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# DISCRETE-TO-CONTINUUM LIMITS OF PARTICLES WITH AN **ANNIHILATION RULE\***

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#### PATRICK VAN MEURS<sup>†</sup> AND MARCO MORANDOTTI<sup>‡</sup>

Abstract. In the recent trend of extending discrete-to-continuum limit passages for gradient 4 flows of single-species particle systems with singular and nonlocal interactions to particles of opposite 5 6 sign, any annihilation effect of particles with opposite sign has been side-stepped. We present the 7 first rigorous discrete-to-continuum limit passage which includes annihilation. This result paves the 8 way to applications such as vortices, charged particles, and dislocations. In more detail, the discrete 9 setting of our discrete-to-continuum limit passage is given by particles on the real line. Particles of 10 the same type interact by a singular interaction kernel; those of opposite sign interact by a regular 11 one. If two particles of opposite sign collide, they annihilate, *i.e.*, they are taken out of the system. 12 The challenge for proving a discrete-to-continuum limit is that annihilation is an intrinsically discrete 13 effect where particles vanish instantaneously in time, while on the continuum scale the mass of the 14 particle density decays continuously in time. The proof contains two novelties: (i) the observation that empirical measures of the discrete dynamics (with annihilation rule) satisfy the continuum 15 evolution equation that only implicitly encodes annihilation, and (ii) the fact that, by imposing a 1617 relatively mild separation assumption on the initial data, we can identify the limiting particle density 18as a solution to the same continuum evolution equation.

19Key words. Particle system, discrete-to-continuum asymptotics, annihilation, gradient flows

AMS subject classifications. 82C22, (82C21, 35A15, 74G10). 20

1. Introduction. A recent trend in discrete-to-continuum limit passages in over-21 damped particle systems with singular and nonlocal interactions (with applications 22to, e.q., vortices [9, 19, 38], charged particles [36], dislocations [18, 27, 30], and dis-23 24 location walls [13, 47, 48]) is to extend such results to two-species particle systems. 25The singularity in the interaction potential imposes the immediate problem that the evolution of the particle system is only defined up to the first collision time between 26 particles of opposite sign. This problem is dealt with by either *regularising* the singu-27lar interaction potential (see [11, 12]) or by limiting the geometry such that particles of 2829 opposite sign cannot collide (see [7, 46]). However, more realistic models of vortices, charged particles, and dislocations include the *annihilation* of particles of opposite 30 sign. While annihilation has been analysed on the discrete scale [40, 41] and contin-31 uum scale [3, 6] separately, there is no rigorous discrete-to-continuum limit passage 32 known between these two scales. 33

34 The main result in this paper establishes the first result on a discrete-to-continuum 35 limit passage in two-species particle systems in one dimension with annihilation.

Below, we first describe the physical context of our main result. Then, we intro-36 duce the discrete and continuum problems. Our main result is the connection between them in terms of the limit passage as the number of particles n tends to  $\infty$ . Then, we 38 put our discrete and continuum problems in the perspective of the literature, and com-39

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ment how our proof combines known techniques with novel ideas. We conclude with
 an exposition of possible extensions to work towards singular interspecies interactions
 and higher dimensions.

**1.1.** Application to plasticity and dislocations. The main application we 43 have in mind is to increase the understanding of the plastic behaviour of metals. 44 Plasticity in metals is the emergent behaviour of large groups of dislocations moving 45 and interacting on microscopic time- and length-scales. Dislocations are stacking 46 faults in the atomic lattice. We keep the description of dislocations concise, and refer 47 to the classical textbooks [21, 24] for a detailed description. In two-dimensional elastic 48 bodies, dislocations are often represented as points in the elastic body at which the 4950 stress has a prescribed singularity. This singularity depends on the orientation of the dislocation, which is described by the so-called Burgers vector. While dislocations themselves exert a stress field, they can also move in response to the stress induced 52by other dislocations in the elastic body. The simplest model to capture such effects 53 is an interacting particle system which fits to the setting in this paper. 54

One of the main unsolved problems in plasticity is how to describe the group 55behaviour of many dislocations in terms of a dislocation density. While there are 56 many different models available in the engineering literature for the dislocation density 57[5, 15, 16, 22, 25, 26, 42], it is not clear which of these models describes the group 58 behaviour of a given collection of dislocations for a given set of parameters. This 59problem arises from a lack of rigour in the derivation of these continuum dislocation 60 61 models from the dynamics of a large group of interacting dislocations (called discrete dislocation models). 62

To resolve this lack of rigour, over the course of two decades a large mathematical community has established rigorous connections between discrete and continuum dislocation models; see [1, 10, 11, 13, 18, 29, 30] for a few examples of different discrete dislocation models and different techniques. The final aim is to lift all the currently required simplifications on the discrete dislocation models without losing the rigorous connection(s) with the related continuum model(s).

In recent years, the simplification that all dislocations have the same Burgers vector is being lifted. This generalisation corresponds to particle systems with mul-70 tiple species. It has the difficulty that dislocations with different Burgers vector may 71 collide in finite time (due to the singular stress they exert). In particular, two (screw) 72 73 dislocations with opposite Burgers vector are known to collide in finite time [23], and disappear upon collision. Such a collision is called annihilation. In the current litera-74ture, the difficulty of including annihilation or other collision rules is side-stepped by 75 either enforcing geometrical restrictions [7, 46], or by introducing an artificial regu-76 77 larisation of the singularity in the stress field (see [12] and [44, Chap. 9]). A common observation in these papers is that, depending on the geometrical restrictions or the 78 regularisation, rigid micro-structures can appear over time which are not recovered by 79 the expected continuum dislocation model. In fact, the simulations in [44, Chap. 9] 80 show that the group behaviour of dislocations can depend on the choice of regulari-81 82 sation, which would imply that the continuum model has to depend on the choice of regularisation. 83

Therefore, to avoid the dependence of the continuum model on the choice of regularisation or geometrical restrictions, we aim to make the first step for including dislocation annihilation in connecting discrete to continuum dislocation models. Our novel result includes an annihilation rule, but sidesteps the additional difficulty that prior to collision, the speed of the colliding dislocations becomes unbounded. To avoid <sup>89</sup> unbounded velocities prior to collision, we replace the singular interaction between

90 dislocations of opposite Burgers vector by a regular one. This choice induces the

91 further restriction of a one-dimensional spatial setting, which is needed to enforce col-

lisions. Indeed, for regular interactions in higher dimensions, dislocations of opposite
Burgers vector need not collide in finite time.

In Section 1.7 we demonstrate how our main result can be used as a stepping stone for considering annihilation with singular interactions between dislocations of opposite Burgers vector.

**1.2.** The discrete problem (particle system with annihilation). We re-97 turn our attention from dislocations to a more general particle system with two species 98 and an annihilation rule. We introduce the related evolution problem by first spec-99 ifying the state of the system, then the related interaction energy, and finally the 100 evolution law. The state of the system is described by  $x \coloneqq (x_1, \ldots, x_n) \in \mathbb{R}^n$  and 101  $b := (b_1, \ldots, b_n) \in \{-1, 0, 1\}^n$ , with  $n \ge 2$  the number of particles. The point  $x_i$  is 102the location of the *i*-th particle, and  $b_i$  is its charge (or Burgers vector, in the setting 103of dislocations). 104

105 To any state (x, b) we assign the interaction energy  $E_n : \mathbb{R}^n \times \{-1, 0, 1\}^n \to$ 106  $\mathbb{R} \cup \{+\infty\}$  by

107 (1.1) 
$$E_n(x;b) \coloneqq \frac{1}{2n^2} \sum_{i=1}^n \left( \sum_{\substack{j=1\\j\neq i\\b_ib_j=1}}^n V(x_i - x_j) + \sum_{\substack{j=1\\b_ib_j=-1}}^n W(x_i - x_j) \right),$$

where V and W are the interaction potentials between particles of equal and opposite charge, respectively. For V and W, we have three choices in mind, all of which are of separate interest:

- (i)  $V(r) = -\log |r|$  and  $W \equiv 0$ . This corresponds to the easiest case in which the two species only interact with their own kind. It is distinct from the single-particle case solely by the annihilation rule which we specify below. We consider this setting as a convenient benchmark problem, but we have no direct application in mind.
- 116 (ii)  $V(r) = -\log |r|$  and W a regularisation of -V (as illustrated in Figure 1). 117 This is a first step to considering the case of positive and negative charges 118 (or positive and negative dislocations) in which W = -V is chosen in a two-119 dimensional setup [40, 41, 43]. After stating our main result for regular W, we 120 comment in Subsection 1.7 on how this result helps in passing to the limit in 121 the particle dynamics corresponding to regular potentials  $W_{\delta}$  which converge 122 to the singular -V as the regularisation parameter  $\delta$  tends to 0.
- (iii)  $V(r) = r \coth r \log |2 \sinh r|$  and W a regularisation of -V. This setting 123corresponds to that of dislocation walls, *i.e.*, infinite arrays of equi-spaced 124 dislocations. The explicit expression for V is found by summing over all 125dislocations in such a wall; see [21, (19-75)] or [46, Sec. 2]. This potential 126V has several pleasant properties: it has a logarithmic singularity at 0, it 127 is decreasing on  $(0, \infty)$ , and it is positive with integrable tails. Discrete-128 129 to-continuum limits of particle systems consisting of interacting dislocation walls are established in [13, 17, 21, 46, 47, 48] for either single-sign scenarios 130or without annihilation. 131

For our analysis, we propose a unified setting which includes the three cases above: we consider a class of potentials V and W which satisfy a certain set of assumptions



FIGURE 1. Plots of  $V(r) = -\log |r|$  and a typical regularisation W of -V.

specified in Assumption 2.1. The crucial assumptions are that the singularity of V at 0 is at most logarithmic, that  $V(r) \to +\infty$  as  $r \to 0$ , that W is regular, and that V and W have at most logarithmic growth at infinity. In view of other typical assumptions in the literature, we *do not* rely on convexity or monotonicity. In Subsection 1.6 we elaborate on the necessity of these assumptions to our main discrete-to-continuum result.

Finally, we make three observations on the structure of (1.1). First, if the *i*-th particle has 0 charge (*i.e.*,  $b_i = 0$ ), then it does not contribute to  $E_n$ . Second, the factor 1/2 in front of the energy is common; it corrects the fact that all interactions are counted twice in the summation. Third, the condition  $j \neq i$  prevents self-interaction.

Equation (1.2) formally describes the dynamics; for a rigorous definition see Problem 4.1 and Definition 4.2.

147 (1.2) 
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_i = -\frac{1}{n}\sum_{j:b_ib_j=1}V'(x_i - x_j) - \frac{1}{n}\sum_{j:b_ib_j=-1}W'(x_i - x_j) & \text{on } (0,T) \setminus T_{\mathrm{col}},\\ \text{annihilation rule at } T_{\mathrm{col}}. \end{cases}$$

Here,  $T_{\text{col}} = \{t_1, \ldots, t_K\}$  is a finite set of collision times, outside of which x(t) is the gradient flow of  $E_n$ . The version of (1.2) in two dimensions and in which  $W(r) = -V(r) = \log |r|$  is discussed in great detail in [41].

Next we explain the "annihilation rule at  $T_{col}$ ". Given that at t = 0 all particles 151are at different positions, (1.2) follows for at least a small time interval simply the 152gradient flow of  $E_n(\cdot; b)$  in which b is constant in time. Since V is a singular, repelling 153interaction potential and W is regular, particles of the same sign will not cross each 154other, but particles of opposite sign may. We call the first time instance at which 155such a crossing happens a *collision time*, and denote it by  $t_1$ . At  $t_1$ , the annihilation 156rule states that those particles of opposite sign which are at the same position are 157158'removed' from the system, and that the system is restarted at time  $t_1$  with the remaining particles at their current positions. It again follows the gradient flow of  $E_n$ 159160 (but now with fewer particles) until the next collision time  $t_2$  at which two particles of opposite sign cross. At  $t_2$ , an analogous annihilation rule is applied. In this manner, 161 $T_{\rm col}$  is constructed. We allow for more than one pair of particles to annihilate at the 162same time instance  $t_k$ . Because of the singularity of V, annihilations that happen at 163

For technical reasons, we encode the removal of particles by putting their charge  $b_i(t)$  from  $\pm 1$  to 0 as opposed to making *n* dependent on *t*. We note that, if particle *i* has zero charge, then

168 •  $x_i(t)$  remains stationary,

169

- the velocity of all other particles does not depend on  $x_i(t)$ , and
- particle *i* cannot annihilate any more with any other particle.

