# ESHELBY’S INCLUSION THEORY IN LIGHT OF NOETHER’S THEOREM

# Eshelby's Inclusion Theory in the Light of Noether's Theorem

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*We dedicate this work to the memory of our maestro Prof. Gaetano Giaquinta (Catania, Italy, 1945-2016), who first taught us Noether's Theorem and showed us its unifying beauty.*

**Abstract**

In a variational setting describing the mechanics of a hyperelastic body with defects or inhomogeneities, we show how the application of Noether's theorem allows for obtaining the classical results by Eshelby. The framework is based on modern differential geometry. First, we present Eshelby's original derivation based on the cut-replace-weld thought experiment. Then, we show how Hamilton's standard variational procedure "with frozen coordinates", which Eshelby coupled with the evaluation of the gradient of the energy density, is shown to yield the strong form of Eshelby's problem. Finally, we demonstrate how Noether's theorem provides the weak form directly, thereby encompassing both procedures that Eshelby followed in his works. We also pursue a declaredly didactic intent, in that we attempt to provide a presentation that is as self-contained as possible, in a modern differential geometrical setting.

**Keywords**: Eshelby stress; energy-momentum tensor; configurational mechanics; inclusion; defect; Noether's theorem; variational principle

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# 1 Introduction

In a classical paper, Eshelby (1951) introduced the concept of *configurational force* as the force required for a region containing a defect in a material body to undergo a *material* virtual displacement. This idea led to the mechanical Maxwell energy-momentum tensor that has been subsequently termed *Eshelby stress* in continuum mechanics (Maugin and Trimarco, 1992). The procedure followed by Eshelby (1951) comprises a set of operations in which the elastic energy in the interior of a region and the net work that the surface tractions exert on the region are evaluated individually. In another work, Eshelby (1975) used Hamilton's standard variational approach of field theory and found his energy-momentum tensor directly, using the components of the regular spatial displacement and of the displacement gradient as the entities called *fields* in the jargon of field theory. In the same paper, Eshelby (1975) also sketched the procedure for the case in which the *fields* are the components of the configuration map, which is the common choice in modern continuum mechanics.

Although initially conceived for a single inclusion or for a discrete set of inclusions, Eshelby's theory naturally applies to *inhomogeneous materials* or materials with continuous distributions of defects. Epstein and Maugin (1990) obtained the Eshelby stress using the concepts of material uniformity and material isomorphism introduced by Noll (1967) for inhomogeneous materials. Gurtin (1995, 2000) reformulated and generalised Eshelby's approach with the method of the varying control volumes and considered the Eshelby stress as the appropriate stress of an *independent* material balance law. The Eshelby stress has been seen as the object capturing inhomogeneities and singularities (e.g., Epstein and Maugin, 1990; Gurtin, 1995, 2000; Epstein and Maugin, 2000; Epstein and Elżanowski, 2007; Verron et al., 2009; Weng and Wong, 2009; Maugin, 2011), or the *driving force* of phenomena of material evolution such as plasticity and growth-remodelling (e.g., Maugin and Epstein, 1998; Epstein and Maugin, 2000; Cermelli et al., 2001; Epstein, 2002; Imatani and Maugin, 2002; Grillo et al., 2003, 2005; Epstein, 2009, 2015; Grillo et al., 2016, 2017; Hamedzadeh et al., 2019), or phase transitions, or evolution of the interfaces among phases (e.g., Gurtin, 1986, 1993; Gurtin and Podio-Guidugli, 1996; Fried and Gurtin, 1994, 2004).

In a didactic spirit, the aim of this work is to reproduce the results of Eshelby (1951, 1975) *directly* by means of the classical Noether's theorem (for a translation into English of Noether's original 1918 paper, see Noether, 1971) for continuum systems, as presented by Hill (1951). The derivation is made using the components of the configuration map as the "fields" and those of the deformation gradient as the "gradients of the fields", while an appropriate "topological" transformation represents the material virtual displacement on the region containing the defect. We would like to emphasise that this work is more than a mere rewrite of Eshelby's findings in a more modern notation. While the relation between Eshelby's work and Noether's theorem has been highlighted in several papers (e.g., Knowles and Sternberg, 1972; Eshelby, 1975; Fletcher, 1976; Edelen, 1981; Golebiewska Herrmann, 1982; Olver, 1984a,b; Huang and Batra, 1996; Kienzler and Herrmann, 2000; Maugin, 2011), to the best of our knowledge, no work in the literature establishes an explicit relation between Eshelby's inclusion theory (and, specifically, the procedure to deal with the presence of the inclusion; Eshelby, 1951, 1975) and Noether's theorem.

