On the l.c.m. of random terms of binary recurrence sequences

ON THE L.C.M. OF RANDOM TERMS OF BINARY RECURRENCE SEQUENCES

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Abstract. For every positive integer n and every $\delta \in [0,1]$, let $B(n,\delta)$ denote the probabilistic model in which a random set $A \subseteq \{1,\dots,n\}$ is constructed by choosing independently every element of $\{1,\dots,n\}$ with probability δ. Moreover, let $(u_k)_{k\geq 0}$ be an integer sequence satisfying $u_k = a_1 u_{k-1} + a_2 u_{k-2}$, for every integer $k \geq 2$, where $u_0 = 0$, $u_1 \neq 0$, and a_1, a_2 are fixed nonzero integers; and let α and β, with $|\alpha| \geq |\beta|$, be the two roots of the polynomial $X^2 - a_1 X - a_2$. Also, assume that α/β is not a root of unity.

We prove that, as $\delta n/\log n \to +\infty$, for every A in $B(n,\delta)$ we have

$$\log \operatorname{lcm}(u_a : a \in A) \sim \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3\log\left|\alpha/\sqrt{(a_1^2, a_2)}\right|}{\pi^2} \cdot n^2$$

with probability $1 - o(1)$, where lcm denotes the lowest common multiple, Li_2 is the dilogarithm, and the factor involving δ is meant to be equal to 1 when $\delta = 1$.

This extends previous results of Akiyama, Tropak, Matiyasevich, Guy, Kiss and Mátyás, who studied the deterministic case $\delta = 1$, and is motivated by an asymptotic formula for $\operatorname{lcm}(A)$ due to Cilleruelo, Rué, Šarka, and Zumalacárregui.

1. Introduction

It is well known that the Prime Number Theorem is equivalent to the asymptotic formula

$$\log \operatorname{lcm}(1,2,\dots,n) \sim n, \tag{1}$$

as $n \to +\infty$, where lcm denotes the lowest common multiple.

For every positive integer n and every $\delta \in [0,1]$, let $B(n,\delta)$ denote the probabilistic model in which a random set $A \subseteq \{1,\dots,n\}$ is constructed by choosing independently every element of $\{1,\dots,n\}$ with probability δ. Motivated by (1), Cilleruelo, Rué, Šarka, and Zumalacárregui [8] proved the following result (see also [5] for a more precise version, and [6, 7, 12] for others results of similar flavor).

Theorem 1.1. *Let A be a random set in $B(n,\delta)$. Then, as $\delta n \to +\infty$, we have*

$$\log \operatorname{lcm}(A) \sim \frac{\delta \log(1/\delta)}{1-\delta} \cdot n,$$

with probability $1 - o(1)$, where the factor involving δ is meant to be equal to 1 for $\delta = 1$.

Let $(u_k)_{k\geq 0}$ be an integer sequence satisfying $u_k = a_1 u_{k-1} + a_2 u_{k-2}$, for every integer $k \geq 2$, where $u_0 = 0$, $u_1 \neq 0$, and a_1, a_2 are two fixed nonzero integers. Moreover, let α and β, with $|\alpha| \geq |\beta|$, be the two roots of the polynomial $X^2 - a_1 X - a_2$. We assume that α/β is not a root of unity, which is a necessary and sufficient condition to have $u_k \neq 0$ for all integers $k \geq 1$.

Akiyama [1] and, independently, Tropak [15] proved the following analog of (1) for the sequence $(u_k)_{k\geq 1}$.

Theorem 1.2. *We have*

$$\log \operatorname{lcm}(u_1, u_2, \dots, u_n) \sim \frac{3\log\left|\alpha/\sqrt{(a_1^2, a_2)}\right|}{\pi^2} \cdot n^2,$$

2010 *Mathematics Subject Classification.* Primary: 11B37, Secondary: 11N37.

Key words and phrases. binary recurrence sequence; lowest common multiple; Lehmer sequence; random sequence.

† C. Sanna is supported by a postdoctoral fellowship of INdAM and is a member of the INdAM group GNSAGA.

as $n \to +\infty$.

