Does sample-time emulation preserve exponential stability?

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ABSTRACT

Whereas classical control theory provides many methods for designing continuous-time feedback controllers, nowadays control algorithms are implemented on digital platforms and have to be designed in sampled time. Approaches to sampled-time control design are based on either discretization of the plant enabling discrete-time controller synthesis, or various redesign methods converting a continuous-time controller into a sampled-time approximation, providing comparable closed-loop system properties. The simplest of redesign approaches, typically used in practice, is the *emulation* of continuous-time feedback by sufficiently fast sampling. In spite of its simplicity, emulation gives rise to an important problem: does emulation at a sufficiently high rate (or, equivalently, with a small sampling time) preserve the stability of the closed-loop system? In this paper, we address this problem for the case of *exponential* stability (local or global). Even for linear systems, the problem of stability preservation becomes non-trivial when sampling is aperiodic. For nonlinear systems, viability of emulation approach is usually proved only under quite restrictive assumptions on the plant and the controller, which, as will be shown, in fact be discarded.

KEYWORDS

Sampled-time system, nonlinear control, stability

1 INTRODUCTION

Whereas classical control theory primarily deals with the centralized architecture "plant-sensor-controller", networked cyberphysical systems are featured by sophisticated architectures, where the functions of sensing, data processing and actuation are spread between numerous digital devices. Communication and computational constraints, imposed by the choice of hardware and requirements of reliability and resilience, often become a "bottleneck" in design of such control systems [24, 41]. Often, a controller should not only operate at a predefined data rate but also obey an external task scheduler, allocating dedicated time slots for data transmission. Unlike classical digital control, primarily assuming *periodic* time sampling [8, 26], the networked architecture can implies the necessity of more complicated *aperiodic* sampling schemes, where the sampling instances cannot be computed in advance.

Methods for digital (sampled-time, discrete-time) control design are most efficient for linear systems that allow an explicit discretization. Design approaches for *nonlinear* sampled-time systems are quite limited and assume, as prerequisite, that an accurate discrete-time approximation of a nonlinear plant is known. Finding of such an approximation is a self-standing non-trivial problem [1, 27, 28, 44]; a discretized model can be considered as a special case of a symbolic model for a continuous-time nonlinear system [36, 37]. At the same time, modern control theory offers many efficient methods of continuous-time controller design such as e.g. feedback linearization, backstepping and forwarding, absolute and robust stability approaches, passivity-based and adaptive control [9, 10, 20, 21, 40]. A natural question arises whether the continuous-time controllers can be redesigned into sampled-time ones. The simplest way of such a redesign is the *emulation* of continuous-time feedback at a sufficiently high sampling rate.

It can be expected that, as the sampling time becomes sufficiently small, the emulation inherits all properties of its continuous-time "ancestor". However, analysis of the resulting hybrid system appears to be non-trivial even in the linear case (except for the periodic sampling) [18, 42]. Nonlinearities in the model of control systems make their analysis even more difficult [18]. The effect of sampling is often represented as a measurement error, which enables to use various methods inspired by robust control and employing input-to-state [29, 30] and input-output [5, 31] stability properties. Alternatively, sampling may be considered as a time-varying "saw-tooth" delay in measurements, making it possible to apply powerful techniques of delay systems theory [11, 12, 25].

As will be discussed, the existing results on stability of nonlinear emulation-based controllers performance impose a number of restrictive assumptions on the continuous-time feedback's structure (such as e.g. the global Lipschitz condition on the feedback mapping). In this paper, we demonstrate that in the case of *exponential* stabilization, most of these requirements can be discarded: emulation at a high sampling rate can provide exponential stability even if the original continuous-time system is formally infeasible (e.g. discontinuous). These advancements in theory of sampled-time systems are based on the ideas from recent works [38, 39].

The paper is organized as follows. Section 2 introduces necessary definitions and the problem setup. In Section 3, we present the main results of the paper (and illustrate them by some examples). The proofs of the main results are given in Section 4.

2 PRELIMINARIES AND PROBLEM SETUP

In this section, the key problem of exponential stability preservation under fast-sampled emulation is formulated. We give the relevant definitions. For the reader's convenience, we also formulate the converse Lyapunov theorems to be used in the subsequent text.

2.1 Emulation of a continuous-time feedback

Consider a nonlinear plant whose state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}^m$ and observed output $y(t) \in \mathbb{R}^l$ obey the equation

$$\frac{d}{dt}x(t) = F(x(t), u(t)), \quad y(t) = H(x(t)). \tag{1}$$

Assume also that a (dynamic) continuous-time controller is given

$$\frac{d}{dt}x_c(t) = F_c(x_c(t), y(t)) \in \mathbb{R}^{n_c}, \quad u(t) = H_c(x_c(t), y(t)). \tag{2}$$

A special case of (2) is the *static* (memoryless) control

$$u(t) = H_c(y(t)). \tag{3}$$

Definition 2.1. **(Emulation)**[1] Consider an increasing sequence $0 = t_0 < t_1 < \ldots < t_n < \ldots$ of sampling instants, $t_k \xrightarrow[k\to\infty]{} \infty$. The following sampled-time feedback law

$$\begin{aligned} w_{k+1} &= w_k + F_c(w_k, y_k)\Delta t_k,\\ u(t) \equiv u_k &= H_c(w_k, y_k)\ \forall t \in [t_k, t_{k+1})\\ y_k &\overset{\Delta}{=} y(t_k), \quad \Delta t_k \overset{\Delta}{=} t_{k+1} - t_k. \end{aligned} \tag{4}$$

is said to be the *emulation*[2] of the feedback (2), corresponding to the sequence (t_k). Similarly, the emulation of (3) is the controller

$$u(t) \equiv u_k = H_c(y_k)\ \forall t \in [t_k, t_{k+1}). \tag{5}$$

The number $\tau_{max} = \sup_k(t_{k+1} - t_k)$ is henceforth referred to as the *maximum sampling time* (MST) of the emulation controller.