We note that each  $b_i(t)$  is a shifted Heaviside functions that jumps at some collision time  $t_k$ .

Next we motivate the applicability of (1.2) by two related examples. The first example is that of dislocations, whose dynamical law naturally includes annihilation effects. The linear relation in (1.2) between the velocity and the gradient of the energy is purely phenomenological, and is, due to its simplicity and lack of consensus for a better alternative, the most commonly used relation in dislocation dynamics models. We refer to [43] for simulations of a generalized version of (1.2) in the context of dislocations.

The second example of a system related to (1.2) is that in [40] and [41, Theo-180 rems 1.3 and 1.4], where the limit of the Ginzburg-Landau equation on the dynamics 181 182 of vortices is studied as the phase-field parameter  $\varepsilon$  tends to 0. In the limiting equation, the vortices are characterised as points with a charge whose dynamics are given 183by the version of (1.2) in which  $W(r) = -V(r) = \log |r|$  and the particles are two-184 dimensional. While detailed properties of the particles trajectories are proven, a pre-185cise solution concept to this version of (1.2) remains elusive. In our one-dimensional 186 187 setting, we establish a solution concept to (1.2) in Definition 4.2 and Proposition 4.5.

1.3. The continuum problem (PDE for the particle density). On the continuum level, the state of the system is described by the nonnegative measures  $\rho^{\pm}$ , which represent the density of the positive/negative particles (including those that are annihilated). We further set

192 
$$\rho := \rho^+ + \rho^- \quad \text{and} \quad \kappa := \rho^+ - \rho^-,$$

and require the total mass of  $\rho$  to be 1. We note that  $\rho^+$  and  $\rho^-$  need not be mutually singular, and thus  $\rho^{\pm} \geq [\kappa]_{\pm}$ , where  $[\kappa]_{\pm}$  denotes the positive/negative part of the signed measure  $\kappa$ . We interpret  $[\kappa]_{\pm}$  as the density of positive/negative particles that have not been annihilated yet.

197 For  $\rho^{\pm}(t)$  we consider the following set of evolution equations

198 (1.3) 
$$\begin{cases} \partial_t \rho^+ = \left( [\kappa]_+ \left( V' * [\kappa]_+ + W' * [\kappa]_- \right) \right)' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \\ \partial_t \rho^- = \left( [\kappa]_- \left( V' * [\kappa]_- + W' * [\kappa]_+ \right) \right)' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \end{cases}$$

where we denote by the prime symbol ' the derivative with respect to the spatial variable. We remark that no annihilation rule is specified; the annihilation is encoded in taking the positive/negative part of  $\kappa$ . Indeed, it is easy to imagine that while the integral of  $\rho = \rho^+ + \rho^-$  is conserved in time, the integral of  $[\kappa]_+ + [\kappa]_- = |\rho^+ - \rho^-|$ may not be conserved.

**1.4.** Main result: discrete-to-continuum limit. Our main theorem (Theorem 5.1) states that the solutions to (1.2) converge to a solution of (1.3) as  $n \to \infty$ . It specifies the concept of solution to both problems, the required conditions on the sequence of initial data of (1.2), and guarantees that the so-constructed solution to (1.3) at time 0 corresponds to the limit of the initial conditions as  $n \to \infty$ . The convergence is uniform in time on [0, T] for any T > 0. The convergence in space is with respect to the weak convergence. As a by-product of Theorem 5.1, we obtain global-in-time existence of a solution  $(\rho^+, \rho^-)$  to (1.3) for which the masses of  $\rho^{\pm}$  are conserved in time.

In order to give effectively an outline of the proof and the motivation for the main assumptions under which Theorem 5.1 holds (Subsection 1.6), we first describe the related literature.

**1.5. Related literature.** We start by relating (1.3) formally to its singular counterpart. Replacing W by -V, we obtain from a formal calculation that the difference of the two equations in (1.3) is given by

219 (1.4) 
$$\partial_t \kappa = \left( |\kappa| (V' * \kappa) \right)'.$$

For  $V(r) = -\log |r|$ , equation (1.4) was introduced by [20] and later proven in [6] to attain unique solutions when posed on  $\mathbb{R}$  with proper initial data.

In the remainder of this subsection, we put our main result Theorem 5.1 in the 2.2.2 perspective of the literature. We start by describing those specifications of [10, 28, 29]223 which are closest to our main result. A specification of [10, Theorems 2.1–2.3] proves a 224'discrete'-to-continuum result from (1.2) to (1.4), in the case where V(r) = -W(r) is 225a regularisation of  $-\log |r|$  on the length-scale 1/n. We put 'discrete' in apostrophes, 226 because their equivalent of (1.2), given by [10, equation (5)], is a Hamilton-Jacobi 227 equation, which includes the solution to (1.2) only if all particles have the same sign. 228 It is not clear if this Hamilton-Jacobi equation relates to (1.2) if the particles have 229opposite sign. 230

As opposed to [10], [29] starts from a different Hamilton-Jacobi equation, which 231 corresponds to the Peierls-Nabarro model [32, 33]. This model is a phase-field model 232 for the dynamics of dislocations which naturally includes annihilation. In this model, 233 opposite to encoding dislocations as points on the line, the dislocations are identified 234by the pulses of the derivative of a multi-layer phase field on the real line. In [29], the 235 width of these pulses is taken to be on the same length-scale as the typical distance 236between neighbouring dislocations. Then, in the joint limit when the regularisation 237 238 length-scale (and thus simultaneously 1/n) tend to 0, an *implicit* Hamilton-Jacobi equation is recovered [29]. In [28, Theorem 1.2] it is shown that this implicit Hamilton-239Jacobi equation converges to (1.4) in the dilute dislocation density limit. While this 240 framework seems promising for a direct 'discrete'-to-continuum result ('discrete' being 241the Peierls-Nabarro model) to (1.3), it only applies to co-dimension 1 objects, *i.e.*, 242 243 particles in 1D and curves in 2D.

Regarding the continuum problem (1.3), we have not found this set of equations in the literature. Nonetheless, we believe the case W = 0 to be of independent interest, since then (1.3) serves as the easiest benchmark problem for future studies on annihilating particles. Also, since our discrete-to-continuum result holds for taking W as a regularisation of -V, we expect that (1.4) can be obtained from (1.3) as the regularisation length-scale tends to 0 (see Subsection 1.7). Therefore, we review the literature on (1.4).

Equation (1.4) as posed on  $\mathbb{R}$  with  $V(r) = -\log |r|$ , or even  $V(r) = |r|^{-a}$  with 0 < a < 1, attains a self-similar solution [6, Theorem 2.4] in which  $\kappa$  has a sign. The self-similar solution is expanding in time (due to the repelling interaction force V'(r)), and describes the long-time behaviour of the unique viscosity solutions to (1.4) [6, Theorem 2.5] for appropriate initial data. Moreover, for  $V(r) = -\log |r|$  and initial condition  $\kappa^{\circ} \in L^1(\mathbb{R})$ , the viscosity solution  $\kappa$  to (1.4) satisfies  $\kappa(t) \in L^p(\mathbb{R})$ for all  $1 \leq p \leq \infty$  [6, Theorem 2.7]. In conclusion, despite (1.4) being the singular counterpart of (1.3), it has a well-defined global-in-time solution concept.

Lastly, we compare our result to that of [3]. There, the authors are interested 259in deriving a gradient flow structure of (1.4) on  $\mathbb{R}^2$  with V having a logarithmic 260singularity at 0 by defining a discrete in time minimising movement scheme and 261 passing to the limit as the time step size tends to 0. The related convergence result is 262[3, Theorem 1.4]. However, the limit equation is not fully characterised as (1.4), since 263in that equation  $|\kappa|$  is replaced by an unknown measure  $\mu \geq |\kappa|$  which is obtained from 264compactness. The connection to our main result is that we faced a similar problem. 265Due to our 1D setup and by a technical assumption on the initial data, we were able 266to characterise the corresponding  $\mu$  as  $|\kappa|$ . 267

**1.6.** Discussion on the proof, assumptions, and possible extensions. We divide this section into several topics regarding the proof, assumptions, and possible extensions of Theorem 5.1 (outlined in Subsection 1.4).

Summary of the proof. A crucial step is the observation that the solution to (1.2), seen as a pair of empirical measures  $\mu_n^{\pm}$ , is a solution to (1.3), *i.e.*,

273 (1.5) 
$$\begin{cases} \partial_t \mu_n^+ = ([\kappa_n]_+ (V' * [\kappa_n]_+ + W' * [\kappa_n]_-))' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \\ \partial_t \mu_n^- = ([\kappa_n]_- (V' * [\kappa_n]_- + W' * [\kappa_n]_+))' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \end{cases}$$

where  $\kappa_n := \mu_n^+ - \mu_n^-$ . The annihilation is completely covered by taking the positive 274and negative part of  $\kappa_n$ . This property is the reason for encoding annihilation in the 275276charges  $b_i(t)$  rather than removing particles from the dynamics. Then, relying on the gradient flow structure underlying (1.2) and the boundedness of W, we find, by 277the usual compactness arguments à la Arzelà-Ascoli, limiting curves  $\rho^{\pm}(t)$ . It then 278remains to pass to the limit  $n \to \infty$  in (1.5). The difficulty is in characterising the 279limit of  $[\kappa_n]_+$ , which only accounts for the particles that have not collided yet. Indeed, 280the convergence of measures is not invariant with respect to taking the positive and 281negative part. It is here that we heavily rely on the one-dimensional setting and 282 on a technical assumption on the initial data (Assumption 2.2), which provides an 283*n*-independent bound on the number of neighbouring pairs of particles with opposite 284sign. This bound allows us to characterise the limit of  $[\kappa_n]_{\pm}$  as  $[\kappa]_{\pm}$ . 285

Motivation for Assumption 2.2. Assumption 2.2 prevents small-scale oscillations between  $\pm 1$  phases. A similar assumption is made in [29], where the initial data for the particles is constructed from the continuum initial datum. While one might expect that small-scale oscillations cancel out on small time scales, the simulations in [45, Chapter 9] suggest otherwise. The problem with such small-scale oscillations is that they cause the limit of  $[\kappa_n]_{\pm}$  to be larger than  $[\kappa]_{\pm}$ , which makes it difficult to characterise the limit as  $n \to \infty$  of (1.5) as (1.3).

Singularity of V. Assuming the singularity of V to be at most logarithmic is needed to apply the discrete-to-continuum limit passage technique in [38].

In fact, we also require that  $V(r) \to \infty$  as  $r \to 0$ , *i.e.*, we do not allow for a regular V. While regular V and W (in particular W = -V) would simplify the equations and many steps in the proof of our main theorem, it may result in two technical difficulties: collision between three or more particles, and the limiting signed measure  $\kappa$  having atoms. These difficulties complicate the convergence proof of  $[\kappa_n]_{\pm}$  to  $[\kappa]_{\pm}$  as  $n \to \infty$ . Since all our intended applications correspond to singular potentials V, we choose to side-step these technical difficulties by simply requiring V to have a singularity at 0. Regularity of W. W being bounded around 0 results in a lower bound on the energy along the evolution, which we need for equicontinuity and thus for compactness of  $\mu_n^{\pm}$ . Also, while passing to the limit  $n \to \infty$  in (1.5), we need W' regular enough (the technique in [38] does not apply for logarithmic W).

Logarithmic tails of V, W. While it would be easier to assume that V is bounded from below and W is globally bounded, we also allow for logarithmic tails to include all three scenarios in Subsection 1.2. The logarithmic tails of V and W result in the energy  $E_n$  to be unbounded from below. However, following the idea in [38] to prove a priori bounds on the moments of  $\mu_n^{\pm}(t)$ , we easily obtain that  $E(\mu_n^{\pm}(t))$  is bounded from below by -C(1+t) for some C > 0 independent of n and t.

Uniqueness of solutions to (1.3). While Theorem 5.1 provides a solution of (1.3) that exists globally in time, we have not investigated uniqueness. We rather interpret (1.3) as a stepping stone for a future convergence result to (1.4), for which a uniqueness result is established in [6].

**1.7. Conclusion and outlook.** We intend our main result to open a new thread
 of research on including annihilation in discrete-to-continuum limits. Here we discuss
 several open ends.

