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Remodelling of Biological Tissues with Fibre Recruitment and Reorientation in the Light of the Theory of Material Uniformity

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Abstract

This study focusses on the remodelling of biological tissues in the framework of the theory of material uniformity. A constitutive evolution model is introduced, including fibre recruitment and reorientation, and subjected to the entropy inequality, which enforces the Second Principle of Thermodynamics. The model is applied to a numerical example describing a pressurised fibre-reinforced cylinder, roughly representing an artery, and is able to capture the major characteristics of remodelling in arteries, as reported in the literature.

Keywords: collagen fibre; recruitment; remodelling; growth; material uniformity

1. Introduction

Growth and remodelling in biological tissues can be 2 studied as anelastic phenomena. Anelastic processes, such 3 as plasticity or growth-remodelling, are accompanied by a 4 change in microstructure resulting in *configurational forces* 5 and residual stresses (e.g., Hoger, 1997; Gurtin, 1999). 6 While plasticity occurs at constant mass, biological tissues 7 not only experience a change in microstructure, but also an increase (growth) or decrease (resorption) of mass. Among 9 the first attempts to approach the problem of growth and 10 remodelling from the continuum mechanical perspective 11 are the seminal works by Cowin and Hegedus (1976) and 12 Hegedus and Cowin (1976) on bone remodelling. Rodriguez 13 et al. (1994) studied growth and remodelling in arteries 14 and used the Bilby-Kröner-Lee decomposition of the de-15 formation gradient F into a growth part F_q and an elastic 16 part F_e . In practice, they considered a residually stressed 17 reference configuration which grows into a stress-free inter-18 mediate (and generally incompatible) configuration, and 19 finally deforms elastically to the current (and compatible) 20 configuration actually attained by the body. Moreover, 21 the fact that the collagen fibres in a biological tissue may 22 be undulated in the reference configuration, and will thus 23 bear stress only after a certain threshold stretch, has been 24 studied as an additional remodelling parameter for the case 25 of aneurysms (Watton et al., 2004; Watton and Hill, 2009). 26 Here we employ the framework proposed by Epstein 27 and Maugin (2000), in which growth and remodelling are 28

seen as the two aspects of an evolution process imply-29 ing a local rearrangement of material inhomogeneities, de-30 scribed in terms of an *implant*, under the light of the *the*-31 ory of material uniformity. In this framework, growth 32 and remodelling are governed by the inhomogeneity rate, 33 $L_P = \dot{P}P^{-1}$, where P^{-1} formally corresponds to the 34 growth tensor F_q of Rodriguez et al. (1994). Specifically, 35 the trace of L_P is often required to be proportional to the 36 source or sink of mass due to growth that features in the 37 local mass balance of the body. Given L_P , the implant 38 tensor P can be determined by integrating the differential 39 equation $\dot{P} = L_P P$. However, the way in which L_P is 40 supplied is not unique. 41

We had previously modelled the effect of the undula-42 tion of the individual fibrils in a collagen fibre (Hamedzadeh 43 et al., 2018) and, in this study, we employ the same mech-44 anism for an entire fibre, and in terms of the theory of 45 material uniformity. Therefore, we introduce the proper 46 material implant describing both reorientation and recruit-47 ment of the fibres in an artery, and solve the benchmark 48 problem previously studied by Grillo et al. (2015) in order 49 to elucidate our results. 50

2. Theory of Uniformity

We follow the *theory of uniformity*, originally introduced by Noll (1967) and further developed by Epstein and Maugin (1990). A material body \mathcal{B} is said to be *uniform* if all of its points are made of the *same* material. This implies that the tangent spaces $T_X \mathcal{B}$ of the points X of \mathcal{B} have been modelled on an archetypal vector space $\mathcal{A} \equiv \mathbb{R}^3$, called precisely the *archetype*, via an isomorphism

$$\boldsymbol{P}(X): \mathcal{A} \to T_X \mathcal{B},\tag{1}$$

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at every point X. In other words, if we look at the microscopic structures surrounding two materially uniform points X and Y, we might not see identical pictures, as one might have been distorted or rotated in a different manner than the other. However, we can pass from X to Y via $P(Y) P^{-1}(X) : T_X \mathcal{B} \to T_Y \mathcal{B}$. For this reason, P is called the material isomorphism.

Now, suppose to have an elastic material with elastic potential $W(X,t) = \hat{W}(\boldsymbol{F}(X,t),X,t)$ depending explicitly on the point X and time t. If the body is uniform, then the elastic potential depends on the point X and time t only through the (in this case, time-dependent) uniformity field \boldsymbol{P} , i.e.,

$$\hat{W}(\boldsymbol{F}(X,t),X,t) = J_{\boldsymbol{P}}^{-1}(X,t)\,\hat{W}(\boldsymbol{F}(X,t)\boldsymbol{P}(X,t)),\quad(2)$$

where \tilde{W} is the elastic potential in the archetype, and J_P^{-1} comes from the theorem of the change of variables (Epstein and Maugin, 1990).