In Section 2, we introduce the notation and give some basic definitions. In particular, we introduce standard and Eshelbian configurations and their *variations*, i.e., displacement fields. The setting is declaredly differential geometrical, although we avoid using differentiable manifolds for simplicity. In Section 3, we review, with our notation and within a suitable geometrical setting, Eshelby's original derivation (Eshelby, 1951) of configurational forces. Similarly, in Section 4, we review Eshelby's variational derivation (Eshelby, 1975). Finally, in Section 5, which is the core of the work, we introduce Noether's theorem, and show how its application renders directly the results of both the previous derivations.

# 2 Theoretical Background

In this section, we illustrate the notation that we employ and report some fundamental results relevant to this work. We generally use index-free notation but sometimes it is useful to show the corresponding expression in index notation. Therefore, we present most expressions in both notations. In index notation, the customary Einstein's summation convention for repeated indices is enforced throughout and a subscript preceded by a comma, as in $f_{,i}$, denotes partial differentiation with respect to its $i$-th argument.

## 2.1 General Notation and Basic Definitions

Here we review some basic definitions of continuum mechanics, in order to elucidate the notation that we employ. The notation is essentially that of Truesdell and Noll (1965) and Marsden and Hughes (1983), with some modifications (Federico, 2012; Federico et al., 2016). We work in a simplified setting based on the use of affine spaces, whose rigorous definition can be found, e.g., in the treatise by Epstein (2010). We could use a presentation in terms of differentiable manifolds (Noll, 1967; Marsden and Hughes, 1983; Epstein, 2010; Segev, 2013), but using affine spaces avoids many of the intricacies of higher-level differential geometry and makes the presentation more intuitive.

An affine space is a set $\mathcal{S}$, called the point space, considered together with a vector space $\mathcal{V}$, called the modelling space, and a mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{V} : (x, y) \mapsto y - x = \boldsymbol{u}$. This means that, at every point $x \in \mathcal{S}$, it is possible to univocally attach the vector given by $\boldsymbol{u} = y - x$, for every point $y \in \mathcal{S}$. The set of all vectors emanating from point $x$ is a vector space denoted $T_x\mathcal{S} = \{\boldsymbol{u} \in \mathcal{V} : \boldsymbol{u} = y - x, \text{ for all } y \in \mathcal{S}\}$ and called *tangent space* to $\mathcal{S}$ at $x$. In the differential geometrical definition, the tangent space $T_x\mathcal{S}$ is the set of the vectors that are each *tangent* at $x$ to one of the infinite possible regular curves $c : [a, b] \to \mathcal{S} : s \mapsto c(s)$ such that $c(s_0) = x$, where $s_0 \in ]a, b[$, i.e., the vectors (see Figure 1)

$$\boldsymbol{u} = \lim_{h\to 0} \frac{c(s_0 + h) - c(s_0)}{h} = c'(s_0) \in T_x\mathcal{S}. \tag{1}$$

For the case of an affine space $\mathcal{S}$, this definition of tangent space $T_x\mathcal{S}$ coincides with that given by the expression $\boldsymbol{u} = y - x$. Indeed, by varying the curve passing by $x$, we obtain all possible "tip points" $y$ of the tangent vectors defined as $\boldsymbol{u} = y - x$. The dual space of $T_x\mathcal{S}$, i.e., the vector space of all linear maps $\boldsymbol{\varphi} : T_x\mathcal{S} \longrightarrow \mathbb{R}$, is denoted $T^\star_x\mathcal{S}$ and is called the *cotangent space* to $\mathcal{S}$ at $x$. The disjoint unions of all tangent and cotangent spaces are called *tangent bundle* $T\mathcal{S}$ and *cotangent bundle* $T^\star\mathcal{S}$, respectively.

