Rigidity and gap results for low index properly immersed self-shrinkers in Rm+1

(Article begins on next page)

23 April 2024

RIGIDITY AND GAP RESULTS FOR LOW INDEX PROPERLY IMMERSED SELF–SHRINKERS IN $\mathbb{R}^{m+1}$

DEBORA IMPERA

ABSTRACT. In this paper we show that the only properly immersed self–shrinkers Σ in $\mathbb{R}^{m+1}$ with Morse index 1 are the hyperplanes through the origin. Moreover, we prove that if Σ is not a hyperplane through the origin then the index jumps and it is at least $m+2$, with equality if and only if Σ is a cylinder $\mathbb{R}^{m-k} \times \mathbb{S}^k(\sqrt{k})$ for some $1 \leq k \leq m-1$.

CONTENTS

1. INTRODUCTION

Let Σ^m be a complete connected orientable m–dimensional Riemannian manifold without boundary isometrically immersed by $x_0 : \Sigma^m \to \mathbb{R}^{m+1}$ in the Euclidean space $\mathbb{R}^{m+1}$. We say that Σ is moved along its mean curvature vector if there is a whole family $x_t = x(\cdot, t)$ of smooth immersions, with corresponding hypersurfaces $\Sigma_t = x_t(\Sigma)$, such that it satisfies the mean curvature flow initial value problem

$$(1) \quad \begin{cases} \frac{\partial}{\partial t} x(p,t) = H(p,t)\nu(p,t) & p \in \Sigma^m \\ x(\cdot,\, t_0) = x_0. \end{cases}$$

Here $H(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the unit normal vector of the hypersurface Σ_t at $x(p,t)$. When possible, we will choose the unit normal ν to be inward pointing.

The short time existence and uniqueness of a solution of (1) was investigated in classical works on quasilinear parabolic equations. Another interesting and more challenging question is what happens to these flows in the long term. Classical examples show that singularities can happen. A major problem in literature has

---

been to study the nature of these singularities and it is a general principle, discovered by Huisken, that the singularities are modeled by self–shrinkers. A connected, isometrically immersed hypersurface $x : \Sigma^m \to \mathbb{R}^{m+1}$ is said to be a *self–shrinker* (based at $0 \in \mathbb{R}^{m+1}$) if the family of surfaces $\Sigma_t = \sqrt{-2t}\Sigma$ flows by mean curvature. Equivalently, a self–shrinker can also be characterized as an isometrically immersed hypersurface whose mean curvature vector field $\mathbf{H}$ satisfies the equation

$$x^\perp = -\mathbf{H},$$

where $(\cdot)^\perp$ denotes the projection on the normal bundle of Σ. Note that we are using the convention $H = \mathrm{Tr}_\Sigma A$, where A denotes the second fundamental form of the immersion defined as

$$AX = -\overline{\nabla}_X \nu,$$

with $\overline{\nabla}$ Levi–Civita connection of $\mathbb{R}^{m+1}$. With this convention, the self–shrinker equation takes the scalar form

$$\langle x, \nu \rangle = -H.$$

Standard examples of self–shrinkers are the hyperplanes through the origin of $\mathbb{R}^{m+1}$, the sphere $\mathbb{S}^m(\sqrt{m})$ and the cylinders $\Sigma = \mathbb{R}^{m-k} \times \mathbb{S}^k(\sqrt{k})$ for some $1 \le k \le m - 1$. Other examples of self–shrinkers are due to Angenent, [2], who constructed a family of embedded self–shrinkers that are topologically $\mathbb{S}^1 \times \mathbb{S}^{m-1}$.

It is well–known that self–shrinkers in $\mathbb{R}^{m+1}$ can be viewed as f–minimal hypersurfaces, that is, critical points of the weighted area functional

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f} d\mathrm{vol}_\Sigma,$$

where $f = |x|^2/2$ (we refer the reader to the papers [3], [9], [14] for more details on f–minimal hypersurfaces). Moreover, we say that a self–shrinker is f–stable if it is a local minimum of the weighted area functional for every compactly supported normal variation. In the instability case, it makes sense to investigate the Morse index, that is, roughly speaking, the maximum dimension of the linear space of compactly supported deformations that decrease the weighted area up to second order.

It was proved by Colding and Minicozzi, [5], that every complete properly immersed self–shrinker is necessarily f–unstable. Equivalently, every properly immersed self–shrinker has Morse index greater than or equal to 1. In the equality case, rigidity results have been proved by Hussey, [13], under the additional assumption of embeddedness. More precisely, he showed that if a complete properly embedded self–shrinker in $\mathbb{R}^{m+1}$ has Morse index 1, then it has to be a hyperplane through the origin. Furthermore, he also proved that if the self-shrinker is not a hyperplane through the origin, then the Morse index jumps and it has to be at least $m + 2$, with equality if and only if the self–shrinker is a cylinder $\mathbb{R}^{m-k} \times \mathbb{S}^k(\sqrt{k})$ for some $1 \le k \le m$. The embeddedness assumption is used repeatedly in the proof of this result to ensure that, besides self–shrinker cylinders, there aren't any others self–shrinkers which are mean convex or, when $m = 2$, that split off a line (this a consequences of Theorems 0.17 and Corollary 10.46 in [5]). In this regard, it is worth

to point out that there are many examples of non-embedded self–shrinkers. Indeed, Abresch and Langer, [1], constructed self–intersecting curves in the plane that are self–shrinkers under the mean curvature flow. Moreover, it is possible to obtain examples of non-embedded self-shrinkers by taking products of these curves with Euclidean factors. Finally Drugan and Kleene, [7], recently constructed in generic dimension infinitely many immersed rotationally symmetric self-shrinkers having the topological type of the sphere (S^n), the plane $(\mathbb{R}^n)$, the cylinder $(\mathbb{R} \times \mathbb{S}^{n-1})$, and the torus $(\mathbb{S}^1 \times \mathbb{S}^{n-1})$.

