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Adaptive hp-FEM for eigenvalue computations

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Abstract We design an adaptive procedure for approximating a selected eigenvalue and its eigenspace for a second-order elliptic boundary-value problem, using an *hp* finite element method. Such iterative procedure judiciously alternates between a stage in which a near-optimal *hp*-mesh for the current level of accuracy is generated, and a stage in which such mesh is sufficiently refined to produce a new, enhanced approximation of the eigenfunctions. We identify conditions on the initial mesh and the operator coefficients under which the procedure yields approximations that converge at a geometric rate independent of any discretization parameter, using a number of degrees of freedom comparable to the smallest number needed to get the achieved accuracy. We detail the second stage for a single eigenvalue, relying on a *p*-robust saturation property.

Keywords *hp*-finite element method · Eigenvalue approximations · Adaptivity · Complexity

Mathematics Subject Classification (2010) 65N30 · 65N25 · 65N50

1 Introduction

Since the pioneering book [24], in which the first a priori error bounds for Galerkin approximations of eigenvalues and eigenfunctions of self-adjoint elliptic operators was established, the achievements of new results concerning the numerical discretization of source-type elliptic boundary-value problems have always been accompanied by companion results about the related eigenvalue problems; we refer to [1] and [5] for overviews on the subject.

In this framework, the analysis of adaptive finite-element methods for eigenvalue computations was started at the beginning of this century, being essentially concerned with *h*-type discretizations. Using reliable and efficient a posteriori error estimators (see e.g. [25,22,15]), a mark-and-refine strategy may be adopted to build successive approximations of a selected eigenvalue and its eigenspace. At first, the convergence of such iterations was established [19,18,13] by proving a suitable contraction property; subsequently, the optimality of the procedure was assessed [14], providing a counterpart of the optimality analysis first developed in [12] for a source-type problem. More recent results concern the adaptive approximation of eigenvalue clusters [17,6], or the availability of guaranteed, fully computable a posteriori error bounds [8] to be used in the adaptive procedure.

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Results concerning *hp*-adaptivity for eigenvalue problems are more limited, being confined to a posteriori estimators (see [20] for a residual-based estimator in the framework of *hp*-DG schemes, and again [8] for an equilibrated-flux estimator which, unlike the former, is p-robust). On the other hand, only recently a rigorous analysis of convergence and optimality for *hp*-adaptive finite-element discretizations of source-type problems has appeared [9,10]. The procedure therein proposed and analyzed consists of a judicious alternation between a stage in which the current error is reduced by a prescribed fraction by suitably refining the current *hp*-mesh (this guarantees convergence), and a stage in which a new *hp*-mesh is built, with the property that the cardinality of the corresponding degrees of freedom is comparable to the minimal cardinality necessary for achieving the current accuracy (this guarantees optimality). To realize the latter stage, a near-optimal *hp*-approximation algorithm [3] is applied, which considers the current numerical solution (as well as the operator coefficients and the right-hand side) as "data" to be approximated at best.

The purpose of the present paper is to adapt the general setting introduced in [9] to the case of *hp*-adaptive approximations of eigenvalue problems. We design an iterative procedure for computing a selected eigenvalue and its eigenspace, which at each step generates a near-optimal *hp*-mesh for the current level of accuracy; such mesh is then sufficiently refined to produce a new, enhanced approximation of the eigenfunctions. We identify conditions on the initial mesh and the operator coefficients under which the procedure yields approximations that converge at a geometric rate independent of any discretization parameter, while growing the number of degrees of freedom in a way comparable to the optimal (i.e., the minimal) number. Our general convergence and optimality result applies to the approximation of eigenvalues of any multiplicity. However, in order to keep the complexity and length of the paper at a moderate level, a realization of the stage in which the approximation error is brought below a given threshold is detailed only for the case of a single eigenvalue; the extension to a multiplicity > 1 will be presented elsewhere. In this stage, we activate an inner loop where a Dörfler-type marking is applied, which is based on a p-robust equilibrated-flux estimator [7,16,8]. For such estimator, we establish a saturation result as in [10] which implies a contraction property for the eigenfunction error in the energy norm. Overall, the number of inner iterations and the complexity (number of activated degrees of freedom) of the proposed realization of this stage is independent of the mesh-size, while depends in a very mild way (i.e., logarithmically) on the largest polynomial degree used on the initial mesh.

The paper is organized as follows. In Sect. 2 we formulate the eigenvalue problem and we introduce the *hp*-partitions and the function spaces built on them, that will be used throughout the paper. In the two following Sects. 3 and 4, we present the two routines that constitute the building blocks of the proposed algorithm. The former one, termed **hp-NEARBEST**, produces, for a given function and a given tolerance, an *hp*-partition and an *hp*-approximation of the function, such that a prescribed error functional is below the tolerance, while the cardinality of the partition is comparable to the minimal cardinality among all partitions that provide a similar accuracy (instance optimality); the routine relies on the construction of [3]. The latter routine, termed **EIGEN**, implements a Galerkin discretization of an eigenvalue problem with given piecewise-polynomial coefficients, and produces an approximation of the eigenfunctions of a selected eigenvalue, with prescribed error bound. The two routines are concatenated in Sect. 5 to create the main algorithm, **hp-AFEM-EIG**, which outputs a sequence of *hp*-partitions and corresponding discrete eigenspaces made of piecewise polynomial functions defined on such partitions. Under reasonable assumptions, we prove that the sequence of discrete eigenspaces converges to the exact eigenspace of the target eigenvalue, and the partitions satisfy an instance optimality property. Finally, in Sect. 6 we describe an implementation of **EIGEN** for an eigenvalue of multiplicity 1, and we discuss its properties.

Throughout the paper, $A \lesssim B$ means $A \leq c\,B$ for some constant $c > 0$ independent of the relevant parameters upon which the non-negative quantities A and B may depend. The symbol $A \simeq B$ means $A \lesssim B$ and $B \lesssim A$.

2 Problem setting

In a polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the following second-order self-adjoint eigenvalue problem:

$$-\nabla\cdot(\nu\nabla w) = \lambda\varrho w \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

with ν, $\varrho \in L^\infty(\Omega)$ satisfying $\nu \geq \nu_0$ and $\varrho \geq \varrho_0$ a.e. in Ω for some constants $\nu_0 > 0$ and $\varrho_0 > 0$. Further assumptions on the coefficients will be made later on. Introducing the bilinear forms

$$a(\nu; u, v) := \int_\Omega \nu\, \nabla u\cdot\nabla v \,\mathrm{dx}, \qquad b(\varrho; u, v) := \int_\Omega uv\, \varrho \,\mathrm{dx}$$

defined in $V := H^1_0(\Omega)$ and $H := L^2(\Omega)$, respectively, the problem can be given the following variational formulation

$$w \in V \;:\quad a(\nu; w, v) = \lambda b(\varrho; w, v) \qquad \forall v \in V. \tag{2.2}$$

It is well known that this problem admits a non-decreasing, unbounded sequence of strictly positive eigenvalues λ_n, $n \geq 1$, with corresponding eigenfunctions $w_n \in V$ satisfying

$$a(\nu; w_n, w_m) = b(\varrho; w_n, w_m) = 0 \qquad \text{whenever } \; n \neq m.$$

We will be interested in approximating the j-th eigenvalue $\lambda_\star := \lambda_j = \cdots = \lambda_{j+M-1}$, which we assume of multiplicity $M \geq 1$, and the corresponding eigenspace $W_\star := \operatorname{span}\{w_j, \ldots, w_{j+M-1}\}$. We suppose the generating functions w_{j+m}, $m = 0, \ldots, M-1$, to be normalized in V.

2.1 *hp*-partitions and *hp*-approximation spaces

In view of the *hp*-adaptive discretization of the spectral problem above, we introduce some notation concerning partitions of the domain and function spaces built on them.

2.1.1 Partitions of the domain

We assume that we are given an essentially disjoint initial partition $\mathcal{K}_0$ of $\bar\Omega$ (the *root* partition) into finitely many closed subdomains, which will be the initial geometric elements. We assume that for each element K that we encounter, there exists a unique way in which K can be split into elements K' and K'', the 'children' of K, such that $K = K' \cup K''$ and $|K' \cap K''| = 0$. The set $\mathfrak{K}$ of all these geometric elements forms an infinite binary 'master tree', having as its roots the elements of the initial partition of $\bar\Omega$. We require that the adopted rule of splitting maintains the ratio between the diameter of an element and the diameter of the inscribed circle uniformly bounded; an example is, for triangles, the 'newest vertex bisection rule'. A subtree of the master tree is a finite subset of $\mathfrak{K}$ that contains all roots and for each element in the subset both its parent and its sibling are in the subset. The leaves of a subtree form an essentially disjoint partition of $\bar\Omega$. The set of all such geometric partitions, or '*h*-partitions', will be denoted as $\mathbb{K}$. For $\mathcal{K}, \widetilde{\mathcal{K}} \in \mathbb{K}$, we call $\widetilde{\mathcal{K}}$ a refinement of $\mathcal{K}$, and write $\mathcal{K} \leq \widetilde{\mathcal{K}}$, when any $K \in \widetilde{\mathcal{K}}$ is either in $\mathcal{K}$ or has an ancestor in $\mathcal{K}$.

Starting from an *h*-partition $\mathcal{K} \in \mathbb{K}$, we obtain an *hp-partition* $\mathcal{D}$ by associating an integer $d \geq 1$ to each element $K \in \mathcal{K}$. This integer will be related to the dimension of certain finite dimensional spaces of functions defined in K; in turn, such spaces will depend on a polynomial degree p, defined by a suitable function $p = p(d)$. A pair $D = (K_D, d_D) \in \mathfrak{K} \times \mathbb{N}$ formed by a geometric element K_D and an integer d_D will be termed an *hp-element*. Thus, a collection $\mathcal{D} = \{D = (K_D, d_D)\}$ of *hp*-elements is an *hp*-partition provided $\mathcal{K}(\mathcal{D}) := \{K_D : D \in \mathcal{D}\} \in \mathbb{K}$; the latter will be the

associated h-partition. The collection of all hp-partitions is denoted as $\mathbb{D}$. The *dimension* of the hp-partition $\mathcal{D}$ is the integer

$$\#\mathcal{D} := \sum_{D\in\mathcal{D}} d_D.$$

For $\mathcal{D}, \widetilde{\mathcal{D}} \in \mathbb{D}$, we call $\widetilde{\mathcal{D}}$ a refinement of $\mathcal{D}$, and write $\mathcal{D} \le \widetilde{\mathcal{D}}$, when both $\mathcal{K}(\mathcal{D}) \le \mathcal{K}(\widetilde{\mathcal{D}})$, and $d_{\widetilde{D}} \ge d_D$, for any $D \in \mathcal{D}$, $\widetilde{D} \in \widetilde{\mathcal{D}}$ with K_D being either equal to $K_{\widetilde{D}}$ or an ancestor of $K_{\widetilde{D}}$.

We will also need to deal with *conforming* hp-partitions, i.e., partitions $\mathcal{D} \in \mathbb{D}$ whose associated h-partitions $\mathcal{K}(\mathcal{D})$ are 'conforming' in the usual finite element sense. We denote by $\mathbb{D}^c \subset \mathbb{D}$ the subset of such partitions. We assume that a mapping $\mathcal{C} : \mathbb{D} \to \mathbb{D}^c$ has been selected, satisfying the property

$$\#\mathcal{C}(\mathcal{D}) = \min\{\#\widetilde{\mathcal{D}} : \widetilde{\mathcal{D}} \in \mathbb{D}^c \text{ and } \widetilde{\mathcal{D}} \ge \mathcal{D}\}, \tag{2.3}$$

and we will set $\mathcal{D}^c := \mathcal{C}(\mathcal{D})$.