75 3. Material Implant for a Single Fibre

The generic fibre is straight with no undulation in the 76 archetype, and the implant P(X, t) rotates the fibre, crimps 77 it and maps it into the tangent space $T_X \mathcal{B}$ at X, as shown 78 in Figure 1. Note that using the implant \boldsymbol{P} is equivalent to 79 assuming the existence of a non-compatible intermediate 80 configuration, which is mapped onto by the *straightening* 81 deformation F_s coming from the multiplicative decompo-82 sition $\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_s$ (Hamedzadeh et al., 2018). 83



Figure 1: Collagen fibre recruitment seen in terms of the theory of uniformity, with the straightened fibre in the archetype.

The archetypal straightened fibre is represented by the vector $\lambda_s \mu$, where μ is a unit vector and λ_s is the straightening stretch needed to map a fibre from its referential crimped state back to the archetypal straight state. The uniformity field P maps the archetypal vector $\lambda_s \mu$ into the unit referential vector M. Application of the polar decomposition theorem to P yields

$$\boldsymbol{P} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{R}\hat{\boldsymbol{U}}(\lambda_s), \quad \boldsymbol{P}^A{}_\beta = \boldsymbol{R}^A{}_\alpha \boldsymbol{U}^\alpha{}_\beta, \qquad (3)$$

where \boldsymbol{R} rotates and shifts the fibre vector $\boldsymbol{\mu} \in \mathcal{A}$ from the archetype to the referential vector $\boldsymbol{M} \in T_X \mathcal{B}$, and $\boldsymbol{U} = \hat{\boldsymbol{U}}(\lambda_s)$ is the crimping experienced by the fibre when passing from the straight archetypal configuration to the undulated referential one. In order to find the expressions of \boldsymbol{R} and \boldsymbol{U} , we need some geometrical preliminaries.

Let **g** be a metric in the archetype \mathcal{A} and $\{\mathbf{a}_{\alpha}\}_{\alpha=1}^{3}$ a 97 **g**-orthonormal basis of \mathcal{A} . Since the body \mathcal{B} is a *trivial* 98 manifold embedded in the affine space $S \equiv \mathbb{E}^3$, we can 99 afford the luxury of choosing Cartesian coordinates $\{Z^{\alpha}\}$, 100 such that the associated basis $\{I_{\alpha}\}_{\alpha=1}^{3}$ coincides with the 101 archetypal basis $\{\mathbf{a}_{\alpha}\}_{\alpha=1}^{3}$ at every tangent space $T_X \mathcal{B}$. We 102 also choose a system of curvilinear coordinates $\{X^A\}$ in 103 the body \mathcal{B} , with associated basis $\{E_A\}_{A=1}^3$. The change 104 of basis and the transformation rule for vectors are 105

$$\boldsymbol{E}_{A} = \frac{\partial Z^{\alpha}}{\partial X^{A}} \boldsymbol{I}_{\alpha}, \qquad W^{A} = \frac{\partial X^{A}}{\partial Z^{\alpha}} W^{\alpha}. \tag{4}$$

Consider the vector $\tilde{\boldsymbol{M}} \in \mathcal{A}$ such that its components are equal to the Cartesian components of $\boldsymbol{M} \in T_X \mathcal{B}$, i.e., $\tilde{M}^{\alpha} = M^{\alpha}$. The orthogonal tensor \boldsymbol{R} is obtained as

$$R^{A}{}_{\beta} = \frac{\partial X^{A}}{\partial Z^{\alpha}} Q^{\alpha}{}_{\beta}, \qquad (5)$$

where $Q^{\alpha}{}_{\beta}$ are the components of the archetypal tensor 109 Q rotating the archetypal direction μ into M. The cor-110 responding matrix $\llbracket Q \rrbracket$ is found as a function of the unit 111 vector $\boldsymbol{\omega} = \boldsymbol{\mu} \times \tilde{\boldsymbol{M}} / \| \boldsymbol{\mu} \times \tilde{\boldsymbol{M}} \|$, which describes the axis of 112 rotation, and the amplitude $\theta = \arccos(\mu, M)$ of the rota-113 tion. Then, the rotation matrix $\llbracket Q \rrbracket$ can be obtained by 114 exponentiating the skew-symmetric matrix $\llbracket \Omega \rrbracket$ associated 115 with the vector $\boldsymbol{\omega}$, i.e., 116

$$\llbracket \boldsymbol{Q} \rrbracket = e^{\llbracket \boldsymbol{\Omega} \rrbracket \boldsymbol{\theta}}, \qquad \Omega^{\alpha}{}_{\gamma} = \epsilon^{\alpha}{}_{\beta\gamma} \omega^{\beta}, \tag{6}$$

which can be conveniently expressed by Rodriguez' formula (Koks, 2006) as

$$Q^{\alpha}{}_{\gamma} = \delta^{\alpha}{}_{\gamma} + (\sin\theta) \,\Omega^{\alpha}{}_{\gamma} + (1 - \cos\theta) \,\Omega^{\alpha}{}_{\beta} \,\Omega^{\beta}{}_{\gamma}.$$
(7)