The aim of this paper is to investigate if hyperplanes through the origin and cylinders remains the only hypersurfaces, among the wider family of properly immersed self–shrinkers in $\mathbb{R}^{m+1}$, having Morse index 1 and $m + 2$ respectively, and if, except for them, every properly immersed self–shrinker has Morse index strictly bigger than $m + 2$. Exploiting the link between stability properties of self–shrinkers and spectral properties of a suitable weighted Schrödinger operator, as well as some basic identities which naturally involve the weighted Laplacian of the self–shrinker, we provide a positive answer to the above mentioned problem. More precisely, we prove the following

Theorem 1.1. *Let Σ^m be a complete properly immersed self–shrinker in $\mathbb{R}^{m+1}$. Then*

(1) $\mathrm{Ind}_f(\Sigma) \geq 1$ *and equality holds if and only if Σ is a hyperplane through the origin;*

(2) *If Σ is non–totally geodesic, then $\mathrm{Ind}_f(\Sigma) \geq m + 2$. Moreover, $\mathrm{Ind}_f(\Sigma) = m + 2$ if and only if $\Sigma = \mathbb{R}^{m-k} \times \mathbb{S}^k(\sqrt{k})$ for some $1 \leq k \leq m - 1$.*

2. Some spectral theory and potential theory on weighted manifolds

2.1. **Some spectral theory for weighted Schrödinger operators.** A weighted manifold is a triple $\Sigma_f^m = (\Sigma^m, \langle\,,\,\rangle, e^{-f} d\mathrm{vol}_\Sigma)$, where $(\Sigma^m, \langle\,,\,\rangle)$ is a complete m–dimensional Riemannian manifold, $f \in C^\infty(\Sigma)$ and $d\mathrm{vol}_\Sigma$ denotes the canonical Riemannian volume form on Σ. In the following we collect some well-know facts about spectral theory on weighted manifolds (see e.g. [17, Chapter 3] for an exhaustive survey on spectral theory on Riemannian manifolds).

Associated to a weighted manifold Σ_f there is a natural divergence form second order diffusion operator, the f–Laplacian, defined on u by

$$\Delta_f u = e^f \operatorname{div}\left(e^{-f} \nabla u\right) = \Delta u - \langle \nabla u, \nabla f \rangle.$$

This is clearly symmetric on $L^2(\Sigma_f)$ endowed with the inner product

$$(u, v)_{L^2(\Sigma_f)} = \int_\Sigma uv e^{-f} d\mathrm{vol}_\Sigma.$$

Given $q \in L^\infty_{loc}(\Sigma)$, consider the *weighted Schrödinger operator*

$$Lu = -\Delta_f u - qu, \quad \forall u \in C^\infty_c(\Sigma).$$

This is again a symmetric linear operator on $L^2(\Sigma_f)$ and we set Q to be the symmetric bilinear form on $L^2(\Sigma_f)$ defined as $Q(u,v) := (Lu,v)_{L^2(\Sigma_f)}$. Recall that L is said to be *bounded from below* if

$$Q(u,u) \geq c\|u\|^2_{L^2(\Sigma_f)}, \quad c \in \mathbb{R}.$$

In particular, when $c \geq 0$, L is said to be *non–negative*.

Given any open, relatively compact domain $\Omega \subset \Sigma$ we define $L_{|\Omega}$ to be the operator L acting on $C_c^\infty(\Omega)$ and we denote by L_Ω its Friedrichs extension. By standard spectral theory, L_Ω has purely discrete spectrum consisting of a divergent sequence of eigenvalues $\{\lambda_k(L_\Omega)\}$. The first eigenvalue of L_Ω is defined by Rayleigh characterization as

$$\lambda_1(L_\Omega) = \inf_{0 \neq u \in C_c^\infty(\Omega)} \frac{Q(u,u)}{\|u\|^2_{L^2(\Sigma_f)}}.$$

Moreover, we define the index of L_Ω, $\mathrm{Ind}(L_\Omega)$, to be the number, counted according to multiplicity, of negative eigenvalues of L_Ω.

