A Study of the Regular Pentagon with a Classic Geometric Approach

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Abstract

In this paper we will consider a regular pentagon and discuss three of its properties, which are linking side, radius, diagonal and apothem to the golden ratio. One of the properties, that regarding the ratio between the diagonal and the radius of the circumscribed circumference is strictly connected to the construction of the regular pentagon with compass and straightedge.

Keywords: **Classic Geometry, Golden Ratio.**

1. Introduction

In a previous paper [1], we have discussed some of the properties of a regular pentagon using an approach based on an analytic method. Here we will discuss three properties of the regular pentagon, determining them with the classic geometry. We will show that: 1) in a regular pentagon, the ratio between diagonal and side is the golden ratio φ, 2) in a regular pentagon, the ratio between apothem and the radius of its circumscribed circumference is half the golden ratio, and 3) in a regular pentagon, the ratio between diagonal and radius is $\sqrt{\varphi^2+1}$, while the ratio between side and radius is $\sqrt{1+1/\varphi^2}$.

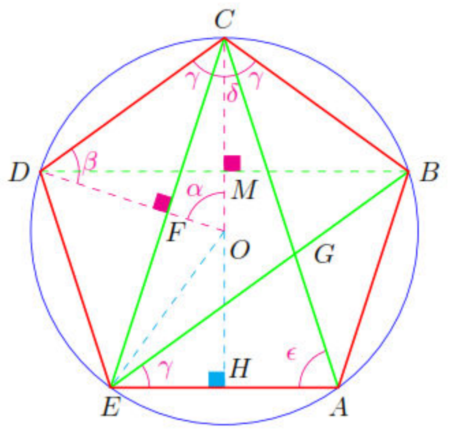

Figure 1. Geometry for the geometric study of the regular pentagon. EA is the side of the pentagon having length *l*. EB is a diagonal, having length *d*. OE is the radius having length *r* and OH is the apothem with length *a*.

2. The classical method and the first property

We have that $\alpha = 360°/5 = 72°$, and $2\beta = 180° - \alpha = 108°$, which implies $\beta = 54°$. From Figure 1 we also have that triangle CFD is rectangular because, due to the symmetry of the regular pentagon, is isometric to triangle BMC, which is clearly rectangular. Thus, $\beta + \gamma + 90° = 180°$, which implies $\gamma =$

36°. Since the pentagon is regular, we have $\hat{D} = \hat{E} = 2\beta = 180°$ and $B\hat{D}C = E\hat{D}A = \gamma$ Therefore, δ = 2β − 2γ = 108° − 72° = 36° = γ. This equivalence, in addition to the consideration that triangles EAC and EAG are also sharing the same angle ε, implies that these triangles are similar. Hence:

$$\frac{AC}{AE} = \frac{AE}{AG} \tag{1}$$

Since triangle EAC is clearly isosceles, so is triangle EAG, thus EA = EG. Moreover, by the symmetry due to the regularity of the pentagon, in a similar way we can prove that $D\hat{E}C = \gamma$ and $C\hat{E}G = \delta$. This implies that also triangle CEG is isosceles, thus EG = GC. Let |AC| = d be the length of the diagonal and |AE| = l be the length of the side of the pentagon, then |EA| = |EG| = |GC| = l and |AG| = |AC| − |GC| = $d - l$. Plugging these values into (1), we have:

$$\frac{d}{l} = \frac{l}{d-l} \tag{2}$$

Since, by construction, $l \neq 0$ and $d - l \neq 0$, we can multiply both sides of (2) by $l \cdot (d - l)$ and get

$$d \cdot (d-l) = l^2 \rightarrow d^2 - ld - l^2 = 0 \rightarrow d = \frac{1 \pm \sqrt{5}}{2} l \tag{3}$$

Since $(1-\sqrt{5})/2 < 0$ and $d,l > 0$, we use just the positive root. So we have:

$$d = \frac{1+\sqrt{5}}{2} l \tag{4}$$

Number $\varphi = (1+\sqrt{5})/2$ is the golden ratio. Therefore, from (4), we have the **first fundamental property of a regular pentagon**, which links the diagonal and the side: **in a regular pentagon, the ratio between the diagonal and the side is the golden ratio φ:**

$$\frac{d}{l} = \frac{1+\sqrt{5}}{2} = \varphi \tag{5}$$

3. The second fundamental property

The second fundamental relationship is the one that links the apothem a of the regular pentagon to the radius. From Figure 1, we apply Pythagoras' theorem both to triangles EHC and EHO, recognizing that |OH| = a, |OE| = |OC| = r and |CH| = $r + a$. These considerations, in addition to the first fundamental property, provide the following system of equations:

$$\begin{aligned} (r+a)^2 + \frac{l^2}{4} &= d^2 \\ a^2 + \frac{l^2}{4} &= r^2 \\ d &= \varphi l \end{aligned} \tag{6}$$

