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Abstract

One dimensional diffusion problems in bounded porous media characterized by the presence of nonlocal interactions are investigated by a fractional calculus approach. Darcy's constitutive equation is assumed of convolution integral type and a power law attenuation function is implemented. Analogies and differences of the flow rate-pressure law with respect to other nonlocal and fractal models are outlined. By means of the continuity relationship, the fractional diffusion equation is then derived. It involves spatial Riemann-Liouville derivatives with non-integer order comprised between 1 and 2. The solution is obtained numerically via fractional finite differences and results are presented both in the transient and in the steady-state regimes. Eventually, the physical meaning of fractional operators is discussed and potential applications of the analysis are suggested.

Keywords: Nonlocal Darcy's law, long-range interactions, fractional diffusion equation

Introduction

Understanding transport problems in porous media emerges nowadays as a primary concern, since it can have a fundamental impact on many different research fields, starting from the optimization of oil extraction to modeling scaffold

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geometry in tissue engineering, up to pharmaceutical and food industries.

The classical diffusion equation is obtained by combining Darcy's constitutive law, stating the proportionality of the flux to the pressure gradient, with the equation of continuity, describing the conservation of the mass. On the other hand, several discrepancies have been found between the related solution, which is generally described by exponential type functions, and experimental data (Nielsen et al., 1962; Bell and Nur, 1978; Roeloffs, 1988). There are several reasons responsible of these deviations: as a matter of fact, fluids can react chemically with the medium increasing or diminishing the pore size, or the can interact with the solid part by carrying some particles which may obstruct some channels (Caputo, 2000). Eventually, the network of channels can result in a complicated interconnected hierarchical geometry.

In order to overcome these drawbacks, different approaches have been proposed since the middle of the last century, mainly based on a modification of constitutive Darcy's relationship. In the framework of soil-water flow theory, for instance, a diffusion coefficient dependent on the water content was firstly considered, leading to a nonlinear partial differential equation known as Richards' equation (Van Dam and Feddes, 2000; Lewandowska and Auriault, 2002; Jacques et al., 2002). Boltzmanns transformation reduces the expression into an ordinary differential equation, allowing the possibility of getting analytical solutions. Nevertheless, in many cases significant deviations from the real behavior were still observed (Taylor et al., 1999; Kunt and Lavallee, 2001). Among different attempts to generalize Richard's equation, let us cite those based on adding the dependence of the diffusion coefficient on time (Guerrini and Swartzendruber, 1992) or space (Pachepsky and Timlin, 1998).

More recent models involve the replacement of the first order time derivative with a fractional one in the final differential equation, to take memory effects into account (Pachepsky et al., 2003; Logvinova and Nel, 2004). A slightly different approach was followed by Gerolymatou et al. (2006), reformulating Richards' equation as a time integral relationship. On the other hand, (Caputo, 2000) applied fractional derivatives (with two different orders, each comprised between 0 and 1) to both members of Darcy's law to consider the temporal variation of the permeability during the process (see also (Caputo and Plastino, 2004)). In the spirit of this approach, it was recently proved that assuming the physical properties of a porous solid varying with a power law is equivalent to consider a dependence of the flux on the time fractional derivative of the pressure with order comprised between 0 and 1 (Deseri and Zingales, 2015; Alaimo and Zingales, 2015).

Regarding engineering applications, an important work to be mentioned is that of (Monteiro et al., 2012) where a mathematical model of the flow in nanoporous rocks was proposed. It is based on the hypothesis that the permeability of the inclusions depends substantially on the pressure gradient. The model, applied to shale oil extraction, showed that the production rate of the oil deposits decays with time following a power law whose exponent lies between $-1/2$ and -1, in agreement with experimental data.

On the contrary, the approach performed by Sen and Ramos (2012) is completely different, since it assumes the flux to be proportional to the pore pressure by means of a spatial convolution integral. By considering the attenuation function of power-law type, nonlocal Darcy's law was rewritten by means of spatial fractional operators. The problem was limited to infinite domains and an interesting interpretation of the entire porous medium as a network of channels with short-, medium-, and long-distance connections was furnished. However, the resulting fractional diffusion equation was not given.