319 W = -V singular. This setting corresponds to charges (or dislocations) on the 320 real line. On the continuum level, see (1.4), this equation is well-understood [6], 321 but on the discrete level we have not found a closed set of equations to describe 322 the discrete counterpart of (1.2) (other than [40, 41], whose results are discussed in 323 Subsection 1.5). Since our main result does allow for -W to be a regularisation  $V_{\delta}$  of 324 V ( $\delta$  denotes the arbitrarily small, but fixed, length-scale of the regularisation), this 325 calls for three interesting limit passages:

(a)  $\delta \to 0$  with n fixed. This limit seems the easiest out of the three. Similar to 326 [40, 41], the idea is to pass to the limit, and *describe* the limit rather than 327 328 posing a closed set of equations for it. One challenge is that in the limiting curves prior to collision at  $t_*$ , the particles' speed blows up as  $\sim 1/\sqrt{t_*-t}$ 329 (this is easily seen by considering only two particles; one positive and one 330 negative). While the resulting curves are not Lipschitz in time, they are  $C^{1/2}$ 331 in time. However, such collisions correspond to  $-\infty$  wells in the energy, which 332 require the development of a proper renormalisation of  $E_n$ . 333

Another challenge is that particles need not collide if they come close, regard-334 less how small  $\delta > 0$  is. To see this, consider two particles with opposite sign 335 and with mutual distance smaller than  $\delta$ . Since  $V_{\delta}$  is regular, the particles 336 will come exponentially close, but they will not collide in finite time. In the 337 case of many particles, such a close pair will only collide if the external force 338 339 (induced by the other particles) acts in the right direction. If it does not collide, then the pair remains in the system (as opposed to the case of singular 340 W), and may even interact with or annihilate other particles that come close. 341 (b) Connecting (1.3) to (1.4) by  $\delta \to 0$ . Taking  $W = -V_{\delta}$  and setting  $\rho_{\delta}^{\pm}$  as a 342 corresponding solution to (1.3), it is impossible to pass directly to the limit in

corresponding solution to (1.3), it is impossible to pass directly to the limit in (1.3) due to the term  $[\kappa_{\delta}]_{\pm}(V'_{\delta}*[\kappa_{\delta}]_{\mp})$ . Instead, the structure of (1.4) in terms of viscosity solutions (see [6]) seems promising. We leave it to future research to find out whether (1.3) enjoys a similar structure, and if not, whether there is a different continuum model for annihilating particles that does.

(c) Connecting (1.2) to (1.4) by a joint limit  $n \to \infty$  and  $\delta_n \to 0$ . This approach fits to the convergence result obtained in [29], where roughly speaking  $\delta_n \sim$ 

Different regularisations of collisions. In the spirit of proving any of the above 353 limit passages, we discuss alternative regularisations other than taking W regular. One idea is 'premature annihilation', where particles are removed from the system 355 when they come  $\delta$ -close, with  $\delta > 0$  a regularisation parameter. This approach is 357 commonly adapted in numerical simulations of discrete systems with an annihilation rule. However, it is not obvious what the limiting equation as  $n \to \infty$  (counterpart 358 of (1.4)) is for  $\delta > 0$  fixed, because we expect the supports of  $[\kappa]_+$  and  $[\kappa]_-$  to be 359 separated by at least  $\delta$ . A third option is to mollify the jump of the charge  $b_i(t)$  from 360  $\pm 1$  to 0, possibly by an additional ODE for  $b_i(t)$ . We have not found a proper rule 361 362 for this that would still allow for a discrete-to-continuum convergence result.

Higher dimensions. In this paragraph we consider the extension to two dimen-363 sions; the discussion easily extends to higher dimensions. The one ingredient in our 364 proof which intrinsically relies on our 1D setting, is the *separation* condition on the 365 366 initial data. This condition limits the collisions to happen only at a finite number of points. In 2D, collisions are bound to happen along curves (or more complicated 367 subsets of  $\mathbb{R}^2$ ), which makes it challenging to characterise the limit of  $[\kappa_n]_{\pm}$ . A similar 368 problem occurred in [3] as discussed in Subsection 1.5. In future research we plan to 369 relax our 'separation' assumption, possibly by considering a different regularisation 370 of collisions. 371

The remainder of the paper is organised as follows. In Section 2 we fix our notation and list the assumptions on V, W and the initial data. In Section 3 we recall known results and provide the preliminaries. In Section 4 we give a rigorous definition of (1.2), show that it attains a unique solution, and establish several properties of it. In Section 5 we state and prove our main result, Theorem 5.1.

2. Notation and standing assumptions. Here we list the symbols and notation which we use in the remainder of this paper:

$\mathcal{B}(\mathbb{R})$	space of Borel sets on $\mathbb{R}$	Section 3
f(a-)	$\lim_{y\uparrow a}f(y)$	
$[f]_{\pm}$	positive or negative part of $f$	
$\mu\otimes u$	product measure; $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$	Section 3
C > 0	constant whose value can possibly change from	
	line to line	
$\mu$	$\boldsymbol{\mu} \coloneqq (\mu^+, \mu^-) \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$	(3.2)
$\mathcal{M}(\mathbb{R})$	space of finite, signed Borel measures on $\mathbb R$	Section 3
$\mathcal{M}_+(\mathbb{R})$	space of the non-negative measures in $\mathcal{M}(\mathbb{R})$	Section 3
$\mathbb{N}$	$\{1, 2, 3, \ldots\}$	
$\mathcal{P}(\mathbb{R})$	space of probability measures;	Section 3
	$\mathcal{P}(\mathbb{R}) = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \mu(\mathbb{R}) = 1\}$	
$\mathcal{P}_2(\mathbb{R})$	probability measures with finite second moment;	Section 3
	$\mathcal{P}_2(\mathbb{R}) = \{ \mu \in \mathcal{P}_2(\mathbb{R}) : \int_{-\infty}^{\infty} x^2 \mathrm{d}\mu(x) < \infty \}$	
V	interaction potential for equally signed particles	Assumption 2.1
W	interaction potential for oppositely signed particles	Assumption 2.1
$W(\mu, u)$	2-Wasserstein distance between $\mu, \nu \in \mathcal{P}(\mathbb{R})$	[2]
$\mathbf{W}(oldsymbol{\mu},oldsymbol{ u})$	2-Wasserstein distance between $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$	(3.3)

Assumption 2.1 lists the standing properties which we impose on V and W.

ASSUMPTION 2.1. We require that the interaction potentials  $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$  and W:  $\mathbb{R} \to \mathbb{R}$  satisfy the following conditions:

382 (2.1a)  $V \in C^1(\mathbb{R} \setminus \{0\}), W \in C^1(\mathbb{R}), V' \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R} \setminus \{0\}), and W' \in \operatorname{Lip}(\mathbb{R}),$ 

- 383 (2.1b) V and W are even;
- 384 (2.1c)  $V(r) \rightarrow +\infty \text{ as } r \rightarrow 0;$
- 385 (2.1d)  $r \mapsto rV'(r)$  and  $r \mapsto rW'(r)$  are in  $L^{\infty}(\mathbb{R})$ .

For convenience, we set  $V'(0) \coloneqq 0$ . Below we list two remarks on Assumption 2.1: • we assume no monotonicity on V or W;

• Condition (2.1d) implies that V has at most a logarithmic singularity (as mentioned in Subsection 1.2), and that V and W have at most logarithmically diverging tails, namely

391 (2.2) 
$$|V(r)| + |W(r)| \le C(|\log |r|| + 1), \text{ for all } r \ne 0.$$

Due to condition (2.1c), and keeping (2.1a) into account, we can sharpen this inequality around 0 by

394 (2.3) 
$$(V+W)(r) \ge -Cr^2$$
, for all  $r \ne 0$ .

The following assumption on the initial data states that no pair of particles of opposite sign should start at the same position.

ASSUMPTION 2.2 (Separation assumption on the initial data  $(x^{\circ}; b^{\circ})$ ). There exist  $-\infty < a_0 \le a_1 \le \ldots \le a_{2L} < +\infty$  such that

399 
$$\{x_i^{\circ}: b_i^{\circ} = 1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-2}, a_{2\ell-1}), \qquad \{x_i^{\circ}: b_i^{\circ} = -1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-1}, a_{2\ell}).$$

The importance of this assumption is clarified later when the limit  $n \to \infty$  is considered, in which the number L is assumed to be n-independent (see also Subsection 1.6). Moreover, we will show in Proposition 4.5 that this assumption is conserved in time.

**3. Preliminary results.** We collect here some basic definitions and known results that will be useful in the sequel.

405 **3.1. Probability spaces and the Wasserstein distance.** On  $\mathcal{P}_2(\mathbb{R})$  (space 406 of probability measures with finite second moment; see Section 2), the square of the 407 2-Wasserstein distance  $W(\mu, \nu)$  with  $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$  is defined as

408 (3.1) 
$$W^2(\mu,\nu) \coloneqq \inf_{\gamma \in \Gamma(\mu,\nu)} \iint_{\mathbb{R}^2} |x-y|^2 \,\mathrm{d}\gamma(x,y),$$

410 where  $\Gamma(\mu, \nu)$  is the set of couplings of  $\mu$  and  $\nu$ , namely,

411 
$$\Gamma(\mu,\nu) \coloneqq \left\{ \gamma \in \mathcal{P}(\mathbb{R}^2) : \gamma(A \times \mathbb{R}) = \mu(A), \ \gamma(\mathbb{R} \times A) = \nu(A) \text{ for all } A \in \mathcal{B}(\mathbb{R}) \right\}.$$

412 We refer to [4] for the basic properties of W. As usual, we set  $\Gamma_{\circ}(\mu, \nu) \subset \Gamma(\mu, \nu)$  as 413 the set of transport plans  $\gamma$  which minimise (3.1). Since we are working with positive and negative particles, we follow [12] by defining a space of probability measures on  $\mathbb{R} \times \{\pm 1\}$ , where  $\mathbb{R} \times \{\pm 1\}$  is endowed with the distance

417 
$$d^2(\bar{x},\bar{y}) := |x-y|^2 + |p-q|, \quad \bar{x} = (x,p) \in \mathbb{R} \times \{\pm 1\}, \ \bar{y} = (y,q) \in \mathbb{R} \times \{\pm 1\}.$$

418 We denote this probability space by  $\mathcal{P}(\mathbb{R} \times \{\pm 1\})$ , and its elements by  $\mu$  or  $(\mu^+, \mu^-)$ , 419 with the understanding that

420 (3.2) 
$$\mu(A^+, A^-) = \mu^+(A^+) + \mu^-(A^-), \quad \text{for all } A^+, A^- \in \mathcal{B}(\mathbb{R}).$$

421 On

427

$$\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}) \coloneqq \left\{ \boldsymbol{\mu} \in \mathcal{P}(\mathbb{R} \times \{\pm 1\}) : \int_{\mathbb{R}} |x|^2 \, \mathrm{d}\boldsymbol{\mu}^{\pm}(x) < +\infty \right\}$$

423 we define the (square of the) 2-Wasserstein distance between  $\mu$  and  $\nu$  as

424 (3.3) 
$$\mathbf{W}^{2}(\boldsymbol{\mu},\boldsymbol{\nu}) \coloneqq \inf_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}(\boldsymbol{\mu},\boldsymbol{\nu})} \iint_{(\mathbb{R}\times\{\pm1\})^{2}} \mathsf{d}^{2}(\bar{x},\bar{y}) \,\mathrm{d}\boldsymbol{\gamma}(\bar{x},\bar{y}),$$

426 where  $\Gamma(\mu, \nu)$  is the set of couplings of  $\mu$  and  $\nu$ , namely,

$$\begin{split} \boldsymbol{\Gamma}(\boldsymbol{\mu},\boldsymbol{\nu}) \coloneqq & \left\{ \boldsymbol{\gamma} \in \mathcal{P}\big( (\mathbb{R} \times \{\pm 1\})^2 \big) : \boldsymbol{\gamma}(A \times (\mathbb{R} \times \{\pm 1\})) = \boldsymbol{\mu}(A), \\ & \boldsymbol{\gamma}((\mathbb{R} \times \{\pm 1\}) \times A) = \boldsymbol{\nu}(A) \text{ for all } A \in \mathcal{B}(\mathbb{R} \times \{\pm 1\}) \right\} \end{split}$$

Since it turns out that (1.3) has a mass-preserving solution  $\rho(t) := (\rho^+(t), \rho^-(t))$ belonging to  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ , for which also the mass of  $\rho^+(t)$  and  $\rho^-(t)$  is conserved in time, we define the corresponding subspace

431 
$$\mathcal{P}_2^m(\mathbb{R}\times\{\pm 1\}) \coloneqq \{\boldsymbol{\mu}\in\mathcal{P}_2(\mathbb{R}\times\{\pm 1\}): \boldsymbol{\mu}^+(\mathbb{R})=m\};$$

432 where  $m \in [0,1]$  is the total mass of the positive particle density. Clearly, if  $\boldsymbol{\mu} \in$ 433  $\mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$ , then  $\boldsymbol{\mu}^-(\mathbb{R}) = 1 - m$ . For any  $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$  we have that

434 (3.4) 
$$\mathbf{W}^{2}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq W^{2}(\boldsymbol{\mu}^{+}, \boldsymbol{\nu}^{+}) + W^{2}(\boldsymbol{\mu}^{-}, \boldsymbol{\nu}^{-}),$$

435 which simply follows by shrinking the set of couplings  $\Gamma(\mu, \nu)$  in (3.3).