The components of the pure stretch \boldsymbol{U} are given by

$$U^{\alpha}{}_{\beta} = (\lambda_s^{-1} - 1)\mu^{\alpha}\mu_{\beta} + \delta^{\alpha}{}_{\beta}, \qquad (8)$$

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where $\mu_{\beta} = \mathfrak{g}_{\beta\gamma} \mu^{\gamma}$ are the components of the covector μ^{\flat} associated with μ via the archetypal metric \mathfrak{g} . Finally, the material implant P is given by

$$P^{A}{}_{\gamma} = \frac{\partial X^{A}}{\partial Z^{\alpha}} Q^{\alpha}{}_{\beta} \left[(\lambda_{s}^{-1} - 1) \mu^{\beta} \mu_{\gamma} + \delta^{\beta}{}_{\gamma} \right], \qquad (9)$$

which can be simplified into

$$P^{A}{}_{\gamma} = \left(\lambda_{s}^{-1} - 1\right) M^{A} \mu_{\gamma} + \frac{\partial X^{A}}{\partial Z^{\alpha}} Q^{\alpha}{}_{\gamma}, \qquad (10)$$

since $Q^{\alpha}{}_{\beta} \mu^{\beta} = \tilde{M}^{\alpha}$ and $(\partial X^{A} / \partial Z^{\alpha}) \tilde{M}^{\alpha} = M^{A}$. For an isochoric implant P (i.e., pure remodelling, no growth, see Epstein and Elzanowski, 2007), the stretch U must be changed into

$$U^{\alpha}{}_{\beta} = (\lambda_s^{-1} - \lambda_s^{1/2})\mu^{\alpha}\mu_{\beta} + \lambda_s^{1/2}\,\delta^{\alpha}{}_{\beta},\tag{11}$$

so that we have

$$P^{A}{}_{\gamma} = \left(\lambda_{s}^{-1} - \lambda_{s}^{1/2}\right) M^{A} \mu_{\gamma} + \lambda_{s}^{1/2} \frac{\partial X^{A}}{\partial Z^{\alpha}} Q^{\alpha}{}_{\gamma}.$$
(12)

129 4. Material Implant for a Distribution of Fibres

We assume that the fibres in our biological tissue have 130 a statistical distribution of orientation. Thus, rather than 131 implanting fibres individually, we can implant a whole fam-132 ily of statistically oriented fibres into a material point X. 133 We also assume that the elastic potential W_f of the fi-134 bres is the sum of an isotropic part \hat{W}_{fi} and an anisotropic 135 part \hat{W}_{fa} . With an abuse of notation, we do not indicate 136 the arguments (X, t) of the tensor fields, and write the 137 anisotropic *ensemble* elastic potential of the fibres (Fed-138 erico and Herzog, 2008) as 139

$$\hat{W}_e(\boldsymbol{C}, X, t) = \int_{\mathbb{S}_X^2 \mathcal{B}} \hat{W}_{fa}(\hat{I}_4, X, t) \ \Psi(\boldsymbol{M}; X, t), \qquad (13)$$

where $\hat{I}_4 = \boldsymbol{C} : (\boldsymbol{M} \otimes \boldsymbol{M})$ is the fourth invariant of the right Cauchy-Green deformation \boldsymbol{C} along the vector \boldsymbol{M} , and the probability distribution Ψ depends explicitly on Xand t. Following the definition (2) of material uniformity, the fibre elastic potential \hat{W}_{fa} is related to its archetypical counterpart by

$$\hat{W}_{fa}(\hat{I}_4, X, t) = J_{P}^{-1} \check{W}_{fa}(\check{I}_4), \qquad (14)$$

where $\check{I}_4 = \mathbf{P}^T \mathbf{C} \mathbf{P} : \boldsymbol{\mu} \otimes \boldsymbol{\mu}$ is the fourth invariant of $\mathbf{P}^T \mathbf{C} \mathbf{P}$ along the vector of $\boldsymbol{\mu}$. Thus, Eq. (13) becomes

$$\hat{W}_e(\boldsymbol{C}, X, t) = J_{\boldsymbol{P}}^{-1} \int_{\mathbb{S}^2} \check{W}_f(\check{I}_4) \,\check{\Psi}(\boldsymbol{\mu}), \qquad (15)$$

where \mathbb{S}^2 denotes the archetypical unit sphere and $\tilde{\Psi}$ is the archetypal probability distribution.