The *bottom of the spectrum of L* on Σ is then defined as

$$\lambda_1^L(\Sigma) = \inf\{\lambda_1(L_\Omega) \; : \; \Omega \subset\subset \Sigma\}.$$

Similarly, the *Morse index of L* on Σ is defined as

$$\mathrm{Ind}^L(\Sigma) := \sup\{\mathrm{Ind}(L_\Omega) \; : \; \Omega \subset\subset \Sigma\}.$$

Adapting to the weighted setting arguments in [10] it is not difficult to prove the following

Proposition 2.1. *Let Σ_f be a weighted manifold and let $L = -\Delta_f - q$, $q \in L_{loc}^\infty(\Sigma)$. The following are equivalent:*

(1) $\mathrm{Ind}^L(\Sigma) < +\infty$;

(2) *There exists a finite dimensional subspace W of the weighted space $L^2(\Sigma_f)$ having an orthonormal basis $\psi_1, \cdots, \psi_k$ consisting of eigenfunctions of L with eigenvalues $\lambda_1, \cdots, \lambda_k$ respectively. Moreover, each λ_i is negative and any function $\phi \in C_c^\infty(\Sigma) \cap W^\perp$ satisfy $Q(\phi,\phi) \geq 0$.*

Finally, in case L is essentially self–adjoint, we can relate the Morse index to the so–called spectral index of L. Towards this aim we first recall that, given a self–adjoint operator $T : \mathcal{D}(T) \to L^2(\Sigma_f)$, its spectral index is defined as

$$\widetilde{\mathrm{Ind}}^T(\Sigma) = \sup\left\{\dim V \; : \; V \subset \mathcal{D}(T), \; (Tu,u)_{L^2(\Sigma_f)} < 0 \; \forall \; 0 \neq u \in V\right\}.$$

If L is essentially self–adjoint, then there is a unique self–adjoint extension L_Σ of L and we can define the *spectral index of L* as the spectral index of its self–adjoint extension, that is

$$\widetilde{\mathrm{Ind}}^L(\Sigma) := \widetilde{\mathrm{Ind}}^{L_\Sigma}(\Sigma).$$

Furthermore, since Σ is complete, if L is also bounded from below by a constant c, then it is essentially self–adjoint. In this case L_Σ corresponds to the Friedrichs extension of L, that is, the self–adjoint extension of L associated to the closure of

the quadratic form Q with respect to the norm $\|\cdot\|_Q$ induced by the inner product $Q(\ ,\ ) + (1-c)(\ ,\ )_{L^2(\Sigma_f)}$. Moreover, it turns out that

$$\overline{C_c^\infty(\Sigma)}^{\|\cdot\|_Q} = \{u \in W^{1,2}(\Sigma_f) : |q|^{1/2}u \in L^2(\Sigma_f)\} =: \mathcal{V}_q,$$

and hence the domain of the operator L_Σ is the space

$$\mathcal{D}(L_\Sigma) = \{u \in \mathcal{V}_q\ :\ L_\Sigma u \in L^2(\Sigma_f)\},$$

where $L_\Sigma u$ is understood in distributional sense.

The relationship between the two concepts of index presented above is clarified by the following

Theorem 2.2. *Let Σ_f be a weighted manifold and let $L = -\Delta_f - q$, $q \in L_{loc}^\infty(\Sigma)$.*

(i) *If L is essentially self–adjoint on $C_c^\infty(\Sigma)$, then $\widetilde{\mathrm{Ind}}^L(\Sigma) = \mathrm{Ind}^L(\Sigma)$;*

(ii) *if $\mathrm{Ind}^L(\Sigma) < +\infty$, then L is bounded from below, essentially self–adjoint on $C_c^\infty(\Sigma)$ and $\mathrm{Ind}^L(\Sigma) = \widetilde{\mathrm{Ind}}^L(\Sigma) < +\infty$.*

The proof of the previous theorem is a consequence of Theorem 3.17 in [17] taking into account that $L = -\Delta_f - q(x)$ is unitarily equivalent to the Schrödinger operator

$$S = -\Delta - [(1/4\langle\nabla f, \nabla f\rangle - 1/2\Delta f) - q(x)] = -\Delta - (p(x) + q(x))$$

under the multiplication map $T(u) = e^{-f/2}u$ of $L^2(\Sigma)$ onto $L^2(\Sigma_f)$ (see for instance [22]).

2.2. f–parabolicity of weighted manifolds.

Following classical terminology in linear potential theory we say that a weighted manifold Σ_f is *f–parabolic* if

$$\begin{cases} \Delta_f u \geq 0 \\ u^* = \sup_M u < +\infty \end{cases} \quad \Rightarrow \quad u \equiv u^*.$$

As a matter of fact, f–parabolicity is related to a wide class of equivalent properties involving the recurrence of the Brownian motion, f–capacities of condensers, the heat kernel associated to the drifted laplacian, weighted volume growth, function theoretic tests, global divergence theorems and many other geometric and potential-analytic properties. Here we limit ourselves to point out the following characterization.

Theorem 2.3. *A weighted manifold Σ_f is f–parabolic if and only if for every vector field X satisfying*

(i) *$|X| \in L^2(\Sigma_f)$,*

(ii) *$(\mathrm{div}_f X)_- \in L_{loc}^1(\Sigma_f)$*

it holds

$$\int_\Sigma \mathrm{div}_f(X)e^{-f}d\mathrm{vol}_\Sigma = 0,$$

where

$$\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f}X).$$

We refer the reader to [17, Theorem 7.27] for a detailed proof of the previous result in the unweighted setting. Although the proof of this theorem can be deduced adapting to the weighted laplacian Δ_f the proof in [17], we provide here a shorter and more direct proof.

