Directions sets: A generalisation of ratio sets

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DIRECTIONS SETS: A GENERALIZATION OF RATIO SETS

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Abstract

For every integer $k \geq 2$ and every $A \subseteq \mathbb{N}$, we define the *k-directions sets* of A as $D^k(A) := \{\boldsymbol{a}/\|\boldsymbol{a}\| : \boldsymbol{a} \in A^k\}$ and $D^{\underline{k}}(A) := \{\boldsymbol{a}/\|\boldsymbol{a}\| : \boldsymbol{a} \in A^{\underline{k}}\}$, where $\|\cdot\|$ is the Euclidean norm and $A^{\underline{k}} := \{\boldsymbol{a} \in A^k : a_i \neq a_j \text{ for all } i \neq j\}$. Via an appropriate homeomorphism, $D^k(A)$ is a generalization of the *ratio set* $R(A) := \{a/b : a, b \in A\}$, which has been studied by many authors. We study $D^k(A)$ and $D^{\underline{k}}(A)$ as subspaces of $S^{k-1} := \{\boldsymbol{x} \in [0,1]^k : \|\boldsymbol{x}\| = 1\}$. In particular, generalizing a result of Bukor and Tóth, we provide a characterization of the sets $X \subseteq S^{k-1}$ such that there exists $A \subseteq \mathbb{N}$ satisfying $D^{\underline{k}}(A)' = X$, where Y' denotes the set of accumulation points of Y. Moreover, we provide a simple sufficient condition for $D^k(A)$ to be dense in S^{k-1}. We conclude leaving some questions for further research.

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1. Introduction

Given $A \subseteq \mathbb{N}$, its *ratio set* is defined as $R(A) := \{a/b : a, b \in A\}$. The study of the topological properties of $R(A)$ as a subspace of $[0, +\infty]$, especially the question of when $R(A)$ is dense in $[0, +\infty]$, is a classical topic and has been considered by many researchers [1–4, 10, 12, 13, 19–23]. More recently, some authors have also studied $R(A)$ as a subspace of the p-adic numbers $\mathbb{Q}_p$ [6, 8, 9, 14, 15, 17].

We consider a further variation on this theme, which stems from the following easy observation: We have that $[0, +\infty]$ is homeomorphic to $S^1 := \{\boldsymbol{x} \in [0,1]^2 : \|\boldsymbol{x}\| = 1\}$ via the map $x \mapsto (1, x)/\|(1, x)\|$, if $x \in [0, +\infty)$, and $+\infty \mapsto (0, 1)$. This sends $R(A)$ onto $D^2(A) := \{\rho(\boldsymbol{a}) : \boldsymbol{a} \in A^2\}$, where $\rho(\boldsymbol{a}) := \boldsymbol{a}/\|\boldsymbol{a}\|$ for each $\boldsymbol{a} \neq \boldsymbol{0}$. Hence, topological questions about $R(A)$ as a subspace of $[0, +\infty]$ are equivalent to questions about $D^2(A)$ as a subspace of S^1. The novelty of this approach is that it can be generalized to higher dimensions. For every integer $k \geq 2$, define the *k-directions sets* of A as

$$D^k(A) := \{\rho(\boldsymbol{a}) : \boldsymbol{a} \in A^k\} \quad \text{and} \quad D^{\underline{k}}(A) := \{\rho(\boldsymbol{a}) : \boldsymbol{a} \in A^{\underline{k}}\},$$

where for every set B we let $B^{\underline{k}} := \{\boldsymbol{b} \in B^k : b_i \neq b_j \text{ for all } i \neq j\}$ denote the set of k-tuples with pairwise distinct entries in B. Put also $S^{k-1} := \{\boldsymbol{x} \in [0,1]^k : \|\boldsymbol{x}\| = 1\}$. We shall study $D^k(A)$ and $D^{\underline{k}}(A)$ as subspaces of S^{k-1}.

