$. A subtree of the master tree is a finite subset of $\mathfrak{K}$ that contains all roots and for each element in the subset both its parent and its sibling are in the subset. The leaves of a subtree form an essentially disjoint partition of $\bar{\Omega}$. The set of all such geometric partitions, or ' $h$-partitions', will be denoted as $\mathbb{K}$. For $\mathcal{K}, \widetilde{\mathcal{K}} \in \mathbb{K}$, we call $\widetilde{\mathcal{K}}$ a refinement of $\mathcal{K}$, and denoted as $\mathcal{K} \leq \widetilde{\mathcal{K}}$, when any $K \in \widetilde{\mathcal{K}}$ is either in $\mathcal{K}$ or has an ancestor in $\mathcal{K}$.

Starting from an $h$-partition $\mathcal{K} \in \mathbb{K}$, we obtain an hp-partition $\mathcal{D}$ by associating an integer $p \in \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$ to each element $K \in \mathcal{K}$. This integer will represent a polynomial degree, which will identify certain finite dimensional spaces of polynomial functions defined in $K$. A pair $D=\left(K_{D}, p_{D}\right) \in \mathfrak{K} \times \mathbb{N}_{0}$ formed by a geometric element $K_{D}$ and an integer $p_{D}$ will be termed an hp-element. Thus, a collection $\mathcal{D}=\left\{D=\left(K_{D}, d_{D}\right)\right\}$ of $h p$-elements is an $h p$-partition provided $\mathcal{K}(\mathcal{D}):=\left\{K_{D}: D \in \mathcal{D}\right\} \in \mathbb{K}$; the latter will be the associated $h$ partition. The collection of all $h p$-partitions is denoted as $\mathbb{D}$. Since $p+1$ is the dimension of the space $\mathbb{P}_{p}(K)$ of the univariate polynomials of degree $\leq p$ in $K$, we define the dimension of the $h p$-partition $\mathcal{D}$ as the integer

$$
\# \mathcal{D}:=\sum_{D \in \mathcal{D}}\left(p_{D}+1\right)
$$

For $\mathcal{D}, \widetilde{\mathcal{D}} \in \mathbb{D}$, we call $\widetilde{\mathcal{D}}$ a refinement of $\mathcal{D}$, and write $\mathcal{D} \leq \widetilde{\mathcal{D}}$, when both $\mathcal{K}(\mathcal{D}) \leq \mathcal{K}(\widetilde{\mathcal{D}})$, and $d_{\widetilde{D}} \geq d_{D}$, for any $D \in \mathcal{D}, \widetilde{D} \in \widetilde{\mathcal{D}}$ with $K_{D}$ being either equal to $K_{\widetilde{D}}$ or an ancestor of $K_{\tilde{D}}$.

### 2.2 Approximation spaces on $h p$-partitions

Let $Z$ be a normed space of vector-valued functions $z: \Omega \rightarrow \mathbb{R}^{m}(m \geq 1)$, which is relevant for our application. For any geometric element $K \in \mathfrak{K}$, let $Z_{K}$ be the space collecting the restrictions $z_{\mid K}$ to $K$ of all functions $z \in Z$. Then, for any geometric partition $\mathcal{K} \in \mathbb{K}$, we define

$$
\begin{equation*}
Z_{\mathcal{K}}:=\left\{z: \Omega \rightarrow \mathbb{R}^{m}: z_{\mid K} \in Z_{K} \forall K \in \mathcal{K}\right\}=\prod_{K \in \mathcal{K}} Z_{K} ; \tag{1}
\end{equation*}
$$

obviously, $Z \subseteq Z_{\mathcal{K}}$. In the sequel, we will work with functions that belong to $Z_{\mathcal{K}}$ for some partition $\mathcal{K} \in \mathbb{K}$; therefore, we set

$$
Z:=\bigcup_{\mathcal{K} \in \mathbb{K}} Z_{\mathcal{K}} .
$$

We assume that for any $K \in \mathfrak{K}$, the space $Z_{K}$ contains all polynomial functions of any degree, and this set of functions is dense in $Z_{K}$. Then, for $p \in \mathbb{N}_{0}$ we assume we have chosen finite dimensional spaces $Z_{K, p} \subset Z_{K}$ of polynomial functions on $K$ of degree related to $p$, satisfying $Z_{K, p} \subset Z_{K, p+1}$ and $Z_{K, p} \subset Z_{K^{\prime}, p} \times Z_{K^{\prime \prime}, p}\left(K^{\prime}\right.$ and $K^{\prime \prime}$ being the children of $K)$. For any $D=\left(K_{D}, p_{D}\right) \in \mathfrak{K} \times \mathbb{N}_{0}$, we set $Z_{D}:=Z_{K_{D}, p_{D}}$. Then, given an $h p$-partition $\mathcal{D}$, we define

$$
\begin{equation*}
Z_{\mathcal{D}}:=\left\{z: \Omega \rightarrow \mathbb{R}^{m}: z_{\mid K_{D}} \in Z_{D} \forall D \in \mathcal{D}\right\}=\prod_{D \in \mathcal{D}} Z_{D} \tag{2}
\end{equation*}
$$

which obviously satisfies $Z_{\mathcal{D}} \subset Z_{\mathcal{K}(\mathcal{D})}$. We will use the notation $z_{\mathcal{D}}$ to indicate a function in $Z_{\mathcal{D}}$. Note that no interelement continuity is imposed in the definition of $Z_{\mathcal{D}}$. Also note that the dimension of $Z_{\mathcal{D}}$ is proportional to the cardinality $\# \mathcal{D}$.

For all $D \in \mathfrak{K} \times \mathbb{N}_{0}$, we assume a local projector $Q_{D}: Z \rightarrow Z_{D}$, and a local error functional $e_{D}=e_{D}(z) \geq 0$, that, for any $z \in \mathcal{Z}$ gives a measure for some function of the
distance between $z_{\mid K_{D}}$ and its local approximation $z_{D}:=Q_{D}(z)$. We assume that this error functional is non-increasing under both ' $h$-refinements' and ' $p$-enrichments', in the sense that
$e_{D^{\prime}}+e_{D^{\prime \prime}} \leq e_{D}$ when $K_{D^{\prime}}, K_{D^{\prime \prime}}$ are the children of $K_{D}$, and $p_{D^{\prime}}=p_{D^{\prime \prime}}=p_{D}$;
$e_{D^{\prime}} \leq e_{D}$ when $K_{D^{\prime}}=K_{D}$ and $p_{D^{\prime}} \geq p_{D}$.
Given any $h p$-partition $\mathcal{D} \in \mathbb{D}$, we define the global projector $Q_{\mathcal{D}}: \mathcal{Z} \rightarrow Z_{\mathcal{D}}$ as $Q_{\mathcal{D}}(z):=\left(z_{D}\right)_{D \in \mathcal{D}}$, and the global error functional

$$
\begin{equation*}
\mathrm{E}_{\mathcal{D}}(z):=\sum_{D \in \mathcal{D}} e_{D}(z) \tag{4}
\end{equation*}
$$

which is a measure for the distance between $z$ and its projection $z_{\mathcal{D}}:=Q_{\mathcal{D}}(z)$. Note that (3) is equivalent to

$$
\begin{equation*}
\mathrm{E}_{\widetilde{\mathcal{D}}}(z) \leq \mathrm{E}_{\mathcal{D}}(z) \quad \forall \widetilde{\mathcal{D}} \geq \mathcal{D} \tag{5}
\end{equation*}
$$

### 2.3 The instance optimal $h p$-approximation algorithm

Herafter, we present the greedy algorithm proposed by P. Binev [5] to produce a nearbest adaptive $h p$-approximation of a function $z \in \mathcal{Z}$, based on the associated local error functionals $e_{D}=e_{D}(z)$ and global error functional $\mathrm{E}_{\mathcal{D}}=\mathrm{E}_{\mathcal{D}}(z)$ introduced above.

Denote by $R \geq 1$ the cardinality of the initial geometric partition $K_{0}$. Using property (3), Binev's algorithm builds a sequence of $h p$-partitions $\mathcal{D}_{N}, N \geq R$, satisfying $\# \mathcal{D}_{N}=$ $N$; the construction is incremental, in that going from $\mathcal{D}_{N}$ to $\mathcal{D}_{N+1}$ one exploits the work already done to build $\mathcal{D}_{N}$. The main feature of the algorithm is its instance optimality, expressed as follows.
Theorem 2.1 ([5]). For $n \geq R$ let

$$
\sigma_{n}:=\inf _{\# \mathcal{D} \leq n} E_{\mathcal{D}}
$$

be the smallest error achievable with an hp-partition of cardinality $\leq n$. Then, the $h p-$ partitions $\mathcal{D}_{N}$ produced by Binev's algorithm yield error functionals $E_{\mathcal{D}_{N}}$ satisfying the bounds

$$
\begin{equation*}
\mathrm{E}_{\mathcal{D}_{N}} \leq \frac{2 N}{N-n+1} \sigma_{n} \quad \forall n \leq N \tag{6}
\end{equation*}
$$

Binev's construction can be easily used to produce an instance optimal $h p$-partition for which the error functional is below a given threshold.

Corollary 2.1 ([6]). Let $B>1$ arbitrary. Given $\varepsilon>0$, let $\mathcal{D} \in \mathbb{D}$ be the first partition in Binev's sequence for which $E_{\mathcal{D}}^{\frac{1}{2}} \leq \varepsilon$. Then, setting $b=\frac{1}{2}\left(1-\frac{1}{B}\right)<1$, it holds

$$
\# \mathcal{D} \leq B \# \hat{\mathcal{D}}
$$

for all partitions $\hat{\mathcal{D}} \in \mathbb{D}$ satisfying $E_{\hat{\mathcal{D}}}^{\frac{1}{2}} \leq b \varepsilon$.

This result motivates the introduction of the following routine, which will constitute one of the two major building blocks of our proposed $h p$-adaptive algorithm.

- $\left[\mathcal{D}, z_{\mathcal{D}}\right]:=\mathbf{h p}-\operatorname{NEARBEST}(\varepsilon, z)$

The routine hp-NEARBEST takes as input $\varepsilon>0$, and $z \in \mathcal{Z}$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $z_{\mathcal{D}} \in Z_{\mathcal{D}}$ such that $\mathrm{E}_{\mathcal{D}}(z)^{\frac{1}{2}} \leq \varepsilon$ and, for some constants $0<b<1<B$, $\# \mathcal{D} \leq B \# \widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\widehat{\mathcal{D}}}(z)^{\frac{1}{2}} \leq b \varepsilon$.
The approximation $z_{\mathcal{D}}$ of the input $z$ is just the element-wise projection given by the operator $Q_{\mathcal{D}}$ associated with the partition $\mathcal{D}$, i.e., we set

$$
\begin{equation*}
z_{\mathcal{D}}:=Q_{\mathcal{D}}(z) \tag{7}
\end{equation*}
$$

## 3 Discontinuous Galerkin $\boldsymbol{h p}$-discretizations

We are interested in solving numerically the model boundary-value problem

$$
\begin{equation*}
A u=f \quad \text { in } \Omega, \quad u=0 \quad \text { on } \partial \Omega \tag{8}
\end{equation*}
$$

with $A u:=-\left(\nu u_{x}\right)_{x}+\xi u$, where $\nu \in L^{\infty}(\Omega)$ satisfies $\operatorname{essinf}_{\Omega} \nu>0, \xi \in L^{2}(\Omega)$ satisfies $\xi \geq 0$ a.e. in $\Omega, f \in L^{2}(\Omega)$. We actually assume that $\nu, \xi$ are piecewise- $H^{1}$ functions, precisely that $\nu_{\mid K}, \xi_{\mid K} \in H^{1}(K)$ for each element $K$ of the initial partition $\mathcal{K}_{0}$ introduced in Sect. 2.1; we will write $\nu, \xi \in H^{1}\left(\Omega ; \mathcal{K}_{0}\right)$. It will be convenient to refer to a triple $g:=$ $(\nu, \xi, f)$ as to a "data" of our problem; we thus have $g \in G(\Omega):=\left(H^{1}\left(\Omega ; \mathcal{K}_{0}\right)\right)^{2} \times L^{2}(\Omega)$. The solution $u \in H_{0}^{1}(\Omega)$ of Problem (8) for given data $g$ will be indicated by $u(g)$.