2.1.2 Approximation spaces on hp-partitions

Given a partition $\mathcal{D} \in \mathbb{D}$, we will have to build on it suitable piecewise polynomial approximations of M functions $v_m \in V$ $(m = 0, \dots, M-1)$ related to the eigenfunctions of our spectral problem, as well as of the coefficients ν and ϱ of the operator. To this end, from now on we assume that the coefficients are smoother than just L^∞-functions in each element of the initial partition; precisely, we assume that $\nu, \varrho \in S(\Omega) := \{v \in L^\infty(\Omega) : v_{|K_0} \in H^\tau(K_0) \quad \forall K_0 \in \mathcal{K}_0\}$ for some $\tau > 1$. Thus, setting $z = ((v_m), \nu, \varrho) \in Z := \big(\prod_{m=0}^{M-1} H^1_0(\Omega)\big) \times S(\Omega) \times S(\Omega)$, it holds $z_{|K} \in Z_K := \big(\prod_{m=0}^{M-1} H^1(K)\big) \times H^\tau(K) \times H^\tau(K)$ for each element K of the master tree $\mathfrak{K}$. For any h-partition $\mathcal{K} \in \mathbb{K}$, we introduce the infinite dimensional space consisting of $(M+2)$-tuples of functions of broken regularity in Ω

$$Z_{\mathcal{K}} := \{z : \Omega \to \mathbb{R}^{M+2} : z_{|K} \in Z_K \quad \forall K \in \mathcal{K}\} = \prod_{K\in\mathcal{K}} Z_K.$$

From now on, let us fix once and for all the mapping $p = p(d)$ relating the dimension parameter d to the polynomial degree p; this function should be non-decreasing and unbounded from above. For instance, when K is a triangle, $p = p(d)$ can be defined as the largest value in $\mathbb{N}$ such that $\dim \mathbb{P}_{p-1}(K) = \frac{1}{2}p(p+1) \le d$. Then, for any element $K \in \mathfrak{K}$ and any integer $d \ge 1$, we introduce the finite-dimensional subspace of Z_K

$$Z_{K,d} := \big(\prod_{m=0}^{M-1} \mathbb{P}_{p(d)}(K)\big) \times \mathbb{P}_{p(d)+\alpha}(K) \times \mathbb{P}_{p(d)+\beta}(K),$$

where α, $\beta \ge 0$ are fixed constant integers. Note that $Z_{K,d} \subseteq Z_{K,d+1}$ and $Z_{K,d} \subset Z_{K',d} \times Z_{K'',d}$ if K', K'' are the children of K. Finally, for any hp-element $D = (K_D, d_D)$ we set $Z_D := Z_{K_D,d_D}$, and for any hp-partition $\mathcal{D} \in \mathbb{D}$ we introduce the (broken) approximation space

$$Z_{\mathcal{D}} := \prod_{D\in\mathcal{D}} Z_D,$$

which is a finite dimensional subspace of $Z_{\mathcal{K}(\mathcal{D})}$.

At last, let us define suitable local and global projection operators and error functionals. For any $r \ge 0$, any element $K \in \mathfrak{K}$ and any polynomial degree $q \ge 0$, let $P^r_{K,q} : H^r(K) \to \mathbb{P}_q(K)$ be the orthogonal projection in the norm of $H^r(K)$. Then, for any hp-element $D = (K_D, d_D)$, we define the operator $Q_D : Z_{K_D} \to Z_D$ by setting

$$Q_D := (P^1_{K_D,p_D}, \dots, P^1_{K_D,p_D}, P^\tau_{K_D,p_D+\alpha}, P^\tau_{K_D,p_D+\beta}).$$

The associated *local error functional* $e_D : Z_D \to \mathbb{R}$, which provides a measure of the squared projection error, is defined for any $z = ((v_m), \mu, \vartheta) \in Z_{K_D}$ as follows

$$\begin{aligned} e_D(z) := \sum_{m=0}^{M-1} \|v_m - P^1_{K_D,p_D} v_m\|^2_{H^1(K_D)} + \\ + \frac{1}{\kappa^2} \left(\|\nu - P^\tau_{K_D,p_D+\alpha}\nu\|^2_{H^\tau(K_D)} + \|\varrho - P^\tau_{K_D,p_D+\beta}\,\varrho\|^2_{H^\tau(K_D)} \right), \end{aligned} \tag{2.4}$$

where $\kappa > 0$ will be chosen small enough later on, and will act as the reciprocal of a penalization parameter. Note that such error functional $e_D = e_D(z)$ is non-increasing under both 'h-refinements' and 'p-enrichments', i.e., it satisfies

$$\begin{aligned} e_{D'} + e_{D''} \le e_D \text{ when } K_{D'}, K_{D''} \text{ are the children of } K_D, \text{ and } d_{D'} = d_{D''} = d_D; \\ e_{D'} \le e_D \text{ when } K_{D'} = K_D \text{ and } d_{D'} \ge d_D. \end{aligned} \tag{2.5}$$

For any *hp*-partition $\mathcal{D} \in \mathbb{D}$, the *global error functional* $\mathrm{E}_\mathcal{D} : Z \to \mathbb{R}$ is defined as

$$\mathrm{E}_\mathcal{D}(z) := \sum_{D\in\mathcal{D}} e_D(z_{|K_D}),$$

and, as a consequence of (2.5), it satisfies

$$\mathrm{E}_{\widetilde{\mathcal{D}}}(z) \le \mathrm{E}_\mathcal{D}(z) \quad \forall \widetilde{\mathcal{D}} \ge \mathcal{D}. \tag{2.6}$$

For the subsequent applications, it is convenient to use a separate, and more concise, notation for eigenfunctions and coefficients. So, il $z = ((v_m), \nu, \varrho)$, we will set $z = (v^\star, \vartheta)$ with $v^\star = (v_m)$ and $\vartheta = (\nu, \varrho)$. If $z \in Z$, then $v^\star \in \mathbb{V} := V^M$ and $\vartheta \in \mathbb{S} := S(\Omega)^2$. On the other hand, if $z_\mathcal{D} \in Z_\mathcal{D}$, then $v^\star_\mathcal{D} \in \mathbb{V}_\mathcal{D} := V^M_\mathcal{D}$ with $V_\mathcal{D} := \prod_{D\in\mathcal{D}} \mathbb{P}_{p_D}(K_D)$, whereas $\vartheta_\mathcal{D} \in \mathbb{S}_\mathcal{D} := \prod_{D\in\mathcal{D}} \big(\mathbb{P}_{p_D+\alpha}(K_D) \times \mathbb{P}_{p_D+\beta}(K_D)\big)$.

Finally, it is convenient to introduce a notation for conforming subspaces defined on *hp*-partitions, namely we set $V^c_\mathcal{D} := V_\mathcal{D} \cap V$, and $\mathbb{V}^c_\mathcal{D} := (V^c_\mathcal{D})^M$. The typical approximation of eigenfunctions and coefficients we are going to build on some partition $\mathcal{D} \in \mathbb{D}$ will be a pair $z_\mathcal{D} = (v^\star_\mathcal{D}, \vartheta_\mathcal{D}) \in \mathbb{V}^c_\mathcal{D} \times \mathbb{S}_\mathcal{D}$.

3 An instance optimal *hp*-approximation algorithm

In this section, we recall the principles of near-best adaptive *hp*-approximation, based on a greedy algorithm proposed by P. Binev [3].

Let us fix a vector of functions $z = (v^\star, \vartheta) \in Z$. For any *hp* element D, let $e_D = e_D(z_{|K_D})$ be the local error functional defined in (2.4), and for any *hp*-partition $\mathcal{D}$ let $E_\mathcal{D} = E_\mathcal{D}(z) = \sum_{D\in\mathcal{D}} e_D$ be the corresponding global error functional.

Denote by $R \ge 1$ the cardinality of the initial geometric partition $\mathcal{K}_0$. Using property (2.5), Binev's algorithm builds a sequence of *hp*-partitions $\mathcal{D}_N$, $N \ge R$, satisfying $\#\mathcal{D}_N = N$; the construction is incremental, in that going from $\mathcal{D}_N$ to $\mathcal{D}_{N+1}$ one exploits the work already done to build $\mathcal{D}_N$. The main feature of the algorithm is its instance optimality, expressed as follows.

Theorem 3.1 ([3]) *For $n \ge R$ let*

$$\sigma_n := \inf_{\#\mathcal{D}\le n} E_\mathcal{D}$$

be the smallest error achievable with an hp-partition of cardinality $\le n$. Then, the hp-partitions $\mathcal{D}_N$ produced by Binev's algorithm yield error functionals $E_{\mathcal{D}_N}$ satisfying the bounds

$$\mathrm{E}_{\mathcal{D}_N} \le \frac{2N}{N-n+1}\sigma_n \quad \forall n \le N. \qquad \square \tag{3.1}$$

Binev's construction can be easily used to produce an instance optimal *hp*-partition for which the error functional is below a given threshold.

Corollary 3.1 ([9]) *Let $B > 1$ arbitrary. Given $\varepsilon > 0$, let $\mathcal{D} \in \mathbb{D}$ be the first partition in Binev's sequence for which $E_{\mathcal{D}} \le \varepsilon^2$. Then, setting $b = \sqrt{\frac{1}{2}(1 - \frac{1}{B})} < 1$, it holds*

$$\#\mathcal{D} \le B\,\#\widehat{\mathcal{D}}$$

for all partitions $\widehat{\mathcal{D}} \in \mathcal{D}$ satisfying $E_{\widehat{\mathcal{D}}} \le (b\varepsilon)^2$. ∎

The proof is by contradiction: the existence of a $\widehat{\mathcal{D}}$ for which $\#\mathcal{D} > B\,\#\widehat{\mathcal{D}}$ would imply, using (3.1), that $\mathcal{D}$ is not the first partition in Binev's algorithm to satisfy $E_{\mathcal{D}} \le \varepsilon^2$; see [9, Corollary 2] for the details. Note that ε is raised to the power 2 as a consequence of the quadratic structure of the functional $\mathrm{E}_{\mathcal{D}}$ (recall (2.4)).

This result motivates the introduction of the following routine, which will constitute one of the two major building blocks of our proposed *hp*-adaptive algorithm for eigenvalue computations.

- $[\mathcal{D}, \vartheta_{\mathcal{D}}] := \textbf{hp-NEARBEST}(\varepsilon, v^\star, \vartheta)$

  The routine **hp-NEARBEST** takes as input $\varepsilon > 0$, and $(v^\star, \vartheta) \in Z$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $\vartheta_{\mathcal{D}} \in \mathbb{S}_{\mathcal{D}}$ such that $\mathrm{E}_{\mathcal{D}}(v^\star, \vartheta)^{\frac{1}{2}} \le \varepsilon$ and, for some constants $0 < b < 1 < B$, $\#\mathcal{D} \le B\,\#\widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\widehat{\mathcal{D}}}(v^\star, \vartheta)^{\frac{1}{2}} \le b\varepsilon$.

The approximation $\vartheta_{\mathcal{D}}$ of the data ϑ is just the element-wise projection given by the operator $Q_{\mathcal{D}}$ associated with the partition $\mathcal{D}$, i.e., we set

$$(v^\star_{\mathcal{D}}, \vartheta_{\mathcal{D}}) := Q_{\mathcal{D}}(v^\star, \vartheta). \tag{3.2}$$

The following L^∞-error estimate will be useful in the sequel.