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Bukor and Tóth [3] characterized the subsets of $[0, +\infty]$ that are equal to $R(A)'$ for some $A \subseteq \mathbb{N}$, where Y' denotes the set of accumulation points of Y. In terms of $D^2(A)$, via the homeomorphism $[0, +\infty] \to S^1$ mentioned above, their result is the following:

Theorem 1.1. *Let $X \subseteq S^1$. Then there exists $A \subseteq \mathbb{N}$ such that $X = D^2(A)'$ if and only if the following conditions are satisfied:*

(i) *X is closed;*

(ii) *$(x_1, x_2) \in X$ implies $(x_2, x_1) \in X$;*

(iii) *if X is nonempty, then $(1, 0) \in X$.*

Note that Theorem 1.1 holds also if $D^2(A)$ is replaced by $D^{\underline{2}}(A)$. Indeed, $D^{\underline{2}}(A) \subseteq D^2(A) \subseteq D^{\underline{2}}(A) \cup \{\rho(1, 1)\}$ and consequently $D^2(A)' = D^{\underline{2}}(A)'$.

Our first result generalizes Theorem 1.1. Before stating it, we need to introduce some notation. Let $\boldsymbol{x} = (x_1, \dots, x_k) \in S^{k-1}$. For every permutation π of $\{1, \dots, k\}$, we put $\pi(\boldsymbol{x}) := (x_{\pi(1)}, \dots, x_{\pi(k)})$. Also, for every $I \subseteq \{1, \dots, k\}$, we say that I *meets* $\boldsymbol{x}$ if there exists $j \in I$ such that $x_j \neq 0$. In such a case, we put $\rho_I(\boldsymbol{x}) := \rho(\boldsymbol{y})$, where $\boldsymbol{y} = (y_1, \dots, y_k)$ is defined by $y_i := x_i$ if $i \in I$, and $y_i := 0$ for $i \notin I$. (This is well defined since $\boldsymbol{y} \neq \boldsymbol{0}$.)

Our first result is the following:

Theorem 1.2. *Let $X \subseteq S^{k-1}$ for some integer $k \geq 2$. Then there exists $A \subseteq \mathbb{N}$ such that $X = D^{\underline{k}}(A)'$ if and only if the following conditions are satisfied:*

(i) *X is closed;*

(ii) *$\boldsymbol{x} \in X$ implies $\pi(\boldsymbol{x}) \in X$, for every permutation π of $\{1, \dots, k\}$;*

(iii) *$\boldsymbol{x} \in X$ implies $\rho_I(\boldsymbol{x}) \in X$, for every $I \subseteq \{1, \dots, k\}$ that meets $\boldsymbol{x}$.*

Note that Theorem 1.2 is indeed a generalization of Theorem 1.1, since $\rho_I(\boldsymbol{x}) \in \{\boldsymbol{x}, (1, 0), (0, 1)\}$ for every $I \subseteq \{1, 2\}$ that meets $\boldsymbol{x} \in S^1$. Furthermore, for $k \geq 3$, Theorem 1.2 is false if $D^{\underline{k}}(A)$ is replaced by $D^k(A)$ (see Remark 2.1 below).

Now we turn our attention to the question of when $D^k(A)$ is dense in S^{k-1}. First, we have the following easy proposition.

Proposition 1.3. *Let $k \geq 2$ be an integer and fix $A \subseteq \mathbb{N}$. We have that $D^k(A)$ is dense in S^{k-1} if and only if $D^{\underline{k}}(A)$ is dense in S^{k-1}.*

Proof. On the one hand, since $D^{\underline{k}}(A) \subseteq D^k(A)$, if $D^{\underline{k}}(A)$ is dense in S^{k-1} then $D^k(A)$ is dense in S^{k-1}. On the other hand, suppose that $D^k(A)$ is dense in S^{k-1}. Then, for every $\boldsymbol{x} \in S^{k-1} \cap \mathbb{R}^{\underline{k}}$, there exists $\boldsymbol{a}^{(n)} \in A^k$ such that $\rho(\boldsymbol{a}^{(n)}) \to \boldsymbol{x}$. Consequently, for all sufficiently large n we have $\boldsymbol{a}^{(n)} \in A^{\underline{k}}$. This implies that $D^{\underline{k}}(A)$ is dense in $S^{k-1} \cap \mathbb{R}^{\underline{k}}$. Since $S^{k-1} \cap \mathbb{R}^{\underline{k}}$ is dense in S^{k-1}, we get that $D^{\underline{k}}(A)$ is dense in S^{k-1}, as desired. □

The next result shows that if $D^k(A)$ is dense in S^{k-1}, for some integer $k \geq 3$ and $A \subseteq \mathbb{N}$, then $D^{k-1}(A)$ is dense in S^{k-2}, but the opposite implication is false.