If we use the third equation of (6) in the first one, we have:

$$(r+a)^2 = \left(\varphi^2 - \frac{1}{4}\right) l^2 \rightarrow l^2 = \frac{4(r+a)^2}{4\varphi^2 - 1} \tag{7}$$

After, using the second equation of (6), we get:

$$a^2(4\varphi^2-1)+(r+a)^2=r^2(4\varphi^2-1) \tag{8}$$

After some calculations, we have that:

$$2\varphi^2a^2+ra-(2\varphi^2-1)r^2=0 \tag{9}$$

A trick we use here consisting of replacing φ^2 with $\varphi + 1$. In fact, if we use $d^2-ld-l^2=0$, assuming $l=1$ so that $d=\varphi$, we have:

$$\varphi^2-\varphi-1=0 \tag{10}$$

and also $\varphi^2=\varphi+1$. With this substitution, (9) becomes:

$$2(\varphi+1)a^2+ra-(2\varphi+1)r^2=0 \tag{11}$$

The reason of this substitution will be clear after computing the discriminant Δ of equation (11). We have:

$$\Delta=r^2+8(\varphi+1)(2\varphi+1)r^2=(16\varphi^2+24\varphi+9)r^2=(4\varphi+3)^2r^2$$

Thus,

$$a=\frac{-r\pm\sqrt{\Delta}}{4(\varphi+1)}=\frac{-1\pm(4\varphi+3)}{4(\varphi+1)}r. \tag{12}$$

Note that the substitution $\varphi^2=\varphi+1$ allowed us to express the discriminant Δ as a perfect square, with a significant simplification in calculation. We have:

$$a=\frac{2\varphi+1}{2(\varphi+1)}r \quad \vee \quad a=-r. \tag{13}$$

We ignore the second solution because the apothem a must be have a positive length. Therefore, using again identity $\varphi^2=\varphi+1$, we have:

$$a=\frac{2\varphi+1}{2(\varphi+1)}r=\frac{\varphi+\varphi+1}{2(\varphi+1)}r=\frac{\varphi+\varphi^2}{2(\varphi+1)}r=\frac{\varphi(\varphi+1)}{2(\varphi+1)}r=\frac{\varphi}{2}r \tag{14}$$

This is the second fundamental property**: in a regular pentagon, the ratio between apothem and radius of its circumscribed circumference is half the golden ratio**.

4. The third property

The third property links side and diagonal to the radius. If we apply Pythagoras' theorem again to triangle EHO taking into account the second fundamental property, we have:

$$\frac{l^2}{4} = r^2 - a^2 = \left(1 - \frac{\varphi^2}{4}\right) r^2 \rightarrow l^2 = (4 - \varphi^2) r^2 \rightarrow l = r\sqrt{4 - \varphi^2} \tag{15}$$

We can apply the first fundamental property, so that $d = \varphi\, l = \varphi\, r\sqrt{4 - \varphi^2} = r\sqrt{\varphi^2 (4 - \varphi^2)}$, and use again identity $\varphi^2 = \varphi + 1$. We obtain:

$$\begin{aligned} d &= r\sqrt{(\varphi + 1)(4 - \varphi - 1)} = r\sqrt{(\varphi + 1)(3 - \varphi)} = r\sqrt{3\varphi - \varphi^2 + 3 - \varphi} \\ &= r\sqrt{-\varphi^2 + 2\varphi + 3} = r\sqrt{-\varphi - 1 + 2\varphi + 3} = r\sqrt{\varphi + 2} \end{aligned} \tag{16}$$

Here we use in (16) identity $\varphi^2 - 1 = \varphi$ (the reason of this articulate choice will be clear soon, when we will provide a geometrical interpretation of the relationship between side and radius involving φ^2 rather than φ):

$$d = r\sqrt{\varphi + 2} = r\sqrt{\varphi^2 + 1} \tag{17}$$

And

$$l = \frac{d}{\varphi} = r\frac{\sqrt{\varphi^2 + 1}}{\varphi} = r\sqrt{\frac{\varphi^2 + 1}{\varphi^2}} = r\sqrt{1 + \frac{1}{\phi^2}} \tag{18}$$

(17) and (18) make up the third fundamental property of the regular pentagon: **in a regular pentagon, the ratio between diagonal and radius is** $\sqrt{\varphi^2 + 1}$ **, while the ratio between side and radius is** $\sqrt{1 + \frac{1}{\varphi^2}}$.