In this paper, we neglect the delays in communication and information processing and assume that the input u_k is computed as soon as the new measurement y_k is received. Notice that the sampling interval Δt_k depends on k (aperiodic sampling) and may be uncertain. In our results, the MST is always supposed to be sufficiently small, whereas the *dwell time* $\inf_k(t_{k+1} - t_k)$ is formally not required to be positive. The requirement $t_k \to \infty$, however, excludes the possibility of Zeno trajectories.

In the subsequent constructions, we will use the following interpolation of w_k between samples

$$w(t) \overset{\Delta}{=} w_k + (t - t_k)F_c(w_k, y_k)\Delta t_k \quad \forall t \in [t_k, t_{k+1}). \tag{6}$$

Obviously, $w(\cdot)$ is a continuous function on $[0, \infty)$ and $w(t_k) = w_k$.

2.2 Exponential stability

In this paper, we are primarily interested in exponential stabilization (local or global) of the system at some predefined equilibrium. We now introduce the relevant definitions.

Given the continuous-time feedback (2), we introduce the state vector of the *closed-loop* system $z(t) = [x(t)^\top, x_c(t)^\top]^\top \in \mathbb{R}^N$, where $N = n + n_c$. For sample-time controller (4), the closed-loop system's state is defined as $z(t) = [x(t)^\top, w(t)^\top] \in \mathbb{R}^N$, where $w(t)$ is the interpolated sampled-time controller's state[3] from (6). In the case of static controller (3) or $u_k = H_c(y(t_k))$, the state of the closed-loop system is $z(t) = x(t)$ and $N = n$.

Without loss of generality, we assume that the system's equilibrium to be stabilized is $z = 0$. Let $B_r \overset{\Delta}{=} \{z \in \mathbb{R}^N : |z| < r\}$ denote the open ball, centered at $z = 0$.

Definition 2.2. **(Exponential stabilization)** A continuous- or sampled-time controller is *locally* exponentially stabilizing if[4]

$$|z(t)| \leq \gamma_1 |z(0)| e^{-\gamma_2 t} \quad \forall t \geq 0\ \ \forall z(0) \in B_r. \tag{7}$$

for some constants $r, \gamma_1, \gamma_2 > 0$. If (7) holds for all $r > 0$ with some γ_1, γ_2 *independent* of r (equivalently, holds for $r = \infty$), the controller is *globally* exponentially stabilizing.

In this paper, we do not consider semi-global exponential stability [22, 34], being a relaxed form of global stability and requiring that (7) holds for all r with $\gamma_i = \gamma_i(r)$. At the same time, some of the results obtained in Section 3 are "semi-global" in the sense that the sampling time depends on the region of initial conditions.

Whereas local stability can be tested via the linearization (which, however, usually gives a very conservative estimate for r), global stability is typically examined by Lyapunov methods [20]. Although analytic synthesis of Lyapunov functions often appears to be difficult, they can often be found via efficient numerical tools such as e.g. SOS programming [13, 33]. Global stability usually guarantees certain robustness of the system against disturbances [20].

2.3 Exponential stability: Lyapunov criteria

We recollect the direct and converse Lyapunov theorems for exponential stability of the nonlinear system

$$\dot{z}(t) = \mathcal{F}(z(t)) \in \mathbb{R}^N,\ \mathcal{F}(0) = 0, \quad t \geq 0. \tag{8}$$

Henceforth we assume that for any initial condition $z(0)$, a solution to the system (8) exists locally (possibly, non-unique), this holds e.g. when $\mathcal{F}$ is continuous [20]. We are interested in the stability of the equilibrium $z = 0$. A well-known sufficient Lyapunov condition, ensuring the local and global exponential stability, is as follows.

Lemma 2.3. *[20, Th. 4.10] Assume that* (8) *admits a C^1-smooth* Lyapunov certificate $V : B_R \to [0, \infty)$ *such that*

$$\alpha_1 |z|^a \leq V(z) \leq \alpha_2 |z|^a,\ V'(z)\mathcal{F}(z) \leq -\alpha_3 |z|^a, \quad \forall z \in B_R. \tag{9}$$

Here $\alpha_i > 0, a > 0$ are constants. Then (7) *holds for*

$$r \overset{\Delta}{=} R\left(\frac{\alpha_1}{\alpha_2}\right)^{1/a}, \quad \gamma_1 \overset{\Delta}{=} \left(\frac{\alpha_2}{\alpha_1}\right)^{1/a}, \quad \gamma_2 = \frac{\alpha_3}{a\alpha_2}. \tag{10}$$

The sketch of the proof of Lemma 2.3 is as follows (see Fig. 1). Given $z(0) \in B_r$, choose $r' \in \left(|z(0)|(\alpha_2/\alpha_1)^{1/a}, R\right)$. Inequalities (9) imply that the solution can never cross the sphere $|z| = r'$ since

$$\min_{|z|=r'} V(z) \geq \alpha_1 (r')^a > \alpha_2 |z(0)|^a \geq V(z(0))$$

and $V(z(t))$ is non-increasing. Hence, $z(t) \in B_{r'} \subset B_R$ for all $t \geq 0$; in particular, the solution remains bounded and is forward complete. Conditions (9) also entail that $\dot{V} \leq -(\alpha_3/\alpha_2)V$ and therefore

$$\alpha_1 |z(t)|^a \leq V(z(t)) \leq V(z(0))e^{-\alpha_3/\alpha_2 t} \leq \alpha_2 |z(0)|^a e^{-\alpha_3 t/\alpha_2}. \blacksquare$$

Usually, the conditions (9) are considered for $a = 2$ (as will be discussed, the condition with $a = 2$ is close to being necessary). In the case $R < \infty$, Lemma 2.3 ensures local exponential stability, for $R = \infty$ (when $B_r = B_R = \mathbb{R}^N$) it guarantees global stability.