Proposition 3.1 *There exists a constant $C_0 > 0$ such that*

$$\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} + \|\varrho - \varrho_{\mathcal{D}}\|_{L^\infty(\Omega)} \le C_0 \kappa\, \mathrm{E}_{\mathcal{D}}(v^\star, \vartheta)^{\frac{1}{2}} \qquad \forall (v^\star, \vartheta) \in Z.$$

Proof Consider the function ν, the argument being identical for ϱ. Let $D \in \mathcal{D}$ be an element such that $\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} = \|\nu - \nu_{\mathcal{D}}\|_{L^\infty(K_D)}$, and let $\hat{K}_D$ be an affine image of K_D with $|\hat{K}_D| = 1$. For convenience, let us set $\phi := (\nu - \nu_{\mathcal{D}})_{|K_D} = \nu_{|K_D} - P^\tau_{K_D, p_D+\alpha}\nu_{|K_D}$, and let $\hat{\phi}$ its affine image on $\hat{K}_D$. By the Sobolev embedding theorem, we have $\|\phi\|_{L^\infty(K_D)} = \|\hat{\phi}\|_{L^\infty(\hat{K}_D)} \le \hat{C}_1 \|\hat{\phi}\|_{H^\tau(\hat{K}_D)}$, where $\hat{C}_1 > 0$ is an absolute constant. Actually, since by construction ϕ has zero-mean in K_D, by a Poincaré-type inequality the norm on the right-hand side can be replaced by the reduced norm $|\!|\!|\hat{\phi}|\!|\!|_{H^\tau(\hat{K}_D)}$ in which the $L^2(\hat{K}_D)$-contribution is missing. It is easily seen that $|\!|\!|\hat{\phi}|\!|\!|_{H^\tau(\hat{K}_D)} \le C_2 |\!|\!|\phi|\!|\!|_{H^\tau(K_D)} \le C_2 \|\phi\|_{H^\tau(K_D)}$ for some absolute constant $C_2 > 0$. In conclusion, setting $\hat{C}_0 = \hat{C}_1 C_2$ and recalling (2.4), we have

$$\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} = \|\nu - \nu_{\mathcal{D}}\|_{L^\infty(K_D)} \le \hat{C}_0 \|\nu - \nu_{\mathcal{D}}\|_{H^\tau(K_D)} \le \hat{C}_0 \kappa\, e_D((v^\star, \vartheta)_{|K_D})^{\frac{1}{2}} \le \hat{C}_0 \kappa\, \mathrm{E}_{\mathcal{D}}(v^\star, \vartheta)^{\frac{1}{2}},$$

which proves the result, with $C_0 = 2\hat{C}_0$. ∎

Remark 3.1 Note that from a call to **hp-NEARBEST** we also obtain an approximation $v^\star_{\mathcal{D}} = (v_{m,\mathcal{D}})$ of the functions in $v^\star = (v_m)$ (which in the subsequent application will be discrete eigenfunctions); however, we prefer to discard such approximations, since in general they are discontinuous at the interelement boundaries. Yet, the output partition is adapted to approximating the functions in $v^\star$ as well, since they enter the definition of the error functional; this will be important for the optimality of the proposed algorithm.

4 Galerkin approximations of the spectral problem

Let $\vartheta_{\mathcal{D}} \in \mathbb{S}_{\mathcal{D}}$ be an approximation of the data ϑ on a *hp*-partition $\mathcal{D} \in \mathbb{D}$. This might be generated by a call to **hp-NEARBEST**, as it will be the case in the sequel.

Keeping into account Proposition 3.1, we may assume that the error between $\vartheta = (\nu, \varrho)$ and $\vartheta_{\mathcal{D}} = (\nu_{\mathcal{D}}, \varrho_{\mathcal{D}})$ in the broken H^τ-norm is sufficiently small to guarantee $\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq \frac{1}{2}\nu_0$ and $\|\varrho - \varrho_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq \frac{1}{2}\varrho_0$. This implies $\nu_{\mathcal{D}} \geq \frac{1}{2}\nu_0$ and $\varrho_{\mathcal{D}} \geq \frac{1}{2}\varrho_0$. As a consequence, the infinite-dimensional spectral problem *with perturbed coefficients*

$$w \in V \ : \quad a(\nu_{\mathcal{D}}; w, v) = \lambda\, b(\varrho_{\mathcal{D}}; w, v) \qquad \forall v \in V, \tag{4.1}$$

admits a non-decreasing, unbounded sequence of strictly positive eigenvalues $\lambda_n^{\mathcal{D}}$, $n \geq 1$, with corresponding eigenfunctions $w_n^{\mathcal{D}} \in V$ satisfying $a(\nu_{\mathcal{D}}; w_n^{\mathcal{D}}, w_m^{\mathcal{D}}) = b(\varrho_{\mathcal{D}}; w_n^{\mathcal{D}}, w_m^{\mathcal{D}}) = 0$ whenever $n \neq m$.

If $\vartheta_{\mathcal{D}}$ is sufficiently close to ϑ, we expect the M perturbed eigenvalues $\lambda_j^{\mathcal{D}} \leq \cdots \leq \lambda_{j+M-1}^{\mathcal{D}}$ to be close to the exact eigenvalue $\lambda_\star$, and the subspace $W^{\mathcal{D}} := \operatorname{span}\{w_j^{\mathcal{D}}, \dots, w_{j+M-1}^{\mathcal{D}}\}$ spanned by the corresponding eigenfunctions to be close to the eigenspace $W_\star$ of $\lambda_\star$. Let us recall that the 'distance', or *gap*, between two subspaces X and Y of V is defined as

$$\hat{\delta}_V(X, Y) := \max(\delta_V(X, Y), \delta_V(Y, X)), \qquad \text{with} \quad \delta_V(X, Y) := \sup_{x \in X, \|x\|_V = 1} \inf_{y \in Y} \|x - y\|_V.$$

Problem (4.1) is discretized by a Galerkin method, built on conforming partitions $\bar{\mathcal{D}} \in \mathbb{D}^c$ that are *refinements* of $\mathcal{D}$, i.e., they satisfy $\bar{\mathcal{D}} \geq \mathcal{D}$. For such a partition, we consider the discrete spectral problem

$$w \in V_{\bar{\mathcal{D}}} \ : \quad a(\nu_{\mathcal{D}}; w, v) = \lambda\, b(\varrho_{\mathcal{D}}; w, v) \qquad \forall v \in V_{\bar{\mathcal{D}}}, \tag{4.2}$$

whose eigenvalues and eigenfunctions, resp., will be denoted by $\lambda_{\bar{\mathcal{D}},n}$ and $w_{\bar{\mathcal{D}},n}$, resp., (they actually depend on $\mathcal{D}$ as well, yet we prefer this simpler notation). In particular, assuming the dimension of $V_{\bar{\mathcal{D}}}$ large enough, we are interested in the M eigenvalues $\lambda_{\bar{\mathcal{D}},j} \leq \cdots \leq \lambda_{\bar{\mathcal{D}},j+M-1}$ and in the subspace $W_{\bar{\mathcal{D}}} := \operatorname{span}\{w_{\bar{\mathcal{D}},j} \dots, w_{\bar{\mathcal{D}},j+M-1}\}$ spanned by the corresponding discrete eigenfunctions.

By choosing a sufficiently refined partition $\bar{\mathcal{D}}$, we are able to bring the distance between the eigenfunction spaces $W^{\mathcal{D}}$ and $W_{\bar{\mathcal{D}}}$ below any prescribed threshold. This may be accomplished by adopting a suitable refinement strategy, based on the information given by some a posteriori error estimator. An explicit realization will be discussed in Sect. 6.

For the moment, we formalize this procedure by assuming the availability of the following routine, which will constitute the second major building block of our proposed *hp*-adaptive algorithm for eigenvalue computations.

- $[\bar{\mathcal{D}}, w_{\bar{\mathcal{D}}}^\star] := \mathbf{EIGEN}(\varepsilon, \mathcal{D}, \vartheta_{\mathcal{D}})$

  The routine **EIGEN** takes as input $\varepsilon > 0$, $\mathcal{D} \in \mathbb{D}$, and data $\vartheta_{\mathcal{D}} \in \mathbb{S}_{\mathcal{D}}$. It outputs a partition $\bar{\mathcal{D}} \in \mathbb{D}^c$ with $\mathcal{D} \leq \bar{\mathcal{D}}$ and a vector $w_{\bar{\mathcal{D}}}^\star = (w_{\bar{\mathcal{D}},j} \dots, w_{\bar{\mathcal{D}},j+M-1}) \in \mathbb{V}_{\bar{\mathcal{D}}}^c$ of eigenfunctions of Problem (4.2), such that $\hat{\delta}_V(W^{\mathcal{D}}, W_{\bar{\mathcal{D}}}) \leq \varepsilon$.

5 The general *hp*-adaptive algorithm

In this section, we concatenate the two routines **hp-NEARBEST** and **EIGEN** introduced above, and we generate an adaptive algorithm with convergence and optimality properties.

To this end, we need the following properties of the global error functional $E_{\mathcal{D}}$.

Proposition 5.1 *i) There exists a constant $C_\star > 0$ with the following property. For all $\mathcal{D} \in \mathbb{D}$ and all $z = (v^\star, \vartheta) \in Z$, let $\vartheta_\mathcal{D} \in \mathbb{S}_\mathcal{D}$ be the projection of ϑ as defined in (3.2), and let $W^\mathcal{D}$ the M-dimensional space spanned by the eigenfunctions $w_j^\mathcal{D}, \dots, w_{j+M-1}^\mathcal{D}$ of the perturbed problem (4.1) with coefficients $(\nu_\mathcal{D}, \varrho_\mathcal{D}) = \vartheta_\mathcal{D}$. Then, it holds*

$$\hat\delta_V(W_\star, W^\mathcal{D}) \le C_\star \kappa \, \mathrm{E}_\mathcal{D}(v^\star, \vartheta)^{\frac12}, \tag{5.1}$$

where κ is the penalization parameter introduced in (2.4).

ii) For all $\mathcal{D} \in \mathbb{D}$, $v^\star, w^\star \in \mathbb{V}$ and $\vartheta \in \mathbb{S}$, it holds

$$|\,\mathrm{E}_\mathcal{D}(w^\star, \vartheta)^{\frac12} - \mathrm{E}_\mathcal{D}(v^\star, \vartheta)^{\frac12}\,| \le \|w^\star - v^\star\|_\mathbb{V}, \tag{5.2}$$

where the norm in $\mathbb{V} = V^M$ is the usual graph norm.

Proof i) Let $T : H \to V$ be the operator associated with the eigenvalue problem (2.2), i.e., such that for any $f \in H$

$$Tf \in V \;:\quad a(\nu; Tf, v) = b(\varrho; f, v) \qquad \forall v \in V, \tag{5.3}$$

which implies $Tw = \lambda^{-1} w$. Similarly, let $T_\mathcal{D} : H \to V$ be the operator associated with the perturbed eigenvalue problem (4.1), i.e., such that for any $f \in H$

$$T_\mathcal{D} f \in V \;:\quad a(\nu_\mathcal{D}; T_\mathcal{D} f, v) = b(\varrho_\mathcal{D}; f, v) \qquad \forall v \in V. \tag{5.4}$$

Then, one easily checks that for any $v \in V$

$$\begin{aligned} a(\nu; Tf - T_\mathcal{D} f, v) &= a(\nu_\mathcal{D} - \nu; T_\mathcal{D} f, v) + b(\varrho - \varrho_\mathcal{D}; f, v) \\ &\le C(\|\nu - \nu_\mathcal{D}\|_{L^\infty(\Omega)} + \|\varrho - \varrho_\mathcal{D}\|_{L^\infty(\Omega)}) \|f\|_{L^2(\Omega)} \|v\|_V, \end{aligned}$$

whence by coercivity and Proposition 3.1

$$\|Tf - T_\mathcal{D} f\|_V \le C'(\|\nu - \nu_\mathcal{D}\|_{L^\infty(\Omega)} + \|\varrho - \varrho_\mathcal{D}\|_{L^\infty(\Omega)}) \|f\|_{L^2(\Omega)} \le C'' \kappa \, \mathrm{E}_\mathcal{D}(v^\star, \vartheta)^{\frac12} \|f\|_{L^2(\Omega)}.$$

The result follows from the bound (see e.g. [1])

$$\hat\delta_V(W_\star, W^\mathcal{D}) \le C \gamma_\mathcal{D}, \qquad \text{with } \gamma_\mathcal{D} := \sup_{v \in W_\star} \frac{\|Tv - T_\mathcal{D} v\|_V}{\|v\|_V}.$$

ii) The bound follows from a double application of the triangle inequality, and from the minimality property of the orthogonal projection.

