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RECONSTRUCTION OF INDEPENDENT SUB-DOMAINS FOR A CLASS OF HAMILTON–JACOBI EQUATIONS AND APPLICATION TO PARALLEL COMPUTING

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Abstract. A previous knowledge of the domains of dependence of a Hamilton–Jacobi equation can be useful in its study and approximation. Information of this nature is, in general, difficult to obtain directly from the data of the problem. In this paper we formally introduce the concept of an *independent sub-domain*, discuss its main properties and provide a constructive implicit representation formula. Through these results, we propose an algorithm for the approximation of these sets that is shown to be relevant in the numerical resolution, *via* parallel computing.

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1. Introduction

A classical, powerful approach to optimal control problems consists of solving a verification partial differential equation of Hamilton–Jacobi type, obtained using the Bellman's Dynamic Programming principle. One remarkable advantage of this approach, compared to the study of optimality conditions, is the ability to provide global minima and closed-loop optimal controls. On the other hand, the study and the approximation of the value function associated to the problem is an unavoidable, and often difficult, technical step. An exceptional achievement was made with the introduction of viscosity solutions, a weak notion of solution proposed by Crandall, Evans and Lions in the 80s, and the subsequent refinements (for a whole presentation of this subject refer to the monographs [3, 5]).

In this paper, we consider a related problem: the detection of a collection of sub-domains where the value function is independent from its value in any other sub-domain. This knowledge is useful for several reasons: it is related to stabilization problems, reconstruction of reachability sets and development of parallel algorithms for fast numerical resolution. The latter point is of special interest. In fact it is well-known that the greatest limitation to the use of the Bellman approach in optimal control comes from the difficulty to solve the

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associated Hamilton–Jacobi equation, even numerically, in a high dimensional context (the so-called *curse of dimensionality*).

A classical strategy to speed up the computation is decomposing the domain into a collection of subsets with the aim to lower the number of nodes to compute in every sub-problem. Then, the problem is solved in parallel on every sub-domain with special conditions imposed in the interface between two different subsets. Unfortunately, a technique of this kind typically requires an iterative process between the parallel resolution on every sub-domain and the interface between them, with a consequent growth of the total complexity. For a whole dissertation on the subject of domain decomposition techniques (DD) we refer to the monograph [22], for Hamilton–Jacobi equations [8,21] and for a parallel version of the Fast Sweeping Methods [11,24]. An alternative approach was proposed in [25], where the authors can handle a decomposition of the domain passing to a quasi variational inequality formulation which is shown to be equivalent to the original problem.

A new direction of research was opened by Ancona and Bressan, in [1]. There, they introduced the original concept of *patchy feedback* with the intention of studying an asymptotic stability problem. Navasca and Krener, in [19], used these ideas to develop a technique of reconstruction for the feedback solution in some special polynomial cases (*patchy solutions*). Those elements in turn inspired the work of Cacace *et al.* [7]. For a special class of Hamilton–Jacobi equations, they propose a preliminary procedure called *patchy decomposition*. This preliminary computation supplies a partition of the domain into sets which can be computed separately, without any exchange of information between the interfaces. The result is achieved using a multi-grid idea. They pre-compute a solution of the problem on a coarse grid, then they obtain an optimal control in a feedback map form using the *synthesis procedure* (*cf.* [3], pp. 10−13). This map is used to detect a decomposition of the domain in accordance with the geometry of the problem. It is shown in some interesting examples, that the error added in this procedure is sufficiently small. Our paper can be considered a development of this idea.

We introduce in a rigorous way the concept of *independent sub-domain* (a different concept from *patchy subset*) and obtain a procedure for its numerical reconstruction without the delicate step on the feedback control. We use the technique developed to propose a parallel algorithm for the approximation of the solution, proving a convergence result and some error estimates for it. With this framework, we enlarge significantly, with respect to the tests performed in [7], the class of equations where the technique is appropriate.

The paper is organized as follows: in Section 2 we introduce the minimum property which is useful for the decomposition and the concept of *independent sub-domains*. In Section 3 we propose an algorithm for their location. The main result of this section is the necessary condition contained in Proposition 3.3 which characterizes the points of the grid belonging to a certain independent domain. Finally in Section 4 we describe a parallel algorithm for the approximation of the solution. The main benefit of our proposal is claimed in Proposition 4.1, where the convergence of the technique and a bound for the error are proved. Through some test examples, we exhibit the benefits of our approach.

2. Formulation of the problem and decomposition property

Let us first of all introduce the classical framework of an *exit problem*. We refer to the general structure of a *differential game*. A generic *optimal control problem* can be viewed as sub-case.

We consider the following *controlled dynamical system*

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & \text{a.e.} \\ y(0) = x, \end{cases} \tag{2.1}$$

where $x \in \Omega$ is an open subset of $\mathbb{R}^n$, $a \in \mathcal{A} := \{a : \mathbb{R}^+ \to A, \textit{measurable}\}$, and $b \in \mathcal{B} := \{b : \mathbb{R}^+ \to B, \textit{measurable}\}$ with A, B compact sets of $\mathbb{R}^m$.

Other data of the problem are the functions *running cost* l and *exit cost* g. Typical hypothesis for the well-posedness of the problem are:

$$\left.\begin{array}{ll} f : (\Omega, A, B) \to \mathbb{R}, & \text{continuous function,} \\ & \text{Lipschitz continuous in the first variable} \\ l : (\Omega, A, B) \to (\rho, +\infty], & \text{is a strictly positive continuous function,} \\ & \text{Lipschitz continuous in the first variable,} \\ g : \bar{\Omega} \to \mathbb{R} & \text{is a continuous function.} \end{array}\right\} \tag{H0}$$

The regularity assumptions on f are enough to guarantee the existence of a solution of the dynamical system $y_x(t, a(t), b(t))$ which is called *trajectory*. The goal is to find the optimum (a $\sup - \inf$ optimum) over $\mathcal{A}$, $\mathcal{B}$ of the functional

$$J_x(a,b) := \int_0^{\tau_x(a,b)} l(y_x(s, a(s), b(s)), a(s), b(s)) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \tau_x(a,b)} g(y_x(\tau_x(a,b))), \quad \lambda \geq 0,$$

where τ is the *time of the first exit from the set* Ω defined as

$$\tau_x(a,b) := \min \{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}.$$

Using the Elliot–Kalton's notion [13] of *non anticipating strategies*, we define the value function of this problem as

$$v(x) := \sup_{\varphi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \varphi(a)), \tag{2.2}$$

where

$$\Phi := \{\varphi : \mathcal{A} \to \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \leq t \text{ implies } \varphi[a](s) = \varphi[\tilde{a}](s) \text{ for all } s \leq t\}.$$

For a simpler presentation, we assume that the *Isaacs' conditions* hold *i.e.* for all $x \in \Omega$ and for all $p \in \mathbb{R}^n$,

$$\min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x,a,b) \cdot p - l(x,a,b)\} = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \{-f(x,a,b) \cdot p - l(x,a,b)\},$$

then the value function of the problem exists, is unique and coincides with v. It is well-known that such function is a *viscosity solution* of the problem

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0, & x \in \Omega \\ v(x) = g(x), & x \in \Gamma \end{cases} \tag{2.3}$$

where the *Hamiltonian* is defined as

$$H(x,p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x,a,b) \cdot p - l(x,a,b)\},$$

and the set Γ is a subset of $\partial\Omega$. To avoid a large number of technicalities and focus on our purposes, we state as hypothesis:

$$\text{the problem (2.3) has an unique Lipschitz continuous viscosity solution } v(x). \tag{H1}$$

This assumption will be essential in the following; conditions to ensure such regularity of the solution have been largely discussed in literature (just to cite some monographs [3,5,9]).