¹⁵⁰ 5. Dissipation Inequality and Evolution Law

An evolution equation is required as an additional differential equation providing the inhomogeneity rate $L_P = \dot{P}P^{-1}$ as a function of all quantities that can act as driving forces of the evolution process, i.e.,

$$\boldsymbol{L}_{\boldsymbol{P}}(X,t) = \hat{\mathcal{F}}(\boldsymbol{P}(X,t),\mathfrak{A}(X,t),X), \quad (16)$$

where \mathfrak{A} represents all possible driving force arguments, such as Eshelby stress, $\mathfrak{E} = W \mathbf{I}^T - \mathbf{F}^T \mathbf{T}$, or Mandel stress, $\mathfrak{M} = \mathbf{F}^T \mathbf{T}, \mathbf{T}$ being the first Piola-Kirchhoff stress. Note that, here, $\hat{\mathcal{F}}$ does not depend on time explicitly, i.e., it is *autonomous* with respect to time.

As shown by Epstein and Maugin (2000) and Epstein and Elzanowski (2007), and mentioned in the Introduction, there are some restrictions that are essential for an appropriate choice of evolution law. First, the evolution law should be invariant with respect to a change of reference configuration. Such an evolution law is said to be *reduced to the archetype* and reads

$$\boldsymbol{L}_{\boldsymbol{P}} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1} = \check{\mathcal{F}}(J_{\boldsymbol{P}} \boldsymbol{P}^T \mathfrak{A} \boldsymbol{P}^{-T}).$$
(17)

Second, the evolution law should satisfy the dissipation inequality, i.e., within a purely mechanical framework and for a hyperelastic material, for which the first Piola-Kirchhoff 169 stress tensor T is given by $T = (\partial \hat{W} / \partial F)(F)$, the dissipation \mathfrak{D} per unit reference volume satisfies (Epstein and Elzanowski, 2007) 172

$$\mathfrak{D} = -\dot{W} + \mathbf{T} : \dot{\mathbf{F}} = -\mathfrak{M} : \mathbf{L}_{\mathbf{P}} \ge 0.$$
(18)

The same result has been found with the BKL decomposi-173 tion in several works on inelastic processes (see e.g., Simo 174 and Hughes, 1986; Simo, 1988; Cleja-Tigoiu and Maugin, 175 2000; Imatani and Maugin, 2002; Grillo et al., 2018; Di Ste-176 fano et al., 2018; Crevacore et al., 2018). Here, we as-177 sume a rate-dependent type of remodelling and reformu-178 late $\mathfrak{D} = \mathfrak{D}(C, P, L_P)$ as a quadratic function of \mathfrak{M} via 179 a Legendre transformation on L_P and enforcing the Prin-180 ciple of Maximum Dissipation (Hackl and Fischer, 2008). 181 Setting $\mathfrak{D} = \check{\mathfrak{D}}(C, P, \mathfrak{M}) = -\mathfrak{M} : \check{\mathbb{K}}(F, P) : \mathfrak{M}$, we have 182

$$\boldsymbol{L}_{\boldsymbol{P}} = -\frac{1}{2} \frac{\partial \tilde{\mathfrak{D}}}{\partial \mathfrak{M}} = -\check{\mathbb{K}}(\boldsymbol{F}, \boldsymbol{P}) : \mathfrak{M},$$
(19)

where $\check{\mathbb{K}}(F, P)$ is a fourth-order tensor with major symme-183 try only. For the purpose of this work, we define $\check{\mathbb{K}}(F, P)$ 184 as $\check{\mathbb{K}}(\mathbf{F}, \mathbf{P}) = k \mathbf{b}_{\mathbf{P}} \otimes \mathbf{c}_{\mathbf{P}}$ (with components $k (\mathbf{b}_{\mathbf{P}})^{AC} (\mathbf{c}_{\mathbf{P}})_{BD}$ iso the "tensor-down" product \otimes is defined in Curnier et al., 186 1995), with k being a positive constant, and $\boldsymbol{b}_{\boldsymbol{P}} = \boldsymbol{P} \, \boldsymbol{\mathfrak{g}}^{-1} \boldsymbol{P}^T$ 187 and $c_P = b_P^{-1}$ being the "left Cauchy-Green tensor" and 188 the "Finger tensor" associated with \boldsymbol{P} , respectively. More-189 over, in order to enforce a deviatoric L_P (no growth), we 190 make it function of the deviatoric Mandel stress \mathfrak{M}_d = 191 $\mathfrak{M} - \frac{1}{3}(\boldsymbol{I}: \mathfrak{M})\boldsymbol{I}^T$, i.e., 192

$$\boldsymbol{L}_{\boldsymbol{P}} = -k \, \boldsymbol{b}_{\boldsymbol{P}} \mathfrak{M}_d \boldsymbol{c}_{\boldsymbol{P}},\tag{20}$$

which can be shown to respect condition (17).

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6. Example: Application to the Arterial Wall

Here, we apply our recruitment-reorientation remodelling framework to the benchmark problem reported by Olsson and Klarbring (2008) and Grillo et al. (2015), with a cylinder reinforced by two families of fibres (mimicking the arterial wall) under plane strain in the plane orthogonal do the direction $X^3 \equiv Z$ of the axis of the cylinder.