In the present work, the model proposed in (Sen and Ramos, 2012) is revisited and generalized to investigate the diffusion process on finite porous domains. The approach represents somehow an extension of the well-established method proposed in the framework of nonlocal elasticity (Carpinteri et al., 2011, 2014) (see also (Di Paola and Zingales, 2008; Drapaca and Sivaloganathan, 2012)) and later implemented also to study wave propagation in nonlocal media (Atanackovic and Stankovic, 2009; Sapora et al., 2013; Challamel et al., 2013; Aksoy , 2016).

Before proceeding, it is worth observing that: i) the present analysis does not involve an explicit connection between the order of the fractional derivative and the fractal geometry (if any) of the medium where diffusion takes place; indeed, some potential connections are suggested on the basis of some recent advances (Carpinteri and Sapora, 2010; Balankin an Elizarraraz , 2012; Zingales, 2014); ii) fractional diffusion equations are strictly related to continuous random walk approaches. They generalize the standard Brownian motion by taking waiting times (which accounts for non-Markovian effects) and anomalous long particle displacements (known as Levy flights, which consider non-Gaussian displacements) into account (Gorenflo et al., 2002; Zoia et al., 2007; Berkowitz et al., 2006); iii) nonlocal diffusion constitutes a broad class of problems of interest in mathematics suited to wide variety of applications, including biological contexts, image processing, particle systems, coagulation models, nonlocal anisotropic approaches for phase transition and mathematical finance using optimal control theory, among others (Andreu-Vaillo et al., 2010). In this context, fractional calculus has proved to be a synthetic and efficient tool to model both memory effects and nonlocal interactions (Scalas et al., 2000; Metzler and Nonnenmacher, 2002; del-Castillo-Negrete, 2006; Magin et al., 2008; Lenzi et al., 2008; Gorenflo and Mainardi, 2009; Evangelista et al., 2011; Tarasov and Trujillo, 2013; Atanackovic et al., 2014; Zingales, 2014; Sobolev, 2014).

Nonlocal Darcy's equation

Let us consider a diffusion process in a one-dimensional porous bar of length l. Assume that the volumetric flow rate per unit area q $[m/s]$ in one point depends on the gradient of the pore pressure p $[N/m^2]$ all over the domain by means of a convolution integral:

$$q(x) = -\frac{k}{\mu}\int_0^l g(y-x)\nabla p(y)\mathrm{d}y, \tag{1}$$

being k the permeability $[m^2]$, μ the fluid viscosity $[Ns/m^2]$, and g an attenuation function. It describes the relationship between non-adjacent points of the medium and it must be a decaying function in space. Equation (1) represents the spatial nonlocal form of Darcy's constitutive equation, and it was firstly proposed in (Sen and Ramos, 2012). Indeed, similar expressions had been put forward even before to investigate Eringen's nonlocal elasticity and nondiffusive transport in magnetically confined plasma (see (Lazar et al., 2006; del-Castillo-Negrete, 2006) an related references).

Different attenuation functions g can be inserted into Eq. (1), leading to different nonlocal models. The attention is focused here on: i) a cone function, as an example of standard nonlocal models (of course other choices, such as bell-shaped or Gaussian functions are possible); ii) a power law expression, leading to a fractional approach.

Let us start by considering the following cone function g:

$$g(\xi)=\begin{cases}\dfrac{1}{l_{ch}}\left(1-\dfrac{|\xi|}{l_{ch}}\right) & \text{for } |\xi|<l_{ch}\\ 0 & \text{for } |\xi|>l_{ch}\end{cases} \tag{2}$$

where l_{ch} is a parameter characteristic of the material and $\xi=y-x$. Of course, if l_{ch} tends to zero the attenuation function (2) tends to Dirac function $\delta(x)$: in this case, the nonlocal constitutive law (1) tends to the local one, $q=-k/\mu\nabla p(x)$. Furthermore, the particular form of (2) has been chosen according to the fact that, if the gradient of the pressure is constant, no differences should be observed from the local model. In other words, the following relationship has to be satisfied:

$$\int_{-\infty}^{+\infty} g(\xi)d\xi=1. \tag{3}$$

The computation of the flow rate generating a constant pressure gradient $\nabla\bar{p}$ is just a matter of integration. By inserting Eq. (2) into Eq. (1), simple analytical manipulations yield:

$$q(x)=\begin{cases}-\dfrac{k\nabla\bar{p}}{2\mu}\left[1+2\left(\dfrac{x}{l_{ch}}\right)-\left(\dfrac{x}{l_{ch}}\right)^2\right] & \text{for } 0<x<l_{ch}\\ -\dfrac{k\nabla\bar{p}}{\mu} & \text{for } l_{ch}<x<l-l_{ch}\\ -\dfrac{k\nabla\bar{p}}{2\mu}\left[1+2\left(\dfrac{l-x}{l_{ch}}\right)-\left(\dfrac{l-x}{l_{ch}}\right)^2\right] & \text{for } l-l_{ch}<x<l\end{cases} \tag{4}$$