Proof of Theorem 2.3. We consider the warped product $\overline{\Sigma} = \Sigma \times_h \mathbb{T}$, where $h := e^{-f}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, so that $\mathrm{vol}(\mathbb{T}) = 1$. We first note that, as proven in [21, Lemma 2.6], the weighted manifold Σ_f is f–parabolic if and only if $\overline{\Sigma}$ is parabolic. Moreover, as a consequence of the Kelvin–Nevanlinna–Royden criterion (see [17, Theorem 7.27]), the parabolicity of $\overline{\Sigma}$ is equivalent to the fact that, for every vector field $Y \in T\overline{\Sigma}$ satisfying

(a) $|Y| \in L^2(\overline{\Sigma})$;
(b) $(\overline{\mathrm{div}})_-(Y) \in L^1_{loc}(\overline{\Sigma})$

it holds

$$\int_{\overline{\Sigma}} \overline{\mathrm{div}}(Y) d\mathrm{vol}_{\overline{\Sigma}} = 0.$$

Here we have denoted by $\overline{\mathrm{div}}$ the divergence with respect to the metric $g_{\overline{\Sigma}} = \pi_\Sigma^*(g_\Sigma) + h^2 \pi_T^*(dt^2)$, where π_Σ and $\pi_{\mathbb{T}}$ denote the projections of $\overline{\Sigma}$ onto Σ and $\mathbb{T}$ respectively. Given a vector field $Y \in T\overline{\Sigma}$, we let $X = (\pi_\Sigma)_*(Y)$. Taking into account that $d\mathrm{vol}_{\overline{\Sigma}} = e^{-f} d\mathrm{vol}_\Sigma dt$ and using the formulas for covariant derivatives on warped products (see [16]), it is not difficult to prove that

$$\mathrm{div}_f(X) = \overline{\mathrm{div}}(Y),$$

$$\int_\Sigma |X|^2 e^{-f} d\mathrm{vol}_\Sigma \leq \int_{\overline{\Sigma}} |Y|^2 d\mathrm{vol}_{\overline{\Sigma}}$$

$$\int_\Sigma (\mathrm{div}_f)_-(X) e^{-f} d\mathrm{vol}_\Sigma = \int_{\overline{\Sigma}} (\overline{\mathrm{div}})_-(Y) d\mathrm{vol}_{\overline{\Sigma}}.$$

Now assume that Σ is f–parabolic and let $X \in T\Sigma$ be a vector field satisfying the integrability conditions (i) and (ii). Define $Y \in T\overline{\Sigma}$ by $Y_{(x,t)} = X_x$. Then $(\pi_\Sigma)_* Y = X$, $|X|^2 = |Y|^2$ and Y satisfies the integrability conditions (a) and (b). Moreover, since $\overline{\Sigma}$ is parabolic, it holds

$$0 = \int_{\overline{\Sigma}} \overline{\mathrm{div}}(Y) d\mathrm{vol}_{\overline{\Sigma}} = \int_\Sigma \mathrm{div}_f(X) e^{-f} d\mathrm{vol}_\Sigma,$$

proving that f–parabolicity is a sufficient condition for the validity of the global (weighted) version of the Stokes theorem.

As for the converse, assume that for every vector field $X \in T\Sigma$ satisfying the integrability conditions (i) and (ii) it holds

$$\int_\Sigma \mathrm{div}_f(X) e^{-f} d\mathrm{vol}_\Sigma = 0.$$

Suppose by contradiction that Σ is not f–parabolic. Thus $\overline{\Sigma}$ is not parabolic and there exists $Y \in T\overline{\Sigma}$ satisfying the integrability conditions in (a) and (b) and such

that

$$\int_{\overline{\Sigma}} \overline{\mathrm{div}}(Y) d\mathrm{vol}_{\overline{\Sigma}} \neq 0.$$

Set $X = (\pi_\Sigma)_*(Y)$. Then X satisfies the integrability conditions (i) and (ii). However

$$\int_{\Sigma} \mathrm{div}_f(X) e^{-f} d\mathrm{vol}_\Sigma = \int_{\overline{\Sigma}} \overline{\mathrm{div}}(Y) d\mathrm{vol}_{\overline{\Sigma}} \neq 0,$$

leading to a contradiction. □

From the geometric point of view, it is well known that f–parabolicity is related to the growth rate of the weighted volume of intrinsic metric objects. Indeed, adapting to the diffusion operator Δ_f standard proofs for the Laplace–Beltrami operator (see for instance [11], [19]), one can prove the following

Proposition 2.4. *Let Σ_f be a weighted manifold. If*

(2) $$\mathrm{vol}_f \left(\partial B_r(o)\right)^{-1} \notin L^1\left(+\infty\right),$$

then Σ_f is f–parabolic.

Here $\partial B_r(o)$ denotes the geodesic ball of radius r centered at a reference point $o \in \Sigma$ and

$$\mathrm{vol}_f \left(\partial B_r\right) = \int_{\partial B_r} e^{-f} d\mathrm{vol}_{m-1},$$

where $d\mathrm{vol}_{m-1}$ denotes the $(m-1)$–dimensional Hausdorff measure.