The condition from Lemma 2.3 is, to some extent, necessary for exponential stability, as shown by the *converse Lyapunov* theorem.

Lemma 2.4. *[20, Th. 4.14] Suppose that the equilibrium $z = 0$ is locally exponentially stable* (7)*, and $\mathcal{F}'(z)$ is continuous and bounded*

[1] Sometimes, emulation is understood in a broader sense as any method of converting a continuous-time algorithm into a sampled-time one [35]. Definition 2.1 deals with the most typical type of emulation considered in control engineering literature.

[2] In this paper, we confine ourselves to the conventional zero-order hold (ZOH) discretization of the control input and use the simple first-order approximation of the dynamic controller (such an approximation is typical in the literature, see e.g. [4]).

[3] System (1),(4) operates on the hybrid time axis [15]; we interpolate its discrete-time part between samples to define the exponential stability in a unified way.

[4] Condition (7) assumes, implicitly, that the solution $z(t)$ exists and is forward complete for any $z(0) \in B_r$; such a solution can however be non-unique.

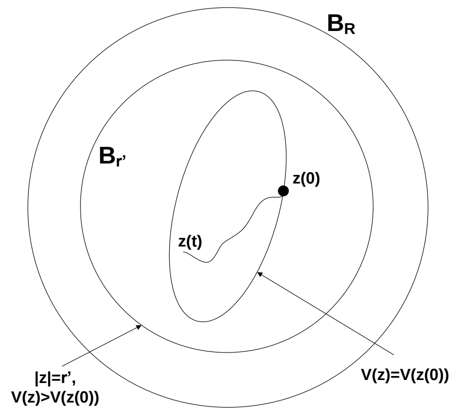

Figure 1: Illustration to the proof of Lemma 2.3

in $B_{\gamma_1 r}$. Then there exists a C^1-smooth function $V : B_r \to [0,\infty)$, such that (9) *holds for any $z \in B_r$ with $a = 2$. Additionally,*

$$|V'(z)| \leq \alpha_4 |z| \quad \forall z \in B_r. \tag{11}$$

Here $\alpha_i > 0$, $i = 1,\dots,4$, are constants. The statement remains valid for global exponential stability $r = \infty$ (in this situation, $\mathcal{F}'$ is supposed to be globally bounded on the whole space $\mathbb{R}^N$).

In the case of the bounded derivative $\mathcal{F}'$, the criterion of *global* exponential stability from Lemma 2.3 is thus not only sufficient but also necessary. Nevertheless, even in this situation a gap between necessary and sufficient conditions exists: the decay rate $\alpha_3/(2\alpha_2)$ guaranteed by the Lyapunov function from Lemma 2.4 can, in general, be less than the actual convergence rate from (7). In the case of local stability $r < \infty$, the Lyapunov function from Lemma 2.4 exists only in B_r and cannot, in general, be defined for a larger set. The reason for this is the construction of the Lyapunov function [20]

$$V(\xi) = \int_0^\delta |z(t|\xi)|^2 dt. \tag{12}$$

Here $z(t|\xi)$ stands for the solution of (8), such that $z(t|0) = \xi$, and $\delta > 0$ is a sufficiently small constant (depending on the upper bound on $|\mathcal{F}'|$ in $B_{\gamma_1 r}$ and the parameters r, γ_1, γ_2). Since the estimate (7) is guaranteed only in B_r, V is in general not well-defined beyond this set (e.g., the solution $z(t|\xi)$ may fail to exist for all $t \in [0,\delta]$). For this reason, the function (12) cannot be used to prove the exponential stability in B_r; the stability in ensured only in the smaller ball.

Analysis of the proof of [20, Theorem 4.14] shows that in the case of global stability $r = \infty$ the function (12) (with properly chosen $\delta > 0$) is globally defined, furthermore, the inequalities (9) and (11) hold with some constants α_2, α_3 and continuous functions $\alpha_1 = \alpha_1(r), \alpha_4 = \alpha_4(r)$. These functions depend, in fact, on the number $L(r) = \max_{|z|\leq\gamma_1 r} |F'(z)|$ and, in general, $\alpha_1(r) \to 0$ and $\alpha_4(r) \to 0$ if $L(r) \to \infty$. Hence, discarding the global boundedness of F', the function V from Lemma 2.4 is defined on $\mathbb{R}^N$ yet allows to prove only semi-global exponential stability [34].

It should be noticed that V from (12), obviously, cannot be found explicitly unless one is able to solve the nonlinear equation (8). Hence the converse Lyapunov theorem itself is a non-constructive result. Nevertheless, Lyapunov functions can often be found analytically (using e.g. the backstepping and feedback linearization methods) [20] or numerically (using e.g. the SOS programming [33]).

