POLITECNICO DI TORINO Repository ISTITUZIONALE

On the number of distinct exponents in the prime factorization of an integer

Original

On the number of distinct exponents in the prime factorization of an integer / Sanna, Carlo. - In: PROCEEDINGS OF THE INDIAN ACADEMY OF SCIENCES. MATHEMATICAL SCIENCES. - ISSN 0253-4142. - STAMPA. - 130:1(2020). [10.1007/s12044-020-0556-y]

Availability: This version is available at: 11583/2802793 since: 2020-05-03T10:42:27Z

Publisher: Springer

Published DOI:10.1007/s12044-020-0556-y

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)



On the number of distinct exponents in the prime factorization of an integer

CARLO SANNAD

Department of Mathematics, Università degli Studi di Genova, Genoa, Italy E-mail: carlo.sanna.dev@gmail.com

MS received 25 February 2019; revised 5 September 2019; accepted 11 November 2019

Abstract. Let f(n) be the number of distinct exponents in the prime factorization of the natural number n. We prove some results about the distribution of f(n). In particular, for any positive integer k, we obtain that

$$#\{n \le x : f(n) = k\} \sim A_k x$$

and

$$#\{n \le x : f(n) = \omega(n) - k\} \sim \frac{Bx(\log\log x)^k}{k!\log x},$$

as $x \to +\infty$, where $\omega(n)$ is the number of prime factors of *n* and A_k , B > 0 are some explicit constants. The latter asymptotic extends a result of Aktaş and Ram Murty (*Proc. Indian Acad. Sci.* (*Math. Sci.*) **127(3)** (2017) 423–430) about numbers having mutually distinct exponents in their prime factorization.

Keywords. Prime factorization; squarefree numbers; powerful number.

2010 Mathematics Subject Classification. Primary: 11N25; Secondary: 11N37, 11N64.

1. Introduction

Let $n = p_1^{a_1} \cdots p_s^{a_s}$ be the factorization of the natural number n > 1, where $p_1 < \cdots < p_s$ are prime numbers and a_1, \ldots, a_s are positive integers. Several functions of the exponents a_1, \ldots, a_s have been studied, including their product [17], their arithmetic mean [2,4,5,7], and their maximum and minimum [11,13,15,18]. See also [3,8] for a more general function.

Let *f* be the arithmetic function defined by f(1) := 0 and $f(n) := #\{a_1, \ldots, a_s\}$ for all natural numbers n > 1. In other words, f(n) is the number of distinct exponents in the prime factorization of *n*. The first values of f(n) are listed in sequence A071625 of OEIS [16].

Our first contribution is a quite precise result about the distribution of f(n).

Theorem 1.1. There exists a sequence of positive real numbers $(A_k)_{k\geq 1}$ such that, given any arithmetic function ϕ satisfying $|\phi(k)| < a^k$ for some fixed a > 1, we have that the series

$$M_{\phi} := \sum_{k=1}^{\infty} A_k \phi(k) \tag{1}$$

converges and

$$\sum_{n \le x} \phi(f(n)) = M_{\phi}x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),$$

for all $x \ge 1$ and $\varepsilon > 0$.

From Theorem 1.1, it follows immediately that all the moments of f are finite and that f has a limiting distribution. In particular, we highlight the following corollary.

COROLLARY 1.1

For each positive integer k, we have

$$#\{n \le x : f(n) = k\} = A_k x + O_{\varepsilon}(x^{1/2 + \varepsilon}),$$

for all $x \ge 1$ and $\varepsilon > 0$.

We also provide a formula for A_k . Before stating it, we need to introduce some notations. Let ψ be the Dedekind function defined by

$$\psi(n) := n \prod_{p \mid n} \left(1 + \frac{1}{p} \right)$$

for each positive integer *n*, and let $(\rho_k)_{k\geq 1}$ be the family of arithmetic functions supported on squarefree numbers and satisfying

$$\rho_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad \rho_{k+1}(n) = \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{n-1} \sum_{\substack{d \mid n \\ d < n}} \rho_k(d) & \text{if } n > 1, \end{cases}$$

for all squarefree numbers n and positive integers k.