Special cases of Theorem 1.2 were previously proved by Matiyasevich, Guy [11], Kiss and Mátyás [10]. Furthermore, Akiyama [2, 3] generalized Theorem 1.2 to sequences having some special divisibility properties, while Akiyama and Luca [4] studied $\operatorname{lcm}(u_{f(1)}, \dots, u_{f(n)})$ when f is a polynomial, $f = \varphi$ (the Euler's totient function), $f = \sigma$ (the sum of divisors function), or f is a binary recurrence sequence.

Motivated by Theorem 1.1, we give the following generalization of Theorem 1.2.

Theorem 1.3. *Let A be a random set in $B(n, \delta)$. Then, as $\delta n / \log n \to +\infty$, we have*

$$\operatorname{lcm}(u_a : a \in A) \sim \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3 \log\left|\alpha/\sqrt{(a_1^2, a_2)}\right|}{\pi^2} \cdot n^2, \tag{2}$$

with probability $1 - o(1)$*, where* $\operatorname{Li}_2(z) := \sum_{k=1}^\infty z^k/k^2$ *is the dilogarithm and the factor involving* δ *is meant to be equal to* 1 *when* $\delta = 1$*.*

When $\delta = 1/2$ all the subsets $A \subseteq \{1, \dots, n\}$ are chosen by $B(n, \delta)$ with the same probability. Hence, Theorem 1.3 together with the identity $\operatorname{Li}_2(\frac{1}{2}) = (\pi^2 - 6(\log 2)^2)/12$ (see, e.g., [16]) give the following result.

Corollary 1.1. *As* $n \to +\infty$*, we have*

$$\operatorname{lcm}(u_a : a \in A) \sim \frac{1}{4}\left(1 - \frac{6(\log 2)^2}{\pi^2}\right) \cdot \log\left|\frac{\alpha}{\sqrt{(a_1^2, a_2)}}\right| \cdot n^2,$$

uniformly for all sets $A \subseteq \{1, \dots, n\}$*, but at most* $o(2^n)$ *exceptions.*

2. Notation

We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables X and Y, we say that "$X \sim Y$ with probability $1 - o(1)$" if $\mathbb{P}\big(|X - Y| \geq \varepsilon |Y|\big) = o_\varepsilon(1)$ for every $\varepsilon > 0$. We write $\operatorname{lcm}(S)$ for the lowest common multiple of the elements of $S \subseteq \mathbb{Z}$, with the convention $\operatorname{lcm}(\varnothing) := 1$. We also let $[a, b]$ and (a, b) denote the lowest common multiple and the greatest common divisor, respectively, of two integers a and b. Throughout, the letters p is reserved for prime numbers, and ν_p denotes the p-adic valuation. As usual, we write $\Lambda(n)$, $\varphi(n)$, $\tau(n)$, and $\mu(n)$, for the von Mangoldt function, the Euler's totient function, the number of divisors, and the Möbius function of a positive integer n, respectively.

3. Preliminaries on Lehmer sequences

Let ζ and η be complex numbers such that $c_1 := (\zeta + \eta)^2$ and $c_2 := \zeta\eta$ are nonzero coprime integers and ζ/η is not a root of unity. Also, assume $|\zeta| \geq |\eta|$. The *Lehmer sequence* $(\widetilde{u}_k)_{k \geq 0}$ associated to ζ and η is defined by

$$\widetilde{u}_k := \begin{cases} (\zeta^k - \eta^k)/(\zeta - \eta) & \text{if } k \text{ is odd}, \\ (\zeta^k - \eta^k)/(\zeta^2 - \eta^2) & \text{if } k \text{ is even}, \end{cases} \tag{3}$$

for every integer $k \geq 0$. It is known that $(\widetilde{u}_k)_{k \geq 1}$ is an integer sequence. For every positive integer m coprime with c_2, let $\varrho(m)$ be the *rank of appearance* of m in the Lehmer sequence $(\widetilde{u}_k)_{k \geq 0}$, that is, the smallest positive integer k such that $m \mid \widetilde{u}_k$. It is known that $\varrho(m)$ exists. Moreover, for every prime number p not dividing c_2, put $\kappa(p) := \nu_p(\widetilde{u}_{\varrho(p)})$.

We need the following properties of the rank of appearance.