Observe also that if $\mathrm{vol}_f(\Sigma) < +\infty$, then condition (2) is automatically satisfied. Hence we conclude that any weighted manifold Σ_f satisfying $\mathrm{vol}_f(\Sigma) < +\infty$ is f–parabolic.

Remark 2.5. It was proved in [4] and [6] that for any complete immersed self-shrinker $x : \Sigma^m \to \mathbb{R}^{m+1}$ the following statements are equivalent:

(a) the immersion x is proper;
(b) Σ has extrinsic polynomial volume growth;
(c) Σ has extrinsic Euclidean volume growth;
(d) Σ has finite f–volume, where $f = |x|^2/2$.

In particular, the equivalence (a)–(d) shows that any complete self–shrinker properly immersed in $\mathbb{R}^{m+1}$ is f–parabolic, with $f = |x|^2/2$.

3. CHARACTERIZATION OF LOW INDEX PROPERLY IMMERSED SELF–SHRINKERS

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a complete self-shrinker and set $f = \frac{|x|^2}{2}$. The function f induces a weighted structure on the self–shrinker that can hence be viewed as a weighted manifold itself. Basic geometric quantities on the self-shrinker satisfy identities which naturally involve the f-Laplacian on Σ_f, as shown in the next

Proposition 3.1. *Let Σ^m be a self–shrinker in $\mathbb{R}^{m+1}$ and let $a \in \mathbb{R}^{m+1}$ be a constant vector. Then*

(1) *the mean curvature H satisfies*
$$\Delta_f H = (1 - \|A\|^2) H.$$

(2) *The function $g_a = \langle \nu, a \rangle$ satisfies*
$$\Delta_f g_a = -\|A\|^2 g_a.$$

(3) *The function $l_a = \langle x, a \rangle$ satisfies*
$$\Delta_f l_a = -l_a.$$

(4) *the squared norm of the second fundamental form satisfies the Simons type formula*
$$\Delta_f \|A\|^2 = 2|\nabla A|^2 + 2\|A\|^2 (1 - \|A\|^2).$$

We refer the reader to [12, Theorem 4.1] and [5, Theorem 5.2] for a proof of the identities listed above.

It turns out that stability properties of self–shrinkers, viewed as critical points of the weighted area functional, are taken into account by spectral properties of the *weighted Jacobi operator L_f*, defined as
$$L_f u = -\Delta_f u - (\|A\|^2 + 1)u.$$

To be more precise, we say that a self–shrinker Σ is *f–stable* if and only if the operator L_f is non–negative, that is if and only if
$$\lambda_1^{L_f}(\Sigma) \geq 0.$$

Furthermore, we define the f–Index of Σ to be the Morse index of the Jacobi operator L_f, that is
$$\operatorname{Ind}_f(\Sigma) := \sup\{\operatorname{Ind}((L_f)_\Omega) \ : \ \Omega \subset\subset \Sigma\}.$$

Remark 3.2. Keeping in mind Theorem 2.2 we see that if $\operatorname{Ind}_f(\Sigma) < +\infty$, then the weighted Jacobi operator L_f is bounded from below, it is essentially self–adjoint and $\operatorname{Ind}_f(\Sigma) = \widetilde{\operatorname{Ind}}^{L_f}(\Sigma)$. Moreover, the domain of the quadratic form $Q(\cdot, \cdot) = (L_f \cdot, \cdot)_{L^2(\Sigma_f)}$ is the space
$$\mathcal{V} = \{u \in W^{1,2}(\Sigma_f) \ : \ \|A\|u \in L^2(\Sigma_f)\}.$$

Furthermore, we point out that, if u is an eigenfunction of the Jacobi operator L_f and $u \in W^{1,2}(\Sigma_f)$, then Lemma 9.15 in [5] implies that $u \in \mathcal{V}$. Finally, if $\operatorname{Ind}_f(\Sigma) < +\infty$, then $\lambda_1^{L_f}(\Sigma) > -\infty$ and one can prove the existence of a positive C^2 function v satisfying $L_f v = \lambda_1^{L_f}(\Sigma)v$ (see Lemma 9.25 in [5]). Then, applying again Lemma 9.15 in [5], it is straightforward to prove that any function $\phi \in W^{1,2}(\Sigma_f)$ belongs to $\mathcal{V}$.

In the following we collect some lemmas that will be essential for the proof of the main result of the paper.

Lemma 3.3. *Let Σ^m be a properly immersed non–totally geodesic self–shrinker in $\mathbb{R}^{m+1}$ satisfying $\mathrm{Ind}_f(\Sigma) < +\infty$. Set*

$$W := \{g_b : \Sigma \to \mathbb{R}, \ g_b(p) = \langle \nu(p), b \rangle \ \forall p \in \Sigma, \ b \in \mathbb{R}^{m+1}\}.$$

Then $\dim W \leq m + 1$ and, if $\dim W = k < m + 1$, we can find $m + 1$ linearly independent constant non–null vectors $\{b_1, \ldots, b_{m+1}\} \in \mathbb{R}^{m+1}$ satisfying the following properties:

(1) $W = \mathrm{span}\{g_{b_1}, \ldots, g_{b_k}\}$;
(2) *Set $U := \mathrm{span}\{g_{b_1}, \ldots, g_{b_k}, l_{b_{k+1}}H, \ldots, l_{b_{m+1}}H\}$. Then*
    (a) *the functions $l_{b_j}H$, $j = k + 1, \ldots, m + 1$, are eigenfunctions of L_f corresponding to the eigenvalue -1;*
    (b) $\dim U = m + 1$;
    (c) $U \subset \mathcal{V}$.
    Moreover, having set

$$V := \{\varphi : \Sigma \to \mathbb{R}, \ \varphi(p) = a + \phi(p) \ \forall p \in \Sigma, \ a \in \mathbb{R}, \ \phi \in U\} \subset \mathcal{V},$$

one has $\dim V = m + 2$.