THEOREM 1.4. *Let $k \geq 3$ be an integer. On the one hand, if $D^k(A)$ is dense in S^{k-1}, for some $A \subseteq \mathbb{N}$, then $D^{k-1}(A)$ is dense in S^{k-2}. On the other hand, there exists $A \subseteq \mathbb{N}$ such that $D^k(A)$ is not dense in S^{k-1} but $D^{k-1}(A)$ is dense in S^{k-2}.*

We also provide a simple sufficient condition for $D^k(A)$ to be dense in S^{k-1}.

THEOREM 1.5. *Let $A \subseteq \mathbb{N}$. If there exists an increasing sequence $a_n \in A$ such that $a_{n-1}/a_n \to 1$, then $D^k(A)$ is dense in S^{k-1} for every integer $k \geq 2$.*

The case $k = 2$ of Theorem 1.5 was proved by Starni [19] (hereafter, we tacitly express all the results about $R(A)$ in terms of $D^2(A)$), who also showed that the condition is sufficient but not necessary.

Let $\mathbb{P}$ be the set of prime numbers. It is known that $D^2(\mathbb{P})$ is dense in S^1 [13, 19] (see also [5, 7, 16, 18] for similar results in number fields). Let p_n be the nth prime number. As a consequence of the Prime Number Theorem, we have that $p_n \sim n \log n$ [11, Theorem 8]. Hence, $p_{n-1}/p_n \to 1$ and thus Theorem 1.5 yields the following:

COROLLARY 1.6. *$D^k(\mathbb{P})$ is dense in S^{k-1}, for every integer $k \geq 2$.*

We leave the following questions to the interested readers:

QUESTION 1.7. *What is a simple characterization of the sets $X \subseteq S^{k-1}$, $k \geq 2$, such that there exists $A \subseteq \mathbb{N}$ satisfying $X = D^k(A)'$?*

QUESTION 1.8. *Strauch and Tóth [20] proved that if $A \subseteq \mathbb{N}$ has lower asymptotic density at least $1/2$, then $D^2(A)$ is dense in S^1. Moreover, they showed that for every $\delta \in [0, 1/2)$ there exists some $A \subseteq \mathbb{N}$ with lower asymptotic density equal to δ and such that $D^2(A)$ is not dense in S^1. How can these results be generalized to $D^k(A)$ with $k \geq 3$?*

QUESTION 1.9. *Bukor, Šalát, and Tóth [4] proved that $\mathbb{N}$ can be partitioned into three sets A, B, C, such that none of $D^2(A)$, $D^2(B)$, $D^2(C)$ is dense in S^1. Moreover, they showed that such a partition is impossible using only two sets. How can these results be generalized to $D^k(A)$ with $k \geq 3$?*

Notation We use $\mathbb{N}$ to denote the set of positive integers. We write vectors in bold and we use subscripts to denote their components, so that $\boldsymbol{x} = (x_1, \dots, x_k)$. Also, we put $\|\boldsymbol{x}\| := \sqrt{x_1^2 + \cdots + x_k^2}$ for the Euclidean norm of $\boldsymbol{x}$. If X is a subset of a topological space T, then X' denotes the set of accumulation points of X. Given a sequence $x^{(n)} \in T$, we write $x^{(n)} \dashrightarrow x$ to mean that $x^{(n)} \to x$ as $n \to +\infty$ and $x^{(n)} \neq x$ for infinitely many n.

2. Proof of Theorem 1.2

ONLY IF PART. Suppose that $X = D^k(A)'$ for some $A \subseteq \mathbb{N}$. We shall prove that X satisfies (i)–(iii). Clearly, X is closed, since it is a set of accumulation points.