Lemma 3.1. *We have:*

(i) $m \mid \widetilde{u}_k$ *if and only if* $(m, c_2) = 1$ *and* $\varrho(m) \mid k$*, for all integers* $m, k \geq 1$*.*

(ii) $\varrho(p^k) = p^{\max(k - \kappa(p), 0)} \varrho(p)$*, for all primes* p *not dividing* $2c_2$ *and all integers* $k \geq 1$*.*

(iii) $\varrho(2^k) = 2^{\max(k-\nu_2(\widetilde{u}_{\varrho(4)}),0)}\varrho(4)$, *for all integers* $k \geq 2$.

Proof. (i) We have $(\widetilde{u}_k, c_2) = 1$ for all integers $k \geq 1$ [13, Lemma 1]. Also, $(\widetilde{u}_k, \widetilde{u}_h) = \widetilde{u}_{(k,h)}$ for all integers $k, h \geq 1$ [13, Lemma 3]. Hence, on the one hand, if $m \mid \widetilde{u}_k$ then $(m, c_2) = 1$ and $m \mid (\widetilde{u}_k, \widetilde{u}_{\varrho(m)}) = \widetilde{u}_{(k,\varrho(m))}$, which in turn implies that $\varrho(m) \mid k$, by the minimality of $\varrho(m)$. On the other hand, if $(c_2, m) = 1$ and $\varrho(m) \mid k$ then $m \mid \widetilde{u}_{\varrho(m)} = \widetilde{u}_{(k,\varrho(m))} = (\widetilde{u}_k, \widetilde{u}_{\varrho(m)})$, so that $m \mid \widetilde{u}_k$.

(ii) If $p \mid \tilde{u}_m$, for some positive integer m, then $p \mid\mid \widetilde{u}_{pm}/\widetilde{u}_m$ [13, Lemma 5]. Hence, it follows by induction on h that $\nu_p(\widetilde{u}_{p^h\varrho(p)}) = \kappa(p) + h$, for every integer $h \geq 0$. At this point, the claim follows easily from (i).

(iii) If $4 \mid \tilde{u}_m$, for some positive integer m, then $2 \mid\mid \widetilde{u}_{pm}/\widetilde{u}_m$ [13, Lemma 5]. The proof proceeds similarly to the previous point. □

Hereafter, in light of Lemma 3.1(i), in subscripts of sums and products the argument of ϱ is always tacitly assumed to be coprime with c_2.

Let us define the cyclotomic numbers $(\phi_k)_{k\geq 1}$ associated to ζ and η by

$$\phi_k := \prod_{\substack{1 \leq h \leq k \\ (h,k)=1}} \left(\zeta - \mathrm{e}^{\frac{2\pi \mathrm{i} h}{k}}\eta\right), \tag{4}$$

for every integer $k \geq 0$. It can be proved that $\phi_k \in \mathbb{Z}$ for every integer $k \geq 3$. Moreover, from (4) it follows easily that

$$\zeta^k - \eta^k = \prod_{d \mid k} \phi_d,$$

which in turn, applying Möbius inversion formula and taking into account (3), gives

$$\phi_k = \prod_{d \mid k} \left(\zeta^d - \eta^d\right)^{\mu(k/d)} = \prod_{d \mid k} \widetilde{u}_d^{\,\mu(k/d)}, \tag{5}$$

for all integers $k \geq 3$. We need the following result about ϕ_k.

Lemma 3.2. *For every integer* $k \geq 13$, *we have*

$$|\phi_k| = \lambda_k \cdot \prod_{\varrho(p) = k} p^{\kappa(p)},$$

where λ_k *is equal to* 1 *or to the greatest prime factor of* $k/(k,3)$.

Proof. Let p be a prime number not dividing c_2. By the definition of $\varrho(p)$, we have that $p \nmid \widetilde{u}_h$ for each positive integer $h < \varrho(p)$. Hence, by (5), we obtain that $\nu_p(\phi_{\varrho(p)}) = \nu_p(\widetilde{u}_{\varrho(p)}) = \kappa(p)$. In particular, $p \mid \phi_{\varrho(p)}$. Let $k \geq 3$ be an integer and suppose that p is a prime factor of ϕ_k. On the one hand, if $\varrho(p) = k$ then, by the previous consideration, $\nu_p(\phi_k) = \kappa(p)$. On the other hand, if $\varrho(p) \neq k$ then $p \mid (\phi_{\varrho(p)}, \phi_k)$. Finally, for $k \geq 13$ and for every integer $h \geq 3$ with $h \neq k$, we have that (ϕ_h, ϕ_k) divides the greatest prime factor of $k/(k,3)$ [13, Lemma 7]. □

We conclude this section with a formula for a sum involving the von Mangoldt function.