The following notation will be useful in the design of a DG discretization of our problem. For any element $K \in \mathfrak{K}$, let $(v, w)_{K}$ denote the $L^{2}$-inner product in $K$, with corresponding norm $\|v\|_{K}$. For any geometric partition $\mathcal{K} \in \mathbb{K}$, let us set

$$
\begin{equation*}
V_{\mathcal{K}}:=\left\{v \in L^{2}(\Omega): v_{\mid K} \in H^{1}(K) \forall K \in \mathscr{K}\right\} \tag{9}
\end{equation*}
$$

For $v \in V_{\mathcal{K}}$, it will be convenient to denote by $\tilde{v}_{x}$ the function in $L^{2}(\Omega)$ such that $\left(\tilde{v}_{x}\right)_{\mid K}=$ $\left(v_{\mid K}\right)_{x}$ for all $K \in \mathcal{K}$; thus, $\left\|\tilde{v}_{x}\right\|_{\Omega}^{2}=\sum_{K \in \mathcal{K}}\left\|\left(v_{\mid K}\right)_{x}\right\|_{K}^{2}$. Let us denote by $\mathcal{E}_{\mathcal{K}}$ the set of all endpoints of elements in $\mathcal{K}$, and let us define the jumps and averages of a piecewise smooth function $\phi$ on $\mathcal{K}$ as follows: if $e \in \mathcal{E}_{\mathcal{K}}$ is shared by two contiguous elements $K^{-}$ and $K^{+}$, then we set

$$
\llbracket \phi \rrbracket_{e}:=\phi_{\mid K^{-}}(e)-\phi_{\mid K^{+}}(e), \quad\{\phi\}_{e}:=\frac{1}{2}\left(\phi_{\mid K^{-}}(e)+\phi_{\mid K^{+}}(e)\right)
$$

whereas if $e$ is the left/right boundary point of $\Omega$, we set $\llbracket \phi \rrbracket_{e}=+/-\phi(e)$ and $\{\phi\}_{e}=\phi(e)$.
For any $h p$-partition $\mathcal{D} \in \mathbb{D}$ let us set

$$
\begin{equation*}
V_{\mathcal{D}}:=\left\{v \in L^{2}(\Omega): v_{\mid K_{D}} \in \mathbb{P}_{p_{D}}\left(K_{D}\right) \quad \forall D \in \mathcal{D}\right\} \subset V_{\mathcal{K}(\mathcal{D})} \tag{10}
\end{equation*}
$$

If $D \in \mathcal{D}$, let $h_{D}:=\left|K_{D}\right|$ denote the size of the element $K_{D}$. If $e \in \mathcal{E}_{\mathcal{D}}:=\mathcal{E}_{\mathcal{K}(\mathcal{D})}$, we define the weight

$$
\begin{equation*}
\sigma_{\mathcal{D}, e}:=\max \left(\frac{p_{D^{-}}^{2}}{h_{D^{-}}}, \frac{p_{D^{+}}^{2}}{h_{D^{+}}}\right) \tag{11}
\end{equation*}
$$

if $e \in K_{D^{-}} \cap K_{D^{+}}$, and $\sigma_{\mathcal{D}, e}:=\frac{p_{D}^{2}}{h_{D}}$ if $e \in \partial \Omega \cap K_{D}$.
It is convenient to introduce the inner product $(\phi, \psi)_{\mathcal{E}_{\mathcal{D}}}:=\sum_{e \in \mathcal{E}_{\mathcal{D}}} \phi_{e} \psi_{e}$ in $\mathbb{R}^{\left|\mathcal{E}_{\mathcal{D}}\right|}$ between two quantities $\phi=\left(\phi_{e}\right)$ and $\psi=\left(\psi_{e}\right)$ indexed in $\mathcal{E}_{\mathcal{D}}$. The corresponding norm will be denoted by $\|\phi\|_{\varepsilon_{\mathcal{D}}}$.

At this point, we are ready to introduce the symmetric bilinear form $a_{\mathcal{D}}$ defined on $V_{\mathcal{D}} \times V_{\mathcal{D}}$ as

$$
\begin{align*}
a_{\mathcal{D}}(w, v):= & \left(\nu \tilde{w}_{x}, \tilde{v}_{x}\right)_{\Omega}+(\xi w, v)_{\Omega} \\
& -\left(\left\{\nu w_{x}\right\}, \llbracket v \rrbracket\right)_{\mathcal{E}_{\mathcal{D}}}-\left(\left\{\nu v_{x}\right\}, \llbracket w \rrbracket\right)_{\mathcal{E}_{\mathcal{D}}}+\gamma\left(\sigma_{\mathcal{D}} \llbracket w \rrbracket, \llbracket v \rrbracket\right)_{\mathcal{E}_{\mathcal{D}}} \tag{12}
\end{align*}
$$

where $\gamma>0$ is a sufficiently large stabilization parameter, as well as the DG-norm defined on $V_{\mathcal{D}}$ as

$$
\begin{equation*}
\|v\|_{\mathcal{D}}:=\left(\left(\nu \tilde{v}_{x}, \tilde{v}_{x}\right)_{\Omega}^{2}+\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket v \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}\right)^{\frac{1}{2}} \tag{13}
\end{equation*}
$$

It is well-known (see $[2,3]$ ) that $a_{\mathcal{D}}$ is a continuous form with respect to the DG-norm, and it is coercive provided $\gamma$ is chosen large enough, with coercivity and continuity constants independent of $\mathcal{D}$; in the sequel, we will assume that this condition is satisfied.

Since $a_{\mathcal{D}}$ depends on the choice of coefficients $\nu$ and $\xi$, and since in the adaptive algorithm we will consider a sequence of DG discretizations with changing (piecewise polynomial) data, sometimes we will prefer the more precise notation $a_{\mathcal{D}}(w, v ; \nu, \xi)$ to indicate the right-hand side of (12).

Problem 8 with data $g=(\nu, \xi, f) \in G(\Omega)$ is then discretized by the following Simmetric Interior Penalty Discontinuous-Galerkin method ([2]):

$$
\begin{equation*}
u_{\mathcal{D}} \in V_{\mathcal{D}} \quad: a_{\mathcal{D}}\left(u_{\mathcal{D}}, v_{\mathcal{D}} ; \nu, \xi\right)=\left(f, v_{\mathcal{D}}\right)_{\Omega} \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}} \tag{14}
\end{equation*}
$$

We will write $u_{\mathcal{D}}=u_{\mathcal{D}}(g)$ when we want to stress the dependence of $u_{\mathcal{D}}$ upon the given data $g$.

### 3.1 Approximation spaces and error functionals

Hereafter, we specify the choice of approximation spaces and error functionals, introduced in a general setting in Sect. 2.2, that is tailored to the discretization problem of interest.

Since we will deal with approximations of a specific solution of Problem 8, and approximations of the corresponding data, our functions $z$ will be of the form $z=(v, g)=$ $(v, \nu, \xi, f)$. Then, a natural choice for the "base" space $Z$ is $Z=H^{1}(\Omega) \times G(\Omega)=$ $H^{1}(\Omega) \times\left(H^{1}\left(\Omega ; \mathcal{K}_{0}\right)\right)^{2} \times L^{2}(\Omega)$. Note that for $\mathcal{K} \in \mathbb{K}$, the local spaces $Z_{K}$ that form the global space $Z_{\mathcal{K}}$ according to (1) are given by $Z_{K}=\left(H^{1}(K)\right)^{3} \times L^{2}(K)$.

For any element $K \in \mathfrak{K}$ and integer $p \in \mathbb{N}_{0}$, we set
$Z_{K, p}=V_{K, p} \times G_{K, p} \quad$ with $\quad V_{K, p}=\mathbb{P}_{p}(K) \quad$ and $\quad G_{K, p}=\mathbb{P}_{p+1}(K) \times \mathbb{P}_{p+1}(K) \times \mathbb{P}_{p-1}(K)$.
Then, for any $\mathcal{D} \in \mathbb{D}$, we define $Z_{\mathcal{D}}$ according to (2); it is easily seen that $Z_{\mathcal{D}}=: V_{\mathcal{D}} \times$ $G_{\mathcal{D}}$, where $V_{\mathcal{D}}$ has been already introduced in (10). We will write $z_{\mathcal{D}}=\left(v_{\mathcal{D}}, g_{\mathcal{D}}\right)=$ $\left(v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}\right)$ for the generic element in $Z_{\mathcal{D}}$.

In order to define the projectors $Q_{D}$, let $\Pi_{K, p}^{0}: L^{2}(K) \rightarrow \mathbb{P}_{p}(K)$ be the $L^{2}$-orthogonal projector, and let $\Pi_{K, p}^{1}: H^{1}(K) \rightarrow \mathbb{P}_{p}(K)$ be the projector such that

$$
\left(\Pi_{K, p}^{1} v\right)_{x}=\Pi_{K, p}^{0} v_{x} \quad \text { and } \quad \int_{K} \Pi_{K, p}^{1} v=\int_{K} v, \quad \forall v \in H^{1}(K)
$$

The latter definition can be extended to functions $v$ that are just piecewise- $H^{1}$ on $K$, by replacing $v_{x}$ with $\tilde{v}_{x}$ in the $L^{2}$-projection. Then, for $z=(v, g)=(v, \nu, \xi, f) \in \mathcal{Z}$ and $D=\left(K_{D}, p_{D}\right)$ we set

$$
Q_{D}(z)=\left(\Pi_{K_{D}, p_{D}}^{1} v_{\mid K_{D}}, \Pi_{K_{D}, p_{D}+1}^{1} \nu_{\mid K_{D}}, \Pi_{K_{D}, p_{D}+1}^{1} \xi_{\mid K_{D}}, \Pi_{K_{D}, p_{D}-1}^{0} f_{\mid K_{D}}\right)
$$

The corresponding local error functional is defined as

$$
\begin{equation*}
e_{D}(z):=e_{1, D}(v)+\frac{1}{\kappa^{2}} \operatorname{osc}_{D}^{2}(g)=: e_{1, D}(v)+\frac{1}{\kappa^{2}}\left(e_{1, D}(\nu)+e_{1, D}(\xi)+e_{0, D}(f)\right), \tag{15}
\end{equation*}
$$

where for $\varphi=v, \nu, \xi$

$$
e_{1, D}(\varphi):=\left\|\left(\mathrm{I}-\Pi_{K_{D}, p_{D}}^{0}\right)\left(\tilde{\varphi}_{x}\right)_{\mid K_{D}}\right\|_{K_{D}}^{2}, \quad e_{0, D}(f):=\frac{h_{D}}{p_{D}}\left\|\left(\mathrm{I}-\Pi_{K_{D}, p_{D}}^{0}\right) f_{\mid K_{D}}\right\|_{K_{D}}^{2}
$$

and $\kappa>0$ is a (sufficiently small) penalization parameter to be chosen later on.
Finally, for a given $h p$-partition $\mathcal{D} \in \mathbb{D}$, the global projector $Q_{\mathcal{D}}: \mathcal{Z} \rightarrow Z_{\mathcal{D}}$ and the global error functional $\mathrm{E}_{\mathcal{D}}(z)=\mathrm{E}_{\mathcal{D}}(v, g)$ are defined as in Sect. 2.2 (see (4)).

We now establish some properties involving the functional $\mathrm{E}_{\mathcal{D}}$, that will be useful in the sequel.
Property 3.1. There exists a constant $C_{0}>0$ such that for any $z=(v, \nu, \xi, f) \in \mathcal{Z}$ and for any partition $\mathcal{D} \in \mathbb{D}$ one has

$$
\left\|\nu-\nu_{\mathcal{D}}\right\|_{L^{\infty}(\Omega)}+\left\|\xi-\xi_{\mathcal{D}}\right\|_{L^{\infty}(\Omega)} \leq C_{0} \kappa \mathrm{E}_{\mathcal{D}}(z)^{\frac{1}{2}}
$$

where $z_{\mathcal{D}}=\left(v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}\right)=Q_{\mathcal{D}}(z)$.
Proof. For any $\mathcal{D} \in \mathbb{D}$ and any $D \in \mathcal{D}$, set $\psi:=\left(\nu-\nu_{\mathcal{D}}\right)_{\mid K_{D}}$. Since by construction $\psi$ vanishes at a point in $K_{D}$, we have for any $x \in K_{D}$

$$
|\psi(x)| \leq h_{D}^{1 / 2}\left\|\psi_{x}\right\|_{K_{D}} \leq|\Omega| e_{1, D}(\nu)^{1 / 2}
$$

from which the bound for $\left\|\nu-\nu_{\mathcal{D}}\right\|_{L^{\infty}(\Omega)}$ easily follows. The coefficient $\xi$ can be treated similarly.