A key property of the value function, that we will use in the following, is the possibility to solve a collection of Hamilton–Jacobi equations obtaining the original solution as the point-wise minimum of such a family. This property was discussed with some differences and with another purpose in the works [17,18].

Consider a decomposition of the set Γ as a union of a collection of subsets, *i.e.* $\Gamma := \bigcup_{i\in\mathcal{I}} \Gamma_i$, with $\mathcal{I} := \{1,\dots,m\} \subset \mathbb{N}$. We call $v_i : \bar{\Omega} \to \mathbb{R}$ a Lipschitz continuous viscosity solution of the problem

$$\begin{cases} \lambda v_i(x) + H\left(x, Dv_i(x)\right) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases} \tag{2.4}$$

where $g_i : \Gamma \to \mathbb{R}$ is a regular function such that

$$\begin{array}{l} g_i(x) = g(x), \text{ if } x \in \Gamma_i, \\ g_i(x) > g(x), \text{ otherwise.} \end{array} \tag{2.5}$$

Also in this case, we assume the existence of a Lipschitz continuous solution of every equation (2.4).

The limiting superdifferential $\partial^L v(x)$ of the continuous function $v(\cdot)$ at x is defined as:

$$\partial^L v(x) := \{p \,|\, \exists \text{ sequences } \; p_i \to p \text{ and } x_i \to x \text{ s.t. } p_i \in D^+ v(x_i) \text{ for each } i\},$$

where $D^+ v(x)$ is the usual Fréchet superdifferential.

The *active indices set* is stated as

$$I(x) \;=\; \{j \in \{1,\dots,m\} \,|\, v_j(x) = \min_{i\in\mathcal{I}} v_i(x)\}, \quad \text{for each } x \in \Omega.$$

We are now ready to state the decomposition result:

Theorem 2.1. *Let* (H0)−(H1) *and the Isaacs' conditions be verified. Define the set* $\Upsilon \subset \Omega$ *as* $\Upsilon := \{x \in \Omega |\, \mathrm{Card}(I(x)) > 1\}$ (*where* $\mathrm{Card}(A)$ *is the cardinality of the set* A) *and the function* $\bar{v} : \Omega \to \mathbb{R}$ *as*

$$\overline{v}(x) := \min_{i\in\mathcal{I}} v_i(x).$$

Under the hypothesis

$$\lambda \bar{v}(x) + H\left(x, \sum_{i\in I(x)} \alpha_i p_i\right) \leq 0, \tag{H2}$$

where $p_i \in \partial^L v_i(x)$ *for each* $i \in I(x)$, $x \in \Upsilon$, *and any convex combination* $\{\alpha_i \,|\, i \in I(x)\}$, *we have that* $\bar{v}$ *is the unique viscosity solution of the problem* (2.3).

Proof. We know that $\overline{v}$ always verifies the boundary conditions from the definition of value function and (2.5). If we show that $\overline{v}$ is both subsolution and supersolution in Ω of the problem (2.3), the thesis follows by the uniqueness of the solution.

We omit the proof that $\bar{v}(x)$ is a supersolution since it is classical in literature. The property of subsolution is less trivial. If $x \notin \Upsilon$, *i.e.* $I(x)$ contains a single index value j, the property is directly verified; so $x \in \Upsilon$. Now, $v_j(\cdot)$ is Lipschitz continuous on a neighborhood of x for each $j \in I(x)$. Since $p \in D^+\bar{v}(x)$, it is certainly the case that $p \in \partial^L \bar{v}(x)$. Using the property that $\bar{v}(x')$ coincides with $\max\{v_j(x') \,|\, j \in I(x')\}$ for x' in some neighborhood of x, we deduce from the max rule for limiting subdifferentials of Lipschitz continuous functions (see, *e.g.* [23]) applied to $-\bar{v}(\cdot)$ the following representation for p:

$$p = \sum_{j\in I(x)} \alpha_j p_j \,,$$

for some convex combination $\{\alpha_j \,|\, j \in I(x)\}$ and vectors $p_j \in \partial^L v_j(x)$, $j \in I(x)$. But then, by (H2),

$$\lambda \bar{v}(x) + H(x,p) = \lambda \bar{v}(x) + H\left(x, \sum_{j\in I(x)} \alpha_j p_j\right) \leq 0\,.$$

This shows that $\overline{u}(x)$ is a subsolution and concludes the proof. □

Remark 2.2. It is quite direct to show that the request (H2) is always verified with the presence of a convex Hamiltonian. As consequence any optimal control problem is included in our framework (in an optimal control problem the associated Hamiltonian is always convex). To pass to this special case, it is sufficient to restrict the set B to a singleton.

Let us now define the concept of independent sub-domains.

Definition 2.3. A subset $\Sigma \subseteq \overline{\Omega}$ is an *independent sub-domain* of the problem (2.2) if, given a point $x \in \Sigma$ and an optimal control $(\overline{a}(t), \bar{\varphi}(\overline{a}(t))$ (*i.e.* $J_x(\overline{a}, \bar{\varphi}(\bar{a})) \leq J_x(a, \bar{\varphi}(a))$ for every choice of $a \in \mathcal{A}$, and $J_x(\overline{a}, \bar{\varphi}(\bar{a})) \geq J_x(\bar{a}, \varphi(\bar{a}))$ for any $\varphi \in \Phi$), the trajectory $y_x(\bar{a}(t), \bar{\varphi}(\bar{a}(t))) \in \Sigma$ for $t \in [0, \tau_x(\bar{a}, \bar{\varphi}(\bar{a}))]$.

It is possible to establish a link between the decomposition result and the concept of independent sub-domain. In particular we show that Theorem 2.1 provides a constructive way to build a independent sub-domains decomposition of Ω.

Proposition 2.4. *Let (H0), (H1), (H2) and the Isaacs' conditions be verified. Given a collection of $n-1$ dimensional subsets $\{\Gamma_i\}_{i=1,\dots,m}$ such that $\Gamma = \cup_{i=1}^m \Gamma_i$, the sets defined as*

$$\Sigma_i := \left\{ x \in \overline{\Omega} \mid v_i(x) = v(x) \right\}, \quad i = 1, \dots, m, \tag{2.6}$$

where v_i and v are defined accordingly to Theorem 2.1, are independent sub-domains of the problem (2.2).