Lemma 3.3. *We have*

$$\sum_{\varrho(m) = r} \Lambda(m) = \varphi(r) \log|\zeta| + O_{\zeta,\eta}(\tau(r)\log(r+1)), \tag{6}$$

and, in particular,

$$\sum_{\varrho(m) = r} \Lambda(m) \ll_{\zeta,\eta} \varphi(r), \tag{7}$$

for every positive integer r.

Proof. Clearly, we can assume $r \geq 13$. Write $m = p^k$, where p is a prime number not dividing c_2 and k is a positive integer. First, suppose that $p > 2$. By Lemma 3.1(ii), we have that $\varrho(m) = p^{\max(k-\kappa(p),0)}\varrho(p)$. Hence, $\varrho(m) = r$ if and only if $k \leq \kappa(p)$ and $\varrho(p) = r$, or $k > \kappa(p)$ and $p^{k-\kappa(p)}\varrho(p) = r$. In the first case, the contribution to the sum in (6) is exactly $\kappa(p)\log p$. In the second case, $p \mid r$ and, since k is determined by p and r, the contribution to the sum in (6) is $\log p$. Using Lemma 3.1(iii), the case $p = 2$ can be handled similarly. Therefore,

$$\sum_{\varrho(m)=r} \Lambda(m) = \sum_{\varrho(p)=r} \kappa(p)\log p + O\left(\sum_{p \mid r} \log p\right) = \log|\phi_r| + O(\log r), \tag{8}$$

where we used Lemma 3.2. Furthermore, from (5) and the the identity $\sum_{d \mid r} \mu(r/d)\, d = \varphi(r)$, it follows that

$$\log|\phi_r| = \varphi(r)\log|\zeta| + O\left(\sum_{d \mid r} \log\left|1 - \left(\tfrac{\eta}{\zeta}\right)^d\right|\right).$$

If $|\eta/\zeta| < 1$ then $\log\left|1 - (\eta/\zeta)^d\right| = O_{\zeta,\eta}(1)$. If $|\eta/\zeta| = 1$ then, since η/ζ is an algebraic number that is not a root of unity, it follows from classic bounds on linear forms in logarithms (see, e.g., [9, Lemma 3]) that $\log\left|1 - (\eta/\zeta)^d\right| = O_{\zeta,\eta}(\log(d+1))$. Consequently,

$$\log|\phi_r| = \varphi(r)\log|\zeta| + O_{\zeta,\eta}(\tau(r)\log(r+1)). \tag{9}$$

Putting together (8) and (9), we get (6). Finally, the upper bound (7) follows since $\tau(k) \leq k^\varepsilon$ and $\varphi(k) \geq k^{1-\varepsilon}$, for all $\varepsilon > 0$ and every integer $k \gg_\varepsilon 1$ [14, Ch. I.5, Corollary 1.1 and Eq. 12]. ∎

4. Further preliminaries

We need two estimates involving the Euler's totient function. Define

$$\Phi(x) := \sum_{n \leq x} \varphi(n),$$

for every $x \geq 1$.

Lemma 4.1. *We have*

$$\Phi(x) = \frac{3}{\pi^2}x^2 + O(x\log x) \quad \textit{and} \quad \sum_{n \leq x} \frac{\varphi(n)}{n} \ll x,$$

for every $x \geq 2$.

Proof. The first formula is well known [14, Ch. I.3, Thm. 4] and implies

$$\sum_{n \leq x} \frac{\varphi(n)}{n} \leq \sum_{n \leq x/2} 1 + \sum_{x/2 < n \leq x} \frac{\varphi(n)}{x/2} \ll x,$$

as desired. ∎

The following lemma is an easy inequality that will be useful later.