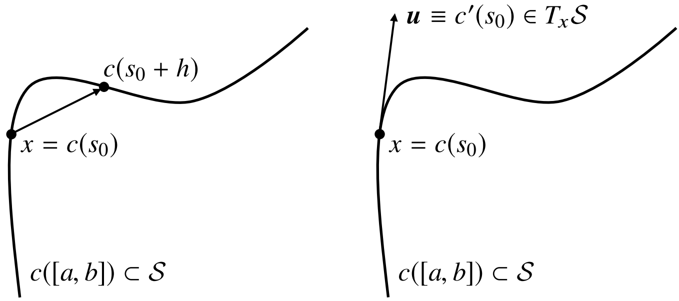


**Figure 1:** Differential geometrical definition of tangent vector at a point $x \in \mathcal{S}$. Left: the *secant* vector $c(s_0 + h) - c(s_0)$ passing by $x = c(s_0)$. Right: the tangent vector $\boldsymbol{u} = c'(s_0)$ at $x = c(s_0)$, obtained as the limit of the secant.

Vector fields and covector fields (or fields of one-forms) on an open set $\mathcal{A} \subseteq \mathcal{S}$ are maps

$$\boldsymbol{u} : \mathcal{A} \subseteq \mathcal{S} \longrightarrow T\mathcal{S} : x \mapsto \boldsymbol{u}(x) \in T_x\mathcal{S}, \tag{2a}$$

$$\boldsymbol{\varphi} : \mathcal{A} \subseteq \mathcal{S} \longrightarrow T^\star\mathcal{S} : x \mapsto \boldsymbol{\varphi}(x) \in T^\star_x\mathcal{S}, \tag{2b}$$

and tensor fields of higher order are defined analogously. Rather than speaking of contractions of vectors and covectors in a specific tangent and cotangent space, we can directly speak of the contractions of vector fields and covector fields in the tangent and cotangent bundle, and we denote the contraction by means of simple juxtaposition, i.e.,

$$\boldsymbol{\varphi}\, \boldsymbol{u} = \boldsymbol{u}\, \boldsymbol{\varphi} = \varphi_a\, u^a. \tag{3}$$

The physical space $\mathcal{S}$ is equipped with a metric tensor $\boldsymbol{g}$, a symmetric and positive definite second-order tensor field defining the scalar product of two vector fields as

$$\boldsymbol{g} : T\mathcal{S} \times T\mathcal{S} \longrightarrow \mathbb{R} : (\boldsymbol{u}, \boldsymbol{v}) \mapsto \langle \boldsymbol{u}, \boldsymbol{v} \rangle \equiv \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = u^a\, g_{ab}\, v^b. \tag{4}$$

We assume use of the Levi-Civita connection, i.e., the covariant derivative associated with the metric tensor $\boldsymbol{g}$ via the Christoffel symbols given by (see, e.g., Marsden and Hughes, 1983)

$$\gamma^a_{bc} = \tfrac{1}{2}\, g^{ad}(g_{cd,b} + g_{bd,c} - g_{bc,d}), \tag{5}$$

which are symmetric in their lower indices, i.e., $\gamma^a_{bc} = \gamma^a_{cb}$. The covariant derivative $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ of the vector field $\boldsymbol{v}$ in the direction of the vector field $\boldsymbol{u}$ has the component expression

$$[\nabla_{\boldsymbol{u}}\boldsymbol{v}]^a \equiv v^a{}_{|b}\, u^b = v^a{}_{,b}\, u^b + \gamma^a_{bc}\, v^c\, u^b. \tag{6}$$

and defines the *gradient* $\operatorname{grad} \boldsymbol{v}$ as the tensor field such that its definition as a linear map is $(\operatorname{grad} \boldsymbol{v})\boldsymbol{u} \equiv \nabla_{\boldsymbol{u}} \boldsymbol{v}$, with components $[\operatorname{grad} \boldsymbol{v}]^a{}_b = v^a{}_{|b}$. The covariant derivative and the gradient of a tensor field of arbitrary order are defined analogously.