The dimensionless flow $q^*=q\mu/(k\nabla\bar{p})$ versus the dimensionless space $x^*=x/l$ is plotted in Fig.1 for different $l^*_{ch}=l_{ch}/l$ values: the flow decreases (in modulus) at the edges, whereas it matches the local solution ($q^*=-1$) on the the central core of the bar. The size of the domain affected by nonlocality depends clearly on the value of l^*_{ch}.

Let us assume now the following power-law expression for function g (Tarasov and Zaslavky, 2006; Atanackovic and Stankovic, 2009; Carpinteri et al., 2011):

$$g(\xi) = \frac{1}{2\Gamma(2-\alpha)|\xi|^{\alpha-1}}, \tag{5}$$

with $1 < \alpha < 2$. In this case α is the material parameter governing the transition from a smooth behavior (lower α) to a sharp one (higher α), and it accounts for non-Gaussian displacements of the particles inside the media. Relation (1) takes thus the following form:

$$q(x) = -\frac{k_\alpha}{\mu} I_{0,l}^{2-\alpha}(\nabla p). \tag{6}$$

The operator $I_{0,l}^{\beta}$ represents the fractional Riesz integral ($\beta > 0$, Samko et al. (1993))

$$I_{0,l}^{\beta} f(x) = \frac{1}{2}\left[I_{0+}^{\beta} f(x) + I_{l-}^{\beta} f(x)\right] = \frac{1}{2\Gamma(\beta)} \int_0^l \frac{f(y)}{|x-y|^{1-\beta}} \mathrm{d}y, \tag{7}$$

where I_{0+}^{β} and I_{l-}^{β} are the left and right Riemann-Liouville fractional integrals, respectively:

$$I_{0+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{f(y)}{(x-y)^{1-\beta}} \mathrm{d}y, \tag{8}$$

$$I_{l-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^l \frac{f(y)}{(y-x)^{1-\beta}} \mathrm{d}y. \tag{9}$$

According to choice (5), the fractional permeability k_α possesses anomalous physical dimensions $[m]^\alpha$. An intriguing possibility, as will be outlined later, would be that of linking them with the fractal features of the medium where diffusion takes place (Chang and Yortos, 1990; Carpinteri and Mainardi , 1997; Yao et al., 2012; Carpinteri et al., 2009). Hereinafter, the following condition will be supposed to hold, for the sake of completeness: $k_\alpha = k$ for $\alpha = 2$. Thus, in this case, Eq. (6) reverts consistently to local Darcy's relationship $q = -k/\mu \nabla p$. Eventually, observe that also in the present fractional approach we can normalize the coordinate r with respect to an intrinsic length l_{ch}, as done in Eq. (2). On the other hand, handling two parameters (α and l_{ch}) governing the transition from a nonlocal behavior to a local behavior would represent a not trivial task, at least at this preliminary stage. For a first attempt in this framework, the reader can refer to (Sumelka and Blaszczyk, 2014).

As done before, let us now consider the flow q associated to a constant pressure gradient $\nabla \bar{p}$:

$$q(x) = -\frac{k_\alpha l^{2-\alpha}}{2\mu\Gamma(3-\alpha)} \nabla \bar{p} \left[\left(\frac{x}{l}\right)^{2-\alpha} - \left(1 - \frac{x}{l}\right)^{2-\alpha} \right] \tag{10}$$

By denoting $q^* = q\mu/(k_\alpha l^{2-\alpha} \nabla \bar{p})$, results are presented in Fig.2 for different α-values. Once again, observe that the flow decreases in correspondence to the bar extremes. Moreover, when $\alpha \to 2$ (as when $l_{ch} \to 0$) the classical local solution ($q^* = -1$) is recovered. Nevertheless, there are some differences with respect to the previous case (Fig.1): whereas the nonlocal model based on a cone attenuation function always provides the local solution at a certain distance from the borders, according to the fractional approach all the structure is affected by non-locality. This is imputable to the long tails of the power law expression (5). Furthermore, it is evident from (10) that the flux increases along the bar length as $l^{2-\alpha}$.