436 **3.2. Weak form of the continuum problem (1.3).** We use the following 437 notation convention. For any  $\rho \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$ , we set

438 (3.5) 
$$\rho \coloneqq \rho^+ + \rho^- \in \mathcal{P}(\mathbb{R}), \qquad \kappa \coloneqq \rho^+ - \rho^- \in \mathcal{M}(\mathbb{R}), \qquad \tilde{\rho}^{\pm} \coloneqq [\kappa]_{\pm} \in \mathcal{M}_+(\mathbb{R}).$$

439 We consider the following weak form of (1.3): given an initial condition  $\rho^{\circ} \in \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ , find  $\rho$  satisfying

$$0 = \int_0^T \int_{\mathbb{R}} \partial_t \varphi^{\pm}(x) \, \mathrm{d}\rho^{\pm}(x) \, \mathrm{d}t$$

$$- \frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} \left( (\varphi^{\pm})'(x) - (\varphi^{\pm})'(y) \right) V'(x-y) \, \mathrm{d}([\kappa]_{\pm} \otimes [\kappa]_{\pm})(x,y) \, \mathrm{d}t$$

$$- \int_0^T \int_{\mathbb{R}} (\varphi^{\pm})'(x) \, (W' \ast [\kappa]_{\mp})(x) \, \mathrm{d}[\kappa]_{\pm}(x) \, \mathrm{d}t,$$

442 for all  $\varphi^{\pm} \in C_c^{\infty}((0,T) \times \mathbb{R})$ , where we have exploited that V' is odd. We seek a 443 solution of (3.6) in AC(0,  $T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$  with  $m = \rho^{\circ,+}(\mathbb{R}) \in [0,1]$ . **3.3. Several topologies and their connections.** Next we define the space of absolutely continuous curves and their metric derivatives. While the following definitions work on any complete metric space, we limit our exposition to  $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$ . For any  $1 \leq p < \infty$ ,  $\operatorname{AC}^p(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  denotes the space of all curves  $\boldsymbol{\mu} : (0, T) \to \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$  for which there exists a function  $f \in \operatorname{L}^p(0, T)$  such that

450 (3.7) 
$$\mathbf{W}(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t)) \leq \int_{s}^{t} |f(r)|^{p} \, \mathrm{d}r, \quad \text{for all } 0 < s \leq t < T.$$

451 We set  $AC(0,T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})) \coloneqq AC^1(0,T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . By [2, Theorem 1.1.2], 452 the metric derivative

453 (3.8) 
$$|\boldsymbol{\mu}'|_{\mathbf{W}}(t) \coloneqq \lim_{s \to t} \frac{\mathbf{W}(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t))}{|s - t|}$$

is defined for any  $\boldsymbol{\mu} \in \operatorname{AC}(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  and for a.e.  $t \in (0, T)$ . Moreover,  $|\boldsymbol{\mu}'|_{\mathbf{W}}$ is a possible choice for f in (3.7).

The following theorem is a simplified version of [31, Theorem 47.1] applied to the metric space  $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W}).$ 

458 LEMMA 3.1 (Ascoli-Arzelà).  $\mathcal{F} \subset C([0,T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  is pre-compact if and 459 only if

460 (i)  $\{\boldsymbol{\mu}(t) : \boldsymbol{\mu} \in \mathcal{F}\}$  is pre-compact in  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$  for all  $t \in [0, T]$ ,

461 (*ii*)  $\forall \varepsilon > 0 \exists \delta > 0$  such that  $\forall \mu \in \mathcal{F}, \forall t, s \in [0,T] : |t-s| < \delta \implies$ 462  $\mathbf{W}(\mu(t), \mu(s)) < \varepsilon.$ 

The following theorem provides a lower semi-continuity result on the  $L^2(0,T)$ norm of the metric derivative. We expect it to be well-known, but we only found it proven in the PhD thesis [45, Lemma 8.2.8].

466 THEOREM 3.2 (Lower semi-continuity of metric derivatives). Let  $\mu_n, \mu : [0, T] \rightarrow$ 467  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ . If  $\mathbf{W}(\mu_n(t), \mu(t)) \rightarrow 0$  as  $n \rightarrow \infty$  pointwise for a.e.  $t \in (0, T)$ , then

468 (3.9) 
$$\liminf_{n \to \infty} \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) \, \mathrm{d}t \ge \int_0^T |\boldsymbol{\mu}'|_{\mathbf{W}}^2(t) \, \mathrm{d}t.$$

469 Proof. We start with several preparations. First, we take a dense subset  $(t_{\ell})_{\ell}$  of 470 [0,T] for which  $\mathbf{W}(\boldsymbol{\mu}_n(t_{\ell}), \boldsymbol{\mu}(t_{\ell})) \to 0$  as  $n \to \infty$  for any  $\ell \in \mathbb{N}$ . Second, without loss 471 of generality, we assume that there exists C > 0 such that for all n

472 (3.10) 
$$\int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) \, \mathrm{d}t \le C.$$

In particular, this means that  $\boldsymbol{\mu}_n$  has a representative in  $\operatorname{AC}^2(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ which is defined for all  $t \in (0, T)$ . Taking this representative, we set  $D_n^{\ell}(t) :=$  $\mathbf{W}(\boldsymbol{\mu}_n(t_{\ell}), \boldsymbol{\mu}_n(t))$ , and obtain from [2, Theorem 1.1.2] that

476 (3.11) 
$$|\boldsymbol{\mu}'_n|_{\mathbf{W}}(t) = \sup_{\ell \in \mathbb{N}} \left| (D_n^{\ell})'(t) \right|$$
 for a.e.  $t \in (0, T)$ .

477 Next we prove (3.9). Firstly, since  $\mathbf{W}(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}(t)) \to 0$  as  $n \to \infty$  for a.e.  $t \in$ 478 (0, T), we have for fixed  $\ell \in \mathbb{N}$  and for a.e.  $t \in (0, T)$  that

479 (3.12) 
$$|D_n^{\ell}(t) - D^{\ell}(t)| \xrightarrow{n \to \infty} 0, \quad \text{where } D^{\ell}(t) \coloneqq \mathbf{W}(\boldsymbol{\mu}(t_{\ell}), \boldsymbol{\mu}(t)).$$

480 Secondly,  $\|D_n^{\ell}\|_{H^1(0,T)}$  and  $\|D^{\ell}\|_{H^1(0,T)}$  are bounded uniformly in n and  $\ell$ . To see 481 this, we have by the definition of the metric derivative and (3.10) that

482 
$$D_n^{\ell}(t) \le \left| \int_{t_{\ell}}^t |\boldsymbol{\mu}_n'|_{\mathbf{W}}(s) \, \mathrm{d}s \right| \le C\sqrt{T}.$$

483 Hence,  $\|D_n^{\ell}\|_{L^2(0,T)}$  is uniformly bounded. With the characterisation of  $|\mu'_n|_{\mathbf{W}}$  in 484 (3.11), we estimate

485 (3.13) 
$$C \ge \int_0^T |\boldsymbol{\mu}'_n|^2_{\mathbf{W}}(t) \, \mathrm{d}t \ge \int_0^T \left( (D_n^\ell)'(t) \right)^2 \mathrm{d}t \quad \text{for all } \ell \in \mathbb{N},$$

and thus  $\|D_n^{\ell}\|_{H^1(0,T)}$  is uniformly bounded. Therefore, in view of (3.12), we have

487 (3.14) 
$$D_n^\ell \rightharpoonup D^\ell$$
 in  $H^1(0,T)$  as  $n \to \infty$ .

488 In particular, we observe from (3.14) that  $D^{\ell} \in H^1(0,T)$  and that

489 
$$C \ge \liminf_{n \to \infty} \|D_n^{\ell}\|_{H^1(0,T)} \ge \|D^{\ell}\|_{H^1(0,T)} \quad \text{for all } \ell \in \mathbb{N}.$$

To establish (3.9), we carefully perform a joint limit passage as  $n \to \infty$  and a maximisation over  $\ell$  in (3.13). With this aim, we take a large fixed  $L \in \mathbb{N}$ , and choose a partition  $\{A_\ell\}_{\ell=1}^L$  of Borel sets of (0,T) such that for all  $\ell = 1, \ldots, L$ ,

493 
$$\left| (D^{\ell})'(t) \right| = \sup_{1 \le \tilde{\ell} \le L} \left| (D^{\tilde{\ell}})'(t) \right| \quad \text{for a.e. } t \in A_{\ell}.$$

494 We estimate

495 
$$\int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) \, \mathrm{d}t \ge \int_0^T \sup_{1 \le \ell \le L} \left( (D_n^\ell)'(t) \right)^2 \mathrm{d}t \ge \sum_{\ell=1}^L \int_{A_\ell} \left( (D_n^\ell)'(t) \right)^2 \mathrm{d}t.$$

496 Using (3.14), we pass to the limit  $n \to \infty$  to obtain

497 
$$\liminf_{n \to \infty} \int_0^T |\boldsymbol{\mu}'_n|^2_{\mathbf{W}}(t) \, \mathrm{d}t \ge \sum_{\ell=1}^L \int_{A_\ell} \left( (D^\ell)'(t) \right)^2 \mathrm{d}t = \int_0^T \sup_{1 \le \ell \le L} \left( (D^\ell)'(t) \right)^2 \mathrm{d}t.$$

498 By using the Monotone Convergence Theorem, we take the supremum over  $L \in \mathbb{N}$  to 499 deduce that

500 
$$\liminf_{n \to \infty} \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) \, \mathrm{d}t \ge \int_0^T \sup_{\ell \in \mathbb{N}} \left( (D^\ell)'(t) \right)^2 \mathrm{d}t$$

501 We conclude by using [2, Theorem 1.1.2] to identify  $\sup_{\ell \in \mathbb{N}} |(D^{\ell})'|$  in  $L^2(0,T)$  by 502  $|\mu'|_{\mathbf{W}}$ .

Next we introduce the *narrow convergence* of measures. For  $\nu_n, \nu \in \mathcal{M}(\mathbb{R})$ , we say that  $\nu_n$  converges in the narrow topology to  $\nu$  (and write  $\nu_n \rightharpoonup \nu$ ) as  $n \rightarrow \infty$  if

505 
$$\int \varphi \, \mathrm{d}\nu_n \xrightarrow{n \to \infty} \int \varphi \, \mathrm{d}\nu.$$

for any bounded test function  $\varphi \in C(\mathbb{R})$ . The following lemma extends this notion for non-negative measures by allowing for discontinuous test functions.

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508 LEMMA 3.3 ([34, Lemma 2.1]). Let  $\nu_n \rightharpoonup \nu$  in  $\mathcal{M}_+(\mathbb{R}^d)$ . Let  $A \in \mathcal{B}(\mathbb{R}^d)$  such 509 that  $\nu(A) = 0$ . Then for every bounded  $\varphi \in C(\mathbb{R}^d \setminus A)$  it holds that

510 
$$\int \varphi \, \mathrm{d}\nu_n \xrightarrow{n \to \infty} \int \varphi \, \mathrm{d}\nu$$

Proofs can be found in [39, Theorems 62-63, chapter IV, paragraph 6] and in [8, 14], or [37] in the case where A is closed.

Finally, we state and prove a lemma which allows us to show that Assumption 2.2 is conserved in the limit as  $n \to \infty$ .

515 LEMMA 3.4 (Narrow topology preserves separation of supports). Let  $(\nu_{\varepsilon})_{\varepsilon>0}$ , 516  $(\rho_{\varepsilon})_{\varepsilon>0} \subset \mathcal{M}_{+}(\mathbb{R})$  converge in the narrow topology as  $\varepsilon \to 0$  to  $\nu$  and  $\rho$ , respectively. 517 If

518 
$$\forall \varepsilon > 0 : \sup(\operatorname{supp} \nu_{\varepsilon}) \leq \inf(\operatorname{supp} \rho_{\varepsilon})$$

519 then also  $\sup(\operatorname{supp} \nu) \leq \inf(\operatorname{supp} \rho)$ .