□

Assumption 5.1. *Let $b < 1 < B$ the constants that appear in the statement of the instance optimality property for the routine* **hp-NEARBEST**. *We assume that the penalization parameter κ in (2.4) is chosen small enough, so that it holds*

$$C_\star \kappa < \frac{b}{2\sqrt{M}}.$$

Let us introduce the parameters and the input data in our algorithm:

Parameters: two real numbers $\xi \in (0,1)$, $\omega > 0$ satisfying $C_\star \kappa < \frac{b}{2\sqrt{M}}(1-\xi)$ and $\omega \in (\frac{\sqrt{2M}}{b}, \frac{1-\xi}{\sqrt{2}C_\star \kappa})$.
Input data: the coefficients $\vartheta = (\nu, \varrho) \in \mathbb{S}$ of the eigenvalue problem (2.1), a set $w_0^\star \in \mathbb{V}$ of M approximations of the exact eigenfunctions $w_j, \dots, w_{j+M-1}$, , and an initial tolerance $\varepsilon_0 \in (0,1)$

such that: *i)* $\hat{\delta}_V(W_\star, W_0) \le \varepsilon_0$, where $W_0 := \operatorname{span} w_0^\star$, and *ii)* $C_0\kappa\omega\varepsilon_0 \le \frac{1}{2}\max(\nu_0, \varrho_0)$, where C_0 is defined in Proposition 3.1.

Algorithm hp-AFEM-EIG$(\vartheta, w_0^\star, \varepsilon_0)$
`for` $i = 1, 2, \ldots$ `do`
$[\mathcal{D}_i, \vartheta_{\mathcal{D}_i}] :=$**hp-NEARBEST**$(\omega\varepsilon_{i-1}, w_{i-1}^\star, \vartheta)$
$[\bar{\mathcal{D}}_i, w_i^\star] :=$ **EIGEN**$(\frac{\xi}{\sqrt{2}}\varepsilon_{i-1}, \mathcal{D}_i, \vartheta_{\mathcal{D}_i})$
$\varepsilon_i := (\sqrt{2}C_\star\kappa\,\omega + \xi)\varepsilon_{i-1}$
`end do`

Note that the constraints imposed on ξ and ω immediately imply $\sqrt{2}C_\star\kappa\,\omega+\xi < 1$ and $b\omega-\sqrt{2M} > 0$. Also note that the assumption on ε_0 and Proposition 3.1 guarantee that all data approximations generated by the algorithm satisfy $\|\nu-\nu_{\mathcal{D}_i}\|_{L^\infty(\Omega)} \le \frac{1}{2}\nu_0$ and $\|\varrho-\varrho_{\mathcal{D}_i}\|_{L^\infty(\Omega)} \le \frac{1}{2}\varrho_0$, thus implying uniform continuity and coercivity properties of the approximate bilinear forms $a(\nu_{\mathcal{D}_i}, u, v)$ and $b(\varrho_{\mathcal{D}_i}; u, v)$ in V and H, respectively.

Theorem 5.1 *Under Assumption* (5.1)*, let* $(w_i^\star)$ *and* $(\mathcal{D}_i)$ *be the sequences of approximate eigenfunctions and hp-partitions produced in* **hp-AFEM-EIG**. *Setting* $W_i := W_{\bar{\mathcal{D}}_i} = \operatorname{span} w_i^\star$*, the following properties hold:*

$$\hat{\delta}_V(W_\star, W_i) \le \varepsilon_i \quad \forall i \ge 0, \qquad \mathrm{E}_{\mathcal{D}_i}(w^\star, \vartheta)^{\frac{1}{2}} \le \sqrt{M}(\omega + \sqrt{2})\varepsilon_{i-1} \quad \forall i \ge 1, \tag{5.5}$$

and

$$\#\mathcal{D}_i \le B\#\mathcal{D} \qquad \text{for any } \mathcal{D} \in \mathbb{D} \text{ with } \mathrm{E}_{\mathcal{D}}(w^\star, \vartheta)^{\frac{1}{2}} \le (\frac{b\omega}{\sqrt{M}} - \sqrt{2})\varepsilon_{i-1}. \tag{5.6}$$

Proof We proceed in several steps.

i) An auxiliary result. In a Hilbert space with norm $\|\cdot\|$, if w satisfies $\|w\| = 1$ and v satisfies $\|w - v\| = \alpha < 1$, then $\hat{v} := \frac{v}{\|v\|}$ satisfies $\|w - \hat{v}\| \le d_\alpha\alpha$, with $d_\alpha := \sqrt{2}\frac{\sqrt{1-\sqrt{1-\alpha^2}}}{\alpha}$. Indeed, the worst case occurs when the half-line tv, $t \ge 0$, is tangent to the ball $B(w, \alpha)$ of center w and radius α; in this case, $w - v$ is orthogonal to v, thus by Pythagoras one easily gets $\|v\|^2 = 1 - \alpha^2$ and $\|w - \hat{v}\|^2 = \|w - v\|^2 + \|\hat{v} - v\|^2 = 2(1 - \sqrt{1 - \alpha^2})$, whence the result. Note that $d_\alpha \in (1, \sqrt{2})$ and is strictly increasing with α.

ii) Proof of the first inequality in (5.5). The bound $\hat{\delta}_V(W_\star, W_0) \le \varepsilon_0$ is valid by assumption. For $i \ge 1$, in order to prove that $\delta_V(W_\star, W_i) \le \varepsilon_i$, we consider any $w \in W_\star$ with $\|w\|_V = 1$. By (5.1) and the property of **hp-NEARBEST**, there exists $v \in W^{\mathcal{D}_i}$ such that $\|w - v\|_V \le C_\star\kappa\,\mathrm{E}_{\mathcal{D}_i}(w_{i-1}^\star, \vartheta)^{\frac{1}{2}} \le C_\star\kappa\omega\varepsilon_{i-1} < 1$. Setting $\hat{v} = v/\|v\|_V$, we have by *i)* $\|w - \hat{v}\|_V \le \sqrt{2}C_\star\kappa\omega\varepsilon_{i-1}$. On the other hand, by the property of **EIGEN**, there exists $w_{\bar{\mathcal{D}}_i} \in W_{\bar{\mathcal{D}}_i}$ such that $\|\hat{v} - w_{\bar{\mathcal{D}}_i}\|_V \le \frac{\xi}{\sqrt{2}}\varepsilon_{i-1}$. We conclude that $\|w - w_{\bar{\mathcal{D}}_i}\|_V \le (\sqrt{2}C_\star\kappa\omega + \frac{\xi}{\sqrt{2}})\varepsilon_{i-1} < \varepsilon_i$.

In order to prove that $\delta_V(W_i, W_\star) \le \varepsilon_i$, we argue in a similar manner: given any $w_{\bar{\mathcal{D}}_i} \in W_{\bar{\mathcal{D}}_i}$ with $\|w_{\bar{\mathcal{D}}_i}\|_V = 1$, one can find $w \in W_\star$ such that $\|w_{\bar{\mathcal{D}}_i} - w\|_V \le (C_\star\kappa\omega + \xi)\varepsilon_{i-1} < \varepsilon_i$.

iii) Proof of the second inequality in (5.5). For any selected eigenfunction w_{j+m} ($m = 0, \ldots, M-1$) in $w^\star$, let $\tilde{w}_{\mathcal{D}_{i-1},j+m} := \operatorname{argmin} \operatorname{dist}_V(w_{j+m}, W_{i-1})$, and let $\hat{w}_{\mathcal{D}_{i-1},j+m} := \tilde{w}_{\mathcal{D}_{i-1},j+m}/\|\tilde{w}_{\mathcal{D}_{i-1},j+m}\|_V$. Let us set $\hat{w}_{i-1}^\star := (\hat{w}_{\mathcal{D}_{i-1},j}, \ldots, \hat{w}_{\mathcal{D}_{i-1},j+M-1})$. Then, by (5.2),

$$\mathrm{E}_{\mathcal{D}_i}(w^\star, \vartheta)^{\frac{1}{2}} \le \mathrm{E}_{\mathcal{D}_i}(\hat{w}_{i-1}^\star, \vartheta)^{\frac{1}{2}} + \|w^\star - \hat{w}_{i-1}^\star\|_{\mathbb{V}}. \tag{5.7}$$

By the first inequality in (5.5) with i replaced by $i - 1$, we have $\|w_{j+m} - \tilde{w}_{\mathcal{D}_{i-1},j+m}\|_V \le \varepsilon_{i-1}$, thus by *i)* $\|w_{j+m} - \hat{w}_{\mathcal{D}_{i-1},j+m}\|_V \le d_{\varepsilon_{i-1}}\varepsilon_{i-1} < \sqrt{2}\varepsilon_{i-1}$. Hence,

$$\|w^\star - \hat{w}_{i-1}^\star\|_{\mathbb{V}} = \left(\sum_{m=0}^{M-1} \|w_{j+m} - \hat{w}_{\mathcal{D}_{i-1},j+m}\|_V^2\right)^{\frac{1}{2}} < \sqrt{2M}\varepsilon_{i-1}.$$

On the other hand, expanding each $\hat{w}_{\mathcal{D}_{i-1},j+m}$ along the computed orthonormal eigenfunctions in $w^\star_{i-1}$ as $\hat{w}_{\mathcal{D}_{i-1},j+m} = \sum_{k=0}^{M-1} \beta_{mk} w_{\mathcal{D}_{i-1},j+k}$ with $\sum_{k=0}^{M-1} \beta_{mk}^2 = 1$, one has for any $D \in \mathcal{D}_i$

$$\begin{aligned}\|\hat{w}_{\mathcal{D}_{i-1},j+m} - P^1_{K_D,p_D}\hat{w}_{\mathcal{D}_{i-1},j+m}\|_{H^1(K_D)} &= \|\sum_{k=0}^{M-1} \beta_{mk}(w_{\mathcal{D}_{i-1},j+k} - P^1_{K_D,p_D} w_{\mathcal{D}_{i-1},j+k})\|_{H^1(K_D)} \\ &\le \sum_{k=0}^{M-1} |\beta_{mk}| \|w_{\mathcal{D}_{i-1},j+k} - P^1_{K_D,p_D} w_{\mathcal{D}_{i-1},j+k}\|_{H^1(K_D)} \\ &\le \left(\sum_{k=0}^{M-1} \|w_{\mathcal{D}_{i-1},j+k} - P^1_{K_D,p_D} w_{\mathcal{D}_{i-1},j+k}\|^2_{H^1(K_D)}\right)^{\frac{1}{2}}.\end{aligned}$$

Squaring and summing over m and D, one immediately obtains $\mathrm{E}_{\mathcal{D}_i}(\hat{w}^\star_{i-1},\vartheta)^{\frac{1}{2}} \le \sqrt{M}\,\mathrm{E}_{\mathcal{D}_i}(w^\star_{i-1},\vartheta)^{\frac{1}{2}}$. Then, (5.7) and the property of **hp-NEARBEST** yield the desired result.

iv) Proof of (5.6). Let $\mathcal{D} \in \mathbb{D}$ with $\mathrm{E}_{\mathcal{D}}(w^\star,\vartheta)^{\frac{1}{2}} \le (\frac{b\omega}{\sqrt{M}} - \sqrt{2})\varepsilon_{i-1}$. Starting from the M computed eigenfunctions in $w^\star_{i-1}$, let us define a vector $\hat{w}^\star$ of M normalized eigenfunctions in $W_\star$, by a minimization procedure similar to the one used in *iii)*. Then, arguments as in *iii)* yield

$$\mathrm{E}_{\mathcal{D}}(w^\star_{i-1},\vartheta)^{\frac{1}{2}} \le \mathrm{E}_{\mathcal{D}}(\hat{w}^\star,\vartheta)^{\frac{1}{2}} + \|\hat{w}^\star - w^\star_{i-1}\|_{\mathbb{V}} < \sqrt{M}\,\mathrm{E}_{\mathcal{D}}(w^\star,\vartheta)^{\frac{1}{2}} + \sqrt{2M}\varepsilon_{i-1} \le b\omega\varepsilon_{i-1}.$$

The desired result follows from the optimality property of **hp-NEARBEST**. ∎

Finally, the bound on the gap between the eigenspaces $W_\star$ and W_i given in (5.5) yields an error estimate for the approximation of the chosen eigenvalue $\lambda_\star$ of Problem (2.1).