Proof. We argue by contradiction using the Dynamic Programming Principle (*cf.* [3]). For a fixed i consider a point $x \in \Sigma_i$. Let us then assume that the trajectory, for an optimal control of the original problem $(\bar{a}, \bar{\varphi}(\bar{a}))$ exits from Σ_i *i.e.* $y_x(\overline{a}(\overline{t}), \bar{\varphi}(\overline{a})) = \overline{x} \notin \Sigma_i$ for a certain $\overline{t} \in [0, \tau_x(\overline{a}(\overline{t}), \bar{\varphi}(\overline{a}))]$. If $\overline{t} = 0$ contradiction comes directly from the definition of Σ_i. If $\overline{t} > 0$ we recall (*Dynamic Programming Principle*)

$$v(x) = \sup_{\varphi \in \Phi} \inf_{a \in \mathcal{A}} \left\{ \int_0^{\overline{t}} l(y_x(a(s), \varphi(a(s))), a(s), \varphi(a(s))) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \overline{t}} v\left(y_x(a(\overline{t}), \varphi(a(\bar{t})))\right) \right\},$$

an analogue formula is obviously valid also for $v_i(x)$. Recalling $v_i(\overline{x}) > v(\overline{x})$,

$$\begin{aligned}
v_i(x) &= \sup_{\varphi \in \Phi} \inf_{a \in \mathcal{A}} \left\{ \int_0^{\overline{t}} l(y_x(a(s), \varphi(a(s))), a(s), \varphi(a(s))) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \overline{t}} v_i\left(y_x(a(\overline{t}), \varphi(a(\bar{t})))\right) \right\} \\
&> \inf_{a \in \mathcal{A}} \left\{ \int_0^{\overline{t}} l(y_x(a(s), \bar{\varphi}(a(s))), a(s), \bar{\varphi}(a(s))) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \overline{t}} v_i\left(y_x(a(s), \bar{\varphi}(a(s)))\right) \right\} \\
&= \int_0^{\overline{t}} l(y_x(\bar{a}(s), \bar{\varphi}(\bar{a}(s))), \bar{a}(s), \bar{\varphi}(\bar{a}(s))) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \overline{t}} v_i\left(\bar{x}\right) \\
&\geq \left\{ \int_0^{\overline{t}} l(y_x(\bar{a}(s), \bar{\varphi}(\bar{a}(s))), \bar{a}(s)) \mathrm{e}^{-\lambda s} \mathrm{d}s + \mathrm{e}^{-\lambda \overline{t}} v(\bar{x}) \right\} = v(x),
\end{aligned}$$

then $v_i(x) > v(x)$, which contradicts again the definition (2.6). □

That property of the trajectories plays an important role in the following; it guarantees the absence of crossing information through the boundary of every independent sub-domain, or using different words, the solution of the problem (2.3) in each sub-domain does not depend on the solution in other sub-domains.

A feature easy to derive from Proposition 2.4 is the connexion of the sets:

Corollary 2.5. *Let (H0), (H1) and (H2) and the Isaacs' conditions be verified. If Γ_i is connected, the respective set Σ_i defined in (2.6) is also connected.*

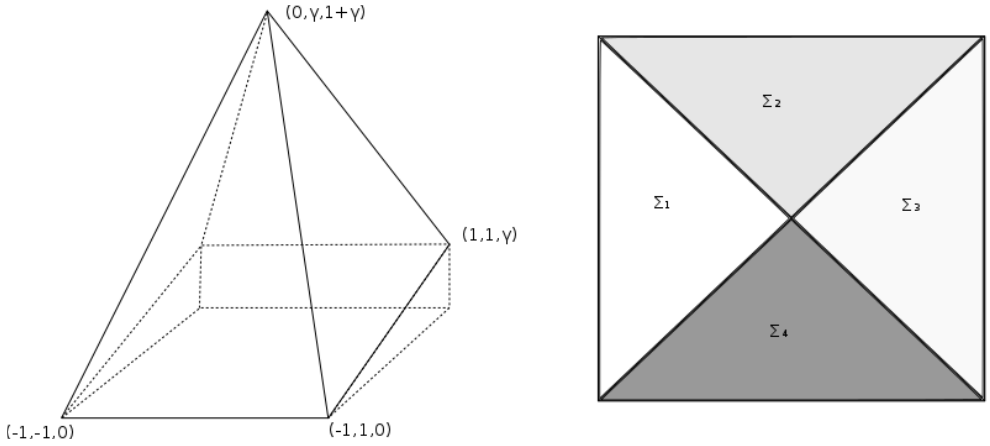

FIGURE 1. Example 2.6, the auxiliary solution u_1 and the independent sub-domains decomposition.

Proof. If Γ_i is connected we can always join two points x, y of the set using the respective optimal trajectories. By Proposition 2.4 such trajectories are contained in the set. □

Let us to give a simple example of an independent sub-domains decomposition:

Example 2.6. The equation considered is, with $\Omega := (-1,1) \times (-1,1)$,

$$\begin{cases} \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1 & x \in \Omega \\ v(x) = 0 & x_1 \in \partial\Omega. \end{cases}$$

Stated $\partial\Omega = \Gamma := \cup_i \Gamma_i = \{\pm 1\} \times (-1,+1) \cup (-1,+1) \times \{\pm 1\}$, we associate to $\Gamma_1 := (-1,-1) \times \{-1\}$ the function $g_1 : \partial\Omega \to \mathbb{R}$ defined as

$$\begin{cases} g_1(x) := 0 & x \in \Gamma_1 \\ g_1(x) := \gamma(1+x_2) & x \in \Gamma \setminus \Gamma_1, \end{cases}$$

for a chosen $\gamma \in \mathbb{R}^+$. The other g_is are defined in a symmetric manner. It is possible to verify that the unique viscosity solution of such a problem is

$$v_1(x) = (1+\gamma) - \max(|x_1 - \gamma|, |x_2|).$$

Finally the original value function $v(x) = 1 - \max(|x_1|, |x_2|)$ is recovered as $v(x) = \min_{i=1,\dots,4} v_i(x)$. The decomposition in independent sub-domains obtained are shown in Figure 1.

3. Independent sub-domains reconstruction

In this section we introduce a numerical technique for the approximation of the independent sub-domains, based on the results of the previous section. The technique is not related to a special numerical scheme, but it needs an *a priori* bound for the approximation; this is necessary to guarantee that the approximation includes the desired set. As example of numerical scheme we consider a semi-lagrangian solver, but the procedure can be easily adapted to other schemes *e.g.* finite differences, finite volumes. For further details about semi-lagrangian techniques we refer to the monograph by Falcone and Ferretti [16].

Let us consider a grid on Ω made by a family of simplices S_j, such that $\overline{\Omega} \in \cup_j S_j$. We denote by x_i, $i = 1, \dots, N$ the nodes of the triangulation,

$$\Delta x := \max_j \operatorname{diam}(S_j) \tag{3.1}$$

the size of the mesh ($\mathrm{diam}(S)$ is the diameter of the set S). Let be G the set of the internal nodes of the grid and $\widehat{G}$ is the set of its boundary points; in the case of a bounded Ω we call also Φ the nodes corresponding to the set $\mathbb{R}^n \setminus \overline{\Omega}$, those nodes typically act as *ghost nodes*. We remark that this discretization space includes the classical case of regular meshes.