*Remark* 1. A scalar is a *tensor of order zero* and thus we find more natural to use the convention adopted by, e.g., Epstein (2010, see page 116) and to consider the gradient of a scalar field $f$ as the *covector* field (or one-form) $\operatorname{grad} f$ such that $(\operatorname{grad} f)(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} f$, as for a tensor of any other order. Accordingly, the components of $\operatorname{grad} f$ are $f_{,a}$. The other possible convention is that adopted by Marsden and Hughes (1983, see page 69), according to which the gradient of $f$ is the vector field with components $g^{ab} f_{,b}$. Note that, in either case, since $f$ is a tensor of order zero, the Christoffel symbols of the connection are *not* involved in the gradient, which is thus connection-independent. There are several advantages in defining the gradient as a covector. First, this definition is *metric-independent*, whereas the vector definition clearly necessitates that a metric tensor $\boldsymbol{g}$ be defined. Second, the covector definition accommodates the analytical mechanical definition of force as a covector field: indeed, an integrable force is the negative of the gradient of a potential energy and is thus consistently represented as a covector field. Finally, with the covector definition of $\operatorname{grad} f$, we have the remarkable chain of identities

$$\nabla f \equiv \operatorname{grad} f \equiv \mathrm{d} f \equiv \mathrm{D} f, \tag{7}$$

where $\mathrm{d}f$ is the *exterior derivative* of $f$, when seen as a zero-form (see, e.g., Epstein, 2010, page 116), and $\mathrm{D}f$ is the *Fréchet derivative* (or tangent map) of $f$, when seen as a point map from $\mathcal{A} \subset \mathcal{S}$ into $\mathbb{R}$.

In the following, the physical space $\mathcal{S}$ is identified with the affine space $\mathbb{E}^3$, which is $\mathbb{R}^3$ considered both as the point space and as the modelling vector space.

## 2.2 Bodies, Configurations and the Deformation Gradient

In the simplified presentation that we adopt, a deformable continuous body $\mathcal{B}$ is identified with one of its placements in the physical space $\mathcal{S}$, and this particular placement is called reference configuration. The body is assumed to be endowed with the material metric $\boldsymbol{G}$, which induces the corresponding Levi-Civita connection, similarly to what seen for the spatial metric $\boldsymbol{g}$.

A *configuration*, or *deformation*, of the body is an *embedding*

$$\phi : \mathcal{B} \to \mathcal{S} : X \mapsto x = \phi(X), \tag{8}$$

i.e., a map such that its codomain-restriction $\phi : \mathcal{B} \to \phi(\mathcal{B})$ is a diffeomorphism, i.e., a continuos and differentiable map, which is invertible, with continuous and differentiable inverse $\Phi \equiv \phi^{-1} : \phi(\mathcal{B}) \to \mathcal{B}$. The configuration $\phi$ maps *material* points $X = (X^1, X^2, X^3)$ in the body $\mathcal{B}$ into *spatial* points $x = (x^1, x^2, x^3)$ in $\mathcal{S}$, i.e., $\phi(X) = x$.

Since we are going to introduce another class of configurations, called *Eshelbian*, we shall refer to the standard definition of configuration given above as to a *conventional configuration*. The set of all $k$-times differentiable conventional configuration maps (with

$k \in \mathbb{N}$) constitutes the *conventional* configuration space $\mathcal{C}$ of the body $\mathcal{B}$. Since $\mathcal{S}$ is an affine space, the space $C^k(\mathcal{B}, \mathcal{S})$ of the $k$-times differentiable maps from $\mathcal{B}$ into $\mathcal{S}$ is an infinite-dimensional affine space. Thus, considering $\mathcal{C}$ as an open set in $C^k(\mathcal{B}, \mathcal{S})$ (Marsden and Hughes, 1983) makes $\mathcal{C}$ an infinite-dimensional trivial manifold. A tangent vector $\boldsymbol{\eta}$ in the functional tangent space $T_\phi\mathcal{C}$ can be thought of as the tangent at $\phi$ to a curve of maps in $\mathcal{C}$ (i.e., a one-parameter family of maps in $\mathcal{C}$), and is a vector field *covering* the configuration $\phi$, i.e.,

$$\boldsymbol{\eta} : \mathcal{B} \to T\mathcal{S} : X \mapsto \boldsymbol{\eta}(X) \in T_{\phi(X)}\mathcal{S} = T_x\mathcal{S}. \tag{9}$$