Theorem 1.2. We have

$$A_k = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{\rho_k(n)}{\psi(n)}$$

for each positive integer k.

Clearly, $f(n) \le \omega(n)$ for all positive integers *n*, where $\omega(n)$ denotes the number of prime factors of *n*. Motivated by a question of Recamán Santos [14], Aktaş and Ram Murty

[1] studied the natural numbers *n* such that all the exponents in their prime factorization are distinct, that is, $f(n) = \omega(n)$. They called such numbers *special numbers* (sequence A130091 of OEIS [16]) and they proved the following.

Theorem 1.3. The number of special numbers not exceeding x is

$$\frac{Bx}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

for all $x \ge 2$, where

$$B := \sum_{\ell} \frac{1}{\ell}$$

and the sum of over natural numbers ℓ that are powerful and special.

Let g be the arithmetic function defined by $g(n) := \omega(n) - f(n)$ for all positive integers n. Hence, by the previous observation, g is a nonnegative function and g(n) = 0 if and only if n is a special number. We prove the following result about g, which extends Theorem 1.3 and it is somehow dual to Corollary 1.1.

Theorem 1.4. For each nonnegative integer k, we have

$$#\{n \le x : g(n) = k\} = \frac{Bx(\log\log x)^k}{k!\log x} \left(1 + O_k\left(\frac{1}{\log\log x}\right)\right),$$

for all $x \ge 3$.

Notation. We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbol \ll , with their usual meaning. Any dependence of the implied constants is explicitly stated. We let ε denote an arbitrary small positive real number, not necessarily the same at each occurrence. We reserve the letter p for prime numbers.

2. Preliminaries

Recall that a natural number *n* is called *powerful* if p | n implies $p^2 | n$, for all primes *p*. For all $x \ge 1$, let $\mathcal{P}(x)$ be the set of powerful numbers not exceeding *x*.

Lemma 2.1. We have $\#\mathcal{P}(x) \ll x^{1/2}$ for every $x \ge 1$.

Proof. See [9].

Lemma 2.2. We have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} \ll \frac{1}{y^{1/2}}, \quad \sum_{\ell \in \mathcal{P}(y)} \frac{1}{\ell^{1/2}} \ll \log y,$$

for all $y \ge 2$.

Proof. By Lemma 2.1 and by partial summation, we have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} = \left. \frac{\#\mathcal{P}(t)}{t} \right|_{t=y}^{+\infty} + \int_{y}^{+\infty} \frac{\#\mathcal{P}(t)}{t^2} \, \mathrm{d}t \ll \int_{y}^{+\infty} \frac{\mathrm{d}t}{t^{1+1/2}} \ll \frac{1}{y^{1/2}}.$$

The proof of the second claim is similar.

We need the following upper bound for the number of prime factors of a natural number.

Lemma 2.3. We have

$$\omega(n) \ll \frac{\log n}{\log \log n}$$

for all integers $n \ge 3$.

Proof. See, for example, [6, Proposition 7.10].

For every $x \ge 1$ and every positive integer *h*, let Q(x; h) denote the number of squarefree numbers not exceeding *x* and relatively prime with *h*.

Lemma 2.4. We have

$$Q(x;h) = \frac{6}{\pi^2} \frac{h}{\psi(h)} x + O(4^{\omega(h)}(x^{1/2} + 1))$$

for all $x \ge 1$ and all positive integers h.

Proof. It follows easily from [10, Eq. 8].

For every $x \ge 1$ and every positive integers *s*, *h*, let $Q_s(x; h)$ denote the number of squarefree numbers not exceeding *x*, having exactly *s* prime factors, and relatively prime with *h*.