We map all the values at the nodes in $V = (V(1), \ldots, V(N))$. By a standard semi-lagrangian discretization [2, 16] of (2.3), it is possible to obtain the following scheme in fixed point form

$$V = T(V), \tag{3.2}$$

where $T : \mathbb{R}^N \to \mathbb{R}^N$ is defined component-wise by

$$[T(V)]_i = \begin{cases} \max\limits_{b\in B} \min\limits_{a\in A} \left\{ \frac{1}{1+\lambda h} \mathbb{I}[V](x_i - hf(x_i, a, b)) - hl(x_i, a, b) \right\} & x_i \in G, \\ g(x_i) & x_i \in \widehat{G}, \\ +\infty & x_i \in \Phi. \end{cases} \tag{3.3}$$

The discrete value function V is extended on the whole space Ω by a linear n-dimensional interpolation, represented by the operator $\mathbb{I}$, as described in [4, 15].

The parameter h is the time step for the forward euler approximation of the controlled ordinary differential equations (2.1). The minimum over A and the maximum over B are evaluated by direct comparison using a discrete approximation of the control space A, B. Generally the fixed point of the equation (3.2) is found through the iterative map $V^{n+1} := T(V^n)$ which is shown to be a contraction.

It is important to recall the following result of convergence for the semi-lagrangian scheme. The proof can be found in [4, 20] for the case of differential games and in [12, 14] for optimal control problems.

Theorem 3.1. *Let v and V be the solutions of, respectively, equations (2.3) and (3.2). Assume verified (H0) and (H1) then*

$$||v - V||_\infty \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx, $q \in \mathbb{R}^+$ depending on the regularity of the problem.

For differential games with a Lipschitz continuous solution, a possible estimate is

$$\|v - V\|_\infty \leq \bar{C} h^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h} \right)^2 \right). \tag{3.4}$$

If the quantity $\frac{\Delta x}{h} = 1$, we have the relation described in Theorem 3.1 with $q = 1/2$. The constant $\bar{C}$ depends on the data of the problem and can be estimated.

In the case of an optimal control problem with $\lambda > 0$, a possible convergence bound is the following

$$\|v - V\|_\infty \leq 2(M_v + M_{v_h}) h^{\frac{1}{2}} + \left(\frac{L_l}{\lambda(\lambda - L_f)} \frac{\Delta x}{h} \right)$$

with M_v, M_{v_h} maxima of the absolute value of the continuous and semidiscrete solution and L_l, L_f Lipschitz constants of the running cost l and the dynamics f. Then Theorem 3.1 holds for $h^2 = \Delta x^3$, $C = 2(M_v + M_{v_h}) + \frac{L_l}{\lambda(\lambda - L_f)}$ and $q = \frac{1}{3}$.

Other examples of error estimates can be found in the literature, even of high order (*i.e.* $q > 1$) in some smooth cases [16].

Using the numerical scheme described above we obtain an approximation of the solution of every decomposed problem (2.4); these discrete solutions are called, in analogy with the continuous case, V_i for $i \in \mathcal{I}$.

A simple observation brings us to the following Lemma:

Lemma 3.2. *Let* (H0), (H1), (H2) *and the Isaacs' conditions be verified. If a node* $x_j \in \Sigma_i$ *then there exists* $C > 0$ *independent from* Δx *and* $q \in \mathbb{R}^+$ *s.t.*

$$|V_i(j) - v(x_j)| \leq C(\Delta x)^q.$$

$V(j)$ *is the* j*-component of the vector* V *(related to the node* x_j*) and the parameters* C *and* q *are the same than in Theorem* 3.1.

Proof. It is sufficient observe that $|V_i(j) - v(x_j)| \leq |V_i(j) - v_i(x_j)| + |v_i(x_j) - v(x_j)|$. Proposition 2.4 and Theorem 3.1 give the estimate. □

We can establish a necessary condition for the nodes of G to belong to a fixed independent sub-domain Σ_i. Let be $B(x, \rho)$ the n-dimensional ball centred in x and of radius ρ.

Proposition 3.3. *Assume* (H0), (H1), (H2) *and the Isaacs' conditions. Let be* $x_j \in G$ *such that, taken an* $\epsilon \in [0, \Delta x)$ *and a direction* $d \in B(0,1)$*, the point* $x = x_j - \epsilon d \in \Omega$ *verifies* $v_i(x) = v(x)$ *for a certain* $i \in \mathcal{I}$*. Then the following estimate holds*

$$|V_i(j) - V(j)| \leq 2(C(\Delta x)^q + M\Delta x) \tag{3.5}$$

with C *and* q *as in Lemma* 3.2 *and* $M := \max\{L_{v_i}, i \in \mathcal{I}\}$ *where* L_{v_i} *is the Lipschitz constant of the function* v_i*.*

Proof. It is sufficient to observe that

$$\begin{aligned}|V_i(j) - V(j)| \leq |V_i(j) - v_i(x_j)| + |v_i(x_j) - v_i(x)| + |v_i(x) - v(x)| \\ + |v(x) - v(x_j)| + |v(x_j) - V(j)| \leq \|V_i(j) - v_i(x_j)\|_\infty + \|v(x_j) - V(j)\|_\infty \\ + |v_i(x_j) - v_i(x)| + |v_i(x) - v(x)| + |v(x) - v(x_j)| \leq 2C(\Delta x)^q + 2M\Delta x + |v_i(x) - v(x)|.\end{aligned}$$

The point $x \in \{x | v_i(x) = v(x)\}$ may not be on the grid G, but since v and v_i are Lipschitz continuous with Lipschitz constant bounded by M it is possible to estimate $|v(x) - v(x_j)|$ and $|v_i(x_j) - v_i(x)|$. From $v_i(x) = v(x)$, (3.5) follows. □

It is worth to note that such condition is verified by the nodes lying in the interior of such set but also by a neighbourhood of the boundary, of thickness depending on the parameters C and M. This criterion is used in the invariant sub-domains reconstruction algorithm; the list of the nodes of G belonging to the approximation of the independent sub-domain Σ_i is denoted $\overline{\Sigma}_i$. Consequently, the relative approximated set is the region delimited by $\overline{\Sigma}_i$.

Let us define $union(X_1, X_2)$ the vector composed by all the elements present in X_1 and X_2.

INDEPENDENT SUB-DOMAINS RECONSTRUCTION ALGORITHM (RA).

– Given a grid G with space step Δx and a collection of vectors $\widehat{G}_i$ such that $union(\widehat{G}_i, i = 1, \dots, m) = \widehat{G}$.

(1) *Resolution of auxiliary problems*
for $i = 1 \dots, m$ solve iteratively the problem
$V_i = T(V_i)$ with T defined as (3.3)
with boundary conditions as (2.5) w.r.t. $\widehat{G}_i$.
end

(2) *Check and reconstruction of the value function*
If necessary, check numerically (H2),
then obtain V as $V := \min_{i=1,\dots,m}\{V_i\}$.

(3) *Reconstruction of the sub-domains*
for $i = 1, \dots, m$,
initialize $\overline{\Sigma}_i = \emptyset$
for $j = 1, \dots, N$,
if $|V_i(j) - V(j)| \leq 2(C(\Delta x)^q + M\Delta x)$
then add x_j to vector $\overline{\Sigma}_i$
end
the i-subset is $\bar{\Sigma}_i$.
end.

Let us remark that, from the computational point of view, the difficult step is only the first one; successive steps are faster and with a negligible complexity. In addition, step (1) is easily performed in parallel, since it consists of a collection of independent problems, reducing the difficulty of resolution.