At this point, let us fix once and for all the data of interest $g_{\star}=\left(\nu_{\star}, \xi_{\star}, f_{\star}\right) \in G(\Omega)$ for Problem (8), and let $u_{\star}:=u\left(g_{\star}\right)$ be the corresponding solution.
Let us set $\nu_{0}:=\operatorname{essinf}_{\Omega} \nu_{\star}>0$.
Assumption 3.1. Let $\mathcal{D}_{0}$ denote the root partition $\mathcal{K}_{0}$ endowed with polynomials of degree 1 in each element. Setting $z_{0}:=\left(0, g_{\star}\right) \in \mathcal{Z}$, we assume that $\mathcal{D}_{0}$ is chosen to satisfy

$$
C_{0} \kappa \mathrm{E}_{\mathcal{D}_{0}}\left(z_{0}\right)^{\frac{1}{2}} \leq \frac{\nu_{0}}{\lambda}, \quad \text { where } \lambda:=2+\frac{1}{\sqrt{2}}|\Omega|
$$

Recalling (5), this assumption together with Property 3.1 guarantees that for any $\mathcal{D} \in \mathbb{D}$ (which trivially satisfies $\mathcal{D} \geq \mathcal{D}_{0}$ ), Problem 8 with approximate data $\nu_{\mathcal{D}}$ and $\xi_{\mathcal{D}}$ is coercive in $H_{0}^{1}(\Omega)$, precisely one has

$$
\begin{equation*}
\left(\nu_{\mathcal{D}} v_{x}, v_{x}\right)_{\Omega}+\left(\xi_{\mathcal{D}} v, v\right)_{\Omega} \geq \frac{\nu_{0}}{2}\left\|v_{x}\right\|_{\Omega}^{2} \quad \forall v \in H_{0}^{1}(\Omega) \tag{16}
\end{equation*}
$$

This easily follows using the bound $\|v\|_{\Omega} \leq \frac{1}{2 \sqrt{2}}|\Omega|\left\|v_{x}\right\|_{\Omega}$.
The following result is fundamental for establishing the convergence of our adaptive algorithm.

Proposition 3.1. i) There exists a constant $C_{\star}>0$ with the following property: for all $\mathcal{D} \in \mathbb{D}$ and all $z \in \mathcal{Z}$ of the form $z=\left(v, g_{\star}\right)$, let $z_{\mathcal{D}}=\left(v_{\mathcal{D}}, g_{\mathcal{D}}\right):=Q_{\mathcal{D}}(z)$, and let $u\left(g_{\mathcal{D}}\right) \in H_{0}^{1}(\Omega)$ be the solution of Problem 8 with data $g_{\mathcal{D}}$; then, it holds

$$
\begin{equation*}
\left\|u_{\star}-u\left(g_{\mathcal{D}}\right)\right\|_{H_{0}^{1}(\Omega)} \leq C_{\star} \kappa \mathrm{E}_{\mathcal{D}}\left(z_{0}\right)^{\frac{1}{2}} \leq C_{\star} \kappa \mathrm{E}_{\mathcal{D}}(z)^{\frac{1}{2}} \tag{17}
\end{equation*}
$$

where $\kappa$ is the penalization parameter introduced in (15).
ii) For all $\mathcal{D} \in \mathbb{D}, v \in V_{\mathcal{K}(\mathcal{D})}, w \in H_{0}^{1}(\Omega)$ and $g \in G(\Omega)$, it holds

$$
\begin{equation*}
\left|\mathrm{E}_{\mathcal{D}}(v, g)^{\frac{1}{2}}-\mathrm{E}_{\mathcal{D}}(w, g)^{\frac{1}{2}}\right| \leq\|v-w\|_{\mathcal{D}} \tag{18}
\end{equation*}
$$

The proof follows step by step the proof of Proposition 3 in [6], to which we refer.

## 4 The adaptive algorithm $h \boldsymbol{p}$-ADFEM

As anticipated in the Introduction, the algorithm we propose consists in alternating between a stage in which a new $h p$-partition is found, which is near-optimal for the current accuracy, and a stage in which this partition is further refined to guarantee a higher accuracy for the corresponding DG discrete solution; the data used in the latter stage to define the DG problem are approximations of the exact data, provided by the former stage.

The first stage will be accomplished by a call to the routine hp-NEARBEST introduced in Sect. 2.3. The second stage will be realized through a routine DG-SOLVE that we present now, postponing to Sect. 5 the detailed description of the underlying algorithm and the analysis of its properties. Essentially, starting from a given $h p$-partition and a corresponding data approximation, several DG problems are solved on subsequently refined partitions, whose generation is driven by an a posteriori error estimator, until a contraction property guarantees that the discretization error is brought below a prescribed threshold. In this stage, optimality is not an issue for the output partition, provided its cardinality remains comparable to that of the input partition.

- $[\overline{\mathcal{D}}, \bar{u}]:=\mathbf{D G - S O L V E}\left(\varepsilon, \mathcal{D}, z_{\mathcal{D}}\right)$

The routine DG-SOLVE takes as input $\varepsilon>0, \mathcal{D} \in \mathbb{D}$, and $z_{\mathcal{D}}=\left(v_{\mathcal{D}}, g_{\mathcal{D}}\right) \in Z_{\mathcal{D}}$. It outputs $\overline{\mathcal{D}} \in \mathbb{D}$ with $\mathcal{D} \leq \overline{\mathcal{D}}$ and $\bar{u}:=u_{\overline{\mathcal{D}}}\left(g_{\mathcal{D}}\right) \in V_{\overline{\mathcal{D}}}$ such that $\left\|u\left(g_{\mathcal{D}}\right)-\bar{u}\right\|_{\overline{\mathcal{D}}} \leq \varepsilon$.

We recall that $u_{\overline{\mathcal{D}}}\left(g_{\mathcal{D}}\right)$ denotes the solution of the following DG problem (see (14)): for $g_{\mathcal{D}}=\left(\nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}\right) \in G_{\mathcal{D}}$,

$$
\begin{equation*}
u_{\overline{\mathcal{D}}} \in V_{\overline{\mathcal{D}}}: a_{\overline{\mathcal{D}}}\left(u_{\overline{\mathcal{D}}}, v_{\overline{\mathcal{D}}} ; \nu_{\mathcal{D}}, \xi_{\mathcal{D}}\right)=\left(f_{\mathcal{D}}, v_{\overline{\mathcal{D}}}\right)_{\Omega} \quad \forall v_{\overline{\mathcal{D}}} \in V_{\overline{\mathcal{D}}} \tag{19}
\end{equation*}
$$

The input function $v_{\mathcal{D}} \in V_{\mathcal{D}}$ may be used in the algorithm to define the starting point of the adaptive iterations.

Assumption 4.1. Let $b<1<B$ be the constants that appear in the statement of the instance optimality property for the routine hp-NEARBEST. We assume that the penalization parameter $\kappa$ in (15) is chosen small enough, so that it holds

$$
C_{\star} \kappa<b .
$$

We are ready to present our algorithm hp-ADFEM. Let us introduce the parameters and the input data.
Parameters: two real numbers $\eta \in(0,1), \omega>0$ satisfying

$$
C_{\star} \kappa<b(1-\eta) \quad \text { and } \quad \omega \in\left(\frac{1}{b}, \frac{1-\eta}{C_{\star} \kappa}\right) .
$$

(Note that such a choice of $\omega$ is equivalent to $b \omega-1>0$ and $C_{\star} \kappa \omega+\eta<1$, which are two quantities that will appear below.)
Input data: $g_{\star} \in G(\Omega), \varepsilon_{0}>0$, and $\bar{u}_{0} \in V_{\overline{\mathcal{D}}_{0}}$ for some $\overline{\mathcal{D}}_{0} \in \mathbb{D}$ such that $\left\|u_{\star}-\bar{u}_{0}\right\|_{\overline{\mathcal{D}}_{0}} \leq \varepsilon_{0}$.

```
Algorithm hp- \(\operatorname{ADFEM}\left(\varepsilon_{0}, \bar{u}_{0}, g_{\star}\right)\)
    for \(i=1,2, \ldots\) do
        \(\left[\mathcal{D}_{i},\left(v_{\mathcal{D}_{i}}, g_{\mathcal{D}_{i}}\right)\right]:=\mathbf{h p}-\operatorname{NEARBEST}\left(\omega \varepsilon_{i-1},\left(\bar{u}_{i-1}, g_{\star}\right)\right)\)
        \(\left[\overline{\mathcal{D}}_{i}, \bar{u}_{i}\right]:=\mathbf{D G - S O L V E}\left(\eta \varepsilon_{i-1}, \mathcal{D}_{i},\left(v_{\mathcal{D}_{i}}, g_{\mathcal{D}_{i}}\right)\right)\)
        \(\varepsilon_{i}:=\left(C_{\star} \kappa \omega+\eta\right) \varepsilon_{i-1}\)
    end do
```

Theorem 4.1. Under Assumptions 3.1 and 4.1, the sequences $\left(\bar{u}_{i}\right)$, $\left(\mathcal{D}_{i}\right)$ produced by hp-ADFEM satisfy the following properties:

$$
\begin{equation*}
\left\|u_{\star}-\bar{u}_{i}\right\|_{\overline{\mathcal{D}}_{i}} \leq \varepsilon_{i} \quad \forall i \geq 0, \quad \mathrm{E}_{\mathcal{D}_{i}}\left(u_{\star}, g_{\star}\right)^{\frac{1}{2}} \leq(\omega+1) \varepsilon_{i-1} \quad \forall i \geq 1 \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\# \mathcal{D}_{i} \leq B \# \mathcal{D} \quad \text { for any } \mathcal{D} \in \mathbb{D} \text { with } \mathrm{E}_{\mathcal{D}}\left(u_{\star}, g_{\star}\right)^{\frac{1}{2}} \leq(b \omega-1) \varepsilon_{i-1} \tag{21}
\end{equation*}
$$

Proof. The bound $\left\|u_{\star}-\bar{u}_{0}\right\|_{\overline{\mathcal{D}}_{0}} \leq \varepsilon_{0}$ is valid by assumption. For $i \geq 1$, the tolerances used for hp-NEARBEST and DG-SOLVE, together with (17) show that

$$
\begin{align*}
\left\|u_{\star}-\bar{u}_{i}\right\|_{\overline{\mathcal{D}}_{i}} & \leq\left\|u_{\star}-u\left(g_{\mathcal{D}_{i}}\right)\right\|_{H_{0}^{1}(\Omega)}+\left\|u\left(g_{\mathcal{D}_{i}}\right)-\bar{u}_{i}\right\|_{\overline{\mathcal{D}}_{i}} \\
& \leq C_{\star} \kappa \mathrm{E}_{\mathcal{D}_{i}}\left(\bar{u}_{i-1}, g_{\star}\right)^{\frac{1}{2}}+\mu \varepsilon_{i-1} \leq\left(C_{\star} \kappa \omega+\mu\right) \varepsilon_{i-1}=\varepsilon_{i} \tag{22}
\end{align*}
$$

The first statement follows for all $i \geq 0$. Using this and (18) implies the second assertion

$$
\mathrm{E}_{\mathcal{D}_{i}}\left(u_{\star}, g_{\star}\right)^{\frac{1}{2}} \leq \mathrm{E}_{\mathcal{D}_{i}}\left(\bar{u}_{i-1}, g_{\star}\right)^{\frac{1}{2}}+\left\|u_{\star}-\bar{u}_{i-1}\right\|_{\overline{\mathcal{D}}_{i-1}} \leq(\omega+1) \varepsilon_{i-1} \quad \forall i \geq 1
$$

Finally, let $\mathcal{D} \in \mathbb{D}$ with $\mathrm{E}_{\mathcal{D}}\left(u_{\star}, g_{\star}\right)^{\frac{1}{2}} \leq(b \omega-1) \varepsilon_{i-1}$. Then, again by $(18), \mathrm{E}_{\mathcal{D}}\left(\bar{u}_{i-1}, g_{\star}\right)^{\frac{1}{2}} \leq$ $b \omega \varepsilon_{i-1}$ and so $\# \mathcal{D}_{i} \leq B \# \mathcal{D}$ because of the optimality property of hp-NEARBEST.

The main result of Theorem 4.1 can be summarized by saying that hp-ADFEM is instance optimal for reducing $\mathrm{E}_{\mathcal{D}}\left(u_{\star}, g_{\star}\right)$ over $\mathcal{D} \in \mathbb{D}$.

## 5 The routine DG-SOLVE

The purpose of this section is the description and analysis of a realization of the routine DG-SOLVE. It is based on an iterative procedure of the form SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE, in which ESTIMATE uses a residual-type estimator, whereas REFINE applies a dyadic splitting of each marked element while preserving the polynomial degree. The procedure satisfies a contraction property, which guarantees the reduction of a suitable "error" by a fixed amount at each iteration. Our construction is strongly inspired by [4], whose arguments are hereafter extended to cover the $h p$-case.