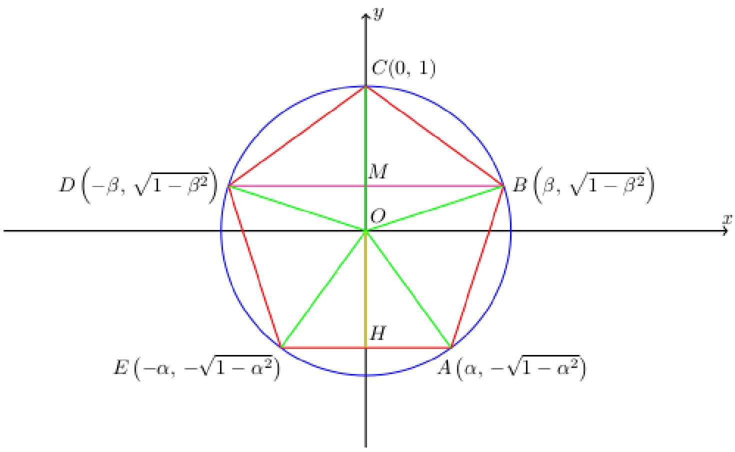

Figure 2. Frame of reference for the analytic geometry.

5. The third property in an analytic approach

The third property can also be computed through the analytic approach to the geometry of regular pentagon. In [1], we used this alternative approach to prove the first and the second property.

Here, we briefly recall the main results of this methodology in order to prove the third property of the regular pentagon. Without loss of generality, we can restrict ourselves to the case where the radius is equal to one, i.e., $r = 1$.
If we choose five vertexes of the regular pentagon like in Figure 2 in terms of unknown coordinates α and β then, after a number of mathematical computations reported in [1], one can find the following values for α and β:

$$\alpha = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}};\ \beta = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}} \tag{19}$$

In order to prove the third property, we manipulate the relationship $\varphi^2 = \varphi + 1$ as follows: we divide both sides by φ^2 and we get

$$1 = \frac{1}{\varphi} + \frac{1}{\varphi^2} \quad , \tag{20}$$

which can also be written as

$$\frac{1}{\varphi} = 1 - \frac{1}{\varphi^2} . \tag{21}$$

Second, we recall that

$$\frac{1}{\varphi} = \frac{2}{\sqrt{5}+1} = \frac{2\left(\sqrt{5}-1\right)}{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)} = \frac{2\left(\sqrt{5}-1\right)}{5-1} = \frac{2\left(\sqrt{5}-1\right)}{4} = \frac{\sqrt{5}-1}{2} . \tag{22}$$

From Figure 2 we have that:

$$l = 2\,\overline{HA} = 2\alpha = \sqrt{\frac{5-\sqrt{5}}{2}} . \tag{23}$$

Using (21) and (22) into (23) we have:

$$l = \sqrt{\frac{5-\sqrt{5}}{2}} = \sqrt{\frac{4+1-\sqrt{5}}{2}} = \sqrt{2+\frac{1-\sqrt{5}}{2}} = \sqrt{2-\frac{1}{\varphi}} = \sqrt{2-\left(1-\frac{1}{\varphi^2}\right)} = \sqrt{1+\frac{1}{\varphi^2}} , \tag{24}$$

which is (18) with $r = 1$ and this furtherly proves the third property of the regular pentagon.

6. The construction of the regular pentagon.

The reason why we wrote the ratio between side and radius like in (18) is because this particular form justifies the construction of the regular pentagon given the radius of the circumscribed circumference. Let us remember that, if we want to construct a pentagon using compass and straightedge, we have to start from a given segment, for instance AB, and bisect it, to find point O. Once we have O, we have the circumference in which we inscribe the pentagon.

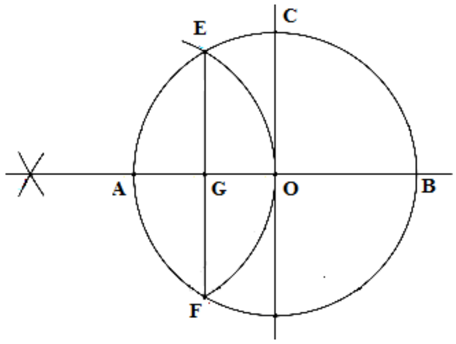

Figure 3. The starting point to construct with compass and straightedge a pentagon inscribed in a circumference is the Vesica Piscis, the figure made by the two arcs EOF and FAE.