Remark 3.4. A delicate phase of the algorithm is the choice of the parameters C and M. Considered that our approximation is done through a necessary condition, all the analytic results are still valid rounding up those values. This is the easiest procedure from the practical point of view. In Section 4.1 we will show that even a not so tight choice produces acceptable approximations of the desired sets, in some situations of interest. Instead the penalization contained in the functions g_is can be done just considering the nodes of $\widehat{G} \setminus \widehat{G}_i$ ghost nodes belonging to Φ. This is compatible with conditions (2.5) for a fixed Δx.

Remark 3.5. It is worth to stress an issue about the stopping criterion used in the iterative resolution (3.2) contained in step (1). It is clear that if, in general, the exact discrete solution is not reached, then the stopping criterion should be compatible with our requests of accuracy. For the case of the semi-Lagrangian approximation, for a $\lambda > 0$, the classical estimate $\|V^n - V^{n+1}\|_\infty \leq \frac{1}{1+\lambda h}\|V^{n-1} - V^{n-2}\|_\infty$ brings us a link between the two successive iterations and the distance (in the L^∞ norm) from the discrete solution as, fixing $h = \Delta x$ (differential game case, (3.4)),

$$\|V^n - V\|_\infty \leq \sum_{t=n}^{\infty} \left(\frac{1}{1+\lambda\Delta x}\right)^t \|V^n - V^{n+1}\|_\infty$$

then a possible stopping criterion is

$$\|V^{n+1} - V^n\|_\infty \leq \epsilon, \quad \epsilon = 2\lambda\Delta x(1+\lambda\Delta x)^{n-1}(C(\Delta x)^q + M\Delta x).$$

The RA builds an approximation of the independent sub-domains. It is guaranteed that such approximation is performed exceeding the desired set, in the sense that $\Sigma_i \subseteq \overline{\Sigma}_i$. Another important property coming from

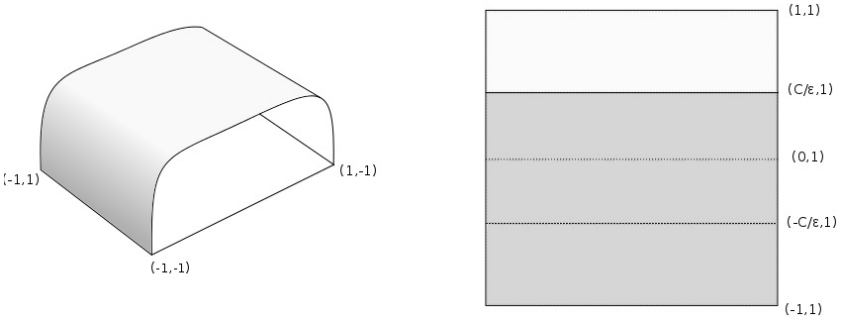

FIGURE 2. Example 3.6, the flatness of the central region depends on δ, in tones of grey the incorrect division coming from numerical uncertainty.

Proposition 3.3 is that, for two discretization steps Δx_1 and Δx_2 such that $\Delta x_1 \geq \Delta x_2$, the approximate independent sub-domains for a given decomposition of Γ satisfy

$$\overline{\Sigma}_i^{\Delta x_1} \supseteq \overline{\Sigma}_i^{\Delta x_2} \tag{3.6}$$

where $\overline{\Sigma}_i^{\Delta x}$ denotes the discrete independent set obtained performing the RA with discretization space step Δx.

A point to discuss is the relation with the decomposition technique proposed in [7]. Despite the analogies, in particular the idea of finding a decomposition in sub-domains which preserves a certain mutual independence, the resulting decomposition can be, in general, slightly different. Let us show it with an example.

Example 3.6. Let us consider the domain $\Omega := (-1, 1) \times \mathbb{R}$ the dynamics

$$f(x, a) := a_1, \quad \lambda := \delta, \quad a = B_2(0, 1),$$

the running cost $l(x, a) \equiv 1$, and the set $\Gamma := \cup_{i=1}^{2} \Gamma_i$, with $\Gamma_1 := \{x_1 = 1\}$, $\Gamma_2 := \{x_1 = -1\}$. Let us impose $g : \Gamma \to \mathbb{R}$ null in Γ_1 and Γ_2. It is possible to check that the solution is (Fig. 2)

$$v(x) = \begin{cases} 1 - \frac{\mathrm{e}^{\delta x_1}}{\mathrm{e}^{\delta}} & \text{for } x_1 \leq 0 \\ -1 + \frac{\mathrm{e}^{\delta x_1}}{\mathrm{e}^{\delta}} & \text{otherwise.} \end{cases}$$

We can notice that for every choice of δ the invariant domains relative to the two boundaries Γ_1 and Γ_2 are respectively $\{x \in \Omega \mid x_1 \geq 0\}$ and $\{x \in \Omega \mid x_1 \leq 0\}$.

For δ sufficiently small a numerical method, although reconstructing correctly the value function, is not able to solve appropriately the synthesis problem. The assignment of the approximated optimal control of the region "too flat" depends on the priority chosen in the computing. So using the *patchy decomposition*, the sub-domains reconstructed is, for example $\{x \in \Omega \mid x_1 + \frac{r}{\delta} \geq 0\}$ and $\{x \in \Omega \mid x_1 + \frac{r}{\delta} \leq 0\}$, $r \in (0, c)$, for a fixed c depending on the computing parameters, which are arbitrarily different (for a generic δ) from the correct division. On the other hand, our implicit reconstruction produces $\{x \in \Omega \mid x_1 + \frac{c}{\delta} \geq 0\}$ and $\{x \in \Omega \mid x_1 - \frac{c}{\delta} \leq 0\}$ which are larger sets containing the correct decomposition. This advantage comes from the fact that our decomposition uses the approximated value function instead the approximated feedback control.

4. Application to parallel computing

In this section we show how the independent sub-domains can be used to compute in parallel the correct solution of the discrete problem (3.2). We prove the convergence of the technique, and provide a bound for the numerical error. Roughly speaking the proposal is based on the reconstruction of a collection of independent

subsets, computed in parallel on a coarse grid, and successively the computation of the solution on every sub-domain on a fine grid, recovering at the end the result, using the minimum property on the regions of overlapping.

Let us state more rigorously the technique. Consider two families of simplices: a *coarse grid* K with space step Δx_K and a *fine grid* G with step Δx_G which both cover the domain Ω, (*i.e.* $\overline{\Omega} \subseteq \cup_j S_j \subseteq \cup_j K_j$). We denote by z_k, $k = 1, \ldots, N_1$ the nodes of the first grid and by x_k, $k = 1, \ldots, N_2$ the nodes of the second grid. Grids are chosen such that $N_1 \ll N_2$. The set of the nodes of K corresponding to Γ is called $\widehat{K}$. The parallel invariant sub-domains based algorithm is the following:

INDEPENDENT-SETS ALGORITHM (ISA).