By means of dimensional analysis, it is possible to prove that the fractional permeability k_α decreases as $l^{1-\alpha}$ ($1 < \alpha < 2$) instead of as l^{-1}, the latter condition holding both for local or other nonlocal models: this means that the pressure increases less than linearly with the bar length, as occurs in the classical case. The interested reader is referred to (Carpinteri and Sapora, 2010), where diffusion problems in fractal media (and more specifically, in a Cantor bar) were investigated, proving that the field variable scales as l^{β}, $\beta = \alpha - 1$ being the non-integer dimension of the fractal set inside the bar where the gradient concentrates. The fractional nonlocal model and fractal model, although different, are thus characterized by the same scaling properties. Further studies are in progress. For recent advances on the relations between fractal geometry and fractional calculus in transport problems, see also Balankin an Elizarraraz (2012); Alaimo and Zingales (2015).

Fractional diffusion equation

Let us now introduce the time variable t and consider the continuity equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{1}{c\phi}\frac{\partial q(x,t)}{\partial x}, \tag{11}$$

where c is the compressibility $[m^2/N]$ and ϕ is the porosity.

By substituting Eq.(6) into (11), we get:

$$\frac{\partial p(x,t)}{\partial t} = \frac{d_\alpha}{2}\left\{D^{\alpha}_{0+}[p(x,t)-p(0,t)] + D^{\alpha}_{l-}[p(x,t)-p(l,t)]\right\}, \tag{12}$$

where $d_\alpha = k_\alpha/\mu c\phi$ is the fractional diffusivity coefficient $[m^{\alpha}/s]$, and D^{β}_{a+} and D^{β}_{b-} are the left and right Riemann-Liouville fractional derivatives with respect to the spatial variable x. They write:

$$D^{\beta}_{0+}f(x) = \sum_{k=0}^{n-1}\frac{f^k(0)}{\Gamma(1+k-\beta)}(x)^{k-\beta} + \frac{1}{\Gamma(n-\beta)}\int_0^x \frac{f^n(y)}{(x-y)^{\beta-n+1}}\mathrm{d}y, \tag{13}$$

$$D^{\beta}_{l-}f(x) = \sum_{k=0}^{n-1}\frac{(-1)^k f^k(l)}{\Gamma(1+k-\beta)}(l-x)^{k-\beta} + \frac{(-1)^n}{\Gamma(n-\beta)}\int_x^l \frac{f^n(y)}{(y-x)^{\beta-n+1}}\mathrm{d}y, \tag{14}$$

n being the smallest integer larger than β, i.e. $n = 2$ in the present case.

Equation (12) represents a fractional differential equation (Podlubny, 1999) in space. Note that, whereas the left fractional derivative coincides always with its integer order counterpart when the order of derivation is an integer number, the right fractional derivative coincides with the corresponding integer derivative only when the order of derivation is even; otherwise, it is equal to its opposite. Thus, the term in the curly brackets (which coincides with the Riesz fractional derivative up to a multiplicative factor, (Samko et al., 1993)) is equal to $2\partial^2 p/\partial x^2$ when $\alpha = 2$ (Eq.(12) reverting to the classical diffusion equation), and vanishes when $\alpha = 1$ (thus leading to a trivial condition providing a constant pressure field in time throughout the body).

Suitable initial and boundary conditions must be assigned to Eq. (12). By analogy of what presented in Carpinteri et al. (2014), they write:

$$p(x, t = 0) = p_0(x), \tag{15}$$

$$p(x = 0, t) = p_0(t), \quad \text{or} \quad q\,(x = 0, t) = \frac{k_\alpha}{\mu} D^{\alpha-1}_{l-}\left[p(x) - p(l)\right]_{x=0} = q_0(t), \tag{16}$$

$$p(x = l, t) = p_l(t), \quad \text{or} \quad q\,(x = l, t) = -\frac{k_\alpha}{\mu} D^{\alpha-1}_{0+}\left[p(x) - p(0)\right]_{x=l} = -q_l(t), \tag{17}$$

In other words, the boundary conditions on the flow rate (16) and (17) are expressed by Caputo's right fractional derivative (with order $\alpha - 1 \in (0, 1]$) evaluated in the left extreme and by Caputo's left fractional derivative (with order $\alpha - 1 \in (0, 1]$) evaluated in the right extreme. Of course, they are integral-type boundary conditions.