Fibre Implant. At each material point, we implant 201 an archetypal distribution with dominant direction $\mu_0 =$ 202 $0 \mathfrak{a}_1 + 0 \mathfrak{a}_2 + 1 \mathfrak{a}_3$ into two families of fibres with equal and 203 opposite angles, γ and $-\gamma$, measured from the Z-direction 204 in the Θ -Z-plane and corresponding to the material direc-205 tions M_{0+} and M_{0-} , as shown in Figure 2. This amounts 206 to defining an implant tensor P and then adapting its ex-207 pression to the two angles γ and $-\gamma$, which gives the im-208 plants P_+ and P_- , respectively. The polar decomposition 209

P = $\mathbf{R}\mathbf{U}$ of the implant (Equation (3)) yields

$$\llbracket \boldsymbol{U} \rrbracket = \begin{bmatrix} \sqrt{\lambda_s} & 0 & 0\\ 0 & \sqrt{\lambda_s} & 0\\ 0 & 0 & \lambda_s^{-1} \end{bmatrix}, \ \llbracket \boldsymbol{R} \rrbracket = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}.$$
(21)

Fibre Orientation Probability. In each family, the fibre orientation follows a *bivariate* von Mises distribution (Holzapfel et al., 2015; Gizzi et al., 2018), in which we set the constants so to normalise it to one, i.e.,

$$\check{\Psi}(\beta,\alpha) = \sqrt{\frac{2b}{\pi}} \frac{\exp(a\cos 2\alpha) \exp(b(1+\cos 2\beta))}{2\pi I_0(a) \operatorname{erfi}(\sqrt{2b})}, \quad (22)$$

where α and β are the archetypical longitude and colatitude angle, erfi is the *imaginary error function* and I_0 is the *Bessel function* of zero kind (see Abramowitz and Stegun, 1964). In this study, we used the values a = -1and b = 5 of the concentration parameters, to obtain fibres mostly laying in the Θ -Z-plane, as illustrated in Figure 2.



Figure 2: Tensors P_+ and P_- , with identical expressions except for the angles γ and $-\gamma$, respectively, implant the two fibre families, described by M_{0+} and M_{0-} , from the archetypal straight state, described by $\lambda_s \mu_0$.

Deformation. We cover the body manifold with a polar chart, denoted by (R, Θ, Z) , in which, $R \in [R_i, R_o], \Theta \in$ $[0, 2\pi], Z \in [0, L]$. Here, R_i and R_o , are the inner and outer radii respectively, Θ is the referential polar angle and L is the length of the cylinder. The current configuration is obtained under the assumption of pure inflation as:

$$(R,\Theta,Z)\mapsto (r,\theta,z)=(\chi^r(R,t),\Theta,Z).$$
 (23)

For convenience, from this point forward, we write $\xi \equiv \chi^r$. Since ξ is a function solely of the radial coordinate R and time, we denote $\xi' \equiv \partial \chi^r / \partial R$. The orthonormal bases for the tangent spaces of the referential and the current configurations are denoted by $\{E_R, E_{\Theta}, E_Z\}$ and $\{e_r, e_{\theta}, e_z\}$, respectively. Thus, the deformation gradient F reads

$$\mathbf{F}(R,t) = \xi'(R,t) \, \mathbf{e}_r \otimes \mathbf{E}^R + \frac{\xi(R,t)}{R} \, \mathbf{e}_\theta \otimes \mathbf{E}^\Theta + \mathbf{e}_z \otimes \mathbf{E}^Z.$$
(24)

Imposing incompressibility, i.e., $J = \det F = 1$, we have

$$\xi'(R,t)\xi(R,t) = R.$$
 (25)

Note that the condition J = 1, together with the restriction $J_P = 1$, amounts to require that also the tensor FP has unitary determinant.

The separable differential equation (25) has solution

$$\xi(R,t) = \sqrt{R^2 + \upsilon(t)},\tag{26}$$

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$$\xi'(R,t) = \frac{R}{\sqrt{R^2 + \upsilon(t)}} = \frac{R}{\xi(R,t)},$$
(27)

so that the matrix representation of ${\boldsymbol F}$ is

$$\llbracket \boldsymbol{F}(R,t) \rrbracket = \begin{bmatrix} \frac{R}{\xi(R,t)} & 0 & 0\\ 0 & \frac{\xi(R,t)}{R} & 0\\ 0 & 0 & 1 \end{bmatrix}.$$
 (28)