Proof. We note first that it is not difficult to prove that if $\dim W = k < m+1$, then we can find $m+1$ linearly independent constant non–null vectors $\{b_1, \ldots, b_{m+1}\} \in \mathbb{R}^{m+1}$ such that $W = \mathrm{span}\{g_{b_1}, \ldots, g_{b_k}\}$ and $g_{b_j} \equiv 0$ for any $j = k + 1, \cdots, m + 1$. This proves part (1). As for part (2a), it suffices to show that the functions $l_{b_j}H$, $j = k + 1, \cdots, m + 1$, satisfy $L_f l_{b_j}H = -l_{b_j}H$. Note that $g_{b_j} \equiv 0$, a simple computation shows that the functions l_{b_j} satisfy

$$\nabla l_{b_j} = b_j^T = b_j, \quad Ab_j \equiv 0.$$

Thus

$$\begin{aligned}
\Delta_f H l_{b_j} &= H \Delta_f l_{b_j} + 2\langle \nabla H, \nabla l_{b_j} \rangle + l_{b_j} \Delta_f H \\
&= -H l_{b_j} - 2\langle A X^T, b \rangle + H l_{b_j}(1 - \|A\|^2) \\
&= -\|A\|^2 H l_{b_j},
\end{aligned}$$

showing that the functions $l_{b_j}H$, $j = k + 1, \cdots, m + 1$, are eigenfunctions of L_f corresponding to the eigenvalue -1.

As for part (2b), let us prove first that the functions $g_{b_1}, \ldots, g_{b_k}, l_{b_{k+1}}H, \ldots, l_{b_{m+1}}H$ are linearly independent. It suffices to show that, for any non-null vectors $b = \alpha_1 b_1 + \cdots + \alpha_k b_k$, $c = \alpha_{k+1}b_{k+1} + \ldots + \alpha_{m+1}b_{m+1}$, the functions g_b and $l_c H$ can not be linearly dependent. Indeed, assume by contradiction that there exists a non–zero constant λ such that $g_b = \lambda l_c H$. Then, in particular

$$-Ab^T = \nabla g_b = \lambda l_c \nabla H + \lambda H \nabla l_c = -\lambda l_c Ax^T + \lambda H c.$$

In particular,

$$0 = -\langle Ab^T, c \rangle = -\langle \lambda l_c A x^T, c \rangle + \lambda H |c|^2 = \lambda H |c|^2.$$

Hence $H \equiv 0$ and Σ must be a hyperplane through the origin, which is absurd.

In order to prove that $U \subset \mathcal{V}$ it suffices to show, taking into account Remark 3.2, that g_{b_i}, $l_{b_j} H \in W^{1,2}(\Sigma_f)$, for any $i = 1, \cdots, k$, $j = k+1, \cdots, m+1$. Clearly, since Σ has finite f-volume, the Cauchy-Schwartz inequality implies that the functions g_{b_i} belong to $L^2(\Sigma_f)$. Moreover, since Σ has finite f–index, Theorem 2 in [14] implies that $\|A\| \in L^2(\Sigma_f)$ and hence, in particular, $|\nabla g_{b_i}| \leq |b_i| \|A\| \in L^2(\Sigma_f)$. As for the functions $l_{b_j} H$, note that Lemma 25 in [18] implies that any properly immersed self-shrinker satisfies $\mathcal{P}(|x|) \in L^1(\Sigma_f)$, with $\mathcal{P}(t)$ any polynomial in t. In particular, the function $|x|^{2q}$ belongs to the space $W^{1,2}(\Sigma_f)$ for any $q \in \mathbb{N}$ and hence, taking into account Remark 3.2, $|x|^{2q} \|A\| \in L^2(\Sigma_f)$. Thus, using the self-shrinker equation, it is straightforward to see that

$$|l_{b_j}| |H| \leq |b_j| |x| |H| \leq |b_j| |x|^2 \in L^2(\Sigma_f)$$
$$|\nabla(l_{b_j} H)| \leq |b_j| |x|^2 \|A\| + |b_j| |x| \in L^2(\Sigma_f).$$

Finally, it only remains to prove that $\dim V = m + 2$. Assume by contradiction that there exists a non-zero constant a and a function $\phi \in U$ such that $\phi = a$. Then

$$0 = \Delta_f \phi = -\|A\|^2 \phi = -\|A\|^2 a,$$

contradicting again the assumption of Σ being non–totally geodesic. $\square$

Lemma 3.4. *Let Σ^m be a properly immersed self–shrinker in $\mathbb{R}^{m+1}$ satisfying $\mathrm{Ind}_f(\Sigma) < +\infty$. Let $\phi \in U$, where U is defined as in the previous Lemma. Then*

$$\int_\Sigma \phi \|A\|^2 e^{-f} d\mathrm{vol}_\Sigma = 0.$$

Proof. Note that the assumption of Σ being properly immersed implies that it is f–parabolic. Let $\phi \in U$. Then $|\nabla \phi| \in L^2(\Sigma_f)$ and the conclusion follows as an application of Theorem 2.3 to the vector field $X = \nabla \phi$, keeping in mind Proposition 3.1 and Lemma 3.3. $\square$

We are now ready to prove the main theorem of this paper.