In the sequel, the input partition $\mathcal{D}$ will be denoted by $\mathcal{D}_{\text {in }}$, whereas the symbol $\mathcal{D}$ will be used to denote any refinement of $\mathcal{D}_{\text {in }}$ generated by the procedure. Similarly, the input function will be denoted by $z_{\text {in }}=\left(v_{\text {in }}, g_{\text {in }}\right)$. To avoid cumbersome notation, we will actually write $g_{\mathrm{in}}=: g=(\nu, \xi, f)$, but we will recall that $g$ is a piecewise polynomial approximation on the input partition $\mathcal{D}_{\text {in }}$ of the given data $g_{\star}=\left(\nu_{\star}, \xi_{\star}, f_{\star}\right) \in G(\Omega)$. Coherently, the exact solution of Problem (8) with input data $g$ will be denoted by $u=$ $u(g)$, whereas for any $h p$-partition $\mathcal{D} \leq \mathcal{D}_{\text {in }}, u_{\mathcal{D}}=u_{\mathcal{D}}(g)$ will be the solution of the corresponding DG Problem (14).

For the analysis of the procedure, following [3], we extend the definition of the DG form $a_{\mathcal{D}}$ given in (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$ to the infinite dimensional space $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ (recall (9)). To this end, we introduce the lifting operator $L_{\mathcal{D}}: V_{\mathcal{K}(\mathcal{D})} \rightarrow V_{\mathcal{D}}$ such that for all $w \in V_{\mathcal{K}(\mathcal{D})}$

$$
\begin{equation*}
L_{\mathcal{D}} w \in V_{\mathcal{D}}:\left(\nu v, L_{\mathcal{D}} w\right)_{\Omega}=(\{\nu \nu v\}, \llbracket w \rrbracket)_{\varepsilon_{\mathcal{D}}} \quad \forall v \in V_{\mathcal{D}} \tag{23}
\end{equation*}
$$

Then, on $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ we define the bilinear form

$$
\begin{align*}
a_{\mathcal{D}}(w, v):= & \left(\nu \tilde{w}_{x}, \tilde{v}_{x}\right)_{\Omega}+(\xi w, v)_{\Omega} \\
& -\left(\nu \tilde{w}_{x}, L_{\mathcal{D}} v\right)_{\mathcal{E}_{\mathcal{D}}}-\left(\nu \tilde{v}_{x}, L_{\mathcal{D}} w\right)_{\mathcal{E}_{\mathcal{D}}}+\gamma\left(\sigma_{\mathcal{D}} \llbracket w \rrbracket, \llbracket v \rrbracket\right)_{\mathcal{E}_{\mathcal{D}}}, \tag{24}
\end{align*}
$$

which is readily seen to coincide with (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$.
The lifting operator satisfies the following stability bound.

Property 5.1. There exists a constant $C_{1}>0$ independent of $\mathcal{D}$ such that

$$
\begin{equation*}
\left\|L_{\mathcal{D}} w\right\|_{\Omega} \leq C_{1}\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket w \rrbracket\right\|_{\varepsilon_{\mathcal{D}}} \quad \forall w \in V_{\mathcal{K}(\mathcal{D})} \tag{25}
\end{equation*}
$$

Proof. If $K$ is any interval of length $h$ and $e$ is one of its endpoints, the inverse inequality $|\phi(e)| \lesssim \frac{p}{h^{1 / 2}}\|\phi\|_{K}$ holds for any $\phi \in \mathbb{P}_{p}(K)$. Then, the result easily follows by choosing $v=L_{\mathcal{D}} w$ in (23).

Using (25), one proves the existence of a constant $\gamma_{0}>0$ independent of $\mathcal{D}$ such that for any $\gamma \geq \gamma_{0}$ the bilinear form $a_{\mathcal{D}}$ is continuous and coercive in $V_{\mathcal{K}(\mathcal{D})}$ with respect to the DG norm $\|v\|_{\mathcal{D}}$, uniformly in $\mathcal{D}$. For future references, let us denote by $0<\alpha_{*} \leq \alpha^{*}$ the coercivity and continuity constants. Since $a_{\mathcal{D}}$ is symmetric, it defines an inner product in $V_{\mathcal{K}(\mathcal{D})}$; the corresponding norm will be denoted by $\|v\|_{a, \mathcal{D}}$ and is uniformly equivalent to the DG norm $\|v\|_{\mathcal{D}}$ introduced in (13).

It is well-known that while the DG-solution $u_{\mathcal{D}} \in V_{\mathcal{D}}$ satisfies the variational equations

$$
\begin{equation*}
a_{\mathcal{D}}\left(u_{\mathcal{D}}, v_{\mathcal{D}}\right)=\left(f, v_{\mathcal{D}}\right)_{\Omega} \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}} \tag{26}
\end{equation*}
$$

the exact solution $u \in H_{0}^{1}(\Omega)$ need not satisfy $a_{\mathcal{D}}(u, v)=(f, v)_{\Omega}$ for all $v \in V_{\mathcal{K}(\mathcal{D})}$ (inconsistency of the DG formulation). However, we do have the partial consistency property

$$
\begin{equation*}
a_{\mathcal{D}}(u, v)=(f, v)_{\Omega} \quad \forall v \in H_{0}^{1}(\Omega) \tag{27}
\end{equation*}
$$

This motivates the introduction of the conforming subspace $V_{\mathcal{D}}^{c}:=V_{\mathcal{D}} \cap H_{0}^{1}(\Omega)$. Then, by subtraction of (26) from (27), we obtain the partial orthogonality property

$$
\begin{equation*}
a_{\mathcal{D}}\left(u-u_{\mathcal{D}}, v_{\mathcal{D}}\right)=0 \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}^{c} \tag{28}
\end{equation*}
$$

It is useful for the sequel to introduce the orthogonal decomposition

$$
\begin{equation*}
V_{\mathcal{D}}=V_{\mathcal{D}}^{c} \oplus V_{\mathcal{D}}^{\perp} \tag{29}
\end{equation*}
$$

where $V_{\mathcal{D}}^{\perp}$ is the orthogonal complement of $V_{\mathcal{D}}^{c}$ with respect to the inner product $a_{\mathcal{D}}(w, v)$. Any $v_{\mathcal{D}} \in V_{\mathcal{D}}$ will be split according to (29) as $v_{\mathcal{D}}=v_{\mathcal{D}}^{c}+v_{\mathcal{D}}^{\perp}$.
Property 5.2. There exists a constant $C_{2}>0$ independent of $\mathcal{D}$ for which the following bound on the $D G$ discretization error holds:

$$
\left\|u-u_{\mathcal{D}}\right\|_{\mathcal{D}} \leq C_{2}\left(\inf _{w_{\mathcal{D}} \in V_{\mathcal{D}}^{c}}\left\|u-w_{\mathcal{D}}\right\|_{H_{0}^{1}(\Omega)}+\left\|u \frac{\perp}{\mathcal{D}}\right\|_{\mathcal{D}}\right)
$$

Proof. For any $w_{\mathcal{D}} \in V_{\mathcal{D}}^{c}$, using (28) we have

$$
\begin{aligned}
a_{\mathcal{D}}\left(u_{\mathcal{D}}-w_{\mathcal{D}}, u_{\mathcal{D}}-w_{\mathcal{D}}\right) & =a_{\mathcal{D}}\left(u_{\mathcal{D}}-w_{\mathcal{D}}, u_{\mathcal{D}}^{c}-w_{\mathcal{D}}\right)+a_{\mathcal{D}}\left(u_{\mathcal{D}}-w_{\mathcal{D}}, u_{\mathcal{D}}^{\perp}\right) \\
& =a_{\mathcal{D}}\left(u-w_{\mathcal{D}}, u_{\mathcal{D}}^{c}-w_{\mathcal{D}}\right)+a_{\mathcal{D}}\left(u_{\mathcal{D}}^{\perp}, u_{\mathcal{D}}-w_{\mathcal{D}}\right) \\
& =a_{\mathcal{D}}\left(u-w_{\mathcal{D}}, u_{\mathcal{D}}-w_{\mathcal{D}}\right)-a_{\mathcal{D}}\left(u_{\mathcal{D}}^{\perp}, u-u_{\mathcal{D}}\right),
\end{aligned}
$$

whence, by the coercivity and continuity of the form $a_{\mathcal{D}}$,

$$
\left\|u_{\mathcal{D}}-w_{\mathcal{D}}\right\|_{\mathcal{D}}^{2} \lesssim\left\|u-w_{\mathcal{D}}\right\|_{\mathcal{D}}\left\|u_{\mathcal{D}}-w_{\mathcal{D}}\right\|_{\mathcal{D}}+\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}\left\|u-u_{\mathcal{D}}\right\|_{\mathcal{D}}
$$

We conclude by the triangle inequality.
We also introduce an approximation operator $\mathbb{I}_{\mathcal{D}}: V_{\mathcal{K}(\mathcal{D})} \rightarrow V_{\mathcal{D}}^{c}$ that will be useful in the sequel. For any $D \in \mathcal{D}$, set $K_{D}=:\left[e_{l}, e_{r}\right]$ and let $\mathcal{P}_{D}: H^{1}\left(K_{D}\right) \rightarrow \mathbb{P}_{p_{D}}\left(K_{D}\right)$ be defined as follows:

$$
\left(\mathcal{P}_{D} v\right)(x):=v\left(e_{l}\right)+\int_{e_{l}}^{x}\left(\Pi_{K_{D}, p_{D}-1}^{0} v_{x}\right)(s) d s
$$

(recall that $\Pi^{0}$ means $L^{2}$-orthogonal projection). Furthermore, consider the Legendre Gauss-Lobatto grid in $K_{D}$ containing $p_{D}+1$ nodes, and let $\psi_{D, e_{l}}$ and $\psi_{D, e_{r}}$ denote the Lagrange basis functions of degree $p_{D}$ on this grid, associated with the boundary nodes. Then, we define $\left(\mathbb{I}_{\mathcal{D}} v\right)_{\mid K_{D}}:=\mathcal{J}_{D} v_{\mid K_{D}}$, where

$$
\begin{equation*}
\mathcal{J}_{D} v:=\mathcal{P}_{D} v-\tau_{e_{l}} \llbracket v \rrbracket_{e_{l}} \psi_{D, e_{l}}+\tau_{e_{r}} \llbracket v \rrbracket_{e_{r}} \psi_{D, e_{r}} \tag{30}
\end{equation*}
$$

with $\tau_{e}=1$ if $e \in \partial \Omega, \tau_{e}=\frac{1}{2}$ otherwise. Checking that $\mathbb{I}_{\mathcal{D}} v \in V_{\mathcal{D}}^{c}$ is straightforward.
Property 5.3. The following error estimates hold for any $v \in H_{0}^{1}(\Omega)$ :

$$
\begin{equation*}
\left\|\left(v-\mathbb{I}_{\mathcal{D}} v\right) \omega_{D}^{-1 / 2}\right\|_{K_{D}} \leq \frac{1}{\left(p_{D}\left(p_{D}+1\right)\right)^{1 / 2}}\left\|v_{x}\right\|_{K_{D}}, \quad\left\|\left(\mathbb{I}_{\mathcal{D}} v\right)_{x}\right\|_{K_{D}} \leq\left\|v_{x}\right\|_{K_{D}} \tag{31}
\end{equation*}
$$

where $\omega_{D}$ is the quadratic bubble function in $K_{D}$, defined as $\omega_{D}(x)=\left(x-e_{l}\right)\left(e_{r}-x\right)$.
The following error estimates hold for any $v \in V_{\mathcal{D}}$ :

$$
\begin{equation*}
\left\|v-\mathbb{I}_{\mathcal{D}} v\right\|_{K_{D}} \lesssim \frac{h_{D}^{1 / 2}}{p_{D}}\left(\llbracket v \rrbracket_{e_{l}}+\llbracket v \rrbracket_{e_{r}}\right), \quad\left\|\left(v-\mathbb{I}_{\mathcal{D}} v\right)_{x}\right\|_{K_{D}} \lesssim \frac{p_{D}}{h_{D}^{1 / 2}}\left(\llbracket v \rrbracket_{e_{l}}+\llbracket v \rrbracket_{e_{r}}\right) . \tag{32}
\end{equation*}
$$

The latter inequality implies the bound

$$
\begin{equation*}
\left\|\tilde{v}_{x}-\left(\mathbb{I}_{\mathcal{D}} v\right)_{x}\right\|_{\Omega} \lesssim\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket v \rrbracket\right\|_{\varepsilon_{\mathcal{D}}} \quad \forall v \in V_{\mathcal{D}} \tag{33}
\end{equation*}
$$

Proof. The first inequality in (31) can be found in [10], whereas the second one is just the stability of the orthogonal projection. The inequalities (32) easily follow from the bounds $\left\|\psi_{D, e}\right\|_{K_{D}} \simeq \frac{h_{D}^{1 / 2}}{p_{D}}$ and $\left\|\left(\psi_{D, e}\right)_{x}\right\|_{K_{D}} \lesssim \frac{p_{D}^{2}}{h_{D}}\left\|\psi_{D, e}\right\|_{K_{D}}$.