– Given a grid K and a collection of vectors $\widehat{K}_i$ such that

$$union(\widehat{K}_i, i = 1, \ldots, M) = \widehat{K}$$

(1) *Reconstruction of the approximated independent sub-domains*
Using RA get a collection $\overline{\Sigma}_i^{\Delta x_k}$, $i = 1, \ldots, M$ subsets of the grid K.
(2) *Projection on the fine grid*
Project $\overline{\Sigma}_i^{\Delta x_K}$ on the grid G getting $\overline{\Sigma}_i^G$ for $i = 1, \ldots, M$,
(3) *Resolution on the fine grid*
for $i = 1, \ldots, M$ solve iteratively the problem on $\overline{\Sigma}_i^G$
$V_i = T(V_i)$ with T defined as (3.3)
end
(4) *Assembly of the final solution*
for $j = 1, \ldots, N_2$
$\overline{V}(j) = \min\{V_i(j) | x_j \in \overline{\Sigma}_i^G\}$
end

Some observations about the algorithm described above:

- Computing of the *RA* at step (1) it is not more expensive than a single computation on the coarse grid. *RA* is an algorithm which can work in parallel, and the number of threads that it needs, is the same requested at step (4).
- The projection is very easy if the grids are chosen to be partially superimposed *i.e.* every point $z_i \in K$ is also a point of the fine grid G; in every case the condition to impose is

$$x_j \in \overline{\Sigma}_i^G \iff x_j \in Con(\{z_j \mid j \in \overline{\Sigma}_i^{\Delta x_K}\});$$

  where $Con(\cdot)$ we mean the union of the simplexes with vertexes in the set. The computational cost of this passage is negligible.
- It is evident from definitions and from (3.6) that

$$\Sigma_i \subseteq \overline{\Sigma}_i^{\Delta x_G} \subseteq \overline{\Sigma}_i^G \equiv \overline{\Sigma}_i^{\Delta x_K}. \tag{4.1}$$

- Resolution on the fine grid is done in *a subsection* of the original domain. New boundaries created by the technique are numerically treated adding a neighborhood of ghost nodes. This does not condition the values inside the region.

This last observations guarantee a delicate point about the convergence of the method, as we show in the following proposition:

Proposition 4.1. *Assume* (H0), (H1), (H2) *and the Isaacs' conditions. Called* $\overline{V}$ *the exact discrete solution of the ISA algorithm* (i.e. *all* $V_i = T(V_i)$ *are verified exactly) and* V *the solution of* (3.2), *then there exists* $C > 0$

and $q \in \mathbb{R}^+$ independent from Δx_G such that

$$\|\overline{V} - v\|_\infty \leq 3C(\Delta x_G)^q + 2M\Delta x_G$$

holds. The parameters C, M and q are the same of Theorem 3.1.

Proof. By observation (4.1) we know that the independent sub-domains eventually obtained on the fine grid, should be subsets of the sub-domains we used in the algorithm. Let us take a $x_j \in G$, through Proposition 3.3 and (4.1) it is assured that there exists at least one index $i \in \{1, \dots, M\}$ such that $v(x_j) = v_i(x_j)$ (solution of (2.4), note that possibly $V_i(j) \neq \bar{V}(j)$ but it can be estimated by (3.5)). Then, using Lemma 3.2 and again Proposition 3.3 on the fine grid

$$|\overline{V}(j) - v(x_j)| \leq |\overline{V}(j) - V_i(j)| + |V_i(j) - v(x_j)| \leq \|V_i - v_i\|_\infty \leq 3C(\Delta x)^q + 2M\Delta x$$

for the arbitrariness of the choice of x_j we have the thesis. □

We observe that in the typical case $q \leq 1$ the convergence is $O(\Delta x^q)$.

4.1. Some examples

In this section we give some examples of problems solved with ISA. We show practically that the procedure of the independent domain approximation RA is computationally cheap and does not add an excessive number of nodes, even when the coarse grid K consists of a small number of points. Moreover we verify that our technique does not add a numerical error with respect to the solution found on the whole domain and we briefly compare the performances of our proposal to the literature. Let us first of all recall the discrete analogue of the L^∞, L^1 norms for a vector X of N elements:

$$\|X\|_{\Delta_\infty} := \max_{j=1,\dots,N} |X(j)|, \qquad \|X\|_{\Delta_1} := \frac{1}{N}\sum_{j=1}^{N} |X(j)|.$$

Example 4.2 (Distance function)**.** Let us start with the very easy case shown in Example 2.6, which is useful to observe some general features. Therefore we consider the Eikonal equation on the set $\Omega := (-1, 1)^2$ with the boundary value fixed to zero on $\Gamma := \partial\Omega$. This equation models the distance from the boundary of such set. Here $\lambda = 1$ corresponds to a nonlinear monotone scaling of the solution (this relation is classically shown through the Kruzkov transform, see [3]), so the correct viscosity solution is the function

$$v(x) = 1 - \frac{\min\{e^{|x_1|}, e^{|x_2|}\}}{e},$$

solution of the equation

$$v(x) + \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1.$$

We consider a uniform decomposition of the set Γ, for example in a 2-treads decomposition $\Gamma_1 := [-1, 1] \times \{-1\} \cup \{-1\} \times [-1, 1]$, $\Gamma_2 := [-1, 1] \times \{1\} \cup \{1\} \times [-1, 1]$, in a 4-treads $\Gamma_1 := [-1, 1] \times \{-1\}$, $\Gamma_2 := \{1\} \times [-1, 1]$, $\Gamma_3 := [-1, 1] \times \{1\}$, $\Gamma_2 := \{-1\} \times [-1, 1]$; *etc.*

In this case it is easy also to give an estimation of the constants introduced above, $M = 1$, $C = 1$ and $\epsilon = 10^{-3}$, $q = 1/2$. Clearly, the precision of the independent sub-domains reconstruction is affected by the discretization step used in the procedure. In Table 1 we report a comparison of such accuracy, in the case of a 4-subsets decomposition. The percentages correspond to the ratio between the maximal extension of an approximated subset $\overline{\Sigma}_i$ and the total area of Ω. In every case the exact decomposition is contained in the approximated one. It is worth noting how, even for a very coarse grid (with 10^2 or 15^2 elements) the technique is able to provide a sufficiently accurate estimate, giving a good reduction of the dimension of the subproblems with a cost of the pre-computing step absolutely negligible. Figure 3 reports the exact decomposition and two approximation sets Σ_1, Σ_3 with $\Delta x = 0.2$.

TABLE 1. Distance function: Comparison of the accuracy of the decomposition with different discretization steps.