520 Proof. We reason by contradiction. Suppose  $M := \sup(\operatorname{supp} \nu) > \inf(\operatorname{supp} \rho) =:$ 521 m. Take a non-decreasing test function  $\varphi \in C_b(\mathbb{R})$  which satisfies

522 
$$\varphi \equiv 0 \text{ on } \left(-\infty, \frac{m+2M}{3}\right], \text{ and } \varphi \equiv 1 \text{ on } [M, \infty).$$

523 Since  $M = \sup(\operatorname{supp} \nu)$ , it holds that  $\int \varphi \, d\nu > 0$ . Hence, from  $\nu_{\varepsilon} \xrightarrow{\varepsilon \to 0} \nu$  we infer 524 that for all  $\varepsilon$  small enough, it also holds that  $\int \varphi \, d\nu_{\varepsilon} > 0$ , and thus

525 
$$\sup(\operatorname{supp}\nu_{\varepsilon}) \ge \frac{m+2M}{3}.$$

526 With a similar argument, we can deduce that  $\inf(\operatorname{supp} \rho_{\varepsilon}) \leq \frac{2m+M}{3}$ , which contradicts 527 with m < M.

4. Definition and properties of the discrete problem (1.2). In this section we give a rigorous definition to the discrete dynamics formally given by (1.2). We start by formulating it as Problem 4.1, which may have several solutions. Then, we define a precise solution concept to Problem 4.1 (see Definition 4.2) which encodes the annihilation rule and selects a unique solution to Problem 4.1. After establishing some properties of the energy  $E_n$  introduced in (1.1), we prove an existence and uniqueness result (see Proposition 4.5). Finally, we state the discrete problem in the language of measures (see Lemma 4.6).

536 PROBLEM 4.1. Given  $(x^{\circ}, b^{\circ}) \in \mathbb{R}^n \times \{\pm 1\}^n$  such that  $x_1^{\circ} < x_2^{\circ} < \ldots < x_n^{\circ}$ , find 537  $(x,b): [0,T] \to \mathbb{R}^n \times \{-1,0,1\}^n$  such that

538 (4.1) 
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_i = -\frac{1}{n} \sum_{j: b_i b_j = 1} V'(x_i - x_j) - \frac{1}{n} \sum_{j: b_i b_j = -1} W'(x_i - x_j) & on \ (0, T) \setminus T_{\mathrm{col}} \\ (x_i(0), b_i(0)) = (x_i^{\circ}, b_i^{\circ}) \end{cases}$$

539 for all i = 1, ..., n, where  $T_{col}$  is the jump set of b.

540 We encode the annihilation rule in the solution concept below. With this aim, 541 we set  $H: \mathbb{R} \cup \{+\infty\} \rightarrow [0, 1]$  as the usual Heaviside function, with  $H(0) \coloneqq 0$  and 542  $H(+\infty) \coloneqq 1$ .

543 544 545 546	DEFINITION 4.2 (Solution to Problem 4.1). We say that $(x,b): [0,T] \to \mathbb{R}^n \times \{-1,0,1\}^n$ is a solution to Problem 4.1 if (a) there exists a vector of collision times $\tau = (\tau_1, \ldots, \tau_n)$ with $\tau_i \in (0,T) \cup \{+\infty\}$ such that, setting
547	$(4.2)  T_{col} \coloneqq \{\tau_i : 1 \le i \le n\} \setminus \{+\infty\} = \{t_1, t_2, \dots, t_K\} \subset (0, T)$
548	with $0 < t_1 < \ldots < t_K < T$ , there holds
549	(4.3) $b_i(t) \coloneqq b_i^{\circ} H(\tau_i - t)  \text{for all } i = 1, \dots, n;$
550 551 552 553	(b) $x \in \operatorname{Lip}([0,T]; \mathbb{R}^n) \cap C^1((0,T) \setminus T_{\operatorname{col}}; \mathbb{R}^n);$ (c) (4.1) is satisfied in the classical sense; (d) setting $t_0 \coloneqq 0$ , for all $k = 1, \ldots, K$ ,
554	$t_k = \inf \{t \in (0,T) : \exists (i,j) \text{ such that }$
558	$b_i(t_{k-1})b_j(t_{k-1}) = -1 \text{ and } x_i(t) = x_j(t) \} > t_{k-1};$
557	(e) at each time $t \in [0,T]$ , there is a bijection
558	$\alpha \colon \{i : b_i^{\circ} = 1,  \tau_i \le t\} \to \{j : b_j^{\circ} = -1,  \tau_j \le t\}$
559	such that $x_i(t) = x_{\alpha(i)}(t)$ .
560	Remark $4.3$ (Comments on Definition $4.2$ ). We collect here some remarks on the
561	notion of solution presented above.
562	• $\tau_i$ is the time at which particle $x_i$ gets annihilated: equation (4.3) describes
563	this by putting to zero the charge $b_i$ at time $\tau_i$ . If $\tau_i = +\infty$ , then it means
564	that the particle $x_i$ does not collide in the time interval $(0, T)$ .
565	• $(t_k)$ is the ordered list of collision times at which at least one collision occurs.
566	• In equation (4.1), both $x_i$ and $b_i$ depend on time. However, on each open
567	component of $(0,T) \setminus T_{col}$ , the charges $b_i$ remain constant.
568	• Since V is singular and W is regular, straight-forward a priori energy esti-
570	positive distance. Hence, the only type of collision that can occur is that
571	of two particles with opposite sign. We prove precise energy estimates in
572	Proposition 4.5.
573	• Property (d) ensures that for each pair of two colliding particles, at least
574	one gets annihilated. Property (e) ensures that both particles are getting
575	annihilated, and that annihilation can only occur for colliding particles with
576	non-zero charge. These two properties are the mathematical formulation of
577	the annihilation process described in Subsection 1.2.
578	• Recalling (4.1), by (4.3), it follows that colliding particles are stationary after
579	collision.
580	With reference to the collision times $t_1 < \ldots < t_K$ in (4.2), we define the set of
581	indices of the particles colliding at $t_k$ and its cardinality by

582 (4.4) 
$$\Gamma_k \coloneqq \{i : \tau_i = t_k\}, \qquad \gamma_k \coloneqq \#\Gamma_k.$$

We observe that  $\gamma_k$  is even for every k and that 583

584 (4.5) 
$$\sum_{k=1}^{K} \gamma_k \le \frac{n}{2}.$$

585We first establish some properties of  $E_n$  defined in (1.1). For convenience, we display 586

587 (4.6) 
$$\frac{\partial}{\partial x_i} E_n(x;b) = \frac{1}{n^2} \sum_{j: b_i b_j = 1} V'(x_i - x_j) + \frac{1}{n^2} \sum_{j: b_i b_j = -1} W'(x_i - x_j),$$

where we rely on the choice V'(0) = 0. We also introduce 588

589 
$$M_k : \mathbb{R}^n \to [0, \infty), \qquad M_k(x) := \frac{1}{n} \sum_{i=1}^n |x_i|^k, \qquad k = 1, 2, \dots$$

which is the k-th moment of the empirical measure related to the particles  $x_1, \ldots, x_n$ . 590

LEMMA 4.4 (Properties of  $E_n$ ). Let  $n \ge 2$ . For any  $x \in \mathbb{R}^n$  and  $b \in \{-1, 0, 1\}^n$ , 591the following properties hold:

(i)  $E_n(x;b) < +\infty$  if and only if  $\forall i \neq j : x_i = x_j \Rightarrow b_i b_j \neq 1;$ (ii)  $E_n + M_2$  is bounded from below; 594

(iii)  $\nabla E_n$  is Lipschitz continuous on the sublevelsets of  $y \mapsto E_n(y; b) + 2M_2(y)$ ; 595

(iv) if  $E_n(x;b) < +\infty$  and if there exists an index pair (I,J) which satisfies 596  $b_I b_J = -1$  and  $x_I = x_J$ , then, there exists C > 0 independent of n such that 597

598 
$$E_n(x;\bar{b}) \le E_n(x;b) + \frac{C}{n}(M_2(x) + x_I^2 + 1),$$

where  $\overline{b}$  is the modification of b in which  $b_I$  and  $b_J$  are put to 0. 599

*Proof.* Property (i) is a direct consequences of the properties of V, W (see As-600 sumption 2.1). Property (ii) is a matter of a simple estimate. Using Assumption 2.1) 601 (in particular (2.2)), some manipulations inspired by [37], and  $r \mapsto r^2 - C \log r$  being 602603 bounded from below, we obtain

$$E_n(x;b) + M_2(x) = \frac{1}{2n^2} \Big( \sum_{\substack{i \neq j \\ b_i b_j = 1}} V(x_i - x_j) + \sum_{\substack{i,j \\ b_i b_j = -1}}^n W(x_i - x_j) + \sum_{\substack{i,j=1 \\ b_i b_j = -1}}^n (x_i^2 + x_j^2) \Big)$$
$$\geq \frac{1}{2n^2} \sum_{i,j=1}^n \Big( -C\Big( [\log |x_i - x_j|]_+ + 1 \Big) + \frac{1}{2} (x_i - x_j)^2 \Big) \geq C.$$

604

$$\geq \frac{1}{2n^2} \sum_{i,j=1} \left( -C(\lfloor \log |x_i - x_j|]_+ + 1) + \frac{1}{2}(x_i - x_j)^2 \right) \geq C.$$

605Property (iii) follows easily from property (ii) by (2.1a) and (2.1c). To prove (iv), we set  $y := x_I = x_J$  and assume for convenience that  $b_I = 1$  and  $b_J = -1$ . Then, we 606 compute 607

$$E_n(x;b) - E_n(x;\bar{b}) = \frac{1}{2n^2} \left( \sum_{\substack{j \neq I \\ b_j = 1}} V(x_I - x_j) + \sum_{\substack{i \neq J \\ b_i = -1}} V(x_i - x_J) \right) \\ + \frac{1}{2n^2} \left( \sum_{\substack{j : b_j = -1 \\ j : b_j = -1}} W(x_I - x_j) + \sum_{\substack{i : b_i = 1 \\ i : b_i = 1}} W(x_i - x_J) \right) - \frac{W(0)}{2n^2} \\ = \frac{1}{2n^2} \left( \sum_{\substack{i = 1 \\ i \neq I, J}}^n |b_i| V(x_i - y) + \sum_{i=1}^n |b_i| W(x_i - y) \right) - \frac{W(0)}{2n^2}$$

608

609

610

$$= \frac{1}{2n^2} \sum_{\substack{i=1\\i \neq I,J}}^n |b_i| (V+W)(x_i - y) + \frac{W(0)}{2n^2}$$
  
$$\geq -\frac{C}{n^2} \sum_{i=1}^n (x_i - y)^2 + \frac{W(0)}{2n^2} \geq -\frac{C}{n} (M_2(x) + y^2 + 1),$$

611 where we have used (2.3).

We now prove that Problem 4.1 has a unique solution. In addition, we establish several properties of it.