Corollary 5.1 *For $i \ge 1$, let $\lambda_{\bar{\mathcal{D}}_i,j+m}$, $0 \le m \le M-1$, be any of the M eigenvalues of Problem* (4.2) *computed in* **EIGEN** *at the i-th iteration of* **hp-AFEM-EIG**. *Then, there exists an index $i_0 > 0$ (depending on $\lambda_\star$) and a constant $C_\sharp > 0$ (independent of $\lambda_\star$), such that the following bound holds for all $i \ge i_0$ and $0 \le m \le M-1$:*

$$\frac{|\lambda_\star - \lambda_{\bar{\mathcal{D}}_i,j+m}|}{\lambda_\star} \le C_\sharp(1+\lambda_\star^{1/2})\varepsilon_i.$$

Proof Let $w_{\bar{\mathcal{D}}_i,j+m} \in W_i$ be an eigenfunction associated with $\lambda_{\bar{\mathcal{D}}_i,j+m}$ and normalized by $\|w_{\bar{\mathcal{D}}_i,j+m}\|_V = 1$. For convenience, from now on we will set $\lambda_i := \lambda_{\bar{\mathcal{D}}_i,j+m}$ and $w_i := w_{\bar{\mathcal{D}}_i,j+m}$. Let $w \in W_\star$ be an eigenfunction associated with $\lambda_\star$ and satisfying $\|w - w_i\|_V \le \varepsilon_i$, which exists by (5.5). Then,

$$\lambda_\star = \frac{a(\nu;w,w)}{b(\varrho;w,w)}, \qquad \lambda_i = \frac{a(\nu_{\mathcal{D}_i};w_i,w_i)}{b(\varrho_{\mathcal{D}_i};w_i,w_i)},$$

which easily gives

$$\lambda_\star - \lambda_i = \frac{b(\varrho_{\mathcal{D}_i};w_i,w_i) - b(\varrho;w,w)}{b(\varrho_{\mathcal{D}_i};w_i,w_i)}\,\lambda_\star + \frac{a(\nu;w,w) - a(\nu_{\mathcal{D}_i};w_i,w_i)}{b(\varrho_{\mathcal{D}_i};w_i,w_i)}. \tag{5.8}$$

In order to bound the right-hand side, let us observe that $|\,\|w\|_V - 1| \le \varepsilon_i \le \varepsilon_0 < 1$. Hence, $a(\nu;w,w) \simeq \|w\|^2_V \simeq 1$; consequently, $b(\varrho;w,w) \simeq \|w\|^2_H \simeq \lambda_\star^{-1}$. We have

$$\begin{aligned}|a(\nu;w,w) - a(\nu_{\mathcal{D}_i};w_i,w_i)| &\le |a(\nu;w,w) - a(\nu;w_i,w_i)| + |a(\nu;w_i,w_i) - a(\nu_{\mathcal{D}_i};w_i,w_i)| \\ &\le \|\nu\|_{L^\infty(\Omega)}\|w+w_i\|_V\|w-w_i\|_V + \|\nu-\nu_{\mathcal{D}_i}\|_{L^\infty(\Omega)}\|w_i\|^2_V \\ &\lesssim \|w-w_i\|_V + \|\nu-\nu_{\mathcal{D}_i}\|_{L^\infty(\Omega)}.\end{aligned}$$

Recalling Proposition 3.1 and the property of **hp-NEARBEST**, we obtain

$$|a(\nu; w, w) - a(\nu_{\mathcal{D}_i}; w_i, w_i)| \lesssim \varepsilon_i + C_0 \kappa \omega \varepsilon_{i-1} \simeq \varepsilon_i. \tag{5.9}$$

Similarly, we have

$$|b(\varrho; w, w) - b(\varrho_{\mathcal{D}_i}; w_i, w_i)| \leq \|\varrho\|_{L^\infty(\Omega)} \|w + w_i\|_H \|w - w_i\|_H + \|\varrho - \varrho_{\mathcal{D}_i}\|_{L^\infty(\Omega)} \|w_i\|_H^2. \tag{5.10}$$

Since $b(\varrho_{\mathcal{D}_i}; w_i, w_i) \simeq \|w_i\|_H^2$, we have for suitable constants $0 < c_1 < 1$ and $c_2 > 1$ and arbitrary $\alpha > 0$

$$c_1 \|w_i\|_H^2 \leq c_2 \|w\|_H^2 + \|\varrho\|_{L^\infty(\Omega)} \big(\alpha(\|w\|_H^2 + \|w_i\|_H^2) + \frac{1}{2\alpha} \|w - w_i\|_H^2\big) + \|\varrho - \varrho_{\mathcal{D}_i}\|_{L^\infty(\Omega)} \|w_i\|_V^2,$$

whence

$$(c_1 - \alpha \|\varrho\|_{L^\infty(\Omega)} - \|\varrho - \varrho_{\mathcal{D}_i}\|_{L^\infty(\Omega)}) \|w_i\|_H^2 \leq (c_2 + \alpha \|\varrho\|_{L^\infty(\Omega)}) \|w\|_H^2 + \frac{1}{2\alpha} \|\varrho\|_{L^\infty(\Omega)} \|w - w_i\|_H^2$$

Choosing α small enough and i_0 large enough so that $C_0 \kappa \omega \varepsilon_{i_0 - 1} < c_1$ and $\varepsilon_{i_0} \leq c_3 \lambda_\star^{-1/2}$ for some $c_3 \in (0, 1]$ to be defined later on, then for all $i \geq i_0$ it holds $\|\varrho - \varrho_{\mathcal{D}_i}\|_{L^\infty(\Omega)} < c_1$ by Proposition 3.1 and $\|w - w_i\|_H \leq \|w - w_i\|_V \leq \varepsilon_i \leq \varepsilon_{i_0} \lesssim \|w\|_H$, which implies

$$\|w_i\|_H \lesssim \|w\|_H;$$

then, going back to (5.10), we conclude that

$$|b(\varrho; w, w) - b(\varrho_{\mathcal{D}_i}; w_i, w_i)| \lesssim \lambda_\star^{-1/2} \varepsilon_i. \tag{5.11}$$

At last, from this inequality we get the existence of constants $c_4, c_5 > 0$ such that

$$\lambda_\star^{-1} \leq c_4 b(\varrho_{\mathcal{D}_i}; w_i, w_i) + c_5 \lambda_\star^{-1/2} \varepsilon_i \leq c_4 b(\varrho_{\mathcal{D}_i}; w_i, w_i) + c_5 \lambda_\star^{-1/2} \varepsilon_{i_0} \leq c_4 b(\varrho_{\mathcal{D}_i}; w_i, w_i) + c_5 c_3 \lambda_\star^{-1}.$$

Choosing c_3 such that $c_5 c_3 < 1$ yields

$$\lambda_\star^{-1} \lesssim b(\varrho_{\mathcal{D}_i}; w_i, w_i). \tag{5.12}$$

The result follows from (5.8)-(5.12). ∎

Remark 5.1 Since **EIGEN** is based on a Galerkin projection (recall (4.2)), by classical results [24] we expect the discrete eigenvalues $\lambda_{\bar{\mathcal{D}}, j+m}$ assciated with the output eigenfunctions $w_{\bar{\mathcal{D}}, j+m}$ to converge quadratically (i.e. proportionally to ε^2) to the corresponding exact eigenvalues $\lambda_{j+m}^{\mathcal{D}}$ of the perturbed problem (4.1). In turn, these eigenvalues approximate the target eigenvalue $\lambda_\star$ of Problem (2.1) with an error bounded by the approximation error of the coefficients, which is proportional to $\kappa\varepsilon$ (remember Proposition 3.1). Thus, choosing κ proportional to ε should guarantee overall quadratic convergence. On the other hand, Banerjee and Osborn [2] have indicated that the use of a suitable quadrature rule guarantees overall quadratic convergence even when the coefficients are nodally interpolated. In our setting, this would correspond to reducing the projection error for ν and ϱ in (2.4) by choosing the constants α and β large enough. The study of a suitable combination of these strategies will be the object of future work.

6 Realizations of the routine EIGEN

This section is devoted to the discussion of a possible realization of the routine **EIGEN**, introduced in Sect. 4.

From now on, we consider the case of a simple eigenvalue λ_j for Problem (2.2), i.e., we set $M = 1$ (the case of multiplicity > 1 will be investigated elsewhere). Then, Proposition 5.1 guarantees that the j-th exact eigenvalue $\lambda_j^{\mathcal{D}}$ of the perturbed problem (4.1) is simple as well, provided the approximate data $\vartheta_{\mathcal{D}}$ on the input partition $\mathcal{D}$ are close enough to the exact data ϑ. In turns, this may be guaranteed within the algorithm **hp-AFEM-EIG** by assuming that the initial tolerance ε_0 be sufficiently small, thanks to Proposition 3.1.

In the sequel, the input partition $\mathcal{D}$ in **EIGEN** will be denoted by $\mathcal{D}_{\rm in}$, whereas the symbol $\mathcal{D}$ will be used to denote any refinement of $\mathcal{D}_{\rm in}$ generated by the procedure. Similarly, to avoid cumbersome notation, we will set $a(u,v) := a(\nu_{\mathcal{D}_{\rm in}}; u, v)$ and $b(u,v) := b(\varrho_{\mathcal{D}_{\rm in}}; u, v)$, where $\vartheta_{\mathcal{D}_{\rm in}} = (\nu_{\mathcal{D}_{\rm in}}, \varrho_{\mathcal{D}_{\rm in}})$ is the input data. It is convenient to introduce the norms $\|v\|_a := \sqrt{a(v,v)}$ and $\|v\|_b := \sqrt{b(v,v)}$, resp., which are uniformly equivalent to the norms in V and H, resp..

The target exact eigenvalue $\lambda_j^{\mathcal{D}_{\rm in}}$ will be denoted by λ, with eigenfunction $w \in V$ satisfying

$$w \in V \ : \quad a(w,v) = \lambda\, b(w,v) \qquad \forall v \in V. \tag{6.1}$$

From now on, we assume that w is normalized to satisfy $\|w\|_b = 1$, whence $\|w\|_a^2 = \lambda$.

Let $\mathcal{D}_{\rm in}^c := \mathcal{C}(\mathcal{D}_{\rm in})$ denote the conforming partition associated with the input partition through the mapping (2.3). In the sequel, we will only consider *conforming* refinements $\mathcal{D} \geq \mathcal{D}_{\rm in}^c$; the associated *conforming* space $V_{\mathcal{D}}^c$ will be simply denoted by $V_{\mathcal{D}}$. For any such partition, we denote by $\lambda_{\mathcal{D}}$ the j-th eigenvalue of the discrete Galerkin eigenvalue problem

$$w_{\mathcal{D}} \in V_{\mathcal{D}} \ : \quad a(w_{\mathcal{D}}, v) = \lambda_{\mathcal{D}}\, b(w_{\mathcal{D}}, v) \qquad \forall v \in V_{\mathcal{D}}, \tag{6.2}$$

with eigenfunction $w_{\mathcal{D}}$ satisfying $\|w_{\mathcal{D}}\|_b = 1$ (hence $\|w_{\mathcal{D}}\|_a^2 = \lambda_{\mathcal{D}}$), and $b(w, w_{\mathcal{D}}) > 0$. We introduce the *eigenfunction error*

$$e_{\mathcal{D}} := w - w_{\mathcal{D}};$$

note that in the present setting the target output condition of **EIGEN**, namely $\hat{\delta}_V(W^{\mathcal{D}_{\rm in}}, W_{\bar{\mathcal{D}}}) \leq \varepsilon$, is equivalent to $\|e_{\bar{\mathcal{D}}}\|_V \leq \varepsilon$. We also introduce the *residual*

$$r_{\mathcal{D}} = r(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) := \lambda_{\mathcal{D}} b(w_{\mathcal{D}}, \cdot) - a(w_{\mathcal{D}}, \cdot) \in V'. \tag{6.3}$$

We now collect some useful relations involving the error. Let us begin with the following representations (see [24, Lemma 6.3] and [13, Lemma 3.1]):

$$\|e_{\mathcal{D}}\|_a^2 \;=\; \lambda \|e_{\mathcal{D}}\|_b^2 + (\lambda_{\mathcal{D}} - \lambda) \;=\; \frac{1}{2}(\lambda_{\mathcal{D}} + \lambda)\|e_{\mathcal{D}}\|_b^2 \;+\; {}_{V'}\langle r_{\mathcal{D}}, e_{\mathcal{D}}\rangle_V, \tag{6.4}$$

which imply the inequalities

$$\lambda_{\mathcal{D}} - \lambda \leq \|e_{\mathcal{D}}\|_a^2 \tag{6.5}$$

and

$$\|e_{\mathcal{D}}\|_a^2 \leq (\lambda_{\mathcal{D}} + \lambda)\|e_{\mathcal{D}}\|_b^2 + \|r_{\mathcal{D}}\|_{V'}^2. \tag{6.6}$$