Lemma 2.5. We have

$$Q_s(x;h) = \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} \left(1 + O_{\delta,s}\left(\frac{\log\log\log(h+15)}{\log\log x}\right)\right)$$

for all $x \ge 3, 0 < \delta < 1$, and for all integers $1 \le h \le x^{\delta}$ and $s \ge 1$.

Proof. For s = 1, the claim follows from the Prime Number theorem, while for h = 1, the claim is a classic result of Landau [12]. Hence, suppose s, h > 1. Also, we can assume that $x \ge 3^{1/(1-\delta)}$. If $n \le x$ is a squarefree number having exactly s prime factors such that (n, h) > 1, then n = pn', where p is a prime number dividing h and $n' \le x/p$ is a squarefree number having exactly s - 1 prime factor. Therefore,

Proc. Indian Acad. Sci. (Math. Sci.) (2020

$$0 \le Q_s(x; 1) - Q_s(x; h) \le \sum_{p \mid h} Q_{s-1}\left(\frac{x}{p}, 1\right) \ll_s \sum_{p \mid h} \frac{x}{p} \frac{(\log \log(x/p))^{s-2}}{\log(x/p)}$$

$$\ll_\delta \frac{x(\log \log x)^{s-2}}{\log x} \sum_{p \mid h} \frac{1}{p} \ll \frac{x(\log \log x)^{s-1}}{\log x} \frac{\log \log \log(h+15)}{\log \log x},$$

where we used the fact that $p \leq x^{\delta}$ and the upper bound

$$\sum_{p \mid h} \frac{1}{p} \le \sum_{p \le \omega(h)} \frac{1}{p} \ll \log \log(\omega(h) + 2) \ll \log \log \log(h + 15),$$

which in turn follows from Mertens' second theorem [6, Theorem 4.5] and the simple bound $\omega(h) \ll \log h$. Consequently,

$$Q_{s}(x;h) = Q_{s}(x;1) + O_{\delta,s}\left(\frac{x(\log\log x)^{s-1}}{\log x}\frac{\log\log\log(h+15)}{\log\log x}\right) \\ = \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} + O_{\delta,s}\left(\frac{x(\log\log x)^{s-1}}{\log x}\frac{\log\log\log(h+15)}{\log\log x}\right),$$

as claimed.

Finally, we need a lemma about certain sums of powers.

Lemma 2.6. *Let* a_0 *be an integer. For all* $x_1, \ldots, x_k > 1$, we have

$$\sum_{a_0 < a_1 < \dots < a_k} \frac{1}{x_1^{a_1} \cdots x_k^{a_k}} = \frac{1}{(x_1 \cdots x_k)^{a_0}} \prod_{j=1}^k \frac{1}{x_j \cdots x_k - 1},$$

where the sum is over all integers a_1, \ldots, a_k satisfying $a_0 < a_1 < \cdots < a_k$.

Proof. We proceed by induction on k. For k = 1, we have

$$\sum_{a_0 < a_1} \frac{1}{x_1^{a_1}} = \frac{1}{x_1^{a_0+1}} \sum_{d=0}^{\infty} \frac{1}{x_1^d} = \frac{1}{x_1^{a_0}} \frac{1}{x_1 - 1},$$
(2)

as claimed. Suppose that the claim is true for k, we shall prove it for k + 1. We have

$$\sum_{a_0 < \dots < a_{k+1}} \frac{1}{x_1^{a_1} \cdots x_{k+1}^{a_{k+1}}} = \sum_{a_0 < \dots < a_k} \frac{1}{x_1^{a_1} \cdots x_k^{a_k}} \sum_{a_k < a_{k+1}} \frac{1}{x_{k+1}^{a_{k+1}}}$$
$$= \sum_{a_0 < \dots < a_{k+1}} \frac{1}{x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} (x_k x_{k+1})^{a_k}} \frac{1}{x_{k+1} - 1}$$

27 Page 6 of 11

$$= \frac{1}{(x_1 \cdots x_{k+1})^{a_0}} \prod_{j=1}^k \frac{1}{x_j \cdots x_{k+1} - 1} \frac{1}{x_{k+1} - 1}$$
$$= \frac{1}{(x_1 \cdots x_{k+1})^{a_0}} \prod_{j=1}^{k+1} \frac{1}{x_j \cdots x_{k+1} - 1},$$

where we used (2), with a_0 and x_1 replaced respectively by a_k and x_{k+1} , and the induction hypothesis.