2.4 Problem in question: stability preservation

In this paper we address the following two problems. The first problem addressed via Lyapunov's converse theorem is the preservation of exponential stability under sample-time emulation.

Problem 1. A continuous-time controller (2) (or (3)) is given that ensures exponential (local or global) stability of the closed-loop system. Does its emulation with sufficiently small MST provide exponential stability of the same type?

As has been mentioned, the Lyapunov function satisfying the inequalities (9) (e.g. quadratic one) can exist in many situations, where the converse Lyapunov theorem is formally inapplicable. This motivates our second problem.

Problem 2. Suppose that the closed-loop system admits the Lyapunov function with properties (9). Does the emulation of this controller whose MST is sufficiently small also provide exponential stability with the same Lyapunov function?

Although the first problem seems quite natural, it has not been, to the best of the authors' knowledge, directly addressed for the general nonlinear system. The second problem is classical and was studied first in the seminal paper [19]. However, most of the existing results consider the Lipschitz continuity of the continuous-time control as a prerequisite to examine the properties of its emulation. We show that most of these restrictions can in fact be discarded.

3 MAIN RESULTS

We start with the assumptions on the plant and the controller, henceforth supposed to hold. For brevity, let

$$\bar{F}(x, x_c) \stackrel{\Delta}{=} F(x, H_c(x_c, H(x))), \quad \bar{F}_c(x, x_c) \stackrel{\Delta}{=} F_c(x_c, H(x)).$$

The closed-loop (continuous-time) system shapes into

$$\dot{z} = \mathcal{F}(z) \stackrel{\Delta}{=} \begin{bmatrix} \bar{F}(x, x_c) \\ \bar{F}_c(x, x_c) \end{bmatrix}, \quad z(t) \stackrel{\Delta}{=} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}. \tag{13}$$

For static controller (3), $z = x$ and $\mathcal{F}(z) = \bar{F}(x) = F(x, H_c(x))$.

Our first assumption is the local Lipschitz[5] property of the *plant* (1) with respect to the state vector.

Assumption 3.1. A locally bounded function $c_2(x_1, x_2, u)$ exists such that for any $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$|F(x_1, u) - F(x_2, u)| \leq c_1(x_1, x_2, u)|x_1 - x_2|. \tag{14}$$

Unlike the plant, the controller need not satisfy the local Lipschitz property, however, we assume the following weaker property on

[5] A function is locally Lipschitz if it is Lipschitz on any compact set.

the closed-loop system's right-hand side. This property implies, in particular, that $\mathcal{F}(0) = 0$, i.e. $z = 0$ is an equilibrium.

Assumption 3.2. A locally bounded[6] function c_1 exists such that

$$|\mathcal{F}(z)| \le c_2(z)|z| \quad \forall z. \tag{15}$$

Assumption 3.1 is natural when dealing with sampled-time control and guarantees the solution uniqueness between sampling instants. Assumption 3.2 guarantees, in particular, that the only solution starting at $z(0) = 0$ is $z(t) \equiv 0$, which is, obviously, necessary for the local exponential stability (7). Without (15), non-equilibrium solutions can start at equilibria, as exemplified by the Cauchy problem $\dot{z} = z^{1/2}, z(0) = 0$ and its solutions $z_1 \equiv 0, z_2(t) = t^2/4$.

The third assumption is very natural and requires that the boundedness of the closed-loop system's state implies that both control inputs and the observed output are also bounded.

Assumption 3.3. The functions H and H_c from (1) and (2) (or (3)) map bounded sets into bounded sets.

Assumption (3.3) holds e.g. if each of the functions H and H_c is continuous or globally bounded, but none of these conditions is necessary. In general, *none* of Assumptions 3.2-3.3 requires the controller be continuous. The plant right-hand side $F(x, u)$ may also be discontinuous in u.

3.1 Stability with a known Lyapunov function

In this subsection, we address Problem 2 formulated in Subsect. 2.4. Along with Assumptions 3.2-3.3, we suppose that a Lyapunov function exists for the closed-loop systems, which satisfies the conditions of Lemma 2.3 are two additional properties.

Assumption 3.4. For some $R \le \infty$, there exists a C^1-function $V : B_R \to \mathbb{R}$ such that a) condition (9) holds for $a = 2$ and some constants $\alpha_1, \alpha_2, \alpha_3 > 0$ and b) $V'(\cdot)$ is locally Lipschitz on B_R, i.e. a locally bounded function $c_4 : B_R \times B_R \to [0, \infty)$ exists such that

$$|V'(z_1) - V'(z_2)| \le c_4(z_1, z_2)|z_1 - z_2|. \tag{16}$$

Notice that, in view of (9), V attains its global minimum $V(0) = 0$ at the origin. Therefore, $V'(0) = 0$ and thus (16) entails a counterpart of condition (15) for the gradient function

$$|V'(z)| \le \tilde{c}_4(z)|z| \ \ \forall z \in B_R, \quad \tilde{c}_4(z) = c_4(z, 0). \tag{17}$$

Assumption 3.4 holds e.g. when V is C^2-smooth on B_R. In particular, it always holds for quadratic Lyapunov function; for such a function, the global Lipschitz condition (16) holds with $c_4 = const$.

We now formulate our first main result.