Constitutive Equations. Following the premises in Section 4, the artery is modelled as hyperelastic with an isotropic matrix contribution \hat{W}_m , an isotropic fibre contribution \hat{W}_{fi} and an anisotropic fibre contribution $\hat{W}_{e\pm}$, 245 integral of the anisotropic fibre contribution $\hat{W}_{fa\pm}$, based on the ensemble potential \hat{W}_e introduced in (13). Thus, 249

$$\hat{W}(\boldsymbol{C}, X) = (1 - \Phi_f) \hat{W}_m(\boldsymbol{C}) + \Phi_f(\hat{W}_{fi}(\boldsymbol{C}) + \hat{W}_{e+}(\boldsymbol{C}, X) + \hat{W}_{e-}(\boldsymbol{C}, X)), \quad (29)$$

where Φ_f is the fibre volumetric fraction, assumed homogeneous through the sample, and 250

$$\hat{W}_m(C) = \frac{1}{2}k_m[\hat{I}_1 - 3],$$
 (30a)

$$\hat{W}_{fi}(C) = \frac{1}{2}k_{fi}[\hat{I}_1 - 3],$$
(30b)

$$\hat{W}_{fa\pm}(\boldsymbol{C}, X) = \frac{1}{4} k_{fa} \,\mathcal{H}(\hat{I}_{4\pm}(X) - 1) [\hat{I}_{4\pm}(X) - 1]^2, \ (30c)$$

where $\hat{I}_1 = \text{tr}(C)$ and the step function \mathcal{H} is needed to "switch-off" fibres with stretch smaller than one. The second Piola-Kirchhoff stress is obtained as $S = 2 \partial \hat{W} / \partial C$ and, in particular, the anisotropic ensemble contribution is given by 256

$$\boldsymbol{S}_{e\pm} = J_{\boldsymbol{P}}^{-1} \int_{\mathbb{S}^2} 2 \, \frac{\partial \check{W}_{fa\pm}}{\partial \check{I}_{4\pm}} \, \frac{\partial \check{I}_{4\pm}}{\partial \boldsymbol{C}} \, \check{\Psi}(\boldsymbol{\mu}), \tag{31}$$

where we used (14) to transform $\hat{W}_{fa\pm}$ into $\check{W}_{fa\pm}$ and

$$\frac{\partial \check{I}_{4\pm}}{\partial C} = \frac{\partial (\boldsymbol{P}_{\pm}^T \boldsymbol{C} \boldsymbol{P}_{\pm} : \boldsymbol{\mu} \otimes \boldsymbol{\mu})}{\partial C} = \boldsymbol{P}_{\pm}^T \underline{\otimes} \boldsymbol{P}_{\pm}^T : \boldsymbol{\mu} \otimes \boldsymbol{\mu}, \quad (32)$$

with components $(\mathbf{P}_{\pm})^{A}{}_{\alpha}(\mathbf{P}_{\pm})^{B}{}_{\beta} \mu^{\alpha} \mu^{\beta}$ (see Curnier et al., 258 1995, for the definition of the "tensor-down" product \otimes). 259

In order to enforce the incompressibility constraint, we employ the pulled-back deviatoric part (see Federico, 2012) of the second Piola-Kirchhoff stress, 260 261 262

$$\boldsymbol{S}_d \equiv \mathrm{Dev}^* \boldsymbol{S} = \boldsymbol{S} - \frac{1}{3} (\boldsymbol{C} : \boldsymbol{S}) \, \boldsymbol{C}^{-1}.$$
 (33)

We emphasise that, since we consider that the elastic potential of the matrix does not evolve and we have two families of fibres with different implants, we only consider the fibre part of the deviatoric Mandel stress as the driving force of evolution, i.e.,

$$\mathfrak{M}_{ed\pm} = \operatorname{Dev}\left(\boldsymbol{C}\boldsymbol{S}_{e\pm}\right) = \boldsymbol{C}\boldsymbol{S}_{e\pm} - \frac{1}{3}(\boldsymbol{I}:\boldsymbol{C}\boldsymbol{S}_{e\pm})\boldsymbol{I}^{T}.$$
 (34)

Equilibrium, Boundary Conditions, Integration. The cylinder is under uniform pressure \wp on the inner boundary $\partial \mathcal{B}_i$ and and zero traction on the outer boundary $\partial \mathcal{B}_o$, and body force and inertial effects are neglected. Thus, the evolution of the tissue is governed by the equation for P, given in (17) and equipped with appropriate initial conditions, and by the boundary value problem

$$Div T = 0, in \mathcal{B}. (35a)$$

$$T N = -J \wp F^{-T}, \qquad \text{on } \partial \mathcal{B}_i, \qquad (35b)$$

$$T N = 0,$$
 on $\partial \mathcal{B}_o,$ (35c)

where N is the normal covector to the boundary $\partial \mathcal{B}$, and the hypothesis of isochoric deformation implies J = 1.

Since we consider an axisymmetric problem, the first 277 Piola-Kirchhoff stress is independent of Θ and Z. Also, 278 the boundary conditions ensure that the matrix associated 279 with the first Piola-Kirchhoff stress is diagonal, i.e., [T] =280 diag $[T_r^R, T_\theta^\Theta, T_z^Z]$. The first Piola-Kirchhoff stress can 281 be expressed as the sum of its hydrostatic and deviatoric 282 components, and in terms of the deviatoric second Piola-283 Kirchhoff stress, as 284

$$\boldsymbol{T} = \boldsymbol{T}_h + \boldsymbol{T}_d = -J \, p \, \boldsymbol{F}^{-T} + \boldsymbol{g} \, \boldsymbol{F} \, \boldsymbol{S}_d. \tag{36}$$

The hydrostatic pressure p is found from (35) (see Grillo et al., 2015).