Numerical solution and discussion of results

If the diffusion problem described by (12) was set on an infinite medium, analytical solutions could be achieved by Laplace-Fourier transforms (Gorenflo and Mainardi, 2009; Atanackovic and Stankovic, 2009). On the other hand, if the analysis refers to finite domains as in the present case, numerical schemes have to be implemented. Different expressions were proposed to approximate fractional operators with order comprised between 1 and 2 (Lynch et al., 2003; Meerschaert et al., 2006; Ortigueira, 2008; Yang et al., 2010): the so-called L2 algorithm by Oldham and Spanier (1974) is here adopted. Let us introduce a partition of the interval $[0,l]$ on the x axis made of n ($n \in N$) intervals of length $\Delta x = l/n$. The generic point of the partition has the abscissa x_i, with $i = 1,...,n+1$ and $x_1 = 0$, $x_{n+1} = l$; that is, $x_i = (i-1)\Delta x$. Hence, for the inner points of the domain ($i = 2,...,n$), the discrete form of Eq.(12) reads ($1 \leq \alpha < 2$):

$$
\begin{aligned}
&\frac{p_{i,j+1} - p_{i,j}}{\Delta t} \approx \frac{d_\alpha}{2} \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \times \\
&\Bigg\{ \frac{2-\alpha}{(i-1)^{\alpha-1}} \left(p_{2,j} - p_{1,j}\right) + \textstyle\sum_{k=0}^{i-2} \left(p_{i-k+1,j} - 2p_{i-k,j} + p_{i-k-1,j}\right) \left[(k+1)^{2-\alpha} - k^{2-\alpha}\right] + \\
&- \frac{2-\alpha}{(n-i+1)^{\alpha-1}} \left(p_{n+1,j} - p_{n,j}\right) + \textstyle\sum_{k=0}^{n-i} \left(p_{i+k+1,j} - 2p_{i+k,j} + p_{i+k-1,j}\right) \left[(k+1)^{2-\alpha} - k^{2-\alpha}\right] \Bigg\},
\end{aligned}
\tag{18}
$$

where $p_{i,j} = p(x_i, t_j)$ and $t_j = j\Delta t$, Δt representing the discrete time step.

Let us now introduce dimensionless time $t^* = t d_\alpha / l^\alpha$. Suppose the following initial shape for the pressure field: $p(x^*, t=0) = p_0 \exp(-(x^* - 0.5)^2/0.01)$, being p_0 a reference pressure, and homogeneous boundary conditions. The space-time dimensionless solution $p^* = p/p_0$ related to Eq. (18) is reported in Fig. 3 for $\alpha = 1.25$ and 2.00.

As can be seen, nonlocal interactions affect the solution, influencing both the shape and the global diffusion velocity (the end of the transient regime results delayed). The situation is described at a fixed time $t^* = 0.031$ in Fig. 4 for different fractional orders α.

Eventually, we consider the case of a constant pressure difference between the bar extremes, $p(x = l, t) > p(x = 0, t)$ in the steady-state regime. The contribution of the left-hand term in Eq. (18) (i.e. the time derivative) vanishes. The solution, in terms of dimensionless pressure gradient $\nabla p^* = \nabla p \times l/(p_l - p_0)$ is plotted in Fig. 5 for different fractional orders α. In the classical local case ($\alpha = 2$), the pressure gradient is obviously constant throughout the body. On the other hand, for fractional orders α, the pressure gradient concentrates at the extremes of the domain due to a lower presence of nonlocal interactions, i.e. to boundary effects. Lower values with respect to the local solution are attained on the central bar, this effect being more pronounced for decreasing fractional orders α.