The vector field $\boldsymbol{\eta}$ is called a (conventional) *displacement field* (and, when compatible with the constraints, but not necessarily attained by the body, it is called a *virtual displacement*). Figure 2 shows the displacement $\boldsymbol{\eta}(X) = \boldsymbol{\eta}(\Phi(x))$ as a tangent vector at $T_x\mathcal{S}$ and an illustration of the configuration space with the displacement field $\boldsymbol{\eta}$ as a tangent vector at $T_\phi\mathcal{C}$.

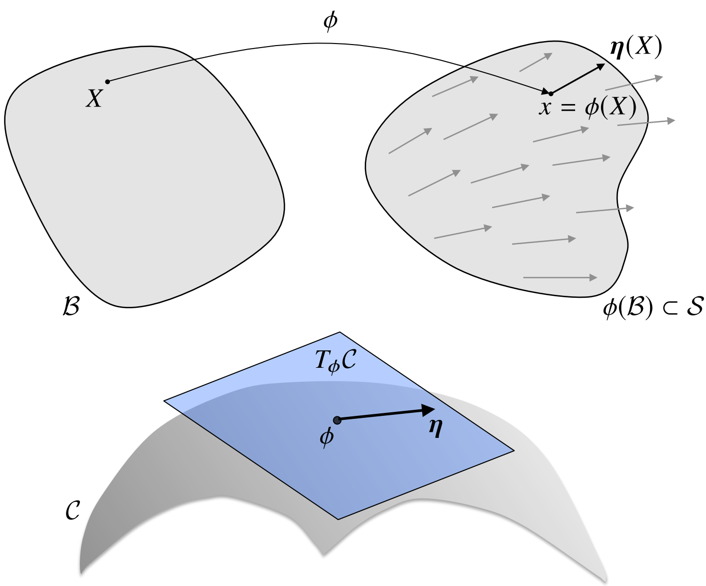


**Figure 2:** A conventional displacement field. Top: The displacement $\boldsymbol{\eta}(X) = \boldsymbol{\eta}(\Phi(x)) \in T_x\mathcal{S}$ as a tangent vector attached at $x = \phi(X)$. Bottom: The displacement field $\boldsymbol{\eta}$ as a tangent vector attached at the configuration $\phi$, which is a point in the configuration space $\mathcal{C}$, here depicted as a surface, for the sake of an intuitive graphical representation.

The deformation gradient at point $X$ is the *tangent map* of $\phi$, i.e., the tensor

$$(T\phi)(X) = \boldsymbol{F}(X) : T_X\mathcal{B} \to T_x\mathcal{S}, \tag{10}$$

with $x = \phi(X)$, expressing the Fréchet derivative of $\phi$ at $X$. Since the existence of the Fréchet derivative of $\phi$ implies the existence of its Gâteaux derivative (or directional derivative), $\boldsymbol{F}(X)$ can be defined through the limit

$$(\partial_{\boldsymbol{M}}\phi)(X) := \lim_{h\to 0}\frac{\phi(X+h\boldsymbol{M})-\phi(X)}{h} = [(T\phi)(X)]\boldsymbol{M} = [\boldsymbol{F}(X)]\boldsymbol{M}, \tag{11}$$

and the Gâteaux derivative $\partial_{\boldsymbol{M}}\phi(X)$ of $\phi$ with respect to *any* tangent vector $\boldsymbol{M} \in T_X\mathcal{B}$ equals the Fréchet derivative $\boldsymbol{F}(X)\boldsymbol{M}$, which is linear in $\boldsymbol{M}$. In components, Equation (11) reads

$$(\partial_{\boldsymbol{M}}\phi)^a(X) = (T\phi)^a{}_B(X)\,M^B = F^a{}_B(X)\,M^B = \phi^a{}_{,B}(X)\,M^B, \tag{12}$$

where we recall that the comma denotes partial differentiation. Note that $\boldsymbol{F}(X)$ is a two-point tensor as it has the *domain leg* in $T_X\mathcal{B}$ and the *codomain leg* in $T_x\mathcal{S}$. As a tensor field, the deformation gradient is

$$\boldsymbol{F} : \mathcal{B} \to T\mathcal{S} \otimes T^\star\mathcal{B}. \tag{13}$$