Proof of Theorem 1.1. Note first that if $\mathrm{Ind}_f(\Sigma) = +\infty$, then the inequality $\mathrm{Ind}_f(\Sigma) \geq m + 2$ is trivially satisfied. Hence assume that $\mathrm{Ind}_f(\Sigma) < +\infty$. We claim that any function $\varphi = a + \phi$, for some $a \in \mathbb{R}$, $\phi \in U$, satisfies

$$\int_\Sigma \varphi L_f \varphi e^{-f} d\mathrm{vol}_\Sigma = - \int_\Sigma \varphi \Delta_f \varphi + (\|A\|^2 + 1)\varphi^2 e^{-f} d\mathrm{vol}_\Sigma < 0.$$

In this case, Lemma 3.3 would imply that either Σ is totally geodesic (and hence a hyperplane through the origin) or $\mathrm{Ind}_f(\Sigma) \geq \dim V = m + 2$.

A straightforward computation shows that

$$\varphi \Delta_f \varphi + (\|A\|^2 + 1)\varphi^2 = \varphi^2 + \|A\|^2 a^2 + a\phi \|A\|.$$

Applying Lemma 3.4 we get

$$\int_\Sigma \phi \|A\|^2 e^{-f} d\mathrm{vol}_\Sigma = 0.$$

Hence

$$- \int_{\Sigma} \varphi \Delta_f \varphi + (\|A\|^2 + 1)\varphi^2 e^{-f} d\mathrm{vol}_{\Sigma} = - \int_{\Sigma} \varphi^2 + \|A\|^2 a^2 + a\phi\|A\| e^{-f} d\mathrm{vol}_{\Sigma}$$

$$= - \int_{\Sigma} \varphi^2 + \|A\|^2 a^2 e^{-f} d\mathrm{vol}_{\Sigma}$$

$$< 0.$$

Finally, assume that $\mathrm{Ind}_f(\Sigma) = m + 2$. Note that, since H belongs to $\mathcal{V}$ (see the proof of Theorem 9.36 in [5]) and $L_f H = -2H$, we get that H does not change sign. To see this, assume by contradiction that H changes sign. Then, by Theorem 9.36 in [5] it follows that the least negative eigenvalue $\lambda_1 := \lambda_1^{L_f}(\Sigma)$ of L_f is strictly less than -2. Let g be an $L^2(\Sigma_f)$ eigenfunction relative to $-\lambda_1$. Since $\mathrm{Ind}_f(\Sigma) = m + 2$ it follows by Proposition 3.1 that $g = aH + \phi$ for some $0 \neq a \in \mathbb{R}$ and for some $\phi \in U$. Then $g \in \mathcal{V}$ and, as a consequence of Lemma 9.25 in [5], g does not change sign. Moreover,

$$-(\lambda_1 + 1 + \|A\|^2)g = \Delta_f g = a\Delta_f H + \Delta_f \phi = -\|A\|^2 g + aH.$$

In particular, $aH = -(\lambda_1 + 1)g$, contradicting the fact that H changes sign.

Since H does not change sign and $\|A\| \in L^2(\Sigma_f)$, we can apply Theorem 8 in [20] with the choices $u = \|A\|$, $v = H$, $a = \|A\|^2 - 1$ to conclude that $\|A\| = CH$ for some $0 \neq C \in \mathbb{R}$ and that $|\nabla A| = |\nabla\|A\||$. These are the key geometric identities to prove our assertion. Indeed, reasoning as in the proof of Theorem 10.1 in [5], we can proceed as follows.

Let $p \in \Sigma$, $\{e_i\}_{i=1}^m$ an orthonormal frame in $T_p\Sigma$ that diagonalizes A_p, then

$$\langle Ae_i, e_j \rangle = \lambda_i \delta_{ij}.$$

We hence have that

(i) For every k, there exists α_k such that $\nabla_{e_k} \langle Ae_i, e_i \rangle = \alpha_k \lambda_i$, $i = 1 \ldots, m$.

(ii) If $i \neq j$ then $\nabla_{e_k} \langle Ae_i, e_j \rangle = 0$.

Since ∇A is fully symmetric, by Codazzi equations, (ii) implies

(ĩi) $\nabla_{e_k} \langle Ae_i, e_j \rangle = 0$ unless $i = j = k$.

If $\lambda_i \neq 0$ and $j \neq i$, then $0 = \nabla_{e_j} \langle Ae_i, e_i \rangle = \alpha_j \lambda_i$, so that $\alpha_j = 0$.

In particular, if $\mathrm{rk}(A_p) \geq 2$, then $\alpha_j = 0$ for every j, and thus, by (i), $(\nabla A)_p = 0$.

We now consider two cases, depending on the rank of A.