3. Proof of Theorem 1.1

We begin by proving that for each positive integer k, there exists $A_k > 0$ such that

$$N_k(x) := \#\{n \le x : f(n) = k\} = A_k x + O_{\varepsilon}(x^{1/2 + \varepsilon}),$$
(3)

for all $x \ge 1$ and $\varepsilon > 0$. Clearly, every natural number *n* can be written in a unique way as $n = m\ell$, where *m* is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If m = 1, then $n = \ell$ is powerful and, by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If m > 1, then f(n) = k is equivalent to $f(\ell) = k - 1$. Also, for each ℓ , there are exactly $Q(x/\ell; \ell) - 1$ choices for m > 1. Therefore, we have

$$N_{k}(x) = \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell) = k-1}} \left(Q\left(\frac{x}{\ell}; \ell\right) - 1 \right) + O(x^{1/2}), \tag{4}$$

for all $x \ge 1$. For each positive integer $\ell \le x$, Lemma 2.3 gives $4^{\omega(\ell)} \ll_{\varepsilon} x^{\varepsilon}$. Consequently, by Lemma 2.4, we obtain

$$Q\left(\frac{x}{\ell};\ell\right) = \frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_{\varepsilon}\left(\frac{x^{1/2+\varepsilon}}{\ell^{1/2}}\right),\tag{5}$$

for all positive integers $\ell \leq x$. By Lemma 2.2, we have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\ell} \ll \frac{1}{x^{1/2}},\tag{6}$$

for all $x \ge 1$. In particular, the series

$$A_k := \frac{6}{\pi^2} \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\psi(\ell)}$$

$$\tag{7}$$

converges. Also, again by Lemma 2.2, we have

$$\sum_{\ell \in \mathcal{P}(x)} \frac{1}{\ell^{1/2}} \ll \log x \ll_{\varepsilon} x^{\varepsilon}.$$
(8)

At this point, putting together (4) and (5), and using (6) and (8), we obtain

Proc. Indian Acad. Sci. (Math. Sci.)

$$\begin{split} N_k(x) &= \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell) = k-1}} \left(\frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_{\varepsilon}\left(\frac{x^{1/2+\varepsilon}}{\ell^{1/2}}\right) \right) + O(x^{1/2}) \\ &= A_k x + O\left(\sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{x}{\psi(\ell)}\right) + O_{\varepsilon}\left(\sum_{\substack{\ell \in \mathcal{P}(x)}} \frac{x^{1/2+\varepsilon}}{\ell^{1/2}}\right) + O(x^{1/2}) \\ &= A_k x + O_{\varepsilon}(x^{1/2+\varepsilon}), \end{split}$$

as desired. Thus (3) is proved.

Now we shall show that

$$A_k \le \frac{6}{\pi^2} \frac{1}{(k-1)!} \tag{9}$$

for all positive integers k. For k = 1, the claim is obvious since $A_1 = 6/\pi^2$. Hence, assume $k \ge 2$. If ℓ is a powerful number such that $f(\ell) = k - 1$, then $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$ for some integers $m_1, \ldots, m_{k-1} \ge 2$ and $2 \le a_1 < \cdots < a_{k-1}$. Consequently,

$$\frac{\pi^2}{6}A_k = \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\ell} < \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \sum_{a=j+1}^{\infty} \frac{1}{m^a}$$
$$= \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \frac{1}{m^j(m-1)} \le \prod_{j=1}^{k-1} \frac{1}{j} = \frac{1}{(k-1)!},$$