Lemma 4.2. *It holds $1 - (1-x)^k \leq kx$, for all $x \in [0,1]$ and all integers $k \geq 0$.*

Proof. The claim is $(1 + (-x))^k \geq 1 + k(-x)$, which follows from Bernoulli's inequality. ∎

5. Proof of Theorem 1.3

Henceforth, all the implied constants may depend by a_1, a_2, and u_1. It is well known that the generalized Binet's formula

$$u_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}\, u_1, \tag{10}$$

holds for every integer $k \geq 0$. We put $\zeta := \alpha/\sqrt{b}$ and $\eta := \beta/\sqrt{b}$, where $b := (a_1^2, a_2)$. Note that indeed $c_1 = a_1^2/b$ and $c_2 = -a_2/b$ are nonzero relatively prime integers, $\zeta/\eta = \alpha/\beta$ is not a root of unity, and $|\zeta| \geq |\eta|$. Moreover, from (3) and (10), it follows easily that

$$u_k = \begin{cases} b^{(k-1)/2} u_1 \widetilde{u}_k & \text{if } k \text{ is odd,} \\ a_1 b^{k/2-1} u_1 \widetilde{u}_k & \text{if } k \text{ is even,} \end{cases}$$

for every integer $k \geq 0$. Therefore, for every $A \subseteq \{1, \dots, n\}$, we have

$$\log \operatorname{lcm}(u_a : a \in A) = \log \operatorname{lcm}(\widetilde{u}_a : a \in A) + O(n).$$

Note that $O(n)$ is a "little oh" of the right-hand side of (2), as $\delta n / \log n \to +\infty$. Hence, it is enough to prove Theorem 1.3 with $\log \operatorname{lcm}(\widetilde{u}_a : a \in A)$ in place of $\log \operatorname{lcm}(u_a : a \in A)$, and this will be indeed our strategy.

Hereafter, let A be a random set in $B(n, \delta)$, and put $L := \operatorname{lcm}(\widetilde{u}_a : a \in A)$ and $X := \log L$. For every positive integer m coprime with c_2, let us define

$$I_A(m) := \begin{cases} 1 & \text{if } \varrho(m) \mid a \text{ for some } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma gives an expression for X in terms of I_A and the von Mangoldt function.

Lemma 5.1. *We have*

$$X = \sum_{\varrho(m) \leq n} \Lambda(m)\, I_A(m).$$

Proof. For every prime power p^k with $p \nmid c_2$, we know from Lemma 3.1(i) that $p^k \mid L$ if and only if $\varrho(p^k) \mid a$ for some $a \in A$ and, in particular, $\varrho(p^k) \leq n$. Hence,

$$X = \sum_{p^k \mid L} \log p = \sum_{\varrho(p^k) \leq n} (\log p)\, I_A\big(p^k\big) = \sum_{\varrho(m) \leq n} \Lambda(m)\, I_A(m),$$

as claimed. □

The next lemma provides two expected values involving I_A and needed in later arguments.

Lemma 5.2. *We have*

$$\mathbb{E}\big(I_A(m)\big) = 1 - (1-\delta)^{\lfloor n/\varrho(m) \rfloor} \tag{11}$$

and

$$\begin{aligned}\mathbb{E}\big(I_A(m) I_A(\ell)\big) = 1 - (1-\delta)^{\lfloor n/\varrho(m) \rfloor} &- (1-\delta)^{\lfloor n/\varrho(\ell) \rfloor} \\ &+ (1-\delta)^{\lfloor n/\varrho(m) \rfloor + \lfloor n/\varrho(\ell) \rfloor - \lfloor n/[\varrho(m), \varrho(\ell)] \rfloor},\end{aligned}$$

for all positive integers m and ℓ with $(m\ell, c_2) = 1$.

Proof. By the definition of I_A, we have

$$\mathbb{E}\big(I_A(m)\big) = \mathbb{P}\big(\exists a \in A : \varrho(m) \mid a\big) = 1 - \mathbb{P}\left(\bigwedge_{t \leq n/\varrho(m)} (\varrho(m) t \notin A)\right) = 1 - (1-\delta)^{\lfloor n/\varrho(m) \rfloor},$$

which is the first claim. On the one hand, by linearity of expectation and by (11), we obtain