Hence, (i) holds. Pick $\boldsymbol{x} \in X$. Then there exists a sequence $\boldsymbol{a}^{(n)} \in A^{\underline{k}}$ such that $\rho(\boldsymbol{a}^{(n)}) \dashrightarrow \boldsymbol{x}$. In particular, this implies that $\|\boldsymbol{a}^{(n)}\| \to +\infty$ and that A is infinite. Let π be a permutation of $\{1, \dots, k\}$. Setting $\boldsymbol{b}^{(n)} := \pi(\boldsymbol{a}^{(n)})$, it follows easily that $\boldsymbol{b}^{(n)} \in A^{\underline{k}}$ and $\rho(\boldsymbol{b}^{(n)}) \dashrightarrow \pi(\boldsymbol{x})$. Consequently, $\pi(\boldsymbol{x}) \in X$ and (ii) holds. Finally, assume that $I \subseteq \{1, \dots, k\}$ meets $\boldsymbol{x}$. Up to passing to a subsequence of $\boldsymbol{a}^{(n)}$, we can assume that each sequence $a_i^{(n)}$, with $i \in \{1, \dots, k\}$, is nondecreasing. Recalling that A is infinite, this implies that we can fix $k - \#I$ distinct $c_i \in A$, with $i \in \{1, \dots, k\} \setminus I$, such that $\boldsymbol{d}^{(n)} \in A^{\underline{k}}$ for every sufficiently large $n \in \mathbb{N}$, where $\boldsymbol{d}^{(n)} \in \mathbb{N}^k$ is defined by $d_i^{(n)} := a_i^{(n)}$ if $i \in I$, and $d_i^{(n)} := c_i$ if $i \notin I$. Since I meets $\boldsymbol{x}$, there exists $j \in I$ such that $x_j \neq 0$, which in turn implies that $a_j^{(n)} \to +\infty$ and consequently $\|\boldsymbol{d}^{(n)}\| \to +\infty$. At this point, it follows easily that $\rho(\boldsymbol{d}^{(n)}) \dashrightarrow \rho_I(\boldsymbol{x})$. Hence, $\rho_I(\boldsymbol{x}) \in X$ and (iii) holds too.

If Part. Suppose that $X \subseteq S^{k-1}$ satisfies (i)–(iii). We shall prove that there exists $A \subseteq \mathbb{N}$ such that $X = D^{\underline{k}}(A)'$. Since X is a closed subset of S^{k-1}, we have that X has a countable dense subset, say $Y := \{\boldsymbol{y}^{(m)} : m \in \mathbb{N}\}$.

Claim 1. *There exists a sequence $\boldsymbol{c}^{(m)}$ such that:*

(c1) $\boldsymbol{c}^{(m)} \in \mathbb{N}^{\underline{k}}$ *for every* $m \in \mathbb{N}$*;*

(c2) $m \mapsto \rho(\boldsymbol{c}^{(m)})$ *is an injection;*

(c3) $\left|\frac{1}{m!} c_i^{(m)} - y_i^{(m)}\right| \to 0$*, for every* $i \in \{1, \dots, k\}$*;*

(c4) $\left\|\rho(\boldsymbol{c}^{(m)}) - \boldsymbol{y}^{(m)}\right\| \to 0$.

Proof. For every $m \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, we define $c_i^{(m)} := \lfloor m!\, y_i^{(m)} \rfloor + s_i^{(m)} + t^{(m)}$, where $\boldsymbol{s}^{(m)} \in \mathbb{N}^k$ and $t^{(m)} \in \mathbb{N}$ will be chosen later. For each $m \in \mathbb{N}$, it is easy to see that we can choose $\boldsymbol{s}^{(m)} \in \{1, \dots, k\}^k$ such that $\boldsymbol{c}^{(m)} \in \mathbb{N}^{\underline{k}}$. (Note that this property does not depend on $t^{(m)}$.) We make this choice so that (c1) holds. Now note that for every fixed $u, v \in \mathbb{R}^+$, with $u \neq v$, the function $\mathbb{R}^+ \to \mathbb{R} : t \mapsto \frac{u+t}{v+t}$ is injective. Therefore, for each $m \in \mathbb{N}$ we can choose $t^{(m)} \in \{1, \dots, m\}$ such that $c_1^{(m)}/c_2^{(m)} \neq c_1^{(\ell)}/c_2^{(\ell)}$ for every positive integer $\ell < m$. In turn, this choice implies that (c2) holds. At this point, both (c3) and (c4) follow easily. □

Define $A := \bigcup_{i=1}^k A_i$, where $A_i := \{c_i^{(m)} : m \in \mathbb{N}\}$ for every $i \in \{1, \dots, k\}$. We claim that $X = D^{\underline{k}}(A)'$.

First, let us prove that $X \subseteq D^{\underline{k}}(A)'$. Pick some $\boldsymbol{x} \in X$. Since Y is a dense subset of X, there exists an increasing sequence of positive integers $(m_n)_{n \in \mathbb{N}}$ such that $\boldsymbol{y}^{(m_n)} \to \boldsymbol{x}$. By the definition of A and by (c1), we have that $\boldsymbol{c}^{(m_n)} \in A^{\underline{k}}$. Moreover, (c2) and (c4) imply that $\rho(\boldsymbol{c}^{(m_n)}) \dashrightarrow \boldsymbol{x}$. Hence, we have $\boldsymbol{x} \in D^{\underline{k}}(A)'$, as desired.