THEOREM 3.5. *Assume that the closed-loop system constituted by plant* (1) *and controller* (2) *or* (3) *satisfies Assumptions 3.2-3.3 and admits a Lyapunov function, satisfying Assumption 3.4. For any constant $\sigma \in (0, 1)$ and constants α_i from* (9)*, denote*

$$\gamma_1 = \sqrt{\frac{\alpha_2}{\alpha_1}}, \quad \gamma_2(\sigma) = \frac{\sigma\alpha_3}{2\alpha_2}. \tag{18}$$

The following statements are valid:

(1) if $R < \infty$, then there exist a constant $\tau_* = \tau_*(\sigma)$ such that any emulation with MST less than τ_* provides local exponential stability (7) *with any $r < R\alpha_1/\alpha_2$ and γ_1 and γ_2 from* (18)*;*

(2) statement (1) retains its validity for $R = \infty$, provided that the Lipschitz properties (14)*,* (15) *and* (16) *hold* globally*, that is, the correspondent functions c_i may be replaced by* constants*;*

(3) if $R = \infty$ but one of c_i is not constant, then for any $r > 0$ there exists $\tau_* = \tau_*(\sigma, r)$ such that any emulation with MST less than τ_* ensures the condition (9) *with the ball B_r.*

Notice that in the local stability case, the region of initial conditions is smaller than one for the continuous-time system (10). Explicit estimates of the functions $\tau_*(\sigma)$ and $\tau_*(\sigma, r)$ can be found, being however rather conservative. Comparing (18) to (10), one can see that the sampled-time controller provides the converge rate arbitrarily close to the rate of continuous-time system as $\sigma \to 1$. The price paid for this is the decrease of admissible MST: $\tau_*(\sigma) \xrightarrow[\sigma \to 1]{} 0$.

Notice that the situation of statement (3) corresponds to the *global* exponential stability of the continuous-time system, however, the stability criterion for the sampled-time system appears to be "semi-global" in the sense that the admissible sampling time depends on the region of initial conditions. In practice, this typically does not matter since this region can always be estimated.

Comparison between Theorem 3.5 and existing results. The result of Theorem 3.5, compared to the results existing in the literature, seems quite counter-intuitive. For the case of static controller (3), stability of sampled-time emulation is usually proved under assumptions of smooth [5, 25, 31], globally Lipschitz [17, 19] or at least locally Lipschitz [43] mapping H_c. The weakest requirement seems to be the continuity of the function $F(x, H_c(x_1))$ in the pair (x, x_1) [32], which is also not required by our assumptions. The result of [32], in fact, does not guarantee preservation of exponential stability, but only bounded (practical) and, under additional assumptions, asymptotic stability under sampled-time control. To the best of the author's knowledge, the only result establishing the preservation of exponential stability under discontinuous control was established in [3]; this result assumes that the plant (1) is either a linear system or a system in the Lur'e form [23], whereas the controller is static and obeys the sectorial inequalities.

Even more surprising is the result for the dynamic controller: the only constraint of the dynamic part F_c is given by (15). We thus discard the Lipschitz condition on the controller, being typical for analysis of sample-time emulations [4, 19]. It should be noticed that the results of [4, 19] in fact do not establish exponential stability under sample-time emulation of a dynamic controller, but only Lyapunov [19] or asymptotic [4] stability.

Notice that in the case of discontinuous mapping F_c or H_c the continuous-time feedback (2) or (3) may become *infeasible* in the sense that even local solvability cannot be established. Remarkably, the emulation controller is feasible and solves the problem of local or global exponential stabilization. This important fact can be used for e.g. discretization of discontinuous algorithms arising in sliding mode control [16, 17] and discontinuous control of nonholonomic systems [2], which problems are however beyond the scope of this paper and will be considered in the future works.

Statement (2) for the case of static controller follows, in fact, from the result of [39, Theorem 2]. The results of [39] however cannot be directly applied to dynamic controllers, also, they require the Lyapunov function to be defined globally and radially unbounded, being inapplicable in the case of local exponential stability.

[6]A function is locally bounded if it is bounded on any compact set

In Section 3.3, we illustrate Theorem 3.5 by analyzing the exponential stability of hybrid Lur'e-type systems, arising as superpositions of linear MIMO blocks and sampled-time nonlinear feedback.

3.2 Exponential stability preservation under sample-time emulation

In this subsection, we use the result of Theorem 3.5 and the converse Lyapunov theorem (Lemma 2.4) in order to address Problem 1 formulated in Subsect. 2.4. The assumptions of the converse theorem and the construction of Lyapunov function (12) impose some restrictions on the closed-loop system. We replace Assumption 3.2 by a stronger requirement of smoothness.

Assumption 3.6. The mapping $\mathcal{F}(z)$ is C^2-smooth and $\mathcal{F}(0) = 0$.

Obviously, Assumption 3.3 implies Assumption 3.2 (for which, in fact, C^1-smoothness of $\mathcal{F}$ is sufficient). We are now ready to formulate our second main result.

Theorem 3.7. *Assume that the plant and controller satisfy Assumptions 3.1, 3.3 and 3.6. Then*

(1) if the equilibrium $z = 0$ is locally exponentially stable (7)*, then the emulation with sufficiently small MST also provides local exponential stability;*
(2) if the equilibrium is globally exponentially stable and $\mathcal{F}'$ is bounded, then a function $\tau_* : (0, \infty) \to (0, \infty)$ and two constants $\gamma_1, \gamma_2 > 0$ exist such that emulation with MST $< \tau_*(r)$ provides condition (7) *for the ball B_r.*

Notice that in the case of local stability, the emulation controller *does not* preserve the region of attraction: the radius r for which (7) holds is smaller than for the continuous-time feedback. Similar to Theorem 3.5, the admissible MST can be estimated explicitly, although the estimate is quite conservative. One can also estimate the convergence rate for the emulation controller. Similar to statement (3) in Theorem 3.5, statement (2) of Theorem 3.7 establishes a "semi-global" criterion of exponential stability in the sense that the emulation sampling time depends on the region of initial conditions. In practice, however, this region can always be estimated.