Corollary 5.1. There exists a constant $C_{3}>0$ independent of $\mathcal{D}$ such that for any $v=v^{c} \oplus v^{\perp} \in V_{\mathcal{D}}=V_{\mathcal{D}}^{c} \oplus V_{\mathcal{D}}^{\perp}$ one has

$$
\left\|v^{\perp}\right\|_{\mathcal{D}} \leq C_{3} \gamma^{1 / 2}\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket v \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}
$$

Proof. One has

$$
\left\|v^{\perp}\right\|_{\mathcal{D}} \simeq\left\|v^{\perp}\right\|_{a, \mathcal{D}}=\inf _{w \in V_{\mathcal{D}}^{c}}\|v-w\|_{a, \mathcal{D}} \simeq \inf _{w \in V_{\mathcal{D}}^{c}}\|v-w\|_{\mathcal{D}} \leq\left\|v-\mathbb{I}_{\mathcal{D}} v\right\|_{\mathcal{D}}
$$

then one concludes by (33).

### 5.1 The residual estimator

Given any $v \in V_{\mathcal{D}}$ and any $D \in \mathcal{D}$, let us define the local residual

$$
\operatorname{res}_{D}(v):=(f-A v)_{\mid K_{D}}
$$

for any $e \in \partial K_{D}$, let us define the jump of the flux at $e$

$$
J_{e}(v)=\llbracket \nu v_{x} \rrbracket_{e}
$$

Then, the (squared) local error estimator is defined as follows

$$
\eta_{D}^{2}(v):=\frac{1}{p_{D}\left(p_{D}+1\right)}\left\|\operatorname{res}_{D}(v) \omega_{D}^{1 / 2}\right\|_{K_{D}}^{2}+\sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{-1} J_{e}^{2}(v)
$$

where $\omega_{D}$ denotes the quadratic bubble function introduced in Property 5.3 above. The (squared) global error estimator is

$$
\eta_{\mathcal{D}}^{2}(v):=\sum_{D \in \mathcal{D}} \eta_{D}^{2}(v)
$$

whereas its restriction to a subset $\mathcal{D}^{\prime} \subseteq \mathcal{D}$ of elements will be denoted by

$$
\eta_{\mathcal{D}}^{2}\left(v ; \mathcal{D}^{\prime}\right):=\sum_{D \in \mathcal{D}^{\prime}} \eta_{D}^{2}(v)
$$

We show that $\eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)$ is a reliable estimator for our DG problem in two steps.
Proposition 5.1. There exists a constant $C_{4}>0$ independent of $\mathcal{D}$ such that

$$
a_{\mathcal{D}}\left(u-u_{\mathcal{D}}, u-u_{\mathcal{D}}\right) \leq C_{4}\left(\eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)+\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}\right)
$$

Proof. We adapt the proof of [4], Lemma 3.1, to our $h p$ setting. Let us split the DG solution as $u_{\mathcal{D}}=u_{\mathcal{D}}^{c}+u_{\mathcal{D}}^{\perp}$ and let us set $e:=u-u_{\mathcal{D}}$ and $w:=u-u_{\mathcal{D}}^{c} \in H_{0}^{1}(\Omega)$, so that $e=w-u \frac{\perp}{\mathcal{D}}$. Then, recalling (27) and (28),

$$
\begin{aligned}
a_{\mathcal{D}}(e, e) & =a_{\mathcal{D}}(e, w)-a_{\mathcal{D}}\left(e, u_{\mathcal{D}}^{\perp}\right)=a_{\mathcal{D}}\left(e, w-\mathbb{I}_{\mathcal{D}} w\right)-a_{\mathcal{D}}\left(e, u_{\mathcal{D}}^{\perp}\right) \\
& =\left(f, w-\mathbb{I}_{\mathcal{D}} w\right)_{\Omega}-a_{\mathcal{D}}\left(u_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}} w\right)-a_{\mathcal{D}}\left(e, u_{\mathcal{D}}^{\perp}\right)
\end{aligned}
$$

Integrating back by parts, we get

$$
a_{\mathcal{D}}\left(u_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}} w\right)=\sum_{D \in \mathcal{D}}\left(A u_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}} w\right)_{K_{D}}+\left(L_{\mathcal{D}} u_{\mathcal{D}}, \nu\left(w-\mathbb{I}_{\mathcal{D}} w\right)_{x}\right)_{\Omega}
$$

whence

$$
a_{\mathcal{D}}(e, w)=\sum_{D \in \mathcal{D}}\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right), w-\mathbb{I}_{\mathcal{D}} w\right)_{K_{D}}+\left(L_{\mathcal{D}} u_{\mathcal{D}}, \nu\left(w-\mathbb{I}_{\mathcal{D}} w\right)_{x}\right)_{\Omega}
$$

Writing $\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right), w-\mathbb{I}_{\mathcal{D}} w\right)_{K_{D}}=\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right) \omega_{D}^{1 / 2},\left(w-\mathbb{I}_{\mathcal{D}} w\right) \omega_{D}^{-1 / 2}\right)_{K_{D}}$ and using (31) as well as (25), we obtain

$$
a_{\mathcal{D}}(e, w) \leq\left(\eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)+C_{1}\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}}\right)\left\|w_{x}\right\|_{\Omega}
$$

where the last norm can be bounded using the coercivity of the form $a_{\mathcal{D}}$ :

$$
\left\|w_{x}\right\|_{\Omega}=\|w\|_{\mathcal{D}} \leq\|e\|_{\mathcal{D}}+\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}} \leq \alpha_{*}^{1 / 2} a_{\mathcal{D}}(e, e)^{1 / 2}+\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}
$$

By Young's inequality, we obtain for a suitable constant $C>0$

$$
a_{\mathcal{D}}(e, w) \leq \frac{1}{4} a_{\mathcal{D}}(e, e)+C\left(\eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)+\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}^{2}+\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}}^{2}\right)
$$

It remains to bound the term $a_{\mathcal{D}}\left(e, u_{\mathcal{D}}^{\perp}\right)$, which is easily done using the continuity of $a_{\mathcal{D}}$ : $a_{\mathcal{D}}\left(e, u_{\mathcal{D}}^{\perp}\right) \leq a_{\mathcal{D}}(e, e)^{1 / 2} a_{\mathcal{D}}\left(u_{\mathcal{D}}^{\perp}, u_{\mathcal{D}}^{\perp}\right)^{1 / 2} \leq a_{\mathcal{D}}(e, e)^{1 / 2}\left(\alpha^{*}\right)^{1 / 2}\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}} \leq \frac{1}{4} a_{\mathcal{D}}(e, e)+\alpha^{*}\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}^{2}$.
We obtain the desired result by invoking Corollary 5.1.

Proposition 5.2. There exists a constant $C_{5}>0$ independent of $\mathcal{D}$ such that for any $\gamma$ large enough, say $\gamma \geq \gamma_{1} \geq \gamma_{0}$, one has

$$
\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}} \leq C_{5} \eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)
$$

Proof. Here, we adapt the proof of [4], Lemma 3.3, to our $h p$ setting. By the coercivity of the form $a_{\mathcal{D}}$ applied to $u_{\mathcal{D}}-\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$, we have

$$
\begin{equation*}
\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2} \leq \alpha_{*}^{-1} a_{\mathcal{D}}\left(u_{\mathcal{D}}-\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}, u_{\mathcal{D}}-\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}\right) \tag{34}
\end{equation*}
$$

since $\llbracket \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}} \rrbracket=0$. For simplicity, let us set $w:=u_{\mathcal{D}}-\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$ and $v:=\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}} \in H_{0}^{1}(\Omega)$. Then,

$$
a_{\mathcal{D}}(w, w)=(f, w)_{\Omega}-a_{\mathcal{D}}(v, w)
$$

and, using $L_{\mathcal{D}} v=0$ several times, we have

$$
\begin{aligned}
a_{\mathcal{D}}(v, w) & =\left(\nu v_{x}, \tilde{w}_{x}\right)_{\Omega}+(\xi v, w)_{\Omega}-\left(L_{\mathcal{D}} u_{\mathcal{D}}, \nu v_{x}\right)_{\Omega} \\
& =\left(\nu \tilde{u}_{\mathcal{D}, x}, \tilde{w}_{x}\right)_{\Omega}+\left(\xi u_{\mathcal{D}}, w\right)_{\Omega}-\left\|\nu^{1 / 2} \tilde{w}_{x}\right\|_{\Omega}^{2}-\left\|\xi^{1 / 2} w\right\|_{\Omega}^{2}-\left(L_{\mathcal{D}} u_{\mathcal{D}}, \nu v_{x}\right)_{\Omega}
\end{aligned}
$$

Using in this identity

$$
\left.\left(\nu v_{x}, \tilde{w}_{x}\right)_{\Omega}=-\sum_{D \in \mathcal{D}}\left(\left(\nu u_{\mathcal{D}, x}\right)_{x}, w\right)_{K_{D}}+\left(\llbracket \nu u_{\mathcal{D}, x} \rrbracket,\{w\}\right)\right)_{\mathcal{E}_{\mathcal{D}}}+\left(\llbracket w \rrbracket,\left\{\left\{\nu u_{\mathcal{D}, x} \rrbracket\right\}\right) \varepsilon_{\mathcal{D}}\right.
$$

and observing that $\left(\llbracket w \rrbracket,\left\{\left\lfloor u_{\mathcal{D}, x}\right\}\right\}\right)_{\mathcal{E}_{\mathcal{D}}}=\left(L_{\mathcal{D}} w, \nu \tilde{u}_{\mathcal{D}, x}\right)_{\Omega}$, we obtain

$$
\begin{align*}
a_{\mathcal{D}}(w, w)= & \sum_{D \in \mathcal{D}}\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right), w\right)_{K_{D}}+\left(J_{\mathcal{D}}\left(u_{\mathcal{D}}\right),\{\{w\}) \varepsilon_{\mathcal{D}}\right.  \tag{35}\\
& \quad+\left\|\nu^{1 / 2} \tilde{w}_{x}\right\|_{\Omega}^{2}+\left\|\xi^{1 / 2} w\right\|_{\Omega}^{2}+\left(L_{\mathcal{D}} u_{\mathcal{D}}, \nu \tilde{w}_{x}\right)_{\Omega}
\end{align*}
$$

By (32) we have

$$
\|w\|_{K_{D}} \leq \sum_{e \in \partial K_{D}} \frac{h_{D}^{1 / 2}}{p_{D}}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right|=\sum_{e \in \partial K_{D}} \frac{h_{D}^{1 / 2}}{p_{D}} \sigma_{\mathcal{D}, e}^{-1 / 2} \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right| \leq \frac{h_{D}}{p_{D}^{2}} \sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right|
$$

whence

$$
\begin{aligned}
\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right), w\right)_{K_{D}} & \leq \frac{h_{D}}{p_{D}^{2}}\left\|\operatorname{res}_{D}\left(u_{\mathcal{D}}\right)\right\|_{K_{D}} \sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right| \\
& \lesssim \frac{1}{p_{D}}\left\|\operatorname{res}_{D}\left(u_{\mathcal{D}}\right) \omega_{D}^{1 / 2}\right\|_{K_{D}} \sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right| \leq \eta_{D}\left(u_{\mathcal{D}}\right) \sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right|
\end{aligned}
$$

where we have used the inverse inequality $\|\phi\|_{K_{D}} \lesssim \frac{p_{D}}{h_{D}}\left\|\phi \omega_{D}^{1 / 2}\right\|_{K_{D}}$ which holds for all polynomials of degree $\simeq p_{D}$, since $\operatorname{res}_{D}\left(u_{\mathcal{D}}\right)$ is such a polynomial. Thus, we obtain

$$
\sum_{D \in \mathcal{D}}\left(\operatorname{res}_{D}\left(u_{\mathcal{D}}\right), w\right)_{K_{D}} \lesssim \eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}
$$

Concerning the second term on the right-hand side of (35), we observe that by construction of $\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$, one has $w(e)=\frac{1}{2} \llbracket u_{\mathcal{D}} \rrbracket_{e}$ at any internal inter-element point $e$, whereas $w(e)=0$ at the boundary points of $\Omega$. Thus,

$$
\begin{aligned}
\left(J_{\mathcal{D}}\left(u_{\mathcal{D}}\right),\{\{w\})\right)_{\mathcal{E}_{\mathcal{D}}} & \lesssim \sum_{e \in \mathcal{E}_{\mathcal{D}}}\left|J_{e}\left(u_{\mathcal{D}}\right)\right|\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right|=\sum_{e \in \mathcal{E}_{\mathcal{D}}} \sigma_{\mathcal{D}, e}^{-1 / 2}\left|J_{e}\left(u_{\mathcal{D}}\right)\right| \sigma_{\mathcal{D}, e}^{1 / 2}\left|\llbracket u_{\mathcal{D}} \rrbracket_{e}\right| \\
& \leq \eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}
\end{aligned}
$$

Finally, using (32) and (25), the three last terms on the right-hand side of (35) can be bounded by $C\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}$. Substituting all the previous bounds in (34), we obtain

$$
\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}}^{2} \lesssim\left(\eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}}+\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}\right)
$$

where the constant implied by the symbol $\lesssim$ is independent of $\gamma$. Therefore, choosing $\gamma$ large enough, we get the desired result.