Evolution Equation. The evolution equation for each of the \dot{P}_{\pm} is obtained from that of $L_{P_{\pm}} = \dot{P}_{\pm}P_{\pm}^{-1}$ by right-multiplying Equation (20) written for each P_{\pm} , by the corresponding P_{\pm}

$$\dot{\boldsymbol{P}}_{\pm} = -k \ J_{\boldsymbol{P}_{\pm}} \ \boldsymbol{P}_{\pm} \ \boldsymbol{\mathfrak{g}}^{-1} \boldsymbol{P}_{\pm}^T \mathfrak{M}_{ed\pm} \boldsymbol{P}_{\pm}^{-T} \mathfrak{g}.$$
(37)

In our example, using (3) and (21), solving Equation (37) for the deviatoric Mandel stress $\mathfrak{M}_{ed\pm}$ of each of the two fibre families, and then summing to obtain the overall deviatoric Mandel stress of the fibres \mathfrak{M}_{ed} , yields

$$\frac{\dot{\lambda_s}}{\lambda_s} = k \left(\mathfrak{M}_{ed}\right)_R^R, \quad (38a)$$
$$\frac{\frac{1}{2}\lambda_s^2(3\cos(2\gamma)-1)\dot{\lambda}_s - \left(\lambda_s^6 - 1\right)\dot{\gamma}\sin(2\gamma)}{\lambda_s^3} = k \left(\mathfrak{M}_{ed}\right)_\Theta^\Theta. \quad (38b)$$

Numerical Algorithm. To study the numerical example discussed in the previous sections, a code is developed in *Wolfram Mathematica*. The main focus of the numerical algorithm in this study is to have high *accuracy* and *precision* as we are studying a model with a simple geometry (isochoric inflation of a hollow cylinder). Although the geometry is simple, the evolution equation (38) makes

Parameter	Value	Symbol
inner radius	$1\mathrm{mm}$	R_i
outer radius	$2\mathrm{mm}$	R_o
internal pressure	$0.02\mathrm{MPa}$	\wp_i
initial angle	$\pi/4$	γ_0
initial λ_s	1.014	λ_{s0}
matrix stiffness	$0.0375\mathrm{MPa}$	k_m
fibre isotropic stiffness	$0.0375\mathrm{MPa}$	k_{fi}
fibre anisotropic stiffness	$0.0375\mathrm{MPa}$	k_{fa}
remodelling stiffness	$5 \times 10^{-8} \mathrm{s/Pa}$	$k^{'}$
fibre volume fraction	0.2	Φ_f

Table 1: Parameters employed in the numerical analysis.

the model computationally heavy. In this numerical study, 302 we have two types of integrals: the surface integral over 303 the unit sphere \mathbb{S}^2 , which describes the fibre distribution, 304 and the integral over the interval bounded by the inner 305 and the outer radii $[R_i, R_o]$. For the surface integral, we 306 use the Lebedev quadrature (Lebedev, 1977), in which the 307 grid points and the corresponding weights are obtained 308 from the exact integration of *spherical harmonics* up to 309 an arbitrary order. The model parameters are given in 310 Table 1. 311

7. Numerical Results

Figure 3 represents the evolution of the straightening 313 stretch λ_s . The behaviour of λ_s is monotonically decreas-314 ing in the radius R throughout the evolution. The dif-315 ference $\lambda_s(R_i, t) - \lambda_s(R_o, t)$ increases monotonically with 316 time. We note that the $\lambda_s(R_o)$ evolves due to the fact that 317 the radial deviatoric Mandel stress of the fibres, \mathfrak{M}_{ed} is not 318 zero (Equation (38)), although the total Mandel stress \mathfrak{M} 319 vanishes due to the boundary conditions. 320

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Figure 3: Evolution of the straightening stretch λ_s with time.

Figure 4 shows the evolution of the behaviour of the angle γ describing the preferred fibres direction with time. After remodelling, the maximum and minimum angles occur at the inner and outer radii, respectively. The differ-322 ence $\gamma(R_i, t) - \gamma(R_o, t)$ is more pronounced in the early cycles and then tends to remain constant with time.



Figure 4: Evolution of the preferred fibre angle γ with time.

Figure 5 shows the evolution of the radial first Piola-327 Kirchhoff stress T_r^R (dashed lines) and circumferential 328 first Piola-Kirchhoff stress T_{θ}^{Θ} (solid lines) as a function 329 of the deformed radius $r = \xi(R, t)$. The remodelling makes 330 the circumferential stress T_{θ}^{Θ} more homogeneous through-331 out the thickness of the tube. The difference $T_{\theta}^{\Theta}(R_i, t)$ – 332 $T_{\theta}^{\Theta}(R_o, t)$ before remodelling is about 23 kPa at t = 0 s 333 and it reduces to 16 kPa at t = 400 s and to 14 kPa at 334 $t = 800 \, \mathrm{s}.$ 335



Figure 5: First Piola-Kirchhoff stresses $T_r{}^R$ (dashed lines) and $T_\theta{}^\Theta$ (solid lines).