Physical meaning of fractional operators

For sufficiently regular functions, Riemann-Liouville fractional derivatives coincide with those defined by Marchaud (Samko et al., 1993), where the derivatives are replaced by the corresponding incremental ratios. In the steady-state regime and in absence of external forces, Eq. (12) can be thus put in the following form:

$$\frac{k_\alpha}{2}\frac{(\alpha-1)}{\Gamma(2-\alpha)}\left[\frac{p(x)-p(0)}{(x)^\alpha}+\frac{\mathrm{p}(x)-p(l)}{(l-x)^\alpha}+\alpha\int_0^l \frac{p(x)-p(y)}{|x-y|^{1+\alpha}}\mathrm{d}y\right]=0. \qquad (19)$$

For the inner points of the domain (i = 2,…, n), the discrete form of Eq. (19) reads:

$$k_{i,1}^{vs}(p_i-p_1)+k_{i,n+1}^{vs}(p_i-p_{n+1})+\sum_{j=1,j\neq i}^{n+1}k_{i,j}^{vv}(p_i-p_j)=0, \qquad (20)$$

It is evident how the nonlocal fractional model is equivalent to a discrete model where two channels appear: the former connecting the inner material pores with the bar edges, ruling the volume-surface long-range interactions, with permeability k^{vs}; the latter connecting the inner material pores with each other, describing the nonlocal interactions between non-adjacent volumes, with permeability k^{vv}. Provided that the indexes are never equal one to the other, the following expressions for the permeabilities hold (i = 1,..., n+1):

$$k^{vs}_{i,1} = k^{vs}_{1,i} = \frac{k_\alpha}{2}\frac{\alpha-1}{\Gamma(2-\alpha)}\frac{A\Delta x}{(x_i-x_1)^\alpha}, \tag{21}$$

$$k^{vs}_{i,n+1} = k^{vs}_{n+1,i} = \frac{k_\alpha}{2}\frac{\alpha-1}{\Gamma(2-\alpha)}\frac{A\Delta x}{(x_{n+1}-x_i)^\alpha}, \tag{22}$$

$$k^{vv}_{i,j} = k^{vv}_{j,i} = \frac{k_\alpha}{2}\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{A(\Delta x)^2}{|x_i-x_j|^{1+\alpha}}, \tag{23}$$

being A the area cross-section. Furthermore, by looking at the boundary conditions (16)-(17), it is possible to state that a fourth set of elements has to be introduced: it is composed by a single channel connecting the bar extremes, with permeability

$$k^{ss}_{1,n+1} = k^{ss}_{n+1,1} = \frac{k_\alpha}{2\,\Gamma(2-\alpha)}\frac{A}{(x_{n+1}-x_1)^{\alpha-1}}. \tag{24}$$

The superscript *ss* for the permeability (24) is used since the element connecting the bar edges can be seen as modeling the interactions between material pores lying on the surface, which, in the simple one-dimensional model under examination, reduce to the two points x= *0*, l. Note that the presence of such a channel was implicitly embedded in the constitutive equation (6). However, since it provides a constant flow contribution throughout the domain, its presence was lost by derivation when passing from Eq. (6) to Eq. (12).

To summarize, the constitutive fractional relationship (6) is equivalent to a discrete pore-channel model with three sets of nonlocal elements. Note that their permeabilities (21)-(24) all decay with the distance, although their decaying velocity is different.

Conclusions

A Darcy's law of convolution integral type, describing the dependence of the flow rate in one point on the gradient of pressure of all nonadjacent points, was assumed. By choosing a power law expression for the attenuation function, modeling the decreasing flow rate along with the distance, the fractional diffusion equation for porous materials was derived. Fractional operators were limited to the space variable. The problem was investigated on finite domains, through fractional finite differences, both in the transient and steady-state regimes. The influence of the fractional order $1 < \alpha \leq 2$ on results was discussed, and a physically-sound interpretation of fractional operators was derived in terms of volume and surface channels with different permeability.

The results presented here may be useful to investigate pressure response of a well reservoir which in general is not homogenous (Chang and Yortos, 1990; Acuna et al., 1995; Yao et al., 2012; Camacho-Velázquez et al., 2008; Yang et al., 2014) and, consequently, is not well described in terms of the usual diffusion equation (Razminia et al., 2015a,b).

Eventually, the following step to extend the present investigation and to analyze the delays of the fluid pressure at the boundary on the flow of fluid through the medium, seems that of further modifying nonlocal Darcy's law (1) by adding a fractional time derivative to the right member. The study can have a great importance in the framework of oil fields, where patterns of mineralization and permeability changes have to be modeled (Caputo, 2000).