$$\begin{aligned}\mathbb{E}\big(I_A(m) I_A(\ell)\big) &= \mathbb{E}\Big(I_A(m) + I_A(\ell) - 1 + \big(1 - I_A(m)\big)\big(1 - I_A(\ell)\big)\Big) \\ &= \mathbb{E}\big(I_A(m)\big) + \mathbb{E}\big(I_A(\ell)\big) - 1 + \mathbb{E}\Big(\big(1 - I_A(m)\big)\big(1 - I_A(\ell)\big)\Big)\end{aligned}$$

$$= 1 - (1-\delta)^{\lfloor n/\varrho(m)\rfloor} - (1-\delta)^{\lfloor n/\varrho(\ell)\rfloor} + \mathbb{E}\big((1 - I_A(m))(1 - I_A(\ell))\big).$$

On the other hand, by the definition of I_A,

$$\mathbb{E}\left(\big(1 - I_A(m)\big)\big(1 - I_A(\ell)\big)\right) = \mathbb{P}\big(\forall a \in A : \varrho(m) \nmid a \text{ and } \varrho(\ell) \nmid a\big)$$

$$= \mathbb{P}\left(\bigwedge_{\substack{k \le n \\ \varrho(m) \,|\, k \text{ or } \varrho(\ell) \,|\, k}} (k \notin A)\right) = (1-\delta)^{\lfloor n/\varrho(m)\rfloor + \lfloor n/\varrho(\ell)\rfloor - \lfloor n/[\varrho(m),\varrho(\ell)]\rfloor},$$

and the second claim follows too. □

Now we give an asymptotic formula for the expected value of X.

Lemma 5.3. *We have*

$$\mathbb{E}(X) = \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3 \log|\zeta|}{\pi^2} \cdot n^2 + O\big(\delta n (\log n)^3\big),$$

for all integers $n \ge 2$. In particular,

$$\mathbb{E}(X) \sim \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3 \log|\zeta|}{\pi^2} \cdot n^2,$$

as $n \to +\infty$, uniformly for $\delta \in (0, 1]$.

Proof. From Lemma 5.1 and Lemma 5.2, it follows that

$$\begin{aligned}\mathbb{E}(X) &= \sum_{\varrho(m) \le n} \Lambda(m)\, \mathbb{E}\big(I_A(m)\big) \\ &= \sum_{\varrho(m) \le n} \Lambda(m) \left(1 - (1-\delta)^{\lfloor n/\varrho(m)\rfloor}\right) \\ &= \sum_{r \le n} \left(1 - (1-\delta)^{\lfloor n/r\rfloor}\right) \sum_{\varrho(m) = r} \Lambda(m).\end{aligned}$$

Consequently, thanks to Lemma 3.3 and Lemma 4.2, we obtain

$$\begin{aligned}\mathbb{E}(X) &= \sum_{r \le n} \left(1 - (1-\delta)^{\lfloor n/r\rfloor}\right) \varphi(r) \log|\zeta| + O\left(\delta n \sum_{r \le n} \frac{\tau(r) \log(r+1)}{r}\right) \\ &= \sum_{r \le n} \left(1 - (1-\delta)^{\lfloor n/r\rfloor}\right) \varphi(r) \log|\zeta| + O\big(\delta n (\log n)^3\big),\end{aligned} \tag{12}$$

where we used the fact that

$$\sum_{r \le n} \frac{\tau(r)}{r} \le \Big(\sum_{s \le n} \frac{1}{s}\Big)^2 \ll (\log n)^2.$$

Note that $\lfloor n/r \rfloor = j$ if and only if $r \in (n/(j+1), n/j]$. Hence,

$$\begin{aligned}\sum_{r \le n} \left(1 - (1-\delta)^{\lfloor n/r\rfloor}\right) \varphi(r) &= \sum_{j \le n} \left(1 - (1-\delta)^j\right) \sum_{n/(j+1) < r \le n/j} \varphi(r) \\ &= \sum_{j \le n} \left(1 - (1-\delta)^j\right) \left(\Phi\left(\frac{n}{j}\right) - \Phi\left(\frac{n}{j+1}\right)\right) \\ &= \delta \sum_{j \le n} (1-\delta)^{j-1}\, \Phi\left(\frac{n}{j}\right) \\ &= \delta \sum_{j \le n} \frac{(1-\delta)^{j-1}}{j^2} \cdot \frac{3}{\pi^2} \cdot n^2 + O\left(\delta \sum_{j \le n} \frac{n}{j} \log\left(\frac{n}{j}\right)\right)\end{aligned} \tag{13}$$

$$= \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3}{\pi^2} \cdot n^2 + O\big(\delta n (\log n)^2\big)\,,$$

where we used Lemma 4.1. Finally, putting together (12) and (13), we get the desired claim. □

The next lemma is an upper bound for the variance of X.