After, we put the point of the compass in A, with the same radius OA. We find points E and F. Let us note that we obtain a figure of the sacred geometry, the Vesica Piscis [2]. From the Figure 4, we can see that:

$$|\mathrm{AO}| = |\mathrm{CO}| = r, \quad |\mathrm{GO}| = |\mathrm{AO}|/2 = r/2.$$

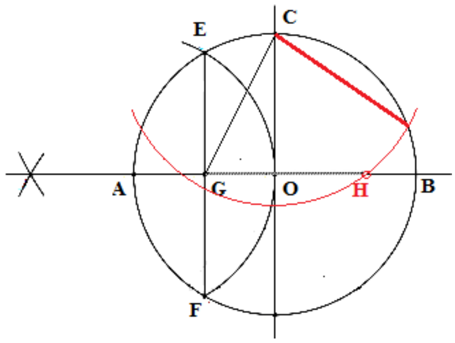

Figure 4. Once we have GC, using a compass we determine the position of H, with |GC| = |GH|. After, using again the compass, with its point in C, we determine one of the sides (in red) of the pentagon.

The final complete construction is reported in Figure 5 (the image was obtained using TikZ, [3,4]) and is the procedure to inscribe a regular pentagon within a given circumference using compass and straightedge only. We prove next that this popular procedure uses property (18). We have:

$$\begin{aligned} |GC| &= \sqrt{|CO|^2 + |GO|^2} = r\sqrt{1+\frac{1}{4}} = r\frac{\sqrt{5}}{2} \\ |GH| &= |GC| = r\frac{\sqrt{5}}{2} \\ |OH| &= |GH| - |GO| = r\frac{\sqrt{5}}{2} - \frac{r}{2} = r\frac{\sqrt{5}-1}{2} \end{aligned} \tag{25}$$

Note that $\frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}-1}{2}\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{5-1}{2(\sqrt{5}+1)} = \frac{2}{\sqrt{5}+1} = \frac{1}{\varphi}$. Then the last Eq. (25) becomes:

$$|OH| = \frac{r}{\varphi} \tag{26}$$

The side l of the regular pentagon is | CH |. Applying Pythagoras' theorem to triangle HOC, we have again property (18):

$$l = |CH| = \sqrt{|CO|^2 + |OH|^2} = r\sqrt{1+\frac{1}{\varphi^2}}$$

(27)

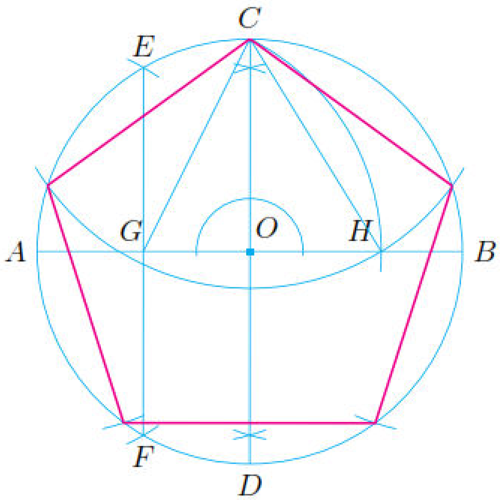

Figure 5. Construction of the regular pentagon given the radius of the circumscribed circumference (image obtained using TikZ [3, 4]).

The interesting thing of construction in Figure 5 is that it does not depend on the inclination of the initial segment AB, but it holds in general, as witnessed in Figure 6.

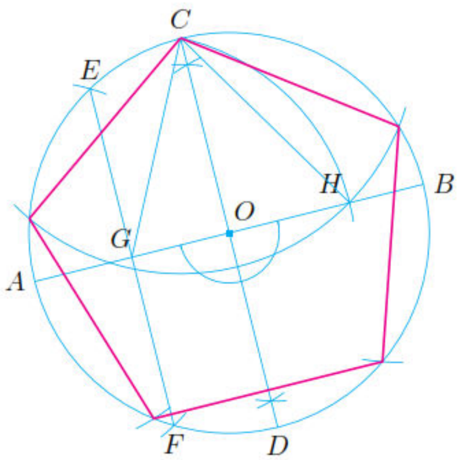

Figure 6. Construction of the regular pentagon given the radius of the circumscribed circumference, when the initial segment is inclined.

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