The deformation gradient $\boldsymbol{F}$ pushes-forward material vector fields $\boldsymbol{M}$ with components $M^A$ into spatial vector fields $\phi_*\boldsymbol{M} = (\boldsymbol{F}\circ\Phi)\,(\boldsymbol{M}\circ\Phi)$ with components $(F^a{}_A\circ\Phi)\,(M^A\circ\Phi)$. The inverse $\boldsymbol{F}^{-1}$ pulls-back spatial vector fields $\boldsymbol{m}$ with components $m^a$ into material vector fields $\phi^*\boldsymbol{m} = (\boldsymbol{F}^{-1}\circ\phi)\,(\boldsymbol{m}\circ\phi)$ with components $((\boldsymbol{F}^{-1})^A{}_a\circ\phi)\,(m^a\circ\phi)$. The transpose $\boldsymbol{F}^T$ pulls-back spatial covector fields $\boldsymbol{\pi}$ with components $\pi_a$ into material covector fields $\phi^*\boldsymbol{\pi} = (\boldsymbol{F}^T\circ\phi)\,(\boldsymbol{\pi}\circ\phi)$ with components $((\boldsymbol{F}^T)_A{}^a\circ\phi)\,(\pi_a\circ\phi) = F^a{}_A\,(\pi_a\circ\phi)$. The inverse transpose $\boldsymbol{F}^{-T}$ pushes-forward material covector fields $\boldsymbol{\Pi}$ with components $\Pi_A$ into spatial covector fields $\phi_*\boldsymbol{\Pi} = (\boldsymbol{F}^{-T}\circ\Phi)\,(\boldsymbol{\Pi}\circ\Phi)$ with components $((\boldsymbol{F}^{-T})_a{}^A\circ\Phi)\,(\Pi_A\circ\Phi) = (\boldsymbol{F}^{-1})^A{}_a\,(\Pi_A\circ\Phi)$.

The determinant $J = \det\boldsymbol{F}$ has the meaning of *volume ratio*, in the spirit of the theorem of the change of variables applied to the transformation from the spatial region $\phi(\mathcal{R}) \subset \mathcal{S}$ to the corresponding material region $\mathcal{R} \subset \mathcal{B}$.

## 2.3 Eshelbian Configurations and Their Tangent Maps

Grillo et al. (2003) introduced the concept of *admissible reference configuration set* of a body as the set of all reference configurations obtained by appl ying a diffeomorphism to the reference configuration $\mathcal{B}$ representing the body (which has some similarities with the idea of boundary reparametresations introduced by Gurtin, 1995). Here, we make use of this concept in a slightly different way.

An *Eshelbian configuration* $\mathcal{Y}$ is a diffeomorphism on the body $\mathcal{B}$. Since we define the body $\mathcal{B}$ as a trivial manifold, i.e., an open subset of the physical space $\mathcal{S}$, the codomain of an Eshelbian configuration $\mathcal{Y}$ should be the whole space $\mathcal{S}$ and the image would be an open set $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B}) \subset \mathcal{S}$. However, if the body $\mathcal{B}$ were a non-trivial manifold, the image $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B})$ would be another non-trivial manifold. To keep the notation as general as possible, we prefer to avoid declaring $\mathcal{S}$ as the codomain of $\mathcal{Y}$. Rather, we consider all admissible diffeomorphisms $\mathcal{Y}$, each with its image $\tilde{\mathcal{B}}$, and we obtain the collection of all admissible reference configurations $\tilde{\mathcal{B}}$, which clearly also contains $\mathcal{B}$ itself (see also Grillo et al., 2003). Then, we consider the union $\mathcal{N} = \cup_{\mathcal{Y}} \tilde{\mathcal{B}}$ of these mutually diffeomorphic sets $\tilde{\mathcal{B}}$, and define the generic Eshelbian configuration as

$$\mathcal{Y} : \mathcal{B} \to \mathcal{N} : X \mapsto \tilde{X} = \mathcal{Y}(X), \tag{14}$$

which has the further notational advantage of not tying $\mathcal{Y}$ to its specific image $\tilde{\mathcal{B}}$.