Corollary 5.2. There exists a constant $C_{6}>0$ independent of $\mathcal{D}$ such that for any $\gamma \geq \gamma_{1}$, one has

$$
a_{\mathcal{D}}\left(u-u_{\mathcal{D}}, u-u_{\mathcal{D}}\right) \leq C_{6} \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)
$$

### 5.2 The adaptive iterations

The routine DG-SOLVE iterates the mapping

$$
\begin{equation*}
\left(\mathcal{D}, u_{\mathcal{D}}, \eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)\right) \rightarrow\left(\mathcal{D}_{*}, u_{\mathcal{D}_{*}}, \eta_{\mathcal{D}_{*}}\left(u_{\mathcal{D}_{*}}\right)\right), \tag{36}
\end{equation*}
$$

where $\mathcal{D}_{*}$ is a refinement of $\mathcal{D}$ obtained by first applying a Dörfler marking to the elements of $\mathcal{D}$ based on the error estimator $\eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right)$, and then performing a dyadic subdivision to the marked elements and its neighbors.

To be precise, let $\vartheta \in(0,1)$ be the Dörfler parameter. Let us order the local error estimators $\eta_{D}\left(u_{\mathcal{D}}\right), D \in \mathcal{D}$, by decreasing value, and let us choose a set $\mathcal{M} \subseteq \mathcal{D}$ of minimal cardinality for which

$$
\begin{equation*}
\eta_{\mathcal{D}}\left(u_{\mathcal{D}} ; \mathcal{N}\right) \geq \vartheta \eta_{\mathcal{D}}\left(u_{\mathcal{D}}\right) \tag{37}
\end{equation*}
$$

Let $\partial \mathcal{M} \subseteq \mathcal{D}$ denote the set of elements $D$ that share an interface with an element in $\mathcal{M}$. Then, we replace each $D=\left(K_{D}, p_{D}\right) \in \mathcal{M} \cup \partial \mathcal{M}$ by the two elements $D^{\prime}=\left(K_{D}^{\prime}, p_{D}\right)$ and $D^{\prime \prime}=\left(K_{D}^{\prime \prime}, p_{D}\right)$, where $K_{D}^{\prime}$ and $K_{D}^{\prime \prime}$ are the two children of $K_{D}$. Thus, the new partition $\mathcal{D}_{*}$ is defined by

$$
\begin{equation*}
\mathcal{D}_{*}=\left\{D^{\prime}, D^{\prime \prime}: D \in \mathcal{M} \cup \partial \mathcal{M}\right\} \cup\{D: D \in \mathcal{D} \backslash(\mathcal{M} \cup \partial \mathcal{M})\} \tag{38}
\end{equation*}
$$

Our aim is to prove that a suitable combination of (squared) DG error and error estimator, i.e.,

$$
\left\|u-u_{\mathcal{D}}\right\|_{a, \mathcal{D}}^{2}+\beta \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)
$$

for some $\beta>0$, is reduced by a fixed rate $\varrho \in(0,1)$ in performing the mapping (36). The proof, which extends [4] to our $h p$-setting, will be based on the following results.
Lemma 5.1. There exists a constant $C_{7}>0$ independent of $\mathcal{D}$ such that for any real $\lambda \in(0,1)$, one has

$$
\eta_{\mathcal{D}_{*}}^{2}\left(u_{\mathcal{D}_{*}}\right) \leq(1+\lambda)\left(1-\frac{\vartheta^{2}}{2}\right) \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)+\frac{C_{7}}{\lambda}\left\|u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right\|_{\mathcal{D}_{*}}^{2}
$$

Proof. We first establish a few results about the Lipschitz continuity of the local error estimators. Assume that $v, w \in V_{\mathcal{D}}$ and let $D \in \mathcal{D}$. By Minkowski's inequality,
$\left|\eta_{D}(v)-\eta_{D}(w)\right| \leq\left(\frac{1}{p_{D}^{2}}\left\|\left(\operatorname{res}_{D}(v)-\operatorname{res}_{D}(w)\right) \omega_{D}^{1 / 2}\right\|_{K_{D}}^{2}+\sum_{e \in \partial K_{D}} \sigma_{\mathcal{D}, e}^{-1}\left|J_{e}(v)-J_{e}(w)\right|^{2}\right)^{1 / 2}$.

One has

$$
\begin{aligned}
\left\|\left(\operatorname{res}_{D}(v)-\operatorname{res}_{D}(w)\right) \omega_{D}^{1 / 2}\right\|_{K_{D}} & \leq\left\|\left(\nu(v-w)_{x}\right)_{x} \omega_{D}^{1 / 2}\right\|_{K_{D}}+\left\|\xi(v-w) \omega_{D}^{1 / 2}\right\|_{K_{D}} \\
& \lesssim p_{D}\left\|\nu(v-w)_{x}\right\|_{K_{D}}+h_{D}\|\xi(v-w)\|_{K_{D}} \\
& \lesssim p_{D}\left\|(v-w)_{x}\right\|_{K_{D}}+h_{D}\|(v-w)\|_{K_{D}}
\end{aligned}
$$

where we have used the inverse inequality $\left\|\phi_{x} \omega_{D}^{1 / 2}\right\|_{K_{D}} \lesssim p_{D}\|\phi\|_{K_{D}}$, which holds for all polynomial $\phi$ of degree $\simeq p_{D}$ in $K_{D}$, as well as the bound $\left\|\omega_{D}^{1 / 2}\right\|_{L^{\infty}\left(K_{D}\right)} \leq h_{D}$.

On the other hand, for each $e \in \partial K_{D}$, let us denote by $D^{\prime}$ the element in $\mathcal{D}$ sharing the interface $e$ with $D$. Then,

$$
\begin{aligned}
\left|J_{e}(v)-J_{e}(w)\right| & \leq\left|\nu(v-w)_{x \mid K_{D}}(e)\right|+\left|\nu(v-w)_{x_{\mid K_{D^{\prime}}}}(e)\right| \\
& \lesssim\left|(v-w)_{x \mid K_{D}}(e)\right|+\left|(v-w)_{x \mid K_{D^{\prime}}}(e)\right| \\
& \lesssim \frac{p_{D}}{h_{D}^{1 / 2}}\left\|(v-w)_{x}\right\|_{K_{D}}+\frac{p_{D^{\prime}}}{h_{D^{\prime}}^{1 / 2}}\left\|(v-w)_{x}\right\|_{K_{D^{\prime}}} \\
& \leq \sigma_{\mathcal{D}, e}^{1 / 2}\left(\left\|(v-w)_{x}\right\|_{K_{D}}+\left\|(v-w)_{x}\right\|_{K_{D^{\prime}}}\right)
\end{aligned}
$$

where we have used the inverse inequality $|\psi(e)| \lesssim \frac{p_{D}}{h_{D}^{1 / 2}}\|\psi\|_{K_{D}}$, which holds for all polynomial $\psi$ of degree $\simeq p_{D}$ in $K_{D}$. We conclude that

$$
\left|\eta_{D}(v)-\eta_{D}(w)\right| \lesssim \mathcal{N}_{D}(v-w), \quad \text { with } \quad \mathcal{N}_{D}^{2}(\phi):=\sum_{D^{\prime}}\left\|\phi_{x}\right\|_{K_{D}}^{2}+\frac{h_{D}^{2}}{p_{D}^{2}}\|\phi\|_{K_{D}}^{2}
$$

where summation is extended to all $D^{\prime} \in \mathcal{D}$ such that $K_{D^{\prime}} \cap K_{D}$ is nonempty; this implies

$$
\begin{equation*}
\eta_{D}^{2}(v) \leq(1+\lambda) \eta_{D}^{2}(w)+\frac{C}{\lambda} \mathcal{N}_{D}^{2}(v-w) \tag{39}
\end{equation*}
$$

for a suitable constant $C>0$ independent of $D$.
We now apply these bounds, with $v=u_{\mathcal{D}_{*}}$ and $w=u_{\mathcal{D}}$, to the partition (38) generated by the refinement procedure. If $D \in \mathcal{M}$, let $D_{m}, m=1,2$ be the two children in which $D$ is split. We have $\omega_{D_{m}}(x) \leq \frac{1}{2} \omega_{D}(x)$ for all $x \in D_{m}$. By definition of refinement, we have $h_{D_{m}}=\frac{1}{2} h_{D}$ as well as $h_{D_{m}^{\prime}}=\frac{1}{2} h_{D^{\prime}}$ for any neighborhood $D^{\prime} \in \mathcal{D}$ of $D$, which implies $\sigma_{\mathcal{D}_{*}, e}^{-1} \leq \frac{1}{2} \sigma_{\mathcal{D}, e}^{-1}$ for any $e \in \partial K_{D}$. Hence, we immediately have $\sum_{m=1}^{2} \eta_{D_{m}}^{2}\left(u_{\mathcal{D}}\right) \leq \frac{1}{2} \eta_{D}^{2}\left(u_{\mathcal{D}}\right)$ and $\sum_{m=1}^{2} \mathcal{N}_{D_{m}}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right) \leq \mathcal{N}_{D}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right)$, whence

$$
\sum_{m=1}^{2} \eta_{D_{m}}^{2}\left(u_{\mathcal{D}_{*}}\right) \leq \frac{1}{2}(1+\lambda) \eta_{D}^{2}\left(u_{\mathcal{D}}\right)+\frac{C}{\lambda} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right)
$$

If $D \in \partial \mathcal{M}$, we can only say that $\sigma_{\mathcal{D}_{*}, e}^{-1} \leq \sigma_{\mathcal{D}, e}^{-1}$ for any $e \in \partial K_{D}$, whence

$$
\sum_{m=1}^{2} \eta_{D_{m}}^{2}\left(u_{\mathcal{D}_{*}}\right) \leq(1+\lambda) \eta_{D}^{2}\left(u_{\mathcal{D}}\right)+\frac{C}{\lambda} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right)
$$

Finally, for any unsplit $D \in \mathcal{D} \backslash(\mathcal{M} \cup \partial \mathcal{M})$, we just have

$$
\eta_{D}^{2}\left(u_{\mathcal{D}_{*}}\right) \leq(1+\lambda) \eta_{D}^{2}\left(u_{\mathcal{D}}\right)+\frac{C}{\lambda} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right)
$$

Summing-up all contributions and using the marking condition, we obtain

$$
\begin{aligned}
\eta_{\mathcal{D}_{*}}^{2}\left(u_{\mathcal{D}_{*}}\right) & \leq(1+\lambda)\left(\eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)-\frac{1}{2} \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}} ; \mathcal{M}\right)\right)+\frac{C}{\lambda} \sum_{D \in \mathcal{D}} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right) \\
& \leq(1+\lambda)\left(1-\frac{\vartheta^{2}}{2}\right) \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)+\frac{C}{\lambda} \sum_{D \in \mathcal{D}} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right)
\end{aligned}
$$

It remains to prove that $\sum_{D \in \mathcal{D}} \mathcal{N}_{D}^{2}\left(u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right) \lesssim\left\|u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right\|_{\mathcal{D}_{*}}^{2}$. Setting now $w:=$ $u_{\mathcal{D}_{*}}-u_{\mathcal{D}}$, we have

$$
\sum_{D \in \mathcal{D}} \mathcal{N}_{D}^{2}(w)=\left\|\tilde{w}_{x}\right\|_{\Omega}^{2}+\sum_{D \in \mathcal{D}} \frac{h_{D}^{2}}{p_{D}^{2}}\|w\|_{K_{D}}^{2}
$$

Writing, for a.e. $x \in \Omega$,

$$
w(x)=\sum_{e \in \mathcal{E}_{\mathcal{D}_{*}}, e<x} \llbracket w \rrbracket_{e}+\int_{\min \Omega}^{x} \tilde{w}_{x}(s) d s=\sum_{e \in \mathcal{E}_{\mathcal{D}_{*}, e<x}} \sigma_{\mathcal{D}_{*}, e}^{-1 / 2} \sigma_{\mathcal{D}_{*}, e}^{-1 / 2} \llbracket w \rrbracket_{e}+\int_{\min \Omega}^{x} \tilde{w}_{x}(s) d s
$$

we have

$$
w^{2}(x) \lesssim\left(\sum_{e \in \mathcal{E}_{\mathcal{D}_{*}}} \sigma_{\mathcal{D}_{*}, e}^{-1}\right) \sum_{e \in \mathcal{E}_{\mathcal{D}_{*}}} \sigma_{\mathcal{D}_{*}, e} \llbracket w \rrbracket_{e}^{2}+|\Omega|\left\|\tilde{w}_{x}\right\|_{\Omega}^{2}
$$

Since $\sum_{e \in \mathcal{E}_{\mathcal{D}_{*}}} \sigma_{\mathcal{D}_{*}, e}^{-1} \leq|\Omega|$, we easily obtain the desired bound.