Next, we introduce an assumption on the initial conforming mesh $\mathcal{D}_{\rm in}^c$, which guarantees that all the discrete eigenvalues we are going to consider are well separated from the exact eigenvalues different from λ (see [24]). To this end, let λ^- (λ^+, resp.) denote the largest (the smallest, resp.) eigenvalue of Problem (6.1) satisfying $\lambda^- < \lambda < \lambda^+$ (with the obvious adaptation if λ is itself the smallest eigenvalue of the problem). Let us assume that $\mathcal{D}_{\rm in}^c$ is sufficiently refined, so that there exists $\alpha \in (0,1)$ for which

$$0 \leq \lambda_{\mathcal{D}_{\rm in}^c}^- - \lambda^- \leq \alpha(\lambda - \lambda^-) \qquad \text{and} \qquad 0 \leq \lambda_{\mathcal{D}_{\rm in}^c} - \lambda \leq \alpha(\lambda^+ - \lambda). \tag{6.7}$$

This has the important consequence that the norm $\|e_{\mathcal{D}}\|_b$ is asymptotically smaller than the norm $\|e_{\mathcal{D}}\|_a$. Indeed, note that for any conforming refinement $\mathcal{D} \geq \mathcal{D}_{\rm in}^c$, the property $\lambda \leq \lambda_{\mathcal{D}} \leq \lambda_{\mathcal{D}_{\rm in}^c}$ (which follows from the minmax principle for the Rayleigh quotient) implies that (6.7) is satisfied with $\mathcal{D}_{\rm in}^c$ replaced by $\mathcal{D}$. In turns, this implies (see [24, pp. 234-235], [14, Lemma 3.3]) the existence of a constant $C > 0$, depending on λ but independent of $\mathcal{D}$, such that

$$\|e_{\mathcal{D}}\|_b \leq C\|w - G_{\mathcal{D}}w\|_b, \tag{6.8}$$

where $G_{\mathcal{D}}w \in V_{\mathcal{D}}$ is the Galerkin projection of w in the a-scalar product, defined by $a(w-G_{\mathcal{D}}w, v) = 0$ for all $v \in V_{\mathcal{D}}$. On the other hand, since we have assumed that Ω is a polygon, there exist constants $r \in (0, 1]$ and $C_{\Omega,r} > 0$ for which

$$f \in L^2(\Omega) \Rightarrow \|z\|_{H^{1+r}(\Omega)} \leq C_{\Omega,r}\|f\|_{L^2(\Omega)} \qquad \text{where } z \in V \text{ solves} \quad a(z, v) = b(f, v) \quad \forall v \in V \tag{6.9}$$

(see, e.g., [21]). Then, it is proven in [14, Lemma 3.4], using an Aubin-Nitsche argument, that for a suitable constant $\bar{C} > 0$ it holds

$$\|w - G_{\mathcal{D}}w\|_b \leq \bar{C}\|h_{\mathcal{D}}\|_\infty^r \|w - G_{\mathcal{D}}w\|_a, \tag{6.10}$$

where $h_{\mathcal{D}} := (h_D)_{D\in\mathcal{D}}$. Concatenating (6.8) and (6.10), and using $\|h_{\mathcal{D}}\|_\infty \leq \|h_{\mathcal{D}_{\rm in}}\|_\infty$, we obtain the following result.

Proposition 6.1 *Under assumption* (6.7)*, there exist constants* $r \in (0, 1]$ *and* $C_\flat > 0$ *(depending on* λ*) such that*

$$\|e_{\mathcal{D}}\|_b \leq C_\flat \|h_{\mathcal{D}_{\rm in}}\|_\infty^r \|e_{\mathcal{D}}\|_a, \qquad \forall \mathcal{D} \in \mathbb{D}^c,\ \mathcal{D} \geq \mathcal{D}_{\rm in}^c. \tag{6.11}$$

An important consequence of this result together with (6.6) is the following 'abstract' a posteriori bound.

Corollary 6.1 *Under assumption* (6.7)*, suppose that* $\mathcal{D}_{\rm in}$ *is such that*

$$(\lambda^+ + \lambda)C_\flat^2 \|h_{\mathcal{D}_{\rm in}}\|_\infty^{2r} \leq \frac{\zeta}{1+\zeta} \qquad \textit{for some fixed } \zeta > 0. \tag{6.12}$$

Then

$$\|e_{\mathcal{D}}\|_a^2 \leq (1+\zeta)\|r_{\mathcal{D}}\|_{V'}^2.$$

Remark 6.1 Since any partition $\mathcal{D}_i$ built inside **hp-AFEM-EIG** is a refinement of the initial partition $\mathcal{K}_0$ (in the sense that $\mathcal{K}(\mathcal{D}_i) \geq \mathcal{K}_0$), assumptions (6.7) and (6.12) may be satisfied by choosing a sufficiently refined $\mathcal{K}_0$ □

At this point, we have to specify how to build a refined partition $\bar{\mathcal{D}}$ from $\mathcal{D}$, in such a way that a suitable measure of the error $e_{\mathcal{D}}$ is reduced by a fixed amount. To this end, we introduce an a posteriori error estimator, which will be used inside an iterative procedure of the form SOLVE → ESTIMATE → MARK → REFINE.

6.1 Equilibrated-flux estimator

A posteriori error estimators based on the construction of local 'equilibrated fluxes' have been proposed in the last years (see e.g. [16] and the reference therein) for their feature of avoiding constants which are not known explicitly; another relevant property, as opposed to classical residual-based estimators (see [23]), is their p-robustness [7]. Recently, an application to the adaptive discretization of eigenvalue problems has been given in [8].

Let us recall a few notation. For the sake of definiteness, hereafter we assume that the geometric elements in which Ω is partitioned are triangles, although quadrilaterals could be considered as well. Let $\mathcal{A}_\mathcal{D}$ be the set of all vertices $\mathbf{a}$ of the partition $\mathcal{D}$. Let $\omega_\mathbf{a} = \omega_{\mathcal{D},\mathbf{a}}$ denote the star centered at $\mathbf{a}$, i.e., the interior of the patch of elements $K_D \in \mathcal{K}(\mathcal{D})$ containing the vertex $\mathbf{a}$. Let $\psi_\mathbf{a} \in V_\mathcal{D}$ be the piecewise linear 'hat function' w.r.t. $\mathcal{K}(\mathcal{D})$, that has value 1 at $\mathbf{a}$ and that is zero at all other vertices; one has $\operatorname{supp}\psi_\mathbf{a} = \overline{\omega_\mathbf{a}}$. Let us set $H^1_*(\omega_\mathbf{a}) := \{v \in H^1(\omega_\mathbf{a})\colon \int_{\omega_\mathbf{a}} v = 0\}$ if $\mathbf{a}$ is an interior vertex, or $H^1_*(\omega_\mathbf{a}) := \{v \in H^1(\omega_\mathbf{a})\colon v = 0 \text{ on } \partial\omega_\mathbf{a} \cap \partial\Omega\}$ if $\mathbf{a}$ is a boundary vertex; both spaces are equipped with the H^1-seminorm.

One localizes the residual $r_\mathcal{D}$ introduced in (6.3) by setting

$$r_{\mathcal{D},\mathbf{a}}(v) := r_\mathcal{D}(\psi_\mathbf{a} v) \qquad \forall v \in V; \tag{6.13}$$

note that $r_{\mathcal{D},\mathbf{a}}(v)$ only depends on $v_{|\omega_\mathbf{a}}$. Then, one introduces the (squared) local error estimators

$$\eta^2_{\mathcal{D},\mathbf{a}}(\lambda_\mathcal{D}, w_\mathcal{D}) := \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_\mathbf{a}))'}, \qquad \mathbf{a} \in \mathcal{A}_\mathcal{D}, \tag{6.14}$$

and the (squared) global error estimator

$$\eta^2_\mathcal{D}(\lambda_\mathcal{D}, w_\mathcal{D}) := \sum_{\mathbf{a}\in\mathcal{A}_\mathcal{D}} \eta^2_{\mathcal{D},\mathbf{a}}(\lambda_\mathcal{D}, w_\mathcal{D}).$$

The reliability and efficiency of such estimator are stated in the following result.

Proposition 6.2 *Under the assumptions of Corollary 6.1, one has*

$$\|e_\mathcal{D}\|^2_a \le 3(1+\zeta)\eta^2_\mathcal{D}(\lambda_\mathcal{D}, w_\mathcal{D}), \qquad \eta^2_\mathcal{D}(\lambda_\mathcal{D}, w_\mathcal{D}) \le C_\lambda \|e_\mathcal{D}\|^2_a,$$

where the constant C_λ only depends upon λ and Ω.

Proof The first inequality is an immediate consequence of Corollary 6.1 and the general bound $\|r_\mathcal{D}\|^2_{V'} \le 3\sum_{\mathbf{a}\in\mathcal{A}_\mathcal{D}} \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_\mathbf{a}))'}$ (see e.g. [10, Proposition 3.1]). In order to prove the second inequality, let us fix any $\mathbf{a} \in \mathcal{A}_\mathcal{D}$ and let $v \in H^1_*(\omega_\mathbf{a})$ be arbitrary. Let us subtract from the relation

$$r_{\mathcal{D},\mathbf{a}}(v) = \lambda_\mathcal{D} b(w_\mathcal{D}, \psi_\mathbf{a} v) - a(w_\mathcal{D}, \psi_\mathbf{a} v)$$

the identity

$$0 = \lambda b(w, \psi_\mathbf{a} v) - a(w, \psi_\mathbf{a} v), \tag{6.15}$$

to get

$$\begin{aligned} r_{\mathcal{D},\mathbf{a}}(v) &= -\lambda b(e_\mathcal{D}, \psi_\mathbf{a} v) + (\lambda_\mathcal{D} - \lambda) b(w_\mathcal{D}, \psi_\mathbf{a} v) + a(e_\mathcal{D}, \psi_\mathbf{a} v) \\ &\le (\lambda \|e_\mathcal{D}\|_{b,\omega_\mathbf{a}} + (\lambda_\mathcal{D} - \lambda)\|w_\mathcal{D}\|_{b,\omega_\mathbf{a}}) \|\psi_\mathbf{a} v\|_{b,\omega_\mathbf{a}} + \|e_\mathcal{D}\|_{a,\omega_\mathbf{a}} \|\psi_\mathbf{a} v\|_{a,\omega_\mathbf{a}} \end{aligned}$$

(with an obvious meaning of $\|\cdot\|_{a,\omega_\mathbf{a}}$ and $\|\cdot\|_{b,\omega_\mathbf{a}}$). Since $\|\psi_\mathbf{a} v\|_{L^2(\omega_\mathbf{a})} \lesssim |\omega_\mathbf{a}|^{1/2} \|\nabla v\|_{L^2(\omega_\mathbf{a})}$ and $\|\nabla(\psi_\mathbf{a} v)\|_{L^2(\omega_\mathbf{a})} \lesssim \|\nabla v\|_{L^2(\omega_\mathbf{a})}$ (see e.g. [16]), we obtain the local efficiency estimate

$$\|r_{\mathcal{D},\mathbf{a}}\|_{(H^1_*(\omega_\mathbf{a}))'} \lesssim (\lambda \|e_\mathcal{D}\|_{b,\omega_\mathbf{a}} + (\lambda_\mathcal{D} - \lambda)\|w_\mathcal{D}\|_{b,\omega_\mathbf{a}}) |\omega_\mathbf{a}|^{1/2} + \|e_\mathcal{D}\|_{a,\omega_\mathbf{a}}.$$