One of the most prominent mechanical aspects of bi-336 ological tissues is the presence of residual stresses. Fung 337 (1983) predicted that the distribution of residual stresses 338 in the arteries is such that the residual circumferential 339 stress (along Θ -axis) is compressive in the interior layers 340 and tensile in the outer ones. The residual second Piola-341 Kirchhoff stresses for our benchmark problem is shown in 342 Figure 6 as a function of the undeformed radius R, at time 343 $t = 800 \,\mathrm{s}$. All three principal residual stresses increase 344 monotonically and the residual circumferential stress $S^{\Theta\Theta}$, 345

in accordance with Fung (1983), is compressive at the inner wall and tensile at the outer wall.



Figure 6: Residual second Piola-Kirchhoff stresses at time t = 800 s.

8. Discussion and Conclusions

In this work we introduced a thermodynamically admissible model for pure remodelling of a fibre-reinforced material representing the arterial wall tissue. The approach is based on the theory of material uniformity, which is described by the material implant P. We proposed a simple evolution law, in which the inhomogeneity rate L_P is linearly related to the *deviatoric Mandel stress* \mathfrak{M}_d .

Using the evolution law (38), we solved a benchmark 356 numerical problem describing a pressurised thick-walled 357 cylinder under plane strain conditions, with uniform in-358 ternal pressure, as in the works by Olsson and Klarbring 359 (2008) and Grillo et al. (2015). We use the same consti-360 tutive laws as in the work by Grillo et al. (2015) but a 361 more realistic fibre orientation probability, with two fam-362 ilies of fibres each obeying a bivariate von Mises distribu-363 tion (Holzapfel et al., 2015; Gizzi et al., 2018) (Figure 2). 364

The results for the remodelling angle are qualitatively 365 similar to those obtained by Grillo et al. (2015). Both 366 models predict that the preferred angle γ increases with 367 time, with values at the inner radius R_i being the largest. 368 Moreover, the dependence on radius and time of the ra-369 dial and circumferential stresses T_r^R and T_{θ}^{Θ} in our model 370 (Figure 5) is similar to that in the paper by Grillo et al. 371 (2015). However, while in Grillo et al. (2015) the cylin-372 der deflates as it becomes stiffer *circumferentially*, in our 373 study the cylinder inflates. This is not surprising, as we 374 have two evolving mechanisms that work simultaneously, 375 namely the relaxation of the fibres (increasing straighten-376 ing stretch λ_s) and the change in fibre angle (increasing 377 preferred angle γ). Indeed, when λ_s increases, it causes a 378 relaxation of the fibres, and the cylinder needs to inflate 379 so that the fibres reach their straightening stretch and are 380 able to bear load.

Other studies considered a change of undulation of the fibres or fibrils and our model is in agreement with these

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findings, despite being fundamentally different in the basic 384 assumptions. Indeed, Humphrey (1999) considers resorp-385 tion and deposition of new fibres and Watton and Hill 386 (2009) and Watton et al. (2009) consider pre-stretch in Z-387 direction. The relaxation effect that our model predicts 388 has been observed by Kamiya and Togawa (1980). In ad-389 dition, the residual stress is compressive in the inner layer 390 and tensile in the outer layer, in agreement with the be-391 haviour described by Fung (1983). 392

It is noteworthy that, in our model, we did not pre-393 scribe the evolution law in accordance to experimental ob-394 servations. Rather, we postulated an evolution law solely 395 based on the conditions of reduction to the archetype (17)396 and of compliance with the dissipation inequality (18). In 397 spite of its relatively simple form, the evolution law could 398 qualitatively reproduce the remodelling behaviour seen in 399 other studies. This indicates that the framework based 400 on the theory of evolution and material uniformity can be 401 a viable and promising paradigm to explore growth and 402 remodelling of biological tissues. 403

This work followed Epstein and Maugin (2000) and Ep-404 stein and Elzanowski (2007), who used the theory of uni-405 formity with a time-dependent implant \boldsymbol{P} , which consti-406 tutes an *internal variable*. In contrast, Grillo et al. (2015) 407 treated the fibre mean angle as a kinematic variable that 408 satisfies a balance of generalised forces, following the same 409 philosophy used by Di Carlo and Quiligotti (2002). Al-410 though different in nature, these two approaches give qual-411 itatively similar results. 412

The proposed model constitutes a step further in the study of growth and remodelling of fibre-reinforced soft biological tissues, in the framework of material implant theory. Even though the numerical example lacks the necessary details to study specific cases such as *hypertension* and *aneurysms*, the agreements of the results with previous studies make this framework promising.

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426 Competing Interests

427 The authors declare no competing interests.

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