Lemma 5.2. There exists a constant $C_{8}>0$ independent of $\mathcal{D}$ such that for any real $\delta \in(0,1)$ and any $\gamma \geq \gamma_{1}$, one has

$$
\left\|u-u_{\mathcal{D}_{*}}\right\|_{a, \mathcal{D}_{*}}^{2} \leq(1+\delta)\left\|u-u_{\mathcal{D}}\right\|_{a, \mathcal{D}}^{2}-\frac{\alpha_{*}}{2}\left\|u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right\|_{\mathcal{D}_{*}}^{2}+\frac{C_{8}}{\delta \gamma}\left(\eta_{\mathcal{D}_{*}}^{2}\left(u_{\mathcal{D}_{*}}\right)+\eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)\right)
$$

Proof. Let us set $w_{*}:=u-u_{\mathcal{D}_{*}}, w:=u-u_{\mathcal{D}}, d:=u_{\mathcal{D}_{*}}-u_{\mathcal{D}}, d^{c}:=u_{\mathcal{D}_{*}}^{c}-u_{\mathcal{D}}^{c}$ and $d^{\perp}:=u_{\mathcal{D}_{*}}^{\perp}-u_{\mathcal{D}}^{\perp}$. Observing that $a_{\mathcal{D}_{*}}\left(w_{*}, d^{c}\right)=0$ by the partial orthogonality property (28), one easily gets

$$
\left\|w_{*}\right\|_{a, \mathcal{D}_{*}}^{2}=a_{\mathcal{D}_{*}}\left(w_{*}, w_{*}\right)=a_{\mathcal{D}_{*}}\left(w_{*}+d^{c}, w_{*}+d^{c}\right)-a_{\mathcal{D}_{*}}\left(d^{c}, d^{c}\right)
$$

Using $u_{\mathcal{D}}=u_{\mathcal{D}}^{c}+u_{\mathcal{D}}^{\perp}$ and $u_{\mathcal{D}_{*}}=u_{\mathcal{D}_{*}}^{c}+u_{\mathcal{D}_{*}}^{\perp}$, one has $w_{*}+d^{c}=w-d^{\perp}$, whence

$$
\begin{aligned}
a_{\mathcal{D}_{*}}\left(w_{*}+d^{c}, w_{*}+d^{c}\right) & =a_{\mathcal{D}_{*}}(w, w)-2 a_{\mathcal{D}_{*}}\left(w, d^{\perp}\right)+a_{\mathcal{D}_{*}}\left(d^{\perp}, d^{\perp}\right) \\
& \leq\|w\|_{a, \mathcal{D}_{*}}^{2}+2\left(\alpha^{*}\right)^{1 / 2}\|w\|_{a, \mathcal{D}_{*}}\left\|d^{\perp}\right\|_{\mathcal{D}_{*}}+\alpha^{*}\left\|d^{\perp}\right\|_{\mathcal{D}_{*}}^{2}
\end{aligned}
$$

where we have used the uniform continuity of the form $a_{\mathcal{D}_{*}}$ with respect to the DG-norm. Using the uniform coercivity and the triangle inequality, we get

$$
a_{\mathcal{D}_{*}}\left(d^{c}, d^{c}\right) \geq \alpha_{*}\left\|d^{c}\right\|_{\mathcal{D}_{*}}^{2} \geq \alpha_{*}\left(\frac{1}{2}\|d\|_{\mathcal{D}_{*}}^{2}-\left\|d^{\perp}\right\|_{\mathcal{D}_{*}}^{2}\right)
$$

Collecting these inequalities and using Young's inequality, we obtain

$$
\begin{equation*}
\left\|w_{*}\right\|_{a, \mathcal{D}_{*}}^{2} \leq(1+\delta)\|w\|_{a, \mathcal{D}_{*}}^{2}-\frac{\alpha_{*}}{2}\|d\|_{\mathcal{D}_{*}}^{2}+\frac{C}{\delta}\left\|d^{\perp}\right\|_{\mathcal{D}_{*}}^{2} \tag{40}
\end{equation*}
$$

At this point, we observe that $\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}_{*}}^{2} \leq 2\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}^{2}$. Indeed, $\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}_{*}}^{2}=\left\|\left(u_{\mathcal{D}}^{\perp}\right)_{x}^{\sim}\right\|_{\Omega}^{2}+$ $\gamma \sum_{e \in \mathcal{E}_{\mathcal{D}_{*}}} \sigma_{\mathcal{D}_{*}, e} \llbracket u_{\mathcal{D}}^{\perp} \rrbracket_{e}^{2}$, but the jumps of $u_{\mathcal{D}}^{\perp}$ occur only at the interfaces $e \in \mathcal{E}_{\mathcal{D}}$, and $\sigma_{\mathcal{D}_{*}, e} \leq 2 \sigma_{\mathcal{D}, e}$ by definition of the refinement strategy. Thus, using Corollary 5.1, we get

$$
\begin{equation*}
\left\|d^{\perp}\right\|_{\mathcal{D}_{*}}^{2} \lesssim\left\|u_{\mathcal{D}_{*}}^{\perp}\right\|_{\mathcal{D}_{*}}^{2}+\left\|u_{\mathcal{D}}^{\perp}\right\|_{\mathcal{D}}^{2} \lesssim \gamma\left\|\sigma_{\mathcal{D}_{*}}^{1 / 2} \llbracket u_{\mathcal{D}_{*}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}_{*}}}^{2}+\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2} \tag{41}
\end{equation*}
$$

It remains to replace $\|w\|_{a, \mathcal{D}_{*}}^{2}$ by $\|w\|_{a, \mathcal{D}}^{2}$. To this end, let us write
$a_{\mathcal{D}_{*}}(w, w)=a_{\mathcal{D}}(w, w)+2\left(L_{\mathcal{D}_{*}} w, \nu \tilde{w}_{x}\right)_{\Omega}-2\left(L_{\mathcal{D}} w, \nu \tilde{w}_{x}\right)_{\Omega}-\gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket w \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}+\gamma\left\|\sigma_{\mathcal{D}_{*}}^{1 / 2} \llbracket w \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}_{*}}}^{2}$.

Using Property 5.1 and the coercivity of the form $a_{\mathcal{D}}$, one gets

$$
\left(L_{\mathcal{D}_{*}} w, \nu \tilde{w}_{x}\right)_{\Omega} \lesssim\left\|\sigma_{\mathcal{D}_{*}}^{1 / 2} \llbracket w \rrbracket\right\|_{\varepsilon_{\mathcal{D} *}} a_{\mathcal{D}}(w, w)^{1 / 2} \lesssim\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\varepsilon_{\mathcal{D}}} a_{\mathcal{D}}(w, w)^{1 / 2}
$$

A similar bound holds for $\left(L_{\mathcal{D}} w, \nu \tilde{w}_{x}\right)_{\Omega}$. Therefore, using once more Young's inequality, we arrive at

$$
\begin{equation*}
\|w\|_{a, \mathcal{D}_{*}}^{2} \leq(1+\delta)\|w\|_{a, \mathcal{D}}^{2}+\frac{C}{\delta} \gamma\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2} \tag{42}
\end{equation*}
$$

Replacing (41)-(42) into (40), we obtain

$$
\begin{aligned}
&\left\|u-u_{\mathcal{D}_{*}}\right\|_{a, \mathcal{D}_{*}}^{2} \leq(1+\delta)^{2}\left\|u-u_{\mathcal{D}}\right\|_{a, \mathcal{D}}^{2}-\frac{\alpha_{*}}{2}\left\|u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right\|_{\mathcal{D}_{*}}^{2} \\
&+\frac{C}{\delta} \gamma\left(\left\|\sigma_{\mathcal{D}_{*}}^{1 / 2} \llbracket u_{\mathcal{D}_{*}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}_{*}}}^{2}+\left\|\sigma_{\mathcal{D}}^{1 / 2} \llbracket u_{\mathcal{D}} \rrbracket\right\|_{\mathcal{E}_{\mathcal{D}}}^{2}\right) .
\end{aligned}
$$

The desired result follows from Proposition 5.2, after replacing $\delta$ by $\delta / 3$.
We are ready to establish the main result of this section.
Theorem 5.1. Consider the mapping (36) defined above. There exist constants $\beta>0$ and $\varrho \in(0,1)$, independent of $\mathcal{D}$, such that, choosing $\gamma>0$ large enough in the definition (24), one has

$$
\left\|u-u_{\mathcal{D}_{*}}\right\|_{a, \mathcal{D}_{*}}^{2}+\beta \eta_{\mathcal{D}_{*}}^{2}\left(u_{\mathcal{D}_{*}}\right) \leq \varrho\left(\left\|u-u_{\mathcal{D}}\right\|_{a, \mathcal{D}}^{2}+\beta \eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)\right)
$$

Proof. Let us simplify our notation by setting $E_{*}^{2}:=\left\|u-u_{\mathcal{D}_{*}}\right\|_{a, \mathcal{D}_{*}}^{2}, E^{2}:=\left\|u-u_{\mathcal{D}}\right\|_{a, \mathcal{D}}^{2}$, $e_{*}^{2}:=\left\|u_{\mathcal{D}_{*}}-u_{\mathcal{D}}\right\|_{\mathcal{D}_{*}}^{2}$ and $\eta_{*}^{2}:=\eta_{\mathcal{D}_{*}}^{2}\left(u_{\mathcal{D}_{*}}\right), \eta^{2}:=\eta_{\mathcal{D}}^{2}\left(u_{\mathcal{D}}\right)$. Then, the inequalities of Lemmas 5.2-5.1 read as follows:

$$
\begin{aligned}
E_{*}^{2} & \leq(1+\delta) E^{2}-\frac{\alpha_{*}}{2} e_{*}^{2}+\frac{C_{8}}{\delta \gamma}\left(\eta_{*}^{2}+\eta^{2}\right) \\
\eta_{*}^{2} & \leq(1+\lambda)\left(1-\frac{\vartheta^{2}}{2}\right) \eta^{2}+\frac{C_{7}}{\lambda} e_{*}^{2}
\end{aligned}
$$

Thus, for any real $\beta>0$,

$$
\begin{aligned}
E_{*}^{2}+\beta \eta_{*}^{2} & \leq(1+\delta) E^{2}-\frac{\alpha_{*}}{2} e_{*}^{2}+\left(\beta+\frac{C_{8}}{\delta \gamma}\right) \eta_{*}^{2}+\frac{C_{8}}{\delta \gamma} \eta^{2} \\
& \leq(1+\delta) E^{2}-\frac{\alpha_{*}}{2} e_{*}^{2}+\left(\beta+\frac{C_{8}}{\delta \gamma}\right)\left((1+\lambda)\left(1-\frac{\vartheta^{2}}{2}\right) \eta^{2}+\frac{C_{7}}{\lambda} e_{*}^{2}\right)+\frac{C_{8}}{\delta \gamma} \eta^{2}
\end{aligned}
$$

Writing $1-\frac{\vartheta^{2}}{2}=\left(1-\frac{\vartheta^{2}}{4}\right)-\frac{\vartheta^{2}}{4}$ and using $E^{2} \leq C_{6} \eta^{2}$ from Corollary (5.2), we easily obtain for $\gamma \geq \gamma_{1}$

$$
\begin{aligned}
E_{*}^{2}+\beta \eta_{*}^{2} \leq & {\left[(1+\delta)-\left(\beta+\frac{C_{8}}{\delta \gamma}\right) \frac{1+\lambda}{C_{6}} \frac{\vartheta^{2}}{4}\right] E^{2}+\left[\left(\beta+\frac{C_{8}}{\delta \gamma}\right) \frac{C_{7}}{\lambda}-\frac{\alpha_{*}}{2}\right] e_{*}^{2} } \\
& +\left[(1+\lambda)\left(1-\frac{\vartheta^{2}}{4}\right)+\frac{C_{8}}{\beta \delta \gamma}\left(1+(1+\lambda)\left(1-\frac{\vartheta^{2}}{4}\right)\right)\right] \beta \eta^{2} \\
=: & \varrho_{1} E^{2}+\varrho_{2} e_{*}^{2}+\varrho_{3} \beta \eta^{2} .
\end{aligned}
$$

At this point, we first choose $\lambda$ sufficiently small to have $(1+\lambda)\left(1-\frac{\vartheta^{2}}{4}\right)<1$. Next, we choose $\delta$ sufficiently small to have $\varrho_{1}<1$ for $\gamma=\gamma_{1}$, hence for any $\gamma \geq \gamma_{1}$. Then, the parameter $\beta>0$ is determined by imposing $\varrho_{2}=0$, which is possible provided $\gamma$ is large enough, say $\gamma \geq \gamma_{2} \geq \gamma_{1}$. Finally, for $\gamma$ even larger, say $\gamma \geq \gamma_{3} \geq \gamma_{2}$, the second addend in $\varrho_{3}$ can be made so small that $\varrho_{3}<1$. In conclusion, the desired result holds for all $\gamma \geq \gamma_{3}$ with $\varrho:=\max \left(\varrho_{1}, \varrho_{3}\right)$.