Notice that the closed-loop system may obey Assumption 3.6, whereas the mapping H_c is non-smooth or even discontinuous, as exemplified by the system

$$\dot{x}(t) = u(t)|x(t)|, \quad u(t) = H_c(x(t)), \quad H_c(x) = \begin{cases} -\frac{1}{|x|}x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The closed-loop system is exponentially stable and is C^2-smooth (being even linear) in spite of the controller's discontinuity at the origin, and Theorem 3.7 is applicable to this simple system.

3.3 Example: a hybrid Lur'e system

Following [3], consider the Lur'e system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(\sigma(t_k)) \in \mathbb{R}^n, \\ \sigma(t) &= Cx(t) \in \mathbb{R}^l, \end{aligned} \qquad t \in [t_k, t_{k+1}), \tag{19}$$

being a hybrid counterpart of the continuous-time Lur'e system

$$\dot{x}(t) = Ax(t) + Bu(t), \; \sigma(t) = Cx(t) \tag{20}$$

$$u(t) = \varphi(\sigma(t)). \tag{21}$$

The Lur'e system is a feedback interconnection of the linear MIMO block (20) (A, B, C are constant matrices) and the nonlinear static controller (21). Its hybrid counterpart (19) arises via the sampled-time emulation of the nonlinear feedback.

Assume that the nonlinear mapping $\varphi : \mathbb{R}^l \to \mathbb{R}^m$ (which may be unknown and, in general, discontinuous) satisfies the inequality[7]

$$\mathcal{S}(y, \varphi(y)) = y^\top Q\varphi(y) + y^\top Py + \varphi(y)^\top R\varphi(y) \geq 0 \quad \forall y \in \mathbb{R}^l. \tag{22}$$

Here $Q, P = P^\top, R = R^\top$ are matrices of appropriate dimensions.

For instance, if $m = \dim \varphi(\sigma) = \dim \sigma = l$, the nonlinearity is decoupled and obeys the sector inequalities

$$\begin{aligned} &\varphi(y) = (\varphi_1(y_1), \dots, \varphi_m(y_m))^\top, \\ &-\infty \leq \mu_1 \leq \frac{\varphi_j(\xi)}{\xi} \leq \mu_2 \leq +\infty \quad \forall j = 1, \dots, m, \end{aligned} \tag{23}$$

then the nonlinearity satisfies the quadratic inequality

$$\mathcal{S}(y, \varphi(y)) = -\sum_{j=1}^{m} \tau_j(y_j - \mu_1^{-1}\varphi_j(y_j))(y_j - \mu_2^{-1}\varphi_j(y_j)) \geq 0$$

with arbitrary constants $\tau_j \geq 0$.

The standard criterion of the global exponential stability of the continuous-time system (22) is based on the quadratic Lyapunov function $V(x) = x^*Hx$, where H obeys the system of LMI[8]

$$\begin{bmatrix} HA + A^\top H + C^\top PC & HB + C^\top Q \\ * & R \end{bmatrix} < 0, \quad H > 0. \tag{24}$$

Indeed, if (24) holds, then, choosing $\varepsilon > 0$ sufficiently small and denoting $u(t) \overset{\Delta}{=} \varphi(y(t))$, one has

$$\dot{V} = 2x^*H(Ax + Bu) + \mathcal{S}(y, u) \leq -\varepsilon V.$$

Theorem 3.5, applied to the linear control system (20) and the static controller (21) ($H_c = \varphi$) entails the following corollary.

Corollary 3.8. *Assume that the LMI* (24) *is feasible and the nonlinear mapping obeys the inequality* (22)*. Then, for sufficiently small MST the hybrid Lur'e system* (19) *has a globally exponentially stable equilibrium $x = 0$.*

To prove Corollary 3.8, notice that $R < 0$ in view of (24), and therefore (22) entails that $|\varphi(y)| \leq C|y|$ for some constant $C > 0$. Hence, $x = 0$ is an equilibrium of the system and Assumptions 3.2,3.3 are valid. Assumptions 3.1 and 3.4 are straightforward since $F(x, u)$ and $V'(x)$ are linear mappings. Furthermore, Lipschitz properties (14), (15) and (16) hold globally. Corollary 3.8 follows now from Theorem 3.5.

In the special case of nonlinearity (23), Corollary 3.8 can be derived from the results established in [3]. The extension of the techniques from [3] to the case of general quadratic constraint (22) is, however, not straightforward, whereas Theorem 3.5 does not rely on a special structure of the nonlinearity. At the same time, the results from [3] provide a constructive frequency-domain estimate for the MST, whereas the estimate from Theorem 3.5 (based on the previous results from [39]) is much less explicit and more

[7] The inequality (22) is often referred to as a *quadratic constraint* [14].