Case 1. *There exists $p \in \Sigma$ such that $\mathrm{rk}(A_p) \geq 2$.* Reasoning as in [5] we deduce that $\mathrm{rk}(A)$ is at least two everywhere and hence $\nabla A \equiv 0$ on Σ. According to a theorem of Lawson, [15], we then obtain that Σ is isometric to a cylinder $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{m-k}$, for some $1 \leq k \leq m$.

Case 2. $\mathrm{rk}(A_p) = 1$. Since $|H| > 0$, the remaining case to consider is when the rank of A is exactly one at every point. Assume without loss of generality that $H > 0$. In this case, reasoning as in Case 2 of the proof of Theorem 10.1 in [5], we are able to conclude that Σ is invariant under the isometric translations in the $(m-1)$–dimensional subspace spanned by a global orthonormal frame for $\mathrm{Ker}(A)$. Therefore Σ is a product of a curve $\Gamma \subset \mathbb{R}^2$ and this $(m-1)$–dimensional

subspace. The curve Γ has to be a smooth complete self-shrinking curve in $\mathbb{R}^2$ with $H > 0$ and finite f-length. In particular, as a consequence of Lemma 10.39 in [5], Γ is bounded and hence, since it is complete, it is a closed self-shrinking curve with $H > 0$. Theorem A in [1] implies then that either Γ is a unit circle or it is a member of the Abresch-Langer family $\{\Gamma_{p,n}\}$ described as follows. If p, n are positive integers satisfying $1/2 < p/n < \sqrt{2}/2$, there is, up to congruence, a unique curve $\Gamma_{p,n}$ having rotation index p and with $2n$ vertices (i.e. critical points of the curvature H). Proposition 2.1 in [8] implies then that either Γ is a unit circle or $n \geq 3$ and the f-stability operator on Γ has at least 4 negative eigenvalues (counted with multiplicity). However this latter possibility cannot occur since, otherwise, we would have that $\mathrm{Ind}_f(\Sigma) = m - 1 + \mathrm{Ind}_f(\Gamma) > m + 2$, contradicting our assumption. Hence Γ must be a unit circle and this concludes the proof.

$\square$

REFERENCES

1. U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986), no. 2, 175–196.

2. S. B. Angenent, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), Progr. Nonlinear Differential Equations Appl., vol. 7, Birkhäuser Boston, Boston, MA, 1992, pp. 21–38.

3. X. Cheng, T. Mejia, and D. Zhou, *Stability and compactness for complete f-minimal surfaces*, Trans. Amer. Math. Soc. **367** (2015), no. 6, 4041–4059.

4. X. Cheng and D. Zhou, *Volume estimates about shrinkers*, Proc. Amer. Math. Soc. **141** (2013), no. 2, 687–696.

5. T. H. Colding and W. P. Minicozzi, *Generic mean curvature flow I; generic singularities*, Ann. of Math. **2** (2012), no. 175, 755–833.

6. Q. Ding and Y. L. Xin, *Volume growth, eigenvalue and compactness for self-shrinkers*, Asian J. Math. **17** (2013), no. 3, 443–456.

7. G. Drugan and S. J. Kleene, *Immersed self-shrinkers*, Preliminary version on arXiv:1306.2383.

8. C. L. Epstein and M. I. Weinstein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. **40** (1987), no. 1, 119–139.

9. J. M. Espinar, *Manifolds with density, applications and gradient Schrödinger operators*, arXiv:1209.6162v6.

10. D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985), no. 1, 121–132.

11. A. A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 135–249.

12. G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299.

13. C. Hussey, *Classification and analysis of low index Mean Curvature Flow self-shrinkers*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–The Johns Hopkins University.

14. D. Impera and M. Rimoldi, *Stability properties and topology at infinity of f-minimal hypersurfaces*, Geom. Dedicata **178** (2015), 21–47.

15. H. B. Lawson, *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 187–197.
16. B. O'Neill, *Semi-Riemannian geometry. with applications to relativity*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983.
17. S. Pigola, M. Rigoli, and A. G. Setti, *Vanishing and finiteness results in geometric analysis. a generalization of the bochner technique*, Progress in Mathematics, vol. 266, Birkhäuser Verlag, Basel, 2008.
18. Stefano Pigola and Michele Rimoldi, *Complete self-shrinkers confined into some regions of the space*, Ann. Global Anal. Geom. **45** (2014), no. 1, 47–65.
19. M. Rigoli and A. G. Setti, *Liouville type theorems for φ-subharmonic functions*, Rev. Mat. Iberoamericana **17** (2001), no. 3, 471–520.
20. M. Rimoldi, *On a classification theorem for self-shrinkers*, Proc. Amer. Math. Soc. **142** (2014), no. 10, 3605–3613.
21. M. Rimoldi and G. Veronelli, *Topology of steady and expanding gradient Ricci solitons via f-harmonic maps*, Differential Geom. Appl. **31** (2013), no. 5, 623–638.
22. A. G. Setti, *Eigenvalue estimates for the weighted Laplacian on a Riemannian manifold*, Rend. Sem. Mat. Univ. Padova **100** (1998), 27–55.

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, via Cozzi 53, I-20125 Milano, ITALY

E-mail address: debora.impera@gmail.com