614 PROPOSITION 4.5. Let  $n \ge 2$ , T > 0, and  $(x^{\circ}, b^{\circ}) \in \mathbb{R}^n \times \{\pm 1\}^n$  be such that 615  $x_1^{\circ} < x_2^{\circ} < \ldots < x_n^{\circ}$ . Then there exists a unique solution (x, b) to Problem 4.1 in the 616 sense of Definition 4.2. Moreover, the following properties are satisfied: 617 (i) there exists  $C \ge 0$  independent of  $r_{\circ}$  such that

617 (i) there exists C > 0 independent of n such that

618 
$$M_2(x(t)) \le Ct + M_2(x^\circ), \quad M_4(x(t)) \le Ct(M_2(x^\circ) + t) + M_4(x^\circ)$$

619 for all  $t \in [0, T]$ ;

620 (*ii*) 
$$\inf_{0 < t < T} \min\{|x_i(t) - x_j(t)| : b_i(t)b_j(t) = 1\} > 0;$$

625 
$$[e(t_k)] \leq \frac{C}{n} \left( \gamma_k M_2(x(t_k)) + \gamma_k + \sum_{i \in \Gamma_k} x_i^2(t_k) \right)$$

626 for every  $k = 1, \ldots, K$ , and

627 (4.8) 
$$\sum_{k=1}^{K} \llbracket e(t_k) \rrbracket \le C(T + M_2(x^\circ) + 1),$$

628 where  $\gamma_k$  and  $\Gamma_k$  are defined in (4.4), and C > 0 is a constant independent 629 of n;

 $\begin{array}{ll} 632 \\ 633 \\ 633 \\ 634 \end{array} (v) there exists an <math>L \in \mathbb{N}$  (independent of n) such that for all  $t \in [0,T)$ , (x(t),b(t)) $\begin{array}{ll} 633 \\ satisfies \ Assumption \ 2.2, \ \text{i.e., there exist} \ -\infty \ < \ a_0(t) \ \leq \ a_1(t) \ \leq \ \dots \ \leq \ a_{2L}(t) < +\infty \ such \ that \end{array}$ 

$$\{x_i(t): b_i(t) = 1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-2}(t), a_{2\ell-1}(t)),$$
  
$$\{x_i(t): b_i(t) = -1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-1}(t), a_{2\ell}(t)).$$

635

636 Proof. Step 1: Construction of (x, b), properties (i) and (ii), and (4.7). We define the counterpart of (4.1) in which no collision occurs, *i.e.*, we seek n trajectories 637 638  $y_i: [0,T] \to \mathbb{R}$  such that  $y_i(0) = x_i^\circ$  and

639 (4.9) 
$$\frac{\mathrm{d}}{\mathrm{d}t}y_i = -\frac{1}{n}\sum_{j: b_i^\circ b_j^\circ = 1} V'(y_i - y_j) - \frac{1}{n}\sum_{j: b_i^\circ b_j^\circ = -1} W'(y_i - y_j) \quad \text{on } (0, +\infty).$$

for all i = 1, ..., n. From (4.6) we observe that (4.9) is the gradient flow of  $E_n(\cdot; b^\circ)$ 640 given by 641

642 (4.10) 
$$\begin{cases} \dot{y}(t) = -n\nabla E_n(y(t); b^\circ), \\ y(0) = x^\circ. \end{cases}$$

From Lemma 4.4 we observe that (4.10) has a unique, classical solution y(t) locally 643 in time. In particular,  $t \mapsto E_n(y(t); b^\circ)$  is non-increasing. 644

Next we show that the solution y can be extended to the complete time interval 645[0,T]. With this aim, we prove that the second moment  $M_2(y(t))$  (and for later use 646 the fourth moment  $M_4(y(t))$  are finite as long as  $t \mapsto y(t)$  exists. We follow the 647 argument in [38]. From (4.9), using (2.1b) and (2.1d), we estimate 648

649 
$$\frac{\mathrm{d}}{\mathrm{d}t}M_2(y(t)) = \frac{2}{n}\sum_{i=1}^n y_i(t)\dot{y}_i(t)$$
  
650 
$$= -\frac{2}{n^2}\sum_{i=1}^n \left(\sum_{j:b_ib_j=1} y_i V'(y_i - y_j) + \sum_{j:b_ib_j=-1} y_i W'(y_i - y_j)\right)$$

651  
652
$$= -\frac{1}{n^2} \sum_{i,j: \ b_i \ b_j = 1} (y_i - y_j) V'(y_i - y_j) - \frac{1}{n^2} \sum_{i,j: \ b_i \ b_j = -1} (y_i - y_j) W'(y_i - y_j) \le C$$

653 Hence,

654 (4.11) 
$$M_2(y(t)) \le M_2(y(0)) + Ct \le M_2(x^\circ) + CT$$
, for all  $t \in [0, T]$ .

Similarly, using the identity  $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$ , we compute 655

656 
$$\frac{\mathrm{d}}{\mathrm{d}t}M_4(y(t)) = \frac{4}{n}\sum_{i=1}^n y_i^3(t)\dot{y}_i(t)$$
657 
$$= -\frac{4}{n^2}\sum_{i=1}^n \left(\sum_{j:\,b_ib_j=1} y_i^3 V'(y_i - y_j) + \sum_{j:\,b_ib_j=-1} y_i^3 W'(y_i - y_j)\right)$$
658 
$$= -\frac{2}{n^2}\sum_{j=1}^n \left(y_j^3 - y_j^3\right)V'(y_i - y_j) - \frac{2}{n^2}\sum_{j=1}^n \left(y_j^3 - y_j^3\right)W'(y_i - y_j)\right)$$

658

659 
$$n^{2} \sum_{i,j:b_{i}b_{j}=1}^{n^{2}} (y_{i}^{2} + y_{i}y_{j} + y_{j}^{2}) + \frac{C}{n^{2}} \sum_{i,j:b_{i}b_{j}=-1}^{n^{2}} (y_{i}^{2} + y_{i}y_{j} + y_{j}^{2}) + \frac{C}{n^{2}} \sum_{i,j:b_{i}b_{j}=-1}^{n^{2}} (y_{i}^{2} + y_{i}y_{j} + y_{j}^{2})$$

661

where we have used (4.11). Hence, 662

663 (4.12) 
$$M_4(y(t)) \le M_4(x^\circ) + CT(M_2(x^\circ) + T),$$
 for all  $t \in [0, T].$ 

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Next we identify  $t_1$  and choose those  $b_i$  that jump at  $t = t_1$  (see (4.3)). For this choice, it is enough to specify the collision times  $\tau_i$  (see (4.2)). We note that

669 
$$t^* := \inf \left\{ t \in (0,T] : \exists (i,j) : b_i^{\circ} b_j^{\circ} = -1 \text{ and } y_i(t) = y_j(t) \right\}$$

is either attained or  $t^* = +\infty$ . If  $t^* \ge T$ , we set x = y and  $\tau_i = +\infty$  for all i, 670 and observe that properties (d) and (e) of Definition 4.2 are satisfied. If  $t^* < T$ , we 671 observe that  $t_1$  in Definition 4.2(d) has to be equal to  $t^*$ . We set  $x|_{[0,t_1]} \coloneqq y|_{[0,t^*]}$ 672 and observe from (4.11) and (4.12) that property (i) is satisfied up to  $t = t_1$ . For the 673 choice of  $\tau_i$ , we follow the algorithm explained in Subsection 1.2, *i.e.*, for each pair 674 of particles that collide at  $t_1$ , we set the corresponding  $\tau_i$  equal to  $t_1$ . We choose the 675 remaining values for  $\tau_i > t_1$  later on in the construction. With this choice for  $\tau_i$ , 676 it follows from the continuity of  $x_i$  that properties (d) and (e) of Definition 4.2 are 677 satisfied by construction. Since  $E_n(x(t)) \leq E_n(x^\circ)$  for all  $t \in [0, t_1)$ , it follows that 678 679 (ii) holds on  $[0, t_1]$ .

Next we show that we can continue the construction above for  $t > t_1$ . First, applying Lemma 4.4(iv)  $\frac{1}{2}\gamma_1$  times (recall from (4.4) that  $\gamma_1$  is even), we find that

682 
$$E_n(x(t_1); b(t_1)) \le E_n(x(t_1); b(t_1-)) + \frac{C}{2n} \left( \gamma_1 M_2(x(t_1)) + \gamma_1 + \sum_{i \in \Gamma_1} x_i^2(t_1) \right).$$

Hence, (4.7) is satisfied for k = 1. Furthermore, we obtain that  $E_n(x(t_1); b(t_1)) < \infty$ , and thus we can continue the construction above for  $t > t_1$  by putting  $x(t_1), b(t_1)$  as the initial condition at  $t = t_1$ .

Iterating over k, this construction identifies all  $\tau_i < T$  (for  $i \notin \bigcup_{k=1}^K \Gamma_k$ , we set  $\tau_i := +\infty$ ) and  $t_k$ , and guarantees that x is piecewise C<sup>1</sup> on  $[t_k, t_{k+1}]$  and globally Lipschitz. In addition, (4.7) holds for all  $k = 1, \ldots, K$ .

Step 2: Uniqueness of (x, b). Let x and  $\tau$  be as constructed in Step 1, and set b 689 accordingly. Since (4.10) has a unique solution, Definition 4.2(d) defines uniquely the 690 time  $t_1$  until which x(t) is uniquely defined. By Definition 4.2(e), b has to be constant 691 on  $[0, t_1)$ . Since x satisfies Property (ii) at  $t = t_1$ , all collisions at  $t_1$  are collisions 692 of two particles with opposite type. Then, from the explanation in Remark 4.3, it is 693 obvious that properties (d) and (e) of Definition 4.2 define uniquely the set of indices 694 i for which  $\tau_i = t_1$ . Hence,  $b(t_1)$  is uniquely determined. We conclude by iterating 695over k. 696

697 Step 3: The remaining Properties (iii)–(v). Estimate (4.7) is already proved; 698 summing over k reads

699 (4.13) 
$$\sum_{k=1}^{K} \llbracket e(t_k) \rrbracket \le \frac{C}{n} \Big( \sum_{k=1}^{K} \gamma_k M_2(x(t_k)) + \sum_{k=1}^{K} \gamma_k + \sum_{k=1}^{K} \sum_{i \in \Gamma_k} x_i^2(t_k) \Big).$$

The first and second sums in the right-hand side above can be easily estimated using (i) and (4.5). We estimate the third sum by using that the sets  $\Gamma_k$  for  $k = 1, \ldots, K$ are disjoint, and that for every  $k = 1, \ldots, K$  and for every  $i \in \Gamma_k$  we have that  $x_i(t) = x_i(t_k)$  for all  $t \ge t_k$ . Hence, the third sum is bounded by  $M_2(x(T))$ . Collecting our estimates, we obtain (4.8) from (4.13). With (iii) proven, we prove (iv) for t = T by the following computation (the case t < T follows by a similar estimate). Setting  $t_{K+1} \coloneqq T$ , we compute

$$\begin{split} E_n(x(T); b(T)) - E_n(x^\circ; b^\circ) &= E_n(x(T); b(T)) - E_n(x(t_K); b(t_K)) \\ &+ \sum_{k=1}^K \left[ \left[ e(t_k) \right] \right] + \left( E_n(x(t_k-); b(t_k-)) - E_n(x(t_{k-1}); b(t_{k-1})) \right) \right] \\ &\leq \sum_{k=1}^{K+1} \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}t} E_n(x(t); b(t)) \, \mathrm{d}t + C(T + M_2(x^\circ) + 1) \\ &= -\sum_{k=1}^{K+1} \frac{1}{n} \int_{t_{k-1}}^{t_k} |\dot{x}(t)|^2 \, \mathrm{d}t + C(T + M_2(x^\circ) + 1) \\ &= -\frac{1}{n} \int_0^T |\dot{x}(t)|^2 \, \mathrm{d}t + C(T + M_2(x^\circ) + 1), \end{split}$$

707

20

where we have used in the second-to-last equality that x(t) satisfies (4.1).

Finally, we prove (v). First, we claim that the strict ordering of the particles 709 $\{x_i(t) : |b_i(t)| = 1\}$  is conserved in time. Clearly, this ordering holds at t = 0. 710 From (ii) it follows that any two particles, say with corresponding indices  $i \neq j$  such 711 that  $b_i(t)b_j(t) = 1$ , can never swap position. Similarly, any pair  $(x_i(t), x_j(t))$  with 712  $b_i(t)b_i(t) = -1$  cannot swap either, because Definition 4.2(d) ensures that  $b_i(t)$  and 713  $b_j(t)$  jump to 0 at the first t at which  $x_i(t) = x_j(t)$ . In fact, as soon as this happens, 714 the particles cease to move (see the last bullet in Remark 4.3 and also the first bullet 715 in Subsection 1.2 regarding the properties of particles with zero charge). 716

717 Next we construct  $a_{\ell}(t)$ . We start with t = 0, and set  $a_0(0), a_1(0), \ldots$  sequentially. 718 We set  $a_0(0) \coloneqq x_1^{\circ} - 1$ , and, if  $b_1^{\circ} = -1$ , we also put  $a_1(0) \coloneqq x_1^{\circ} - 1$ . For each pair of 719 consecutive particles  $x_i^{\circ}, x_{i+1}^{\circ}$  of opposite sign, we define a new point

720 
$$a_{\ell}(0) \coloneqq \frac{1}{2} (x_i^{\circ} + x_{i+1}^{\circ}).$$

If the current value of  $\ell$  is odd, we define  $L := (\ell + 1)/2$  and set  $a_{2L}(0) \coloneqq x_n^0 + 1$ . If *l* is even, we define  $L := (\ell + 2)/2$  and set  $a_{2L-1}(0) \coloneqq a_{2L}(0) \coloneqq x_n^0 + 1$ .