Squaring and summing-up over $\mathbf{a}$, and using (6.5), we get

$$\sum_{\mathbf{a}\in\boldsymbol{\mathcal{A}}_{\mathcal{D}}} \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_{\mathbf{a}}))'} \lesssim (\lambda^2\|e_{\mathcal{D}}\|_b^2 + (\lambda_{\mathcal{D}}-\lambda)^2\|w_{\mathcal{D}}\|_b^2) \max_{\mathbf{a}\in\boldsymbol{\mathcal{A}}_{\mathcal{D}}} |\omega_{\mathbf{a}}| + \|e_{\mathcal{D}}\|_a^2$$
$$\lesssim (\lambda^2\|e_{\mathcal{D}}\|_b^2 + (\lambda_{\mathcal{D}}-\lambda)\|e_{\mathcal{D}}\|_a^2)|\Omega| + \|e_{\mathcal{D}}\|_a^2,$$

and we conclude using (6.11). ∎

The local estimators $\eta^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}})$ defined in (6.14) are obviously not computable in practice. However, following [7,16], it is proven in [8] that one can find piecewise-polynomial vector fields $\boldsymbol{\sigma}_{\mathbf{a}}$ in certain Raviart-Thomas spaces over the star $\omega_{\mathbf{a}}$, such that a suitable expression defined in terms of their L^2-norms, say $\tilde{\eta}_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}})$, is uniformly equivalent to the dual norm of $r_{\mathcal{D},\mathbf{a}}$. In other words, one can define computable equivalent local estimators

$$\tilde{\eta}^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) \simeq \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_{\mathbf{a}}))'} = \eta^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}}), \qquad \mathbf{a}\in\mathcal{A}_{\mathcal{D}}. \tag{6.16}$$

We recall that $\tilde{\eta}^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}})$ can be efficiently computed by solving local finite-dimensional saddle-point problems, whose data depend upon $\lambda_{\mathcal{D}}$ and $w_{\mathcal{D}}$ restricted to the star $\omega_{\mathbf{a}}$; we refer to [8, Sect. 4] for the details.

Next, we extend to the eigenvalue problem some results given in [10] for a source problem, and we introduce a saturation property, that will be crucial in defining the refinement $\bar{\mathcal{D}}$ of the partition $\mathcal{D}$. To this end, let $\mathcal{D}_{\mathbf{a}}$ denote the set of elements $D\in\mathcal{D}$ such that $\mathbf{a}\in K_D$, and let us set $p_{\mathbf{a}} := \max\{p_D : D\in\mathcal{D}_{\mathbf{a}}\}$. Furthermore, for any integer $q\geq 0$, let us define $\mathbb{P}_q(\mathcal{D}_{\mathbf{a}}) := \{v : \omega_{\mathbf{a}}\to\mathbb{R} : v_{|K_D}\in\mathbb{P}_q(K_D) \text{ for all } D\in\mathcal{D}_{\mathbf{a}}\}$.

Proposition 6.3 *There exists a non-decreasing function $q:\mathbb{N}\to\mathbb{N}$ satisfying $p<q(p)\lesssim p$ for any p, such that*

$$\|r_{\mathcal{D},\mathbf{a}}\|_{(H^1_*(\omega_{\mathbf{a}}))'} \simeq \|r_{\mathcal{D},\mathbf{a}}\|_{(H^1_*(\omega_{\mathbf{a}})\cap\mathbb{P}_{q(p_{\mathbf{a}})}(\mathcal{D}_{\mathbf{a}}))'}, \qquad \mathbf{a}\in\boldsymbol{\mathcal{A}}_{\mathcal{D}}.$$

Proof Let $\mathcal{E}^{\rm int}_{\mathbf{a}}$ denote the collection of the edges of the elements in $\mathcal{D}_{\mathbf{a}}$ which are internal to $\omega_{\mathbf{a}}$. Then, one has for any $v\in H^1_*(\omega_{\mathbf{a}})$

$$r_{\mathbf{a}}(v) = \sum_{D\in\mathcal{D}_{\mathbf{a}}}\int_{K_D} r_D\psi_{\mathbf{a}}v + \sum_{e\in\mathcal{E}^{\rm int}_{\mathbf{a}}}\int_e r_e\psi_{\mathbf{a}}v,$$

with $r_D := \lambda_{\mathcal{D}}\varrho_{\mathcal{D}_{\rm in}}w_{\mathcal{D}} + \nabla\cdot(\nu_{\mathcal{D}_{\rm in}}\nabla w_{\mathcal{D}})$ and $r_e := [\![\nu_{\mathcal{D}_{\rm in}}\nabla w_{\mathcal{D}}\cdot\mathbf{n}]\!]_e$. Recall that the coefficients $\nu_{\mathcal{D}_{\rm in}}$ and $\varrho_{\mathcal{D}_{\rm in}}$ are piecewise polynomials on the input partition $\mathcal{D}_{\rm in}$ of the routine **EIGEN**; hence, we may assume that on the star ω_a they belong to some $\mathbb{P}_{p_{\mathbf{a},\rm in}}(\mathcal{D}_{\mathbf{a}})$ with $p_{\mathbf{a},\rm in}\leq p_{\mathbf{a}}$ (since, as we will see below, we never decrease the polynomial degree in **EIGEN**). Consequently, each r_D is a polynomial of degree at most $p_{\mathbf{a},\rm in}+p_{\mathbf{a}}\leq 2p_{\mathbf{a}}$, and each r_e is a polynomial of degree at most $p_{\mathbf{a},\rm in}+p_{\mathbf{a}}-1<2p_{\mathbf{a}}$. Therefore, we are in the same situation considered in [10, Theorem 6.3], with $p_{\mathbf{a}}$ replaced by $2p_{\mathbf{a}}$. The desired result follows from that theorem, after noticing that its assumptions have been numerically checked in Sect. 7 of the cited paper. We also refer to [11], where a property similar to any of these assumptions has been proven for a quadrangular element; the proof indicates that a function of the form $q(p_{\mathbf{a}}) := (1+c)2p_{\mathbf{a}}$ for any fixed $c>1$ guarantees the validity of the result. ∎

6.2 The contraction property

The refined partition $\bar{\mathcal{D}} \geq \mathcal{D}$ is defined through a star-based Dörfler-type marking followed by an increment of the polynomial degree in the elements belonging to the marked stars. Suppose that all the local estimators $\tilde{\eta}^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}})$ introduced in (6.16) have been computed. Let us fix a constant $\theta \in (0,1]$, and let us find a subset $\mathcal{M}_{\mathcal{D}} \subseteq \mathcal{A}_{\mathcal{D}}$ of minimal cardinality such that

$$\sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \tilde{\eta}^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) \geq \theta \sum_{\mathbf{a}\in\mathcal{A}_{\mathcal{D}}} \tilde{\eta}^2_{\mathcal{D},\mathbf{a}}(\lambda_{\mathcal{D}}, w_{\mathcal{D}}). \tag{6.17}$$

Definition 6.1 For any $D = (K_D, p_D) \in \mathcal{D}$, set $\bar{D} = (K_{\bar{D}}, p_{\bar{D}})$ with $K_{\bar{D}} = K_D$ and

$$p_{\bar{D}} := \begin{cases} p_D & \text{if} \quad \mathcal{M}_{\mathcal{D}} \cap K_D = \emptyset, \\ \max\limits_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}\cap K_D} q(p_{\mathbf{a}}) \,+\, 1, & \text{otherwise.} \end{cases}$$

Then, the new conforming partition $\bar{\mathcal{D}}$ is the collection of such $\bar{D}$.

Let $V_{\bar{\mathcal{D}}} \supset V_{\mathcal{D}}$ denote the conforming space associated with the partition $\bar{\mathcal{D}}$. Let $\lambda_{\bar{\mathcal{D}}}$ denote the j-th eigenvalue of the discrete problem analogous to (6.2) but with $V_{\mathcal{D}}$ replaced by $V_{\bar{\mathcal{D}}}$, and let $w_{\bar{\mathcal{D}}} \in V_{\bar{\mathcal{D}}}$ denote the corresponding eigenfunction (such that $\|w_{\bar{\mathcal{D}}}\|_b = 1$ and $b(w, w_{\bar{\mathcal{D}}}) > 0$). Let us set for convenience $e_{\bar{\mathcal{D}},\mathcal{D}} := w_{\bar{\mathcal{D}}} - w_{\mathcal{D}}$. Then, we have $\lambda \leq \lambda_{\bar{\mathcal{D}}} \leq \lambda_{\mathcal{D}}$, inequality (6.5) has the counterpart

$$\lambda_{\mathcal{D}} - \lambda_{\bar{\mathcal{D}}} \leq \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_a, \tag{6.18}$$

and, under the same assumption, Proposition 6.1 extents to $e_{\bar{\mathcal{D}},\mathcal{D}}$ as well (see [14, Lemmas 3.3–3.4]), i.e.,

$$\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_b \leq C_\flat \|h_{\mathcal{D}_{\rm in}}\|^r_\infty \|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a, \qquad \forall \bar{\mathcal{D}}, \mathcal{D} \in \mathbb{D}^c,\ \bar{\mathcal{D}} \geq \mathcal{D} \geq \mathcal{D}^c_{\rm in}. \tag{6.19}$$

Next property provides an inequality analogous to the second one in Proposition 6.2.

Proposition 6.4 *Under assumptions* (6.7) *and* (6.12)*, there exists a constant* $\bar{C}_\lambda > 0$ *such that*

$$\sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_{\mathbf{a}})\cap\mathbb{P}_{q(p_{\mathbf{a}})}(\mathcal{D}_{\mathbf{a}}))'} \leq \bar{C}_\lambda \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_a.$$

Proof Following the proof of Proposition 6.2, we observe that (6.15) is now replaced by

$$0 = \lambda_{\bar{\mathcal{D}}} b(w_{\bar{\mathcal{D}}}, \psi_{\mathbf{a}} v) - a(w_{\bar{\mathcal{D}}}, \psi_{\mathbf{a}} v)$$

for $\mathbf{a} \in \mathcal{M}_{\mathcal{D}}$ and $v \in H^1_*(\omega_{\mathbf{a}}) \cap \mathbb{P}_{q(p_{\mathbf{a}})}(\mathcal{D}_{\mathbf{a}})$; indeed, the definition of $p_{\bar{D}}$ given above guarantees that $\psi_{\mathbf{a}} v \in V_{\bar{\mathcal{D}}}$. Then, proceeding as in that proof, we arrive at

$$\sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \|r_{\mathcal{D},\mathbf{a}}\|^2_{(H^1_*(\omega_{\mathbf{a}})\cap\mathbb{P}_{q(p_{\mathbf{a}})}(\mathcal{D}_{\mathbf{a}}))'} \lesssim (\lambda^2_{\bar{\mathcal{D}}} \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_b + (\lambda_{\mathcal{D}} - \lambda_{\bar{\mathcal{D}}}) \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_a)|\Omega| + \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_a,$$

and the conclusion follows from (6.19) and the assumptions. ∎

We now prove that the given definition of $\bar{\mathcal{D}}$ guarantees the validity of a *saturation condition.*

Proposition 6.5 *Under assumptions* (6.7) *and* (6.12)*, there exists a constant* $C_S > 0$ *depending only on* λ *and* Ω*, such that*

$$\|e_{\mathcal{D}}\|^2_a \leq C_S \|e_{\bar{\mathcal{D}},\mathcal{D}}\|^2_a.$$

Proof The first inequality in Proposition 6.2 together with (6.16) yields

$$\|e_{\mathcal{D}}\|_a^2 \lesssim \sum_{\mathbf{a}\in\mathcal{A}_{\mathcal{D}}} \eta_{\mathcal{D},\mathbf{a}}^2(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) \lesssim \sum_{\mathbf{a}\in\mathcal{A}_{\mathcal{D}}} \tilde{\eta}_{\mathcal{D},\mathbf{a}}^2(\lambda_{\mathcal{D}}, w_{\mathcal{D}}).$$

On the other hand, using (6.17), again (6.16) and Proposition 6.3, we get

$$\begin{aligned}\sum_{\mathbf{a}\in\mathcal{A}_{\mathcal{D}}} \tilde{\eta}_{\mathcal{D},\mathbf{a}}^2(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) &\le \theta^{-1} \sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \tilde{\eta}_{\mathcal{D},\mathbf{a}}^2(\lambda_{\mathcal{D}}, w_{\mathcal{D}}) \\ &\lesssim \sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \|r_{\mathcal{D},\mathbf{a}}\|_{(H^1_*(\omega_{\mathbf{a}}))'}^2 \lesssim \sum_{\mathbf{a}\in\mathcal{M}_{\mathcal{D}}} \|r_{\mathcal{D},\mathbf{a}}\|_{(H^1_*(\omega_{\mathbf{a}})\cap\mathbb{P}_{q(p_{\mathbf{a}})}(\mathcal{D}_{\mathbf{a}}))'}^2\end{aligned}$$

and we conclude by Proposition 6.4. ∎

The saturation result just stated can be suitably perturbed in order to be applied in the forthcoming proof of the contraction property.