N. of variables	Δx_K	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	1×10^{-3}	50%	
7^2	0.28	2×10^{-3}	43%	
10^2	0.2	4×10^{-3}	38%	
15^2	0.133	2×10^{-2}	35%	25%
20^2	0.1	5×10^{-2}	33%	
30^2	0.06	1.01	30%	
40^2	0.05	3	29%	
50^2	0.04	11	28.3%	

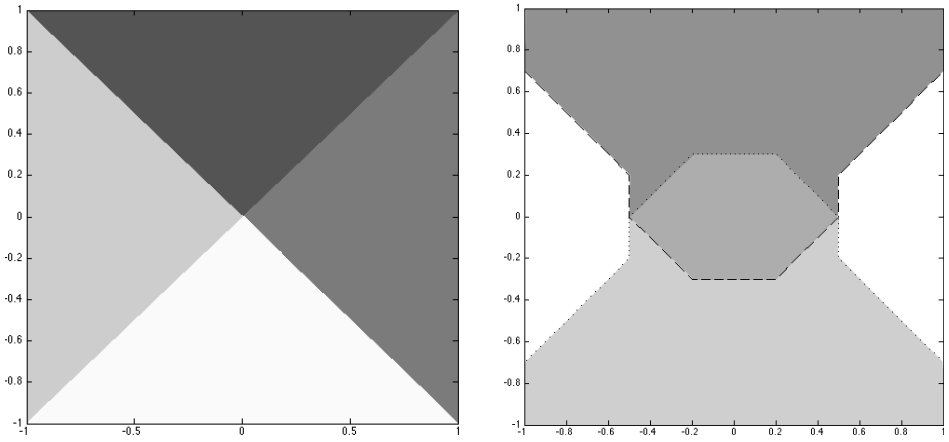

FIGURE 3. Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a coarse grid of 15^2 points (the third tone of grey in the centre is the superposition area between the two sets).

After the decomposition, the problem can be solved separately, and possibly at the same time on each $\overline{\Sigma}_i$. It is consequential a large gain in term of computational cost. We show, in Table 2, the time of computation of the resolution in the whole domain ND, compared with a standard domain decomposition methods DD (4 equal sub-domains, no superposition, as described in [8]) where we show also the number of iterations between the sub-domains necessary to reach the solution and our algorithm ISA. Our performances are comparable to a single iteration of the DD, which consists of a resolution on a part of the grid containing the 25% of the nodes of Ω. The ISA is performed on a approximated independent sub-domains decomposition obtained using RA on a 20^2 grid; accordingly as shown in Table 1 the dimension of such decomposed domain is approximately 33% of the original one.

Table 3 shows the experimental error in various *fine grids*, in the case of the original problem (solved on the whole domain Ω) or in presence of independent domain decomposition. The error introduced in not affected by the discretization step of the *coarse grid*.

Example 4.3 (Van Der Pol oscillator)**.** A well-known example is the *Van Der Pol oscillator*, here formulated as target problem. We consider $\Gamma := \partial B(0,\rho)$ (in this case $\rho = 0.2$) and $\Omega := (-1,1)^2 \setminus \bar{B}(0,\rho)$. The dynamics of the nonlinear system is the following:

$$f(x,a) = \begin{pmatrix} x_2 \\ (1-x_1^2)x_2 - x_1 + a \end{pmatrix}.$$

TABLE 2. Distance function: Comparison between the efficiency of the various methods (ND no decomposition, DD domain decomposition, ISA Independent set decomposition).

N. of variables	Δx_K	Time ND	Time (it) DD	Time ISA
25^2	0.08	0.13	0.06 (2)	0.035
50^2	0.04	7.02	1.2 (2)	0.68
75^2	0.026	57.5	12.2 (3)	4.8
100^2	0.02	1.5×10^3	65.3 (3)	16.6
200^2	0.01	1.9×10^5	1.2×10^4 (5)	3×10^3
300^2	0.006	$>10^6$	1.8×10^5 (11)	4.6×10^4

TABLE 3. Distance function: Approximation error Error in norm $\Delta\infty$ (and Δ_1).

	50^2	100^2	200^2
original	$1.2 \times 10^{-2}(1.1 \times 10^{-2})$	$6.5 \times 10^{-3}(3.6 \times 10^{-3})$	$2.5 \times 10^{-3}(1.6 \times 10^{-3})$
2-subsets	$1.2 \times 10^{-2}(7.2 \times 10^{-3})$	$6.5 \times 10^{-3}(3.7 \times 10^{-3})$	$2.5 \times 10^{-3}(1.4 \times 10^{-3})$
4-subsets	$9 \times 10^{-3}(7.2 \times 10^{-3})$	$4.6 \times 10^{-3}(3.6 \times 10^{-3})$	$1.4 \times 10^{-3}(1.3 \times 10^{-3})$
8-subsets	$9 \times 10^{-3}(7.2 \times 10^{-3})$	$4.6 \times 10^{-3}(3.6 \times 10^{-3})$	$1.4 \times 10^{-3}(1.3 \times 10^{-3})$

The others parameters of the system are:

$$\Omega = (-1,1)^2, \quad A = [-1,1], \quad \lambda = 1, \quad l(x,y,a) = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad g(x) \equiv 0.$$

For this problem we do not have an analytic formula for the solution, then we consider exact (in the error estimation) the numerical solution computed on a grid of 400^2 elements.

We consider a division of the target in "slices of a cake", meaning that a 2-parts division is $\Gamma_1 := \{x \in B(0,0.2) | x_2 \geq 0\}$, $\Gamma_2 := \{x \in B(0,0.2) | x_2 \leq 0\}$ and a 4-parts, $\Gamma_1 := \{x \in B(0,0.2) | x_1 \geq 0, x_2 \geq 0\}$, $\Gamma_2 := \{x \in B(0,0.2) | x_1 \geq 0, x_2 \leq 0\}$, $\Gamma_3 := \{x \in B(0,0.2) | x_1 \leq 0, x_2 \leq 0\}$, $\Gamma_4 := \{x \in B(0,0.2) | x_1 \leq 0, x_2 \geq 0\}$, *etc.*

In this case the choice of the constants is less elementary: we overestimate them as $C = 1$, $M = 1$, $\epsilon = 10^{-3}$, $q = \frac{3}{4}$ (it is possible to check this choice numerically).

Figure 4 shows a comparison between the exact division in sub-domains and two approximated sets ($\overline{\Sigma}_1$, $\overline{\Sigma}_3$).

Table 4 contains the accuracy of the 4-independent subset reconstruction with various discretization steps. In this case it is possible to see an inherent limitation of the effectiveness of such decomposition for parallel computing: the exact division in independent subset is not balanced, then the reduction of dimension in the greater subset is less considerable. In some cases this problem could even nullify the efficacy of the method (we can obtain a decomposition in some empty sets and the whole Ω), we discuss this point in the conclusions section.

In Table 6 we show as the decomposition does not affect the convergence to the solution.

Example 4.4 (A pursuit-evasion game)**.** The following example is a decomposable differential game. We consider a pursuit evasion game, where two agents have the opposite goal to reduce/postpone the time of capture. The dynamics are the following:

$$f(x,a,b) := \begin{pmatrix} f_1(x)(a_1 - b_1) \\ f_2(x)(a_2 - b_2) \end{pmatrix}$$

where the functions f_1, f_2 are $f_1(x) := x_2^2 + 1$ and $f_2(x) := 1$. The running cost is $l(x,a,b) := x_1^2 + 0.1$. This is a modification of the classical pursuit-evasion game on a plane presented here to emphasize another aspect of our technique. The controls are taken in the unit ball for the pursuer $A = B(0,1)$ and $B = B(0,1/2)$ for the

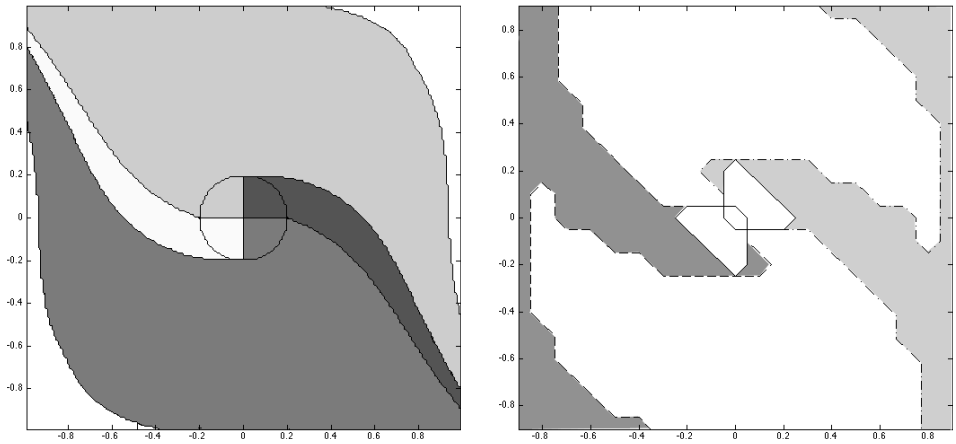

FIGURE 4. Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets found with a coarse grind of 15^2 points.