Since the strict ordering of the particles  $\{x_i(t): |b_i(t)| = 1\}$  is conserved in time, 723 we can construct  $a_{\ell}(t)$  analogously, but for a time-dependent  $L_t$ . Next we show how 724to modify this construction such that  $L_t$  can be chosen independently of t. Because 725 of the ordering of  $\{x_i(t) : |b_i(t)| = 1\}$  and that its cardinality is non-increasing in 726time, the numbers of pairs of consecutive particles  $x_i(t), x_{i+1}(t)$  of opposite non-zero 727 charge is also non-increasing in time. Hence,  $t \mapsto L_t$  is non-increasing in time. In 728case  $L_t < L$ , we modify the construction of  $a_\ell(t)$  above simply by adding a surplus of 729 points  $a_{\ell}(t)$  which all equal  $a_{2L_{\ell}}(t)$ . 730

731 Next we establish several properties of the empirical measures associated to the 732 solution (x; b) of Problem 4.1 with initial condition  $(x^{\circ}, b^{\circ})$  as in Proposition 4.5. 733 With this aim, we set

734 (4.14) 
$$n^{\pm} := \#\{i : b_i^\circ = \pm 1\}$$

as the number of positive/negative particles at time 0, and note that  $n^+ + n^- = n$ . The empirical measures associated to (x(t); b(t)) are

737 (4.15) 
$$\mu_n^{\circ,\pm} \coloneqq \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{x_i^\circ}, \qquad \mu_n^{\pm}(t) \coloneqq \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{x_i(t)},$$

which both have total mass equal to  $n^{\pm}/n$  for all  $t \in [0, T)$ . As in (3.5), we also set

739 (4.16) 
$$\kappa_n(t) \coloneqq \frac{1}{n} \sum_{i=1}^n b_i^{\circ} \delta_{x_i(t)}, \qquad \mu_n(t) \coloneqq \frac{1}{n} \sum_{i=1}^n \delta_{x_i(t)}, \qquad \tilde{\mu}_n^{\pm}(t) \coloneqq [\kappa_n(t)]_{\pm}.$$

T40 LEMMA 4.6 (Proposition 4.5 in terms of measures). Given the setting as in T41 Proposition 4.5 with (x,b) the solution to (4.1), let  $\boldsymbol{\mu}_n \coloneqq (\mu_n^+, \mu_n^-), \ \tilde{\boldsymbol{\mu}}_n \coloneqq (\tilde{\mu}_n^+, \tilde{\mu}_n^-),$ T42 and  $\kappa_n$  as constructed from (x,b) through (4.15) and (4.16). Then,

743 (i) 
$$\tilde{\mu}_n^{\pm}(t) = \frac{1}{n} \sum_{i=1}^n [b_i(t)]_{\pm} \delta_{x_i(t)};$$

744 (ii)  $\boldsymbol{\mu}_n \in \mathrm{AC}^2(0,T;\mathcal{P}_2^m(\mathbb{R}^2))$  with  $m = n^+/n$  (see (4.14)), and

745 (4.17) 
$$|\boldsymbol{\mu}'_n|^2_{\mathbf{W}}(t) \le \frac{1}{n} \sum_{i=1}^n \left(\frac{\mathrm{d}}{\mathrm{d}t} x_i(t)\right)^2 \text{ for all } 0 < t < T;$$

746 (iii)  $\boldsymbol{\mu}_n$  is a solution to (1.3) with initial condition  $\boldsymbol{\mu}_n^{\circ} = (\mu_n^{\circ,+}, \mu_n^{\circ,-}).$ 

747 Proof. Property (i) is a corollary of Proposition 4.5. Indeed, Proposition 4.5(v) 748 implies that  $[\kappa_n(t)]_{\pm} \geq \frac{1}{n} \sum_{i=1}^n [b_i(t)]_{\pm} \delta_{x_i(t)}$ , while Definition 4.2(e) implies that 749  $|\kappa_n(t)|(\mathbb{R}) \leq \frac{1}{n} \sum_{i=1}^n |b_i(t)|$ . We conclude (i).

Next we prove (ii). From the definition of  $\boldsymbol{\mu}_n$  in (4.15) we observe that  $\boldsymbol{\mu}_n(t) \in \mathcal{P}_2^m(\mathbb{R}^2)$  for all 0 < t < T. Hence, (3.4) applies, and we obtain

752 (4.18) 
$$\mathbf{W}^{2}(\boldsymbol{\mu}_{n}(s),\boldsymbol{\mu}_{n}(t)) \leq W^{2}(\boldsymbol{\mu}_{n}^{+}(s),\boldsymbol{\mu}_{n}^{+}(t)) + W^{2}(\boldsymbol{\mu}_{n}^{-}(s),\boldsymbol{\mu}_{n}^{-}(t))$$

for all  $0 < s \le t < T$ . To estimate the right-hand side, we let  $0 < s \le t < T$  be given, and introduce the coupling

755 
$$\gamma_n^{\pm} \coloneqq \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{(x_i(s), x_i(t))} \in \Gamma\left(\mu_n^{\pm}(s), \mu_n^{\pm}(t)\right).$$

756 By definition of the Wasserstein distance (3.1), we obtain

757 (4.19) 
$$W^2(\mu_n^{\pm}(s), \mu_n^{\pm}(t)) \leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \, \mathrm{d}\gamma_n^{\pm}(x, y) = \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \left( x_i(s) - x_i(t) \right)^2.$$

Finally, using in sequence the estimates (3.8), (4.18), and (4.19), we conclude (4.17). Since  $x \in \text{Lip}([0,T];\mathbb{R}^n)$ , we obtain that  $\boldsymbol{\mu}_n \in \text{AC}^2(0,T;\mathcal{P}_2^m(\mathbb{R}^2 \times \{\pm 1\})).$ 

760 Next we prove (iii). We rewrite (4.1) as

761 
$$\dot{x}_i(t) = -b_i(t) (V' * \tilde{\mu}_n^+(t) + W' * \tilde{\mu}_n^-(t)) (x_i(t)), \quad \text{for } i \text{ such that } b_i^\circ = 1,$$

$$\dot{x}_{i}(t) = -b_{i}(t)(W' * \tilde{\mu}_{n}^{+}(t) + V' * \tilde{\mu}_{n}^{-}(t))(x_{i}(t)), \quad \text{for } i \text{ such that } b_{i}^{\circ} = -1$$

Let  $\varphi \in C_c^{\infty}((0,T) \times \mathbb{R})$  be any test function. Since  $x_i$  is Lipschitz, the Fundamental

T

765 Theorem of Calculus applies, and thus we obtain, using (i),

766 
$$0 = \frac{1}{n} \sum_{i:b_i^\circ = 1} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \varphi(t, x_i(t)) \,\mathrm{d}t$$

$$= \frac{1}{n} \sum_{i:b_i^\circ = 1} \left[ \int_0^T \partial_t \varphi(t, x_i(t)) \, \mathrm{d}t + \int_0^T \varphi'(t, x_i(t)) \, \dot{x}_i(t) \, \mathrm{d}t \right]$$

769 770

$$= \int_0^T \int_{\mathbb{R}} \partial_t \varphi \, \mathrm{d}\mu_n^+ \mathrm{d}t - \int_0^T \frac{1}{n} \sum_{i:b_i=1}^{T} \varphi'(x_i) \left( V' * \tilde{\mu}_n^+ + W' * \tilde{\mu}_n^- \right)(x_i) \, \mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{R}} \partial_t \varphi \, \mathrm{d}\mu_n^+ \mathrm{d}t - \int_0^T \int_{\mathbb{R}} \varphi' \left( V' * [\kappa_n]_+ + W' * [\kappa_n]_- \right) \mathrm{d}[\kappa_n]_+ \mathrm{d}t,$$

where  $\varphi'$  denotes the partial derivative with respect to the spatial variable. Since  $\varphi$  is arbitrary and V' is odd, we conclude that  $\mu_n^+$  satisfies (3.6). From a similar argument, it follows that also  $\mu_n^-$  satisfies (3.6).

5. Statement and proof of the main convergence theorem. In this section, we state and prove our main convergence theorem.

THEOREM 5.1 (Discrete-to-continuum limit). Let the potentials V and W satisfy Assumption 2.1. Let  $(x^{n,\circ}, b^{n,\circ})_n$  be a sequence of initial conditions such that

- (i)  $E_n(x^{n,\circ}; b^{n,\circ})$  is bounded uniformly in n,
- (ii)  $(\boldsymbol{\mu}_n^{\circ})_n$  (see (4.15)) has bounded fourth moment uniformly in n,
- (iii) there exists an  $L \in \mathbb{N}$  independent of n such that Assumption 2.2 is satisfied for all n.

Then for every T > 0 the curves  $\boldsymbol{\mu}_n \in \operatorname{AC}^2(0,T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  determined by the solution  $(x^n, b^n)$  to Problem 4.1 through (4.15) for each n, converge in measure uniformly in time along a subsequence to a solution  $\boldsymbol{\rho}$  of (3.6), whose initial condition  $\boldsymbol{\rho}^\circ$  is the limit of  $(\boldsymbol{\mu}_n^\circ)_n$  along the same subsequence.

The proof is divided in three steps. In the first step we use compactness of  $\boldsymbol{\mu}_n(t)$  to extract a subsequence  $n_k$  along which  $\boldsymbol{\mu}_n(t)$  converges to some  $\boldsymbol{\rho}(t)$ . In the remaining two steps we pass to the limit in (3.6) as  $k \to \infty$  to show that the limiting curve  $\boldsymbol{\rho}(t)$ also satisfies (3.6). Step 2 contains the main novelty; relying on Assumption 2.2 with an  $n_k$ -independent number L, we prove that  $[\kappa_{n_k}(t)]_{\pm} \rightharpoonup [\kappa(t)]_{\pm}$  as  $k \to \infty$  pointwise in t.

792 Proof. Step 1:  $\mu_n$  converges along a subsequence  $n_k \to \infty$  in  $C([0,T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  to  $\rho \in AC^2(0,T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$  with  $m := \rho^{\circ,+}(\mathbb{R})$ . We prove this statement 794 by means of the Ascoli-Arzelà Theorem (see Lemma 3.1) applied to the metric space 795  $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W}).$ 

First, we show that, for fixed  $t \in [0, T]$ , the sequence  $(\boldsymbol{\mu}_n(t))_n$  is pre-compact in  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ . From the assumption on the initial data and Proposition 4.5(i) we observe that the second and fourth moments of the measures  $\mu_n(t)$  defined in (4.16), given by

800 
$$M_2(x^n(t)) = \int_{\mathbb{R}} y^2 \, \mathrm{d}\mu_n(t)(y), \qquad M_4(x^n(t)) = \int_{\mathbb{R}} y^4 \, \mathrm{d}\mu_n(t)(y),$$

are bounded uniformly in n and  $t \in [0, T]$ . Then, from [47, Lemma B.3] and [2, Proposition 7.1.5] we find that  $(\boldsymbol{\mu}_n(t))_n$  is pre-compact in the Wasserstein distance **W**.

22

Second, we show that the sequence  $(\boldsymbol{\mu}_n)_n \subset C([0,T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  is equicon-804 tinuous (*i.e.*,  $(\boldsymbol{\mu}_n)_n$  satisfies Lemma 3.1(ii)). For any  $0 \leq s < t \leq T$ , we estimate 805

806 (5.1) 
$$\mathbf{W}^{2}(\boldsymbol{\mu}_{n}(t),\boldsymbol{\mu}_{n}(s)) \leq \left(\int_{s}^{t} |\boldsymbol{\mu}_{n}'|_{\mathbf{W}}(r) \,\mathrm{d}r\right)^{2} \leq (t-s) \int_{0}^{T} |\boldsymbol{\mu}_{n}'|_{\mathbf{W}}^{2}(r) \,\mathrm{d}r.$$

To estimate the last integral above, we use the estimates in Lemma 4.6(ii) and Propo-807 sition 4.5(iv) to obtain 808

$$(5.2) \quad \int_0^T |\boldsymbol{\mu}'_n|^2_{\mathbf{W}}(r) \, \mathrm{d}r \leq \frac{1}{n} \int_0^T \sum_{i=1}^n \left(\frac{\mathrm{d}}{\mathrm{d}t} x_i^n(r)\right)^2 \mathrm{d}r = \frac{1}{n} \int_0^T |\dot{x}^n(r)|^2 \, \mathrm{d}r \\ \leq C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ}; b^{n,\circ}) - E_n(x^n(T); b^n(T)).$$

810 Since, by Lemma 4.4(ii) and Proposition 4.5(i), we have

$$E_n(x^n(T); b^n(T)) = [E_n(x^n(T); b^n(T)) + M_2(x^n(T))] - M_2(x^n(T))$$
  
 
$$\geq -C - [\tilde{C}T + M_2(x^{n,\circ})],$$

we obtain from (5.2) that 812

813 (5.3) 
$$\int_0^T |\boldsymbol{\mu}'_n|^2_{\mathbf{W}}(r) \, \mathrm{d}r \le C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ}; b^{n,\circ}).$$