Now let us prove that $D^{\underline{k}}(A)' \subseteq X$. Pick $\boldsymbol{x} \in D^{\underline{k}}(A)'$. Then there exists a sequence $\boldsymbol{a}^{(n)} \in A^{\underline{k}}$ such that $\rho(\boldsymbol{a}^{(n)}) \dashrightarrow \boldsymbol{x}$. Up to passing to a subsequence, we can assume that there exist some $j_1, \dots, j_k \in \{1, \dots, k\}$ such that $\boldsymbol{a}^{(n)} \in A_{j_1} \times \cdots \times A_{j_k}$ for every

$n \in \mathbb{N}$. In turn, this implies that there exists a sequence $\boldsymbol{m}^{(n)} \in \mathbb{N}^k$ such that $a_i^{(n)} = c_{j_i}^{(m_i^{(n)})}$ for every $n \in \mathbb{N}$ and $i \in \{1, \dots, k\}$. Thanks to (ii), without loss of generality, we can reorder the entries of $\boldsymbol{a}^{(n)}$. Hence, up to reordering and up to passing to a subsequence, we can assume that there exists $h \in \{1, \dots, k\}$ such that $y_{j_1}^{(m_1^{(n)})}, \dots, y_{j_h}^{(m_h^{(n)})} \neq 0$ and $y_{j_{h+1}}^{(m_{h+1}^{(n)})} = \cdots = y_{j_k}^{(m_k^{(n)})} = 0$ for every $n \in \mathbb{N}$. Similarly, again up to reordering and up to passing to a subsequence, we can assume that there exists $\ell \in \{1, \dots, h\}$ such that $m_1^{(n)} = \cdots = m_\ell^{(n)} > m_{\ell+1}^{(n)} \geq \cdots \geq m_h^{(n)}$ for every $n \in \mathbb{N}$. In particular, since $\boldsymbol{a}^{(n)} \in A^{\underline{k}}$ for every $n \in \mathbb{N}$, we get that $j_1, \dots, j_\ell$ are pairwise distinct. Let π be any permutation of $\{1, \dots, k\}$ such that $\pi(i) = j_i$ for all $i \in I := \{1, \dots, \ell\}$. Note that I meets $\pi(\boldsymbol{y}^{(m_1^{(n)})})$ for every $n \in \mathbb{N}$. Put $\boldsymbol{z}^{(n)} := \rho_I(\pi(\boldsymbol{y}^{(m_1^{(n)})}))$ for every $n \in \mathbb{N}$. Hence, by (ii) and (iii) we have that $\boldsymbol{z}^{(n)} \in X$ for every $n \in \mathbb{N}$. Thanks to (c3), we have that $\left|\frac{1}{m_1^{(n)}!} a_i^{(n)} - y_{j_i}^{(m_1^{(n)})}\right| \to 0$ for each $i \in I$, and $\frac{1}{m_1^{(n)}!} a_i^{(n)} \to 0$ for each $i \in \{1, \dots, k\} \setminus I$, as $n \to +\infty$. As a consequence, $\left\|\rho(\boldsymbol{a}^{(n)}) - \boldsymbol{z}^{(n)}\right\| \to 0$, which in turn implies that $\boldsymbol{z}^{(n)} \to \boldsymbol{x}$. Finally, since X is closed by (i), we obtain that $\boldsymbol{x} \in X$, as desired.

The proof is complete.