Corollary 6.2 *Under assumptions* (6.7) *and* (6.12)*, one has*

$$\|e_{\mathcal{D}}\|_a^2 \le C_S(1+\zeta)(\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2 - \lambda_{\bar{\mathcal{D}}}\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_b^2).$$

Proof Since $\lambda_{\bar{\mathcal{D}}} \le \lambda + \lambda^+$ by (6.7), we use (6.12) and (6.19) to get $\lambda_{\bar{\mathcal{D}}}(1+\zeta)\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_b^2 \le \zeta\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2$. Applying Proposition 6.5, we obtain

$$\|e_{\mathcal{D}}\|_a^2 \le C_S(1+\zeta)\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2 - C_S\zeta\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2 \le C_S(1+\zeta)\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2 - C_S\lambda_{\bar{\mathcal{D}}}(1+\zeta)\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_b^2,$$

which is the desired result. ∎

We are ready to state the crucial contraction property concerning the mapping $\mathcal{D} \mapsto \bar{\mathcal{D}}$.

Theorem 6.1 *Under assumptions* (6.7) *and* (6.12)*, there exists a constant* $\rho \in (0,1)$ *depending only on* λ *and* Ω*, such that*

$$\|e_{\bar{\mathcal{D}}}\|_a^2 - \lambda\|e_{\bar{\mathcal{D}}}\|_b^2 \le \rho^2(\|e_{\mathcal{D}}\|_a^2 - \lambda\|e_{\mathcal{D}}\|_b^2) \qquad \forall \mathcal{D}\in\mathbb{D}^c,\ \mathcal{D} \ge \mathcal{D}^c_{\rm in},$$

where $\bar{\mathcal{D}} \ge \mathcal{D}$ *is defined in Definition 6.1.*

Proof At first, note that the differences on both sides are positive, due to the assumptions and inequality (6.11); actually, we have

$$\|e_{\mathcal{D}}\|_a^2 \simeq (\|e_{\mathcal{D}}\|_a^2 - \lambda\|e_{\mathcal{D}}\|_b^2) \qquad \text{uniformly in } \mathcal{D}. \tag{6.20}$$

Our target result is a consequence of the following quasi-orthogonality property (see [14, Lemma 3.1]):

$$\|e_{\bar{\mathcal{D}}}\|_a^2 - \lambda\|e_{\bar{\mathcal{D}}}\|_b^2 = (\|e_{\mathcal{D}}\|_a^2 - \lambda\|e_{\mathcal{D}}\|_b^2) - (\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_a^2 - \lambda_{\bar{\mathcal{D}}}\|e_{\bar{\mathcal{D}},\mathcal{D}}\|_b^2).$$

Indeed, we use Corollary 6.2, in which it is not restrictive to assume that $C := C_S(1+\zeta) > 1$, to get

$$\|e_{\bar{\mathcal{D}}}\|_a^2 - \lambda\|e_{\bar{\mathcal{D}}}\|_b^2 \le (1-C^{-1})\|e_{\mathcal{D}}\|_a^2 - \lambda\|e_{\mathcal{D}}\|_b^2 \le (1-C^{-1})(\|e_{\mathcal{D}}\|_a^2 - \lambda\|e_{\mathcal{D}}\|_b^2),$$

whence the result with $\rho := (1-C^{-1})^{1/2}$. ∎

6.3 The definition of **EIGEN**

The iterative procedure that produces the output of **EIGEN** can be defined as follows. Let us set $\mathcal{D}^{(0)} := \mathcal{D}^c_{\rm in}$ and, recursively, $\mathcal{D}^{(\ell)} := \overline{\mathcal{D}^{(\ell-1)}}$ for $\ell = 1, 2, \ldots$. Then, under all the previous assumptions, we have by Theorem 6.1

$$\|e_{\mathcal{D}^{(\ell)}}\|_a^2 - \lambda \|e_{\mathcal{D}^{(\ell)}}\|_b^2 \le \rho^{2\ell} (\|e_{\mathcal{D}^{(0)}}\|_a^2 - \lambda \|e_{\mathcal{D}^{(0)}}\|_b^2).$$

Recalling (6.20) and the equivalence of $\|\cdot\|_a$ and $\|\cdot\|_V$, there exists a constant $K > 0$ such that

$$\|e_{\mathcal{D}^{(\ell)}}\|_V \le K \rho^\ell \, \|e_{\mathcal{D}^{(0)}}\|_V. \tag{6.21}$$

Assuming that an estimate of $\|e_{\mathcal{D}^{(0)}}\|_V$ is available, say $\|e_{\mathcal{D}^{(0)}}\|_V \le \delta_0$, the procedure may be terminated at iteration $\ell_\star$, where $\ell_\star$ is the smallest ℓ such that $K\rho^\ell \delta_0 \le \varepsilon$.

Definition 6.2 The output $[\bar{\mathcal{D}}, w^\star_{\bar{\mathcal{D}}}]$ of the routine **EIGEN**$(\varepsilon, \mathcal{D}, \vartheta_{\mathcal{D}})$ introduced in Sect. 4 is defined as follows: $\bar{\mathcal{D}} := \mathcal{D}^{(\ell_\star)}$, $w^\star_{\bar{\mathcal{D}}} = w_{\mathcal{D}^{(\ell_\star)}}$.

When **EIGEN** is used inside the loop of Algorithm **hp-AFEM-EIG**, an estimate on the initial error $\|e_{\mathcal{D}^{(0)}}\|_V$ can be obtained as follows. Suppose **EIGEN** is called at the i-th iteration, after $[\mathcal{D}_i, \vartheta_{\mathcal{D}_i}]$:=**hp-NEARBEST**$(\omega\varepsilon_{i-1}, w^\star_{i-1}, \vartheta)$; in this case, $\mathcal{D}^{(0)} = \mathcal{D}^c_{\rm in} := \mathcal{D}^c_i$. By (5.5), we know that

$$\|w^\star - w^\star_{i-1}\|_V \le \varepsilon_{i-1}. \tag{6.22}$$

On the other hand, recalling that $w = w^{\mathcal{D}_i} \in V$ is the exact j-th eigenfunction of Problem (4.1) with data $(\nu_{\mathcal{D}}, \varrho_{\mathcal{D}}) := (\nu_{\mathcal{D}_i}, \varrho_{\mathcal{D}_i}) = \vartheta_{\mathcal{D}_i}$, (5.1) tells us that

$$\|w^\star - w\|_V \le C_\star \kappa \, \mathrm{E}_{\mathcal{D}_i}(w^\star_{i-1}, \vartheta)^{\frac{1}{2}} \le C_\star \kappa \omega \varepsilon_{i-1}. \tag{6.23}$$

Finally, we recall from [9, Theorem 4] that there exists a constant $C_{\mathcal{D}_i} > 0$ such that

$$\inf_{v \in V^c_{\mathcal{D}^c_i}} \|w^\star_{i-1} - v\|_V \le C_{\mathcal{D}_i} \, \mathrm{E}_{\mathcal{D}_i}(v^\star, \vartheta)^{\frac{1}{2}} \le C_{\mathcal{D}_i} \omega \varepsilon_{i-1}, \tag{6.24}$$

with $C_{\mathcal{D}_i} \simeq (1 + \log p_{\mathcal{D}_i})^{\frac{3}{2}}$ for $p_{\mathcal{D}_i} := \max_{D \in \mathcal{D}_i} p_D$. Concatenating the three last inequalities via the triangle inequality, and recalling that $V^c_{\mathcal{D}^c_i}$ is nothing but $V_{\mathcal{D}^{(0)}}$ according to the notation used in the present section, we deduce that

$$\inf_{v \in V_{\mathcal{D}^{(0)}}} \|w - v\|_V \le (1 + (C_\star \kappa + C_{\mathcal{D}_i})\omega)\varepsilon_{i-1}.$$

This allows us to find the desired upper bound δ_0 for $\|w - w_{\mathcal{D}^{(0)}}\|_V$, since it is shown in [1, formulas (8.21) and (8.45)], that there exists a constant $C > 0$ depending only on λ and Ω such that

$$\|w - w_{\mathcal{D}^{(0)}}\|_V \le C \inf_{v \in V_{\mathcal{D}^{(0)}}} \|w - v\|_V.$$

Recalling that the target accuracy in the call of **EIGEN** at iteration i is proportional to ε_{i-1}, we conclude that the number $\ell_\star$ of iterations inside **EIGEN** grows at most like a multiple of $\log C_{\mathcal{D}_i}$, i.e., it exhibits a very mild dependence on the largest polynomial degree in $\mathcal{D}_i = \mathcal{D}(\mathcal{D}_i)$.

At last, let us discuss the complexity of **EIGEN**, expressed in terms of the cardinality of the partitions built during one call, say the i-th one, from **hp-AFEM-EIG**. Recall that $\mathcal{D}^{(0)} := \mathcal{D}^c_i = \mathcal{C}(\mathcal{D}_i)$, where $\mathcal{C}$ satisfies (2.3). As already noted in [9], without further assumptions on the construction of $\mathcal{D}_i$, the cardinality of $\mathcal{C}(\mathcal{D}_i)$ cannot be controlled by that of $\mathcal{D}_i$, although counterexamples are rather pathological (see [9, Sect. 5.1]). Work is in progress on a realization of **hp-NEARBEST** which outputs near-optimal *conforming* partitions if the root partition $\mathcal{K}_0$ is

conforming [4]. On the other hand, we are able to control the growth of $\#\mathcal{D}^{(\ell)}$ with respect to $\#\mathcal{D}^{(0)}$. To this end, first observe that while going from $\mathcal{D}$ to $\bar{\mathcal{D}}$ according to Definition 6.1, one has

$$\#\bar{\mathcal{D}} \leq C_{\#}\, \#\mathcal{D} \tag{6.25}$$

for a constant $C_{\#} > 0$ independent of the initial partition. Indeed, it is easily seen from the assumptions on the mapping $p \mapsto q(p)$ that

$$\forall \bar{D} \in \bar{\mathcal{D}}, \quad \exists D' \in \mathcal{D} \qquad \text{such that} \quad K_{\bar{D}} \cap K_{D'} \neq \emptyset \quad \text{and} \quad p_{\bar{D}} \lesssim p_{D'}.$$

Then, the existence of $C_{\#}$ follows from the fact that each geometric element in $\mathcal{D}$ intersects at most a fixed number of other elements, depending on the root partition $\mathcal{K}_0$. Applying (6.25), we get $\#\mathcal{D}^{(\ell)} \leq C_{\#}^{\ell} \#\mathcal{D}^{(0)}$ for any $\ell > 0$. From this inequality, recalling that $\ell_\star$ can be bounded by a multiple of $\log C_{\mathcal{D}_i}$ where $C_{\mathcal{D}_i} \simeq (1 + \log p_{\mathcal{D}_i})^{\frac{3}{2}}$, we deduce that the cardinality of the output partition $\mathcal{D}^{(\ell_\star)}$ of **EIGEN** satisfies

$$\#\mathcal{D}^{(\ell_\star)} \leq \widetilde{C}_{\mathcal{D}_i} \#\mathcal{D}^{(0)},$$

where $\widetilde{C}_{\mathcal{D}_i}$ may grow with $\mathcal{D}_i$ at most as a (fixed) power of $1 + \log p_{\mathcal{D}_i}$.

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