Lemma 5.4. *We have*

$$\mathbb{V}(X) \ll \delta n^3 \log n,$$

for all integers $n \geq 2$.

Proof. On the one hand, by Lemma 5.1, we have

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}\big(X^2\big) - \mathbb{E}(X)^2 \\ &= \sum_{\varrho(m),\, \varrho(\ell) \leq n} \Lambda(m)\Lambda(\ell)\big(\mathbb{E}\big(I_A(m)I_A(\ell)\big) - \mathbb{E}\big(I_A(m)\big)\mathbb{E}\big(I_A(\ell)\big)\big)\,.\end{aligned}$$

On the other hand, from Lemma 5.2 and Lemma 4.2, it follows that

$$\begin{aligned}&\mathbb{E}\big(I_A(m)I_A(\ell)\big) - \mathbb{E}\big(I_A(m)\big)\mathbb{E}\big(I_A(\ell)\big) \\ &\quad = (1-\delta)^{\lfloor n/\varrho(m)\rfloor + \lfloor n/\varrho(\ell)\rfloor - \lfloor n/[\varrho(m),\varrho(\ell)]\rfloor}\big(1-(1-\delta)^{\lfloor n/[\varrho(m),\varrho(\ell)]\rfloor}\big) \leq \frac{\delta n}{[\varrho(m),\varrho(\ell)]}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{V}(X) &\leq \delta n \sum_{\varrho(m),\, \varrho(\ell) \leq n} \frac{\Lambda(m)\,\Lambda(\ell)}{[\varrho(m),\varrho(\ell)]} = \delta n \sum_{r,s \leq n} \frac{1}{[r,s]} \sum_{\varrho(m) = r} \Lambda(m) \sum_{\varrho(\ell) = s} \Lambda(\ell) \\ &\ll \delta n \sum_{r,s \leq n} \frac{\varphi(r)\,\varphi(s)}{[r,s]} = \delta n \sum_{r,s \leq n} (r,s)\, \frac{\varphi(r)\,\varphi(s)}{rs},\end{aligned} \tag{14}$$

where we used Lemma 3.3 and the identity $[r,s] = rs/(r,s)$. At this point, writing $r = dr'$ and $s = ds'$, where $d := (r,s)$, we obtain

$$\begin{aligned}\sum_{r,s \leq n} (r,s)\, \frac{\varphi(r)\,\varphi(s)}{rs} &= \sum_{d \leq n} d \sum_{\substack{r',s' \leq n/d \\ (r',s')=1}} \frac{\varphi(dr')\,\varphi(ds')}{d^2 r' s'} \leq \sum_{d \leq n} d \Bigg(\sum_{t \leq n/d} \frac{\varphi(t)}{t}\Bigg)^2 \\ &\ll \sum_{d \leq n} d \left(\frac{n}{d}\right)^2 \ll n^2 \log n,\end{aligned} \tag{15}$$

where we used Lemma 4.1 and the inequality $\varphi(dm) \leq d\varphi(m)$, holding for every integer $m \geq 1$. Finally, putting together (14) and (15), we get the desired claim. □

Proof of Theorem 1.3. By Chebyshev's inequality, Lemma 5.3, and Lemma 5.4, we have

$$\mathbb{P}\big(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)\big) \leq \frac{\mathbb{V}(X)}{\big(\varepsilon \mathbb{E}(X)\big)^2} \ll \frac{\log n}{\varepsilon^2 \delta n} = o_\varepsilon(1),$$

as $\delta n / \log n \to +\infty$. Hence, again by Lemma 5.3, we have

$$X \sim \frac{\delta \operatorname{Li}_2(1-\delta)}{1-\delta} \cdot \frac{3 \log|\zeta|}{\pi^2} \cdot n^2,$$

with probability $1 - o(1)$, as desired. □

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