[8] Under natural assumptions of observability and controllability of the triple (A, B, C), the solvability of (24) is equivalent to a frequency-domain condition, involving the coefficients P, Q, R and the transfer function of the linear block $W(s) = C(sI-A)^{-1}B$. The corresponding result is known as the Kalman-Yakubovich-Popov lemma [14].

conservative. Corollary 3.8 may be also extended to more general hybrid Lur'e systems examined in [3, 6]

$$\dot{x}(t) = Ax(t) + B_0\varphi_0(\sigma_0(t)) + B\varphi(\sigma(t_k)) \in \mathbb{R}^n,$$
$$\sigma(t) = Cx(t) \in \mathbb{R}^l, \qquad t \in [t_k, t_{k+1})$$
$$\sigma_0(t) = C_0x(t) \in \mathbb{R}^{l_0},$$

arising as a feedback interconnection of a continuous-time Lur'e system and a sampled-time feedback. Another possible extension is the case where coefficients of the linear part are also uncertain, e.g. belong to a polytope of matrices that admits a common quadratic Lyapunov function [31].

Notice that sectorial nonlinearities *need not* be continuous. For this reason, Corollary 3.8 cannot be derived from most of the existing results on stability under aperiodic sampling [4, 18, 31]. For discontinuous nonlinearities, the continuous-time Lur'e system (20),(21) may fail to have a classical solution, at the same time, the hybrid system (19) is always feasible.

4 PROOFS OF THE MAIN RESULTS

4.1 A technical lemma

We start with a technical result similar in spirit to Lemma 2 in [39], however, omitting the restrictive assumption of radially unbounded Lyapunov function and applicable to the local stability analysis.

For the moment, we consider only *static* controller (3) and assume that for this controller all Assumptions 3.1-3.4 hold, in particular, it provides local or global exponential stability. To examine the emulation of this controller (namely, the inter-sampling behavior), introduce an auxiliary system where the control is "frozen"

$$\dot{x}(t) = F(x(t), u_*), \quad x(0) = x_*, \quad u_* = H_c(x_*). \tag{25}$$

In view of Assumption 3.1, system (25) has a unique solution (which, however, is not guaranteed to be forward complete). We denote the corresponding solution by $x(t, x_*)$. Obviously, $x(t, 0) = 0$ due to Assumption 3.2.

In view of Assumption 3.4, for any $x_* \in B_R$ one has

$$V'(x_*)F(x_*, u_*) = V'(x_*)F(x_*, H_c(x_*)) \leq -\varkappa V(x_*), \ \varkappa \overset{\Delta}{=} \frac{\alpha_3}{\alpha_2}. \tag{26}$$

Therefore, for any fixed $\sigma \in (0, 1)$ there exists a maximal time interval $\Delta_\sigma = \Delta_\sigma(x_*)$, on which, first, the solution is well defined and, second the "relaxed form" of inequality (26) holds as follows

$$V'(x(t))F(x(t, x_*), u_*) \leq -\sigma\varkappa V(x_*) \quad \forall t \in \Delta_\sigma. \tag{27}$$

Let $\Delta_\sigma = [0, t_\sigma(x_*))$; it is possible that $t_\sigma(x_*) = \infty$. In particular, we obviously have $t_\sigma(0) = 0$. Henceforth $\sigma \in (0, 1)$ is fixed.

Lemma 4.1. *Let the system* (1),(3) *obeys Assumptions 3.1-3.4, and the latter assumption holds in the finite ball B_R, $R < \infty$. Then, the function $t_\sigma(\cdot)$ is uniformly positive on B_{r_0}, whenever $r_0 < R(\alpha_1/\alpha_2)^{1/2}$.*

$$\inf_{x_* \in B_{r_0}} t_\sigma(x_*) > 0. \tag{28}$$

The latter statement retains its validity for $R = r_0 = \infty$, provided that the functions c_1, c_2, c_4 in Assumptions 3.1-3.4 are all constant.

The proof of Lemma 4.1 retraces the proof of Lemma 2 in [39] with the following important difference. In [39], the boundedness of the solution $x(t, x_*)$ on the interval Δ_σ was established using the radial unboundedness of the function V, which is no longer assumed. However, retracing the argument from the proof of Lemma 2.3, one shows that the solution starting at the *closed* ball $x_* \in \bar{B}_{r_0} = \{x : |x| \leq r_0\}$ cannot leave open ball $B_{r'} \subseteq B_R$, which contains the sublevel set $B(x_*) \overset{\Delta}{=} \{x \in B_R : V(x) \leq V(x_*)\}$ (Fig. 1). The union of latter sets over all $x_* \in \bar{B}_{r_0}$ is compact, which enables to use all constructions from [39]. Notice also that the inequality [39, Equation (33)] also holds in view of (17) and (15). This allows to retrace the proof from [39] without principal changes.

4.2 Proof of Theorem 3.5, static feedback case

The proof in the case of static controller is now straightforward from Lemmas 4.1 and 2.3.

Consider first the local stability case $R < \infty$ and consider a solution of the closed loop system (1),(5), starting at $x(0) = x_0 \in B_r$ (recall that $z = x$ in the case of static controller), where $r < R\alpha_1/\alpha_2$. Since $r_0 = r\sqrt{\alpha_2/\alpha_1} < R\sqrt{\alpha_2/\alpha_1}$, we have $\tau_*(\sigma) \overset{\Delta}{=} \inf_{|x|<r_0} t_\sigma(x) > 0$. We assume that the emulation (5) has MST less than t_σ.