The fractional model proposed herein is intrinsically multiscale. As it is well known, the structure of shales deposit has been reported to be multiscale, i.e., ranging from the nanoscale up to the global scale of a deposit. We argue that the fractional mathematical modeling of the flow of fluids and gas in nanoporous geomaterials can create a new branch of subterranean fluid mechanics. Our future steps will include comparison with real field data.

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List of Figures

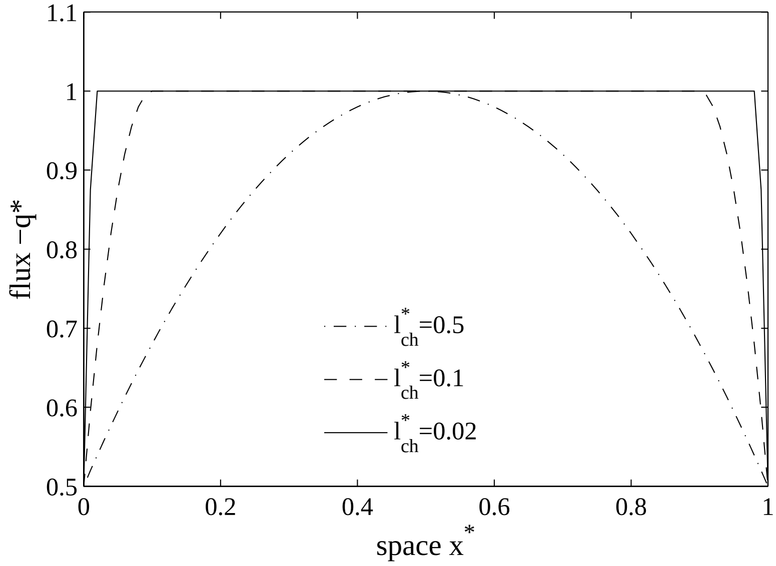

Figure 1: Nonlocal diffusion according to a cone attenuation function: dimensionless flow field providing a constant pressure gradient, for different l^*_{ch} values.

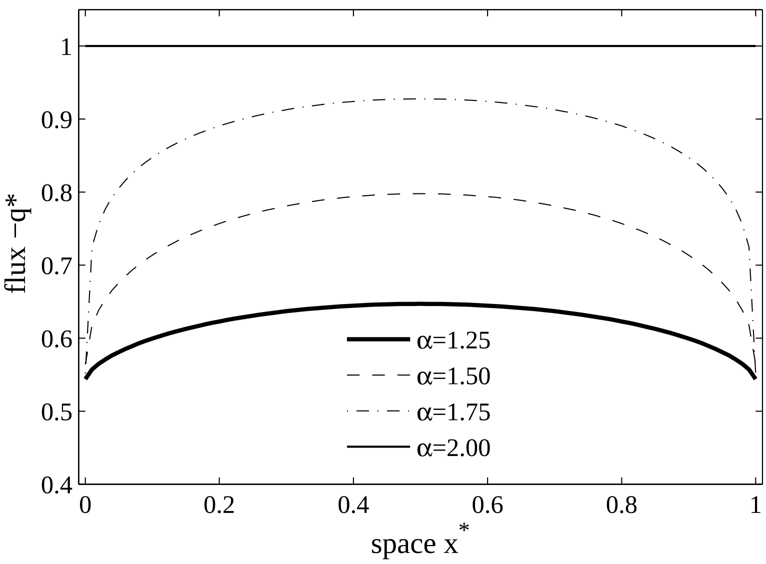

Figure 2: Nonlocal diffusion according to a power-law attenuation function (fractional model): dimensionless flow field providing a constant pressure gradient for different α values.