where we used the facts that

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1$$

and

$$\sum_{m=2}^{\infty} \frac{1}{m^{j}(m-1)} < \frac{1}{2^{j}} + \frac{1}{3^{j} \cdot 2} + \sum_{n=3}^{\infty} \frac{1}{n^{j+1}}$$
$$< \frac{1}{2^{j}} + \frac{1}{3^{j} \cdot 2} + \int_{2}^{+\infty} \frac{dt}{t^{j+1}} = \frac{1}{2^{j}} + \frac{1}{3^{j} \cdot 2} + \frac{1}{j^{2^{j}}} < \frac{1}{j}$$

for all integers $j \ge 2$. Thus (9) is proved.

Now let ϕ be an arithmetic function satisfying $|\phi(k)| < a^k$ for all positive integers k, where a > 1 is some constant. From (9) it follows that series (1) converges. Define

$$y := 2a + \lfloor C \log x / \log \log(x+2) \rfloor,$$

where C > 0 is some absolute constant. Since $f(n) \le \omega(n)$ for all positive integers *n*, by Lemma 2.3, we can choose *C* sufficiently large so that $f(n) \le y$ for all natural numbers $n \le x$. Moreover, from (9) and $y \ge 2a$, we get that

$$\sum_{k>y} A_k \phi(k) \ll \sum_{k>y} \frac{a^k}{(k-1)!} < \frac{a^{y+1}}{y!} \sum_{j=0}^{\infty} \left(\frac{a}{y}\right)^j \ll_a \frac{a^y}{y!} \ll_a \frac{1}{x^{1/2}}$$
(10)

and

$$a^{y}y \ll_{a,\varepsilon} x^{\varepsilon}, \tag{11}$$

for all $x \ge 1$. Therefore, putting together (3), (10) and (11), we have

$$\sum_{n \le x} \phi(f(n)) = \sum_{k \le y} N_k(x)\phi(k) = \sum_{k \le y} (A_k\phi(k)x + O_\varepsilon(\phi(k)x^{1/2+\varepsilon}))$$
$$= M_\phi x + O\left(\sum_{k > y} A_k\phi(k)x\right) + O_\varepsilon(a^y y x^{1/2+\varepsilon})$$
$$= M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),$$

for all $x \ge 1$ and $\varepsilon > 0$. The proof is complete.

4. Proof of Theorem 1.2

Recall that A_k is defined by (7). For k = 1, the claim is obvious, since $f(\ell) = 0$ if and only if $\ell = 1$. Hence, assume $k \ge 2$. If ℓ is a powerful number such that $f(\ell) = k - 1$, then ℓ can be written in a unique way as $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$, where $1 < a_1 < \cdots < a_{k-1}$ are integers and $m_1, \ldots, m_{k-1} > 1$ are pairwise coprime squarefree numbers. Therefore, from (7) and Lemma 2.6, we obtain

$$\frac{\pi^2}{6}A_k = \sum_{m_1,\dots,m_{k-1}}\sum_{\substack{1 < a_1 < \dots < a_{k-1} \\ \psi(m_1^{a_1} \cdots m_{k-1}^{a_{k-1}})}} \frac{1}{\psi(m_1^{a_1} \cdots m_{k-1})}$$
$$= \sum_{m_1,\dots,m_{k-1}}\frac{m_1 \cdots m_{k-1}}{\psi(m_1 \cdots m_{k-1})} \sum_{\substack{1 < a_1 < \dots < a_{k-1} \\ 1 < a_1 < \dots < a_{k-1}}} \frac{1}{m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}}$$
$$= \sum_{m_1,\dots,m_{k-1}}\frac{1}{\psi(m_1 \cdots m_{k-1})} \prod_{j=1}^{k-1}\frac{1}{m_j \cdots m_{k-1} - 1},$$

where, here and in the rest of the proof, in summation subscripts m_1, \ldots, m_{k-1} are meant to be pairwise coprime, squarefree and greater than 1. At this point, it is enough to prove that