Corollary 5.3. Denote by $\left\{\left(\mathcal{D}_{k}, u_{\mathcal{D}_{k}}, \eta_{\mathcal{D}_{k}}\left(u_{\mathcal{D}_{k}}\right)\right): k \geq 0\right\}$ the sequence produced by iterating the mapping (36) from the input partition $\mathcal{D}_{0}:=\mathcal{D}_{\text {in }}$. Then,

$$
\left\|u-u_{\mathcal{D}_{k}}\right\|_{\mathcal{D}_{k}}^{2} \leq \alpha_{*}^{-1} \varrho^{k}\left(\left\|u-u_{\mathcal{D}_{0}}\right\|_{a, \mathcal{D}_{0}}^{2}+\beta \eta_{\mathcal{D}_{0}}^{2}\left(u_{\mathcal{D}_{0}}\right)\right)
$$

The latter result guarantees that the target accuracy $\left\|u-u_{\mathcal{D}_{k}}\right\|_{\mathcal{D}_{k}}^{2} \leq \varepsilon^{2}$ of DG-SOLVE can be matched provided the iterations are stopped at a sufficiently large $k$. In particular, if there exists a constant $C_{9}>0$ such that

$$
\begin{equation*}
\left\|u-u_{\mathcal{D}_{0}}\right\|_{a, \mathcal{D}_{0}}^{2}+\beta \eta_{\mathcal{D}_{0}}^{2}\left(u_{\mathcal{D}_{0}}\right) \leq C_{9} \varepsilon^{2} \tag{43}
\end{equation*}
$$

then the number $K$ of iterations in DG-SOLVE is bounded independently of $\varepsilon$. In this case, since the mapping (36) at most doubles the cardinality of the partition, i.e., $\left|\mathcal{D}_{*}\right| \leq 2|\mathcal{D}|$, we conclude that the cardinality of the output partition $\mathcal{D}_{\text {out }}:=\mathcal{D}_{K}$ is uniformly bounded by the cardinality of the input partition $\mathcal{D}_{\text {in }}$, precisely

$$
\left|\mathcal{D}_{\text {out }}\right| \leq 2^{K}\left|\mathcal{D}_{\text {in }}\right|
$$

Remark 5.1. (Arithmetic complexity) According to [5], if $N:=\# \mathcal{D}$ denotes the cardinality of the current $h p$-partition, the arithmetic complexity of hp-NEARBEST is $O\left(N^{2}\right)$ (or $O(N \log N)$ in some specific situations). On the other hand, DG-SOLVE performs a bounded numbers of solutions of DG problems, which can be achieved in linear complexity.

### 5.3 Initialization

Let us discuss a possible strategy to fulfill (43). Recall that we enter DG-SOLVE at iteration $i$ of hp-ADFEM with input partition $\mathcal{D}_{i}$ and data $g_{\mathcal{D}_{i}}$. This means that, with the notation of hp-ADFEM, condition (43) reads

$$
\begin{equation*}
\left\|u\left(g_{\mathcal{D}_{i}}\right)-u_{\mathcal{D}_{i}}\right\|_{a, \mathcal{D}_{i}}^{2}+\beta \eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right) \leq C_{9} \varepsilon_{i}^{2} \tag{44}
\end{equation*}
$$

The first term on the left-hand side can be bounded from above by using the uniform continuity of the form $a_{\mathcal{D}_{i}}$ and the bounds given in Property 5.2, Corollary 5.1 and Proposition 5.2. This yields

$$
\left\|u\left(g_{\mathcal{D}_{i}}\right)-u_{\mathcal{D}_{i}}\right\|_{a, \mathcal{D}_{i}}^{2}+\beta \eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right) \leq C_{10} \inf _{w_{\mathcal{D}_{i}} \in V_{\mathcal{D}_{i}}^{c}}\left\|u\left(g_{\mathcal{D}_{i}}\right)-w_{\mathcal{D}_{i}}\right\|_{H_{0}^{1}(\Omega)}^{2}+C_{11} \eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right)
$$

for constants $C_{10}, C_{11}>0$ independent of $\mathcal{D}_{i}$. We now show that the infimum on the right-hand side can be bounded by a multiple of $\varepsilon_{i}^{2}$.
Property 5.4. There exists a constant $C_{12}>0$ independent of $\mathcal{D}_{i}$ such that

$$
\inf _{w_{\mathcal{D}_{i}} \in V_{\mathcal{D}_{i}}^{c}}\left\|u\left(g_{\mathcal{D}_{i}}\right)-w_{\mathcal{D}_{i}}\right\|_{H_{0}^{1}(\Omega)} \leq C_{12} \varepsilon_{i}
$$

Proof. For simplicity, set again $u:=u\left(g_{\mathcal{D}_{i}}\right)$. Then, for any $w_{\mathcal{D}_{i}} \in V_{\mathcal{D}_{i}}^{c}$, let us write $u-w_{\mathcal{D}_{i}}=\left(u-u_{\star}\right)+\left(u_{\star}-\bar{u}_{i-1}\right)+\left(\bar{u}_{i-1}-w_{\mathcal{D}_{i}}\right)$. Using (17), we get

$$
\begin{equation*}
\left\|u-u_{\star}\right\|_{H_{0}^{1}(\Omega)}=\left\|u\left(g_{\star}\right)-u\left(g_{\mathcal{D}_{i}}\right)\right\|_{H_{0}^{1}(\Omega)} \leq C_{\star} \kappa \mathrm{E}_{\mathcal{D}_{i}}\left(\bar{u}_{i-1}, g_{\star}\right)^{\frac{1}{2}} \leq C_{\star} \kappa \omega \varepsilon_{i-1} . \tag{45}
\end{equation*}
$$

On the other hand, recalling (20), we have

$$
\begin{equation*}
\left\|u_{\star}-\bar{u}_{i-1}\right\|_{\overline{\mathcal{D}}_{i-1}} \leq \varepsilon_{i-1} \tag{46}
\end{equation*}
$$

Let us define $w_{\mathcal{D}_{i}}$ as follows. Set $\psi:=\left(\bar{u}_{i-1}\right)_{x}^{\sim} \in L^{2}(\Omega)$ and let $q \in L^{2}(\Omega)$ be the piecewise polynomial function such that $q_{\mid K_{D}}=\Pi_{K_{D}, p_{D}-1}^{0} \psi_{\mid K_{D}}$ for all $D \in \mathcal{D}_{i}$. Notice that, recalling the definition (15), we have

$$
\|\psi-q\|_{\Omega}^{2}=\sum_{D \in \mathcal{D}_{i}}\|\psi-q\|_{K_{D}}^{2} \leq \sum_{D \in \mathcal{D}_{i}} e_{D}\left(\bar{u}_{i-1}, g_{\star}\right)=\mathrm{E}_{\mathcal{D}_{i}}\left(\bar{u}_{i-1}, g_{\star}\right) \leq \omega^{2} \varepsilon_{i-1}^{2}
$$

On the other hand, it holds

$$
\int_{\Omega} q=\int_{\Omega} \psi=\sum_{D \in \overline{\mathcal{D}}_{i-1}} \int_{K_{D}} \bar{u}_{i-1, x}=-\sum_{e \in \mathcal{E}_{\overline{\mathcal{D}}_{i-1}}} \llbracket \bar{u}_{i-1} \rrbracket \rrbracket_{e}=-\sum_{e \in \mathcal{E}_{\overline{\mathcal{D}}_{i-1}}} \sigma_{\overline{\mathcal{D}}_{i-1}, e}^{-1 / 2} \sigma_{\overline{\mathcal{D}}_{i-1}, e}^{1 / 2} \llbracket \bar{u}_{i-1} \rrbracket_{e},
$$

whence

$$
\left(\int_{\Omega} q\right)^{2} \leq\left(\sum_{e \in \mathcal{E}_{\overline{\mathcal{D}}_{i-1}}} \sigma_{\overline{\mathcal{D}}_{i-1}, e}^{-1}\right)\left\|\sigma_{\overline{\mathcal{D}}_{i-1}}^{1 / 2} \llbracket \bar{u}_{i-1} \rrbracket\right\|_{\overline{\mathcal{D}}_{\overline{\mathcal{D}}_{i-1}}}^{2} \leq|\Omega|\left\|\sigma_{\overline{\mathcal{D}}_{i-1}}^{1 / 2} \llbracket \bar{u}_{i-1} \rrbracket\right\|_{\overline{\mathcal{D}}_{\overline{\mathcal{L}}_{i-1}}}^{2} \leq \frac{|\Omega|}{\gamma} \varepsilon_{i-1}^{2}
$$

by (46). Therefore, if we set

$$
w_{\mathcal{D}_{i}}(x)=\int_{x_{0}}^{x} q(s) d s-\left(x-x_{0}\right) \int_{\Omega} q
$$

where $x_{0}=\min \Omega$, we realize $w_{\mathcal{D}_{i}} \in V_{\mathcal{D}_{i}}^{c}$ and $\left\|\left(\bar{u}_{i-1}\right)_{x}^{\sim}-w_{\mathcal{D}_{i}, x}\right\|_{\Omega} \leq C \varepsilon_{i-1}$. This concludes the proof, since $\varepsilon_{i-1} \simeq \varepsilon_{i}$.

By Property 5.4, we get the bound

$$
\left\|u\left(g_{\mathcal{D}_{i}}\right)-u_{\mathcal{D}_{i}}\right\|_{a, \mathcal{D}_{i}}^{2}+\beta \eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right) \leq C_{13} \varepsilon_{i}^{2}+C_{11} \eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right)
$$

At this point, we may proceed as follows. Assume that we have chosen, once and for all, an absolute constant $\hat{C}>0$. We check the validity of

$$
\eta_{\mathcal{D}_{i}}^{2}\left(u_{\mathcal{D}_{i}}\right) \leq \hat{C} \varepsilon_{i}^{2} .
$$

- In the affirmative case, $u_{\mathcal{D}_{i}}$ does satisfy condition (44), and we can start the iterations of DG-SOLVE.
- In the negative case, we discard $u_{\mathcal{D}_{i}}$ and compute $\hat{u}_{\mathcal{D}_{i}}^{c} \in V_{\mathcal{D}_{i}}^{c}$, the (continuous) Galerkin approximation of $u\left(g_{\mathcal{D}_{i}}\right)$ on the partition $\mathcal{D}_{i}$. For such an approximation, it is known that the residual estimator is both reliable and efficient; hence, resorting once more to Property 5.4,

$$
\eta_{\mathcal{D}_{i}}\left(\hat{u}_{\mathcal{D}_{i}}^{c}\right) \simeq\left\|u\left(g_{\mathcal{D}_{i}}\right)-\hat{u}_{\mathcal{D}_{i}}^{c}\right\|_{a} \simeq\left\|u\left(g_{\mathcal{D}_{i}}\right)-\hat{u}_{\mathcal{D}_{i}}^{c}\right\|_{H_{0}^{1}(\Omega)} \leq C_{12} \varepsilon_{i}
$$

Therefore, condition (44) is satisfied with $u_{\mathcal{D}_{i}}$ replaced by $\hat{u}_{\mathcal{D}_{i}}^{c}$, and we start the iterations of DG-SOLVE from this approximation.

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