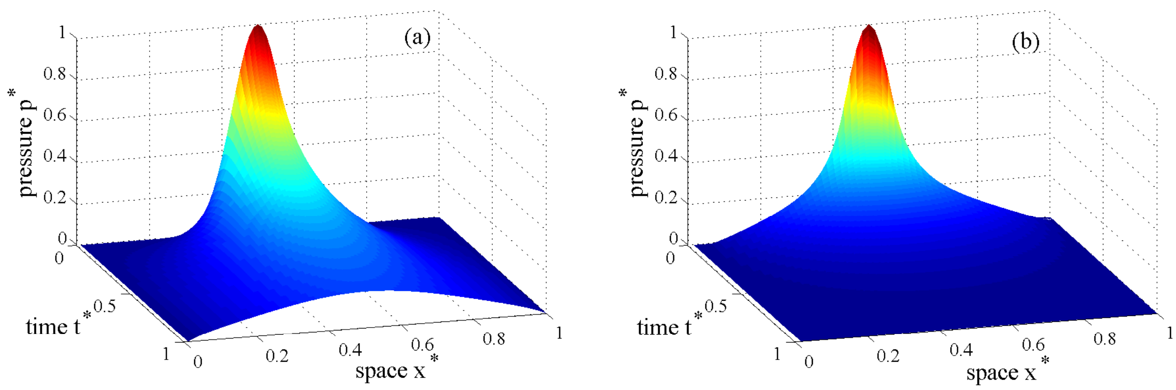

Figure 3: Dimensionless pressure field related to: (a) a nonlocal model ($\alpha = 1.25$); (b) a local one ($\alpha = 2.00$).

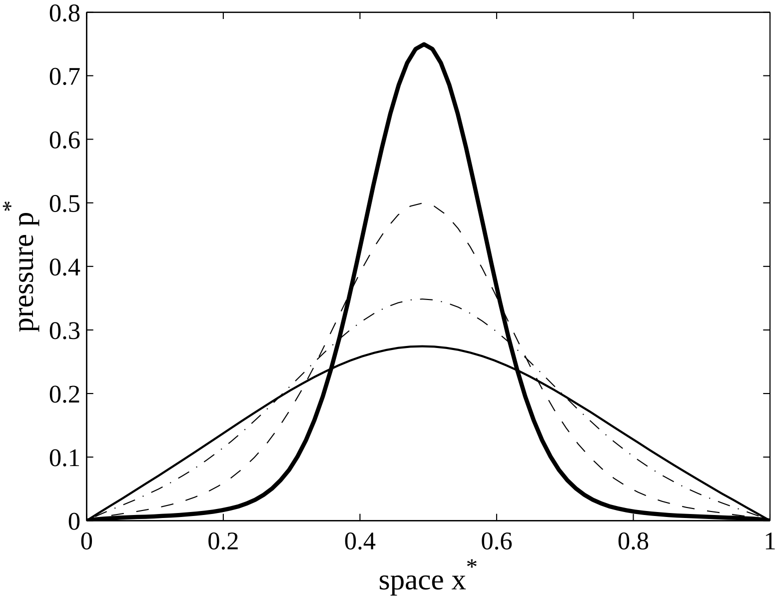

Figure 4: Dimensionless pressure field at $t^* = 0.031$ for different α-orders.

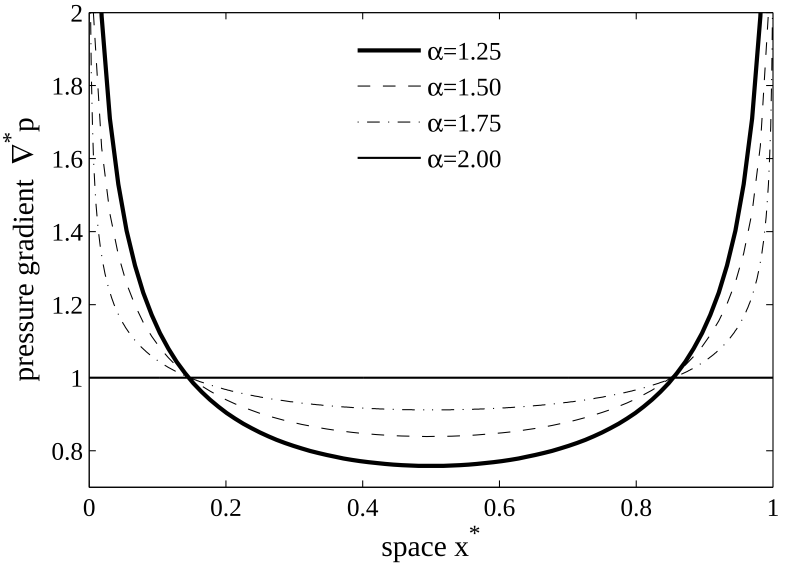

Figure 5: Dimensionless pressure gradient in the steady-state regime for different fractional orders α.