Analogously to the case of a conventional configuration, the tangent map of an Eshelbian configuration at point $X$ is the tensor

$$(T\mathcal{Y})(X) : T_X\mathcal{B} \to T_{\tilde{X}}\mathcal{N}, \tag{15}$$

with $\tilde{X} = \mathcal{Y}(X)$. Again, $(T\mathcal{Y})(X)$ is the Fréchet derivative of $\mathcal{Y}$ at $X$ and, since $\mathcal{Y}$ is a diffeomorphism, $(T\mathcal{Y})(X)$ can be computed by means of the Gâteaux derivative of $\mathcal{Y}$ at $X$, i.e.,

$$(\partial_{\boldsymbol{M}}\mathcal{Y})(X) = \lim_{h\to 0} \frac{\mathcal{Y}(X + h\,\boldsymbol{M}) - \mathcal{Y}(X)}{h} = [(T\mathcal{Y})(X)]\boldsymbol{M}. \tag{16}$$

The *material identity map* is the particular case of Eshelbian configuration obtained by considering that $\mathcal{B} \subset \mathcal{N}$, and is defined as

$$\mathcal{X} : \mathcal{B} \to \mathcal{B} : X \mapsto X = \mathcal{X}(X), \tag{17}$$

with the component representation

$$\mathcal{X}^A : \mathcal{B} \to \mathbb{R} : X \mapsto X^A = \mathcal{X}^A(X) \equiv \mathcal{X}^A(X^1, X^2, X^3). \tag{18}$$

Its tangent map is clearly the (material) identity tensor in $T\mathcal{B}$, i.e.,

$$T\mathcal{X} = \boldsymbol{I} : T\mathcal{B} \to T\mathcal{B}, \qquad (T\mathcal{X})^A{}_B = \mathcal{X}^A{}_{,B} = \delta^A{}_B. \tag{19}$$

Also in the case of Eshelbian configurations, we can exploit the affine structure of $\mathcal{S}$: since all sets $\tilde{\mathcal{B}}$ are open subsets of $\mathcal{S}$, also $\mathcal{N} = \cup_{\mathcal{Y}} \tilde{\mathcal{B}} \subseteq \mathcal{S}$ is an open set, and thus we can define the space of all Eshelbian configurations as an open subset $\mathcal{M}$ of the infinite-dimensional affine space $C^k(\mathcal{B}, \mathcal{N})$, which makes $\mathcal{M}$ an infinite-dimensional trivial manifold.

*Remark* 2. In our setting, in which the physical space $\mathcal{S}$ is an affine space and a body $\mathcal{B}$ is a subset of $\mathcal{S}$, the distinction between a conventional configuration $\phi : \mathcal{B} \to \mathcal{S}$ and an Eshelbian configuration $\mathcal{Y} : \mathcal{B} \to \mathcal{N}$ seems to fade out, because $\mathcal{N} = \cup_{\mathcal{Y}} \tilde{\mathcal{B}} \subseteq \mathcal{S}$. However this is not the case, as will become clear from the explanation given in Section 3 (see also Figures 3 and 4). Moreover, when $\mathcal{B}$ is a general manifold, the distinction is fundamental. In this case, while a conventional configuration $\phi$ remains an embedding of $\mathcal{B}$ in $\mathcal{S}$, i.e., it gives $\mathcal{B}$ a placement $\phi(\mathcal{B}) \subset \mathcal{S}$, an Eshelbian configuration transforms the manifold $\mathcal{B}$ into *a different* manifold $\tilde{\mathcal{B}}$.