Remark 2.1. *We note that for $k \geq 3$ the statement of Theorem 1.2 is false if $D^{\underline{k}}(A)$ is replaced by $D^k(A)$. In fact, fix an integer $k \geq 3$ and let X be the subset of S^{k-1} containing all the permutations of $\boldsymbol{\eta} := \rho(1, \sqrt{2}, 0, \dots, 0)$ and $\rho(1, 0, \dots, 0)$ (and nothing else). It follows by Theorem 1.2 that there exists $A \subseteq \mathbb{N}$ such that $X = D^{\underline{k}}(A)'$. For the sake of contradiction, let us suppose that there exists $B \subseteq \mathbb{N}$ such that $X = D^k(B)'$. Since $\boldsymbol{\eta} \in X$, there exists a sequence $\boldsymbol{b}^{(n)} \in B^k$ such that $\rho(\boldsymbol{b}^{(n)}) \rightarrow\!\!\!\!\cdot\;\, \boldsymbol{\eta}$. Let $\boldsymbol{c}^{(n)} \in \mathbb{N}^k$ be the sequence defined by $c_i^{(n)} = b_1^{(n)}$ if $i \neq 2$, and $c_i^{(n)} := b_2^{(n)}$ if $i = 2$. We obtain that $\boldsymbol{c}^{(n)} \in B^k$ and $\rho(\boldsymbol{c}^{(n)}) \rightarrow\!\!\!\!\cdot\;\, \boldsymbol{\theta}$, where $\boldsymbol{\theta} := \rho(1, \sqrt{2}, 1, \dots, 1)$. (Here we have used that η_1/η_2 is irrational and consequently $\rho(\boldsymbol{c}^{(n)}) \neq \boldsymbol{\theta}$.) Therefore, $\boldsymbol{\theta} \in D^k(B)' = X$, which is a contradiction.*

3. Proof of Theorem 1.4

Let $k \geq 3$ be an integer and let $A \subseteq \mathbb{N}$. Suppose that $D^k(A)$ is dense in S^{k-1}. We shall prove that $D^{k-1}(A)$ is dense in S^{k-2}. For every $\boldsymbol{x} \in S^{k-2}$, let $f_k(\boldsymbol{x}) \in S^{k-1}$ be defined by $f_k(\boldsymbol{x}) := \rho(x_1, \dots, x_{k-1}, 0)$. Since $D^k(A)$ is dense in S^{k-1}, we have that there exists a sequence $\boldsymbol{a}^{(n)} \in A^k$ such that $\rho(\boldsymbol{a}^{(n)}) \to f_k(\boldsymbol{x})$. In turn, this implies that $\rho(\boldsymbol{b}^{(n)}) \to \boldsymbol{x}$, where $\boldsymbol{b}^{(n)} \in A^{k-1}$ is defined by $b_i^{(n)} := a_i^{(n)}$ for $i \in \{1, \dots, k-1\}$. Hence, $D^{k-1}(A)$ is dense in S^{k-2}, as desired.

Now given an integer $k \geq 3$, we shall prove that there exists $A \subseteq \mathbb{N}$ such that $D^{k-1}(A)$ is dense in S^{k-2}, but $D^k(A)$ is not dense in S^{k-1}. Let $X := \{\boldsymbol{x} \in S^{k-1} : x_i = 0 \text{ for some } i\}$. Clearly, X satisfies conditions (i)–(iii) of Theorem 1.2, and consequently there exists $A \subseteq \mathbb{N}$ such that $D^{\underline{k}}(A)' = X$. Therefore, $D^{\underline{k}}(A)$ is not dense in S^{k-1} and, in light of Proposition 1.3, $D^k(A)$ is not dense in S^{k-1} as well. Finally, for every $\boldsymbol{x} \in S^{k-2}$ we have $f_k(\boldsymbol{x}) \in X$, and the same reasonings of the previous paragraph show that $D^{k-1}(A)$ is dense in S^{k-2}.

4. Proof of Theorem 1.5

Suppose that there exists an increasing sequence $a_n \in A$ such that $a_{n-1}/a_n \to 1$. Fix an integer $k \geq 2$ and pick $\boldsymbol{x} \in S^{k-1}$ with $x_1, \dots, x_k > 0$. Clearly, for every integer $m \geq a_1 \,/\, \min\{x_1, \dots, x_k\}$ there exist integers $m_1, \dots, m_k \geq 2$ such that $a_{m_i - 1} \leq m x_i < a_{m_i}$ for each $i \in \{1, \dots, k\}$. Hence, for every $i \in \{1, \dots, k\}$, we have that

$$x_i < \frac{a_{m_i}}{m} \leq \frac{a_{m_i}}{a_{m_i - 1}}\, x_i,$$

which, since $m_i \to +\infty$ as $m \to +\infty$, yields that $a_{m_i}/m \to x_i$ as $m \to +\infty$. Putting $\boldsymbol{a}^{(m)} := (a_{m_1}, \dots, a_{m_k})$, it follows that $\rho(\boldsymbol{a}^{(m)}) \to \boldsymbol{x}$. Therefore, $D^k(A)$ is dense in S^{k-1}, as claimed.

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