814 By the assumptions on the initial data, the right-hand side is bounded uniformly in n. Hence, the right-hand side in (5.1) is bounded by C(t-s), and thus  $(\boldsymbol{\mu}_n)_n$  is 815 equicontinuous. 816

From the pre-compactness of  $(\boldsymbol{\mu}_n(t))_n$  and the equicontinuity of  $(\boldsymbol{\mu}_n)_n$ , we obtain 817 from Lemma 3.1 the existence of a subsequence  $n_k$  along which  $(\boldsymbol{\mu}_n)_n$  converges 818 in  $C([0,T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  to some limiting curve  $\rho \in C([0,T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . In 819 fact, combining the lower semi-continuity obtained in Theorem 3.2 with (5.3), we 820 obtain that  $\rho \in AC^2(0,T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . Moreover, since the total mass of  $\mu_n^+(t)$  is 821 conserved in time, and since the narrow topology conserves mass, we conclude that 822  $\rho(t) \in \mathcal{P}_2^n(\mathbb{R} \times \{\pm 1\})$  for all  $t \in [0,T]$ . This completes the proof of Step 1. For later 823 use, we set as in (3.5)824

811

$$\rho \coloneqq \rho^+ + \rho^-, \qquad \kappa \coloneqq \rho^+ - \rho^-, \qquad \tilde{\rho}^\pm \coloneqq [\kappa]_\pm.$$

Step 2:  $\tilde{\boldsymbol{\mu}}_{n_k}(t) \rightarrow \tilde{\boldsymbol{\rho}}(t)$  as  $k \rightarrow \infty$  pointwise for all  $t \in [0,T]$ . We set  $\tilde{\mu}_{n_k}^{\pm} = [\kappa_{n_k}]_{\pm}$ 826 as in (4.16) and  $\tilde{\mu}_{n_k}$  as in Lemma 4.6. We keep  $t \in [0,T]$  fixed, and remove it from 827 the notation in the remainder of this step. The structure of the proof of Step 2 is to 828 show by compactness that  $(\tilde{\mu}_{n_k})_k$  has a converging subsequence, and to characterise 829 the limit as  $\tilde{\rho}$ . Since  $\tilde{\rho}$  is independent of the choice of subsequence, we then conclude 830 that the full sequence  $(\tilde{\mu}_{n_k})_k$  converges to  $\tilde{\rho}$ . Keeping this in mind, in the following 831 832 we omit all labels of subsequences of n.

Since the second moments of  $\tilde{\mu}_n$  are obviously bounded by  $M_2(x^n)$ , the sequence 833 834  $(\tilde{\boldsymbol{\mu}}_n)$  is tight, and thus, by Prokhorov's Theorem,  $(\tilde{\boldsymbol{\mu}}_n)$  converges narrowly along a subsequence to some  $\tilde{\mu} \in \mathcal{M}_+(\mathbb{R} \times \{\pm 1\}).$ 835

We claim that  $\tilde{\mu}$  does not have atoms. We reason by contradiction. Suppose that 836  $\tilde{\mu}^+$  has an atom at y of mass  $\alpha > 0$  (the case of  $\tilde{\mu}^-$  can be treated analogously). 837 Then, setting  $B_{\eta}(y)$  as the ball around y with radius  $\eta$ , we infer from  $\tilde{\mu}_n^+ \rightharpoonup \tilde{\mu}^+$  that 838

 $\liminf_{n\to\infty} \tilde{\mu}_n^+(B_\eta(y)) \ge \alpha > 0$  for any  $\eta > 0$ . By choosing  $\eta > 0$  small enough, 839 the contribution of the particles in  $B_{\eta}(y)$  to the energy  $E_n(x^n; b^n)$  can be made 840 arbitrarily large, which contradicts with the uniform bound on  $E_n(x^n; b^n)$  given by 841 Proposition 4.5(iv). 842

In the remainder of this step we show that  $\tilde{\mu}^{\pm} = [\kappa]_{\pm}$ , regardless of the choice of 843 the subsequence. It is enough to show that 844

845 (5.4) 
$$[\kappa]_{\pm} \le \tilde{\mu}^{\pm}$$

$$[\kappa]_{\pm}(\mathbb{R}) \ge \tilde{\mu}^{\pm}(\mathbb{R})$$

848 Regarding (5.4), we obtain from Step 1 that

849 
$$\tilde{\mu}_n^+ - \tilde{\mu}_n^- = \kappa_n \rightharpoonup \kappa \quad \text{as } n \to \infty.$$

Hence,  $\tilde{\mu}^+ - \tilde{\mu}^- = \kappa$ , which implies (5.4). To prove (5.5), we let  $\{a_\ell^n\}_{\ell=0}^{2L}$  be as in 850 Proposition 4.5(v), and set 851

852 
$$\tilde{\mu}_n^{\ell} := \begin{cases} \tilde{\mu}_n^+|_{(a_{\ell-1}^n, a_{\ell}^n)} & \ell \text{ odd} \\ \tilde{\mu}_n^-|_{(a_{\ell-1}^n, a_{\ell}^n)} & \ell \text{ even} \end{cases}$$

for all  $\ell \in \{1, \ldots, 2L\}$ . By construction, 853

854 
$$\sum_{\ell=1}^{L} \tilde{\mu}_{n}^{2\ell-1} = \tilde{\mu}_{n}^{+} \text{ and } \sum_{\ell=1}^{L} \tilde{\mu}_{n}^{2\ell} = \tilde{\mu}_{n}^{-}$$

Together with  $\tilde{\mu}_n \rightharpoonup \tilde{\mu}$ , we conclude that  $(\tilde{\mu}_n^{\ell})_n$  are tight for any  $\ell$ , and thus, applying 855

Prokhorov's Theorem once more, each sequence  $(\tilde{\mu}_n^{\ell})_n$  converges along a subsequence 856 in the narrow topology to some  $\tilde{\mu}^{\ell} \in \mathcal{M}_+(\mathbb{R})$ . In particular, from  $\tilde{\mu}_n \rightharpoonup \tilde{\mu}$  and 857

858
$$\tilde{\mu}_n^- = \sum_{\ell=1}^L \tilde{\mu}_n^{2\ell} \rightharpoonup \sum_{\ell=1}^L \tilde{\mu}^{2\ell},$$

we infer that  $\tilde{\mu}^- = \sum_{\ell=1}^L \tilde{\mu}^{2\ell}$ . By a similar argument, it follows that  $\tilde{\mu}^+ = \sum_{\ell=1}^L \tilde{\mu}^{2\ell-1}$ . Finally, since  $\sup(\sup \tilde{\mu}_n^\ell) < \inf(\sup \tilde{\mu}_n^{\ell+1})$  for all  $1 \le \ell \le 2L - 1$ , we obtain from Lemma 3.4 that  $\sup(\sup \tilde{\mu}^\ell) < \inf(\sup \tilde{\mu}^{\ell+1})$  for all  $1 \le \ell \le 2L - 1$ . Hence, there exists  $A := \{a_\ell\}_{\ell=1}^{2L-1}$  such that 859 860 861 862

864 
$$\operatorname{supp} \tilde{\mu}^{+} \cap \operatorname{supp} \tilde{\mu}^{-} = \left(\bigcup_{\ell=1}^{L} \operatorname{supp} \tilde{\mu}^{2\ell-1}\right) \cap \left(\bigcup_{k=1}^{L} \operatorname{supp} \tilde{\mu}^{2k}\right)$$
865 
$$= \bigcup_{\ell=1}^{L} \bigcup_{k=1}^{L} \left(\operatorname{supp} \tilde{\mu}^{2\ell-1} \cap \operatorname{supp} \tilde{\mu}^{2k}\right) = \bigcup_{\ell=1}^{2L-1} \left(\operatorname{supp} \tilde{\mu}^{\ell} \cap \operatorname{supp} \tilde{\mu}^{\ell+1}\right) \subset$$

866

863

Since  $\tilde{\mu}^{\pm}$  does not have atoms,  $\tilde{\mu}^{\pm}(A) = 0$ . Together with  $\tilde{\mu}^{+} - \tilde{\mu}^{-} = \kappa$ , it is easy 867 to construct a Hahn decomposition of  $\kappa$  (see, e.g., [35, Theorem 6.14]). We conclude 868 (5.5).869

A.

Step 3:  $\rho$  is a solution to (1.3). To ease notation, we replace  $n_k$  by n. We show 870that  $\rho$  satisfies (3.6). With this aim, let  $\varphi^{\pm} \in C_c^{\infty}((0,T) \times \mathbb{R})$  be arbitrary. We recall 871

872 from Lemma 4.6(iii) that  $\mu_n$  satisfies

(5.6) 
$$0 = \int_0^T \int_{\mathbb{R}} \partial_t \varphi^{\pm}(x) \, \mathrm{d}\mu_n^{\pm}(x) \mathrm{d}t - \int_0^T \int_{\mathbb{R}} (\varphi^{\pm})'(x) \, (W' \ast [\kappa_n]_{\mp})(x) \, \mathrm{d}[\kappa_n]_{\pm}(x) \mathrm{d}t - \frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} \left( (\varphi^{\pm})'(x) - (\varphi^{\pm})'(y) \right) V'(x-y) \, \mathrm{d}([\kappa_n]_{\pm} \otimes [\kappa_n]_{\pm})(x, y) \mathrm{d}t.$$

We show that we can pass to the limit in all three terms separately. From Step 1 it follows that  $\mu_n \rightharpoonup \rho$ , and thus the limit of the first integral equals

876 
$$\int_0^T \int_{\mathbb{R}} \partial_t \varphi^{\pm}(x) \, \mathrm{d} \rho^{\pm}(x) \, \mathrm{d} t$$

Regarding the other two integrals in (5.6), we recall from Step 2 that  $[\kappa_n(t)]_{\pm} \rightarrow [\kappa(t)]_{\pm}$  as  $n \to \infty$  pointwise for all  $t \in [0,T]$ . Then, for the second term, since  $(x,y) \mapsto (\varphi^{\pm})'(x) W'(x-y)$  is bounded and continuous on  $\mathbb{R}^2$ , we obtain that

880 
$$\int_{\mathbb{R}} (\varphi^{\pm})'(x) \left( W' \ast [\kappa_n]_{\mp} \right)(x) \operatorname{d}[\kappa_n]_{\pm}(x) = \iint_{\mathbb{R}^2} (\varphi^{\pm})'(x) W'(x-y) \operatorname{d}([\kappa_n]_{\pm} \otimes [\kappa_n]_{\mp})(x,y)$$

881 converges, as  $n \to \infty$ , to

882 
$$\iint_{\mathbb{R}^2} (\varphi^{\pm})'(x) \, W'(x-y) \, \mathrm{d}([\kappa]_{\pm} \otimes [\kappa]_{\mp})(x,y) = \int_{\mathbb{R}} (\varphi^{\pm})'(x) \, (W' \ast [\kappa]_{\mp})(x) \, \mathrm{d}[\kappa]_{\pm}(x).$$

Finally, we pass to the limit in the third integral in (5.6). We employ Lemma 3.3 with d = 2 and  $\Delta = \{(y, y) : y \in \mathbb{R}\}$  the diagonal in  $\mathbb{R}^2$ . To show that the conditions of Lemma 3.3 are satisfied, we observe from the fact that  $r \mapsto rV'(r)$  is bounded and belongs to  $C(\mathbb{R} \setminus \{0\})$ , it holds that  $(x, y) \mapsto [(\varphi^{\pm})'(x) - (\varphi^{\pm})'(y)]V'(x-y)$  is bounded and belongs to  $C(\mathbb{R}^2 \setminus \Delta)$ . Moreover, by Step 2,  $([\kappa]_{\pm} \otimes [\kappa]_{\pm})(\Delta) = (\tilde{\mu}^{\pm} \otimes \tilde{\mu}^{\pm})(\Delta) = 0$ . Hence, by Lemma 3.3 we can pass to the limit in the third term in (5.6), whose limit reads

890

$$-\frac{1}{2}\int_0^T\iint_{\mathbb{R}\times\mathbb{R}}\left((\varphi^{\pm})'(x)-(\varphi^{\pm})'(y)\right)V'(x-y)\,\mathrm{d}([\kappa]_{\pm}\otimes[\kappa]_{\pm})(x,y)\mathrm{d}t.$$

Combining the three limits above, and recalling the time regularity of  $\rho$  from Step 1, we conclude that  $\rho$  is a solution to (1.3).

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