TABLE 4. Van Der Pol: Comparison of the accuracy of the decomposition with different discretization steps.

N. of variables	Δx_K	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	1.4×10^{-3}	62%	
10^2	0.2	0.011	55%	
20^2	0.1	0.103	47%	42.2%
30^2	0.06	1.47	45%	
40^2	0.05	5.6	44.6%	
50^2	0.04	16.3	44.1%	

TABLE 5. A pursuit-evasion game: Comparison of the accuracy of the decomposition with different discretization steps.

N. of variables	Δx_K	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	10^{-3}	60%	
10^2	0.2	0.008	46%	
30^2	0.06	1.38	38%	25%
50^2	0.04	15.9	36.1%	

evader. The capture occurs when the trajectory is driven to touch the small ball $B(0,\rho)$, ($\rho = 0.2$, in this case), then the set Γ is, as in the previous example $\Gamma := \partial B(0, 0.2)$.

It is possible to show that the Hamilton–Jacobi equation associated to this problem verifies the decomposability condition (H2): consider the norm

$$\|p\|_* := \max_{a\in B(0,1)} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} a^T \cdot p,$$

the Hamiltonian associated is equivalent to

$$H(x,p) := \|p\|_* - \frac{\|p\|_*}{2} - (x_1^2 + 0.1) = \frac{\|p\|_*}{2} - (x_1^2 + 0.1)$$

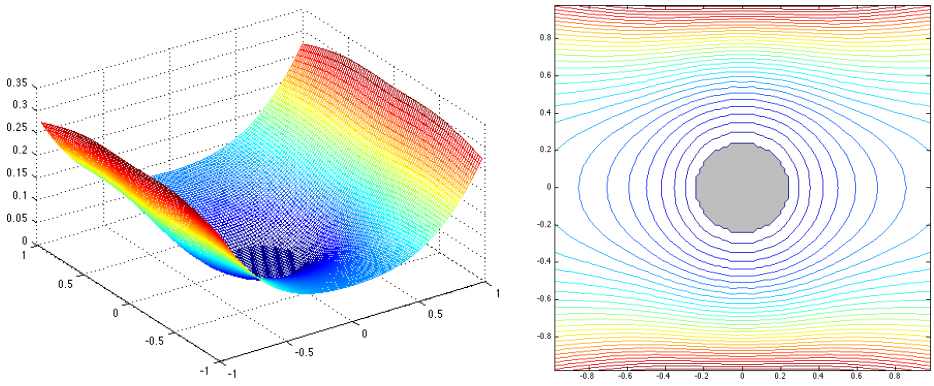

FIGURE 5. A pursuit-evasion game: Approximated value function of the differential game presented in Example 4.4.

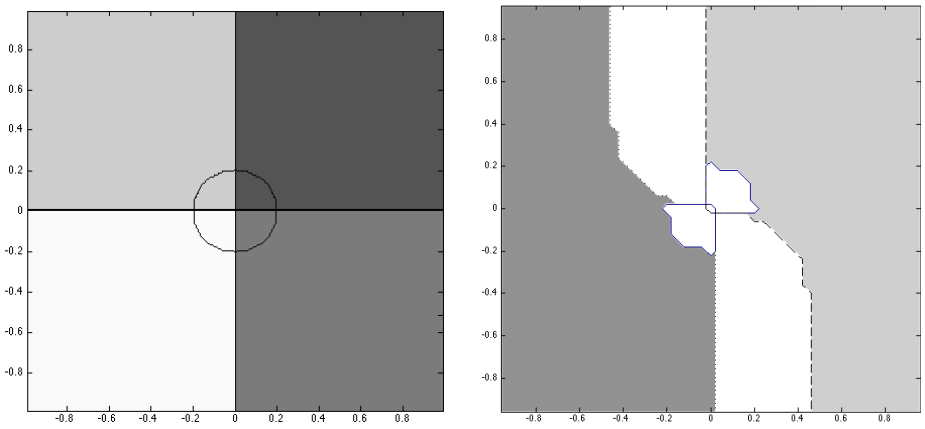

FIGURE 6. A pursuit-evasion game: Exact decomposition and two (of the four) approximated independent subsets found with a coarse grid of 40^2 points.

TABLE 6. Van Der Pol: Approximation error error in norm $\Delta\infty$ (and Δ_1).

	50^2	100^2	200^2
original	0.09 (0.07)	0.03 (0.01)	0.01 (6×10^{-3})
2-subsets	0.09 (0.07)	0.03 (0.01)	0.01 (6×10^{-3})
4-subsets	0.09 (0.07)	0.03 (0.01)	0.01 (6×10^{-3})
8-subsets	0.09 (0.07)	0.03 (0.01)	0.01 (6×10^{-3})

evidently convex everywhere with respect to p; then (H2) is automatically verified.

The value function of the game is shown in Figure 5. The function is very flat along the axis $x_2 = 0$, this produces a critical effect in the sets approximation, shown in Figure 6 and in Table 6, in this test, the parameter are estimated as $C = 1$, $M = 3$, $q = 1/2$. The convergence to the exact division in independent subsets is very slow.

5. Conclusions

We have shown a constructive procedure to obtain a decomposition in independent subsets of the domain of a Hamilton–Jacobi equation verifying condition (H2). These independent subsets have the property of being computed independently from each other. The new method resumes some general ideas already presented in [7], clarifying the theoretical background, enlarging the class of equations where the technique is relevant, proving the convergence of the parallel algorithm ISA and producing some estimates for the error. Those estimates are asymptotically coincident with the resolution in the whole domain and confirmed by tests.

A detailed evaluation of the performances of ISA is still an open question postponed to a forthcoming work. We can expect results similar to [7] since our pre-computing step gives (as shown in the Sect. 4.1) a division into sub-domains sufficiently close to a partition.

Some further improvements can be adapted to the technique. The critical occurrence shown in Example 4.3, about the balance in size of the subsets, can be solved with a recursive refinement of the division of Γ, producing a more balanced division. The case presented in example 4.4 is more critical. Here, for the moment, it is impossible to obtain a satisfactory reduction of the dimension of the decomposed domains without solving the problem on a sufficiently fine grid.

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