$$\sum_{n=m_1\cdots m_{k-1}} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1} = \rho_k(n)$$

for all squarefree numbers n > 1. We proceed by induction on k. For k = 2, the claim is true since

$$\frac{1}{n-1} = \frac{\rho_1(1)}{n-1} = \frac{1}{n-1} \sum_{\substack{d \mid n \\ d < n}} \rho_1(d) = \rho_2(n),$$

for all squarefree numbers n > 1. Assuming that the claim is true for k, we shall prove it for k + 1. We have

$$\sum_{n=m_1\cdots m_k} \prod_{j=1}^k \frac{1}{m_j\cdots m_k - 1} = \frac{1}{n-1} \sum_{m_1|n} \sum_{\substack{m_1|n \ n/m_1 = m_2\cdots m_k \ j=2}} \prod_{j=2}^k \frac{1}{m_j\cdots m_k - 1}$$
$$= \frac{1}{n-1} \sum_{\substack{m_1|n \ d < n}} \rho_k(n/m_1)$$

for all squarefree numbers n > 1, as desired. The proof is complete.

5. Proof of Theorem 1.4

We have to count the number of positive integers $n \le x$ such that g(n) = k. As in the proof of Theorem 1.1, every *n* can be written in a unique way as $n = m\ell$, where *m* is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If m = 1, then $n = \ell$ is powerful and by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If m > 1, then

$$\omega(m) = \omega(n) - \omega(\ell) = g(n) + f(n) - f(\ell) - g(\ell) = k + 1 - g(\ell).$$

In particular, $1 \le \omega(m) \le k+1$. Assume *x* sufficiently large, and put $y := (\log x)^2$. Then, by Lemma 2.2, the number of $n \le x$ such that $\ell > y$ is at most

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{x}{\ell} \ll \frac{x}{y^{1/2}} = \frac{x}{\log x}.$$

Therefore,

$$M_{k}(x) := \#\{n \le x : g(n) = k\} = \sum_{s=1}^{k+1} \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} Q_{s}\left(\frac{x}{\ell};\ell\right) + O\left(\frac{x}{\log x}\right).$$
(12)

For each nonnegative integer r, put

$$B_r := \sum_{\substack{\ell \in \mathcal{P} \\ g(\ell) = r}} \frac{1}{\ell}.$$

Note that, in light of Lemma 2.2, the series defining B_r converges and, more precisely,

$$\sum_{\substack{\ell \in \mathcal{P}(y)\\g(\ell)=r}} \frac{1}{\ell} = B_r + O\left(\frac{1}{y^{1/2}}\right) = B_r + O\left(\frac{1}{\log x}\right).$$
(13)

Clearly, we can assume x sufficiently large so that $x/y \ge 3$ and $y \le x^{\delta/(1+\delta)}$, for some fixed $0 < \delta < 1$. Hence, applying Lemma 2.5, we obtain

$$Q_s\left(\frac{x}{\ell};\ell\right) = \frac{x(\log\log(x/\ell))^{s-1}}{\ell(s-1)!\log(x/\ell)} \left(1 + O_k\left(\frac{\log\log\log(\ell+15)}{\log\log(x/\ell)}\right)\right)$$

Proc. Indian Acad. Sci. (Math. Sci.)

(2020) 130:27

$$= \frac{x(\log\log x)^{s-1}}{\ell(s-1)!\log x} \left(1 + O_k\left(\frac{\log\ell}{\log x}\right)\right) \left(1 + O_k\left(\frac{\log\log\log(\ell+15)}{\log\log x}\right)\right)$$
$$= \frac{x(\log\log x)^{s-1}}{\ell(s-1)!\log x} \left(1 + O_k\left(\frac{\log(\ell+1)}{\log\log x}\right)\right),$$

for all positive integers $s \le k + 1$ and $\ell \le y$. Consequently,

$$\sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} \mathcal{Q}_s\left(\frac{x}{\ell};\ell\right)$$