Notice that the ball B_r is *not* a forward invariant set. Nevertheless, we are going to show that the solution $x(t)$ never leaves the larger ball B_{r_0}. Notice first that $V(x_0) \leq \alpha_2|x_0|^2 < r^2\alpha_2$. Since $u(t) = H_c(x_0)$ for $t \in [0, t_1)$, we have $x(t) = x(t, x_0)$ for $t \in [0, t_1)$ (where $x(t, x_*)$ is the solution to (25)). Recalling that $t_1 \leq t_\sigma(x_0)$, one obtains that (27) holds and, in particular, $V(x(t))$ is non-increasing for $t \in [0, t_1)$, so that $V(x(t)) \leq r^2\alpha_2$ for all $t \in [0, t_1]$, entailing that $|x(t)| < r\sqrt{\alpha_2/\alpha_1} \leq r_0$. We have proved that $x(t) \in B_{r_0}$ for $t \in [t_0, t_1]$ and $V(x_1) = V(x(t_1)) \leq r^2\alpha_2$. Retracing this argument and noticing that $x(t) = x(t, x_n)$ for $t \in [t_n, t_{n+1})$, one proves via induction on n that $x_n = x(t_n) \in B_{r_0}$, $t_{n+1} - t_n < t_\sigma(x_n)$ and thus in view of (27) the function $V(x(t))$ is non-increasing on $[t_n, t_{n+1})$, so that $V(x(t)) \leq r^2\alpha_2$ and thus $x(t) \in B_{r_0}$.

Therefore, along any solution starting in B_r one has

$$\dot{V}(x(t)) \leq -\sigma\frac{\alpha_3}{\alpha_2}V(x(t)), \tag{29}$$

which guarantees the exponential stability (7) with γ_i defined in (18). This finishes the proof of statement (1) for a static controller.

In the situation of statement (2), the proof is even simpler since $\tau_*(\sigma) = \inf_{x\in\mathbb{R}^n} t_\sigma(x) > 0$. By construction, if $t_{n+1} - t_n < \tau_*(\sigma)$, every solution of the closed loop system satisfies (29).

Statement (3) is obtained by applying statement (1) to the finite ball B_R, where $R = r(\alpha_2/\alpha_1)$.

4.3 Proof of Theorem 3.5, dynamic feedback

The result for the dynamic controller is proved by the following trick. Consider an augmented plant

$$\dot{x}(t) = F(x, u), \quad \dot{x}_c(t) = v \in \mathbb{R}^{n_c}, \tag{30}$$

where x_c is a new state variable. Consider also a new *static* continuous-time controller as follows

$$u(t) = H_c(x_c, H(x)), \quad v(t) = F_c(x_c, H(x)), \tag{31}$$

which obviously leads to the same closed-loop dynamics (13). It can be easily shown that the closed-loop *sampled-time* system also remains the same, and the new "augmented" plant and new "augmented" controller satisfy all Assumptions 3.1-3.3. The result now follows from the static feedback case.

4.4 Proof of Theorem 3.7

Since the continuous-time closed-loop system satisfies the condition (7) in some ball B_r, the Lyapunov function (12) with properly chosen $\delta = \delta(\gamma_1, \gamma_2)$ satisfies the conditions (9) [20, Theorem 4.14]. (From the proof in [20] it may be noticed that the lower bound α_1 depends on the number $L(r) = \max_{|z|<\gamma_1 r} |\mathcal{F}'(z)|$, for this reason, the boundedness of $\mathcal{F}'$ on $B_{\gamma_1 r}$ is important).

Since $\mathcal{F}$ is C^2-smooth, the solution $z(t|z_0)$ is C^2-smooth in z_0, therefore, the function V defined by (12) is also C^2-smooth on B_r (see e.g. [7, Ch. 1,Sec 7]). In particular, the gradient $V'(z)$ is locally Lipschitz (16). In view of smoothness, the closed-loop system also obeys Assumption 3.2, where $c_2(z) = \max_{\xi \in B_{|z|}} |\mathcal{F}'(\xi)|$.

We thus have a continuous-time closed-loop system obeying Assumption 3.1-3.3 and the Lyapunov function satisfying Assumption 3.4. Fixing an arbitrary $\sigma \in (0, 1)$ statements (1) and (2) are immediate from, respectively, statements (1) and (3) of Theorem 3.5.

CONCLUSIONS

In this paper, we address two problems related to sampled-time emulation of continuous-time exponentially stabilizing controllers. We show, first, that if the continuous-time system admits a Lyapunov function with proper set of properties (which is, in practice, a prerequisite to prove the stability of a nonlinear system), then the emulation with small sampling time provides exponential stability of the closed-loop system. Surprisingly, this result requires very mild assumptions on the controller, which can be even infeasible (with discontinuous right-hand side). Second, using the converse Lyapunov theorem, we establish a qualitative result: if a general nonlinear control system is exponentially stable and has sufficiently smooth right-hand side, then its emulation at a sufficiently high sampling rate is also exponentially stable. In the case of global exponential stability, however, the emulation in general provides "semi-global" stability in the sense that the sampling time depends on the region of initial conditions.

The results of the paper can be applied to discretization of sliding-mode and other nonlinear controllers, which is a subject of ongoing research. Important extension to be presented in the future works are concerned with robustness to disturbances and communication delays in the sampled-time emulation controllers.

Another extension, being a subject of ongoing studies, concerns alternative types of stability. Similar to the previous work [39], we require the closed-loop system to satisfy Assumption 3.2, which excludes the possibility of finite-time and fixed-time stabilization. Sampled-time emulations of finite-time stabilizing controllers thus do not satisfy the conditions of Theorems 3.5 and 3.7, and their analysis remains a non-trivial open problem.

ACKNOWLEDGEMENTS

The paper has been partially supported by Russian Foundation for Basic Research (RFBR), grants 18-38-20037 and 20-01-00619. The author is grateful to anonymous reviewers for their useful suggestions that have enhanced the paper's readability.

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