$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} \frac{1}{\ell} \left(1 + O_k\left(\frac{\log(\ell+1)}{\log\log x}\right)\right)$$

$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} \left(B_{k+1-s} + O\left(\frac{1}{\log x}\right) + O_k\left(\frac{1}{\log\log x}\right)\right)$$

$$= \frac{x(\log\log x)^{s-1}}{(s-1)!\log x} \left(B_{k+1-s} + O_k\left(\frac{1}{\log\log x}\right)\right), \quad (14)$$

where we used (13) and the fact that the series

$$\sum_{\ell \in \mathcal{P}} \frac{\log(\ell+1)}{\ell}$$

converges. Thus, putting together (12) and (14), and noting that $B_0 = B$, we obtain

$$M_k(x) = \frac{Bx(\log\log x)^k}{k!\log x} \left(1 + O_k\left(\frac{1}{\log\log x}\right)\right),$$

as desired. The proof is complete.

Acknowledgements

The author is thankful to the anonymous referee for carefully reading the paper and providing useful suggestions. The author is a member of the INdAM group GNSAGA and, during the preparation of this work, was supported by a postdoctoral fellowship of INdAM.

References

- Aktaş K and Ram Murty M, On the number of special numbers, *Proc. Indian Acad. Sci. (Math. Sci.)* 127(3) (2017) 423–430
- [2] Cao H Z, On the average of exponents, Northeast. Math. J. 10(3) (1994) 291-296
- [3] Cao H Z, Functions involving the number of prime factors of a natural number, Acta Math. Sinica (Chin. Ser.) 39(5) (1996) 602–608
- [4] De Koninck J-M, Sums of quotients of additive functions, Proc. Amer. Math. Soc. 44 (1974) 35–38
- [5] De Koninck J-M and Ivić A, Sums of reciprocals of certain additive functions, *Manuscripta Math.* 30(4) (1979/80) 329–341
- [6] De Koninck J-M and Luca F, Analytic number theory, Graduate Studies in Mathematics, vol. 134, American Mathematical Society, Providence, RI (2012) Exploring the anatomy of integers

- [7] Duncan R L, On the factorization of integers, Proc. Amer. Math. Soc. 25 (1970) 191-192
- [8] Duncan R L, Some applications of the Turán–Kubilius inequality, Proc. Amer. Math. Soc. 30 (1971) 69–72
- [9] Golomb S W, Powerful numbers, Amer. Math. Monthly 77 (1970) 848-855
- [10] Hazlewood D G, On *k*-free integers with small prime factors, *Proc. Amer. Math. Soc.* **52** (1975) 40–44
- [11] Kátai I and Subbarao M V, On the maximal and minimal exponent of the prime power divisors of integers, *Publ. Math. Debrecen* 68(3–4) (2006) 477–488
- [12] Landau E, Sur quelques problèmes relatifs à la distribution des nombres premiers, *Bull. Soc. Math. France* 28 (1900) 25–38
- [13] Niven I, Averages of exponents in factoring integers, Proc. Amer. Math. Soc. 22 (1969) 356-360
- [14] Recamán Santos B, Consecutive numbers with mutually distinct exponents in their canonical prime factorization, http://mathoverflow.net/questions/201489
- [15] Sinha K, Average orders of certain arithmetical functions, J. Ramanujan Math. Soc. 21(3) (2006) 267–277
- [16] Sloane N J A, The On-Line Encyclopedia of Integer Sequences, http://oeis.org
- [17] Suryanarayana D and Sitaramachandra Rao R, The number of square-full divisors of an integer, Proc. Amer. Math. Soc. 34 (1972) 79–80
- [18] Suryanarayana D and Sitaramachandra Rao R, On the maximum and minimum exponents in factoring integers, Arch. Math. (Basel) 28(3) (1977) 261–269

COMMUNICATING EDITOR: Sanoli Gun