A tangent vector $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$ is a vector field

$$\boldsymbol{U} : \mathcal{B} \to T\mathcal{B} : X \mapsto \boldsymbol{U}(X) \in T_X\mathcal{B}, \tag{20}$$

and is called a *material displacement* field. When an Eshelbian configuration $\mathcal{Y} : \mathcal{B} \to \mathcal{N}$, is defined as a *perturbation* of the material identity $\mathcal{X}$, i.e.,

$$\mathcal{Y}(X) = \mathcal{X}(X) + h\,\boldsymbol{U}(X) = X + h\,\boldsymbol{U}(X), \quad \mathcal{Y}^A(X) = \mathcal{X}^A(X) + h\,U^A(X) = X^A + h\,U^A(X), \tag{21}$$

where $h \in \mathbb{R}$ is a smallness parameter and $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$, it is called an "infinitesimal transformation of the coordinates", in the language of field theory. Omitting the argument $X$, we can write

$$\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}, \qquad \mathcal{Y}^A = \mathcal{X}^A + h\,U^A. \tag{22}$$

The tangent map of $\mathcal{Y}$ in Equation (22) is expressed by

$$T\mathcal{Y} = T\mathcal{X} + h\,\mathrm{Grad}\,\boldsymbol{U} = \boldsymbol{I} + h\,\mathrm{Grad}\,\boldsymbol{U}, \quad (T\mathcal{Y})^A{}_B = (T\mathcal{X})^A{}_B + h\,U^A{}_{|B} = \delta^A{}_B + h\,U^A{}_{|B}, \tag{23}$$

where $\boldsymbol{I}$ is the material identity tensor and $\mathrm{Grad}\,\boldsymbol{U}$, with components $U^A{}_{|B}$, is the gradient (or covariant derivative) of $\boldsymbol{U}$. For $h \to 0$, the Jacobian determinant of $T\mathcal{Y}$ is

$$\begin{aligned} \det(T\mathcal{Y}) &= \det(\boldsymbol{I} + h\,\mathrm{Grad}\,\boldsymbol{U}) = 1 + h\,\mathrm{Tr}\,(\mathrm{Grad}\,\boldsymbol{U}) + o(h) \\ &= 1 + h\,\mathrm{Div}\,\boldsymbol{U} + o(h) = 1 + h\,U^A{}_{|A} + o(h). \end{aligned} \tag{24}$$

## 2.4 Conventions on Forces and Stresses

As mentioned in Remark 1, in the analytical mechanics / field theory approach, followed by, e.g., Hill (1951) and Eshelby (1975), forces are regarded as *covector* fields, acting on velocity or displacement vector fields. Thus, the contraction of a force with a velocity or displacement is given precisely by (3). Consequently, the first leg of the stress (the "force leg") is a *covector*, while the second leg (the "area leg") is a *vector*. Indeed, in the expression of Cauchy's theorem, the traction vectors relative to the spatial and material elements of area are given by

$$\boldsymbol{t_n} = \boldsymbol{\sigma}\,\boldsymbol{n}, \quad \mathbf{t}_{\boldsymbol{N}} = \boldsymbol{P}\boldsymbol{N}, \qquad (\boldsymbol{t_n})_a = \sigma_a{}^b\,n_b, \quad (\mathbf{t}_{\boldsymbol{N}})_a = P_a{}^B\,N_B. \tag{25}$$

In Equation (25), $\boldsymbol{n}$ is the normal covector to a surface element at the spatial point $x = \phi(X)$ in the current configuration, $\boldsymbol{N}$ is the normal covector to the corresponding surface element at the material point $X$ in the reference configuration and the first Piola-Kirchhoff stress is related to Cauchy stress by means of the backward Piola transformation

$$\boldsymbol{P} = J\,(\boldsymbol{\sigma} \circ \phi)\,\boldsymbol{F}^{-T}, \qquad P_a{}^B = J\,(\sigma_a{}^b \circ \phi)\,(\boldsymbol{F}^{-T})_b{}^B. \tag{26}$$

Equations (25) and (26) show that the tractions $\boldsymbol{t_n}$ and $\mathbf{t}_{\boldsymbol{N}}$ are indeed *covectors* if the Cauchy stress $\boldsymbol{\sigma}$ and the first Piola-Kirchhoff stress $\boldsymbol{P}$, respectively, are treated as "mixed" tensors (we remark that $\mathbf{t}_{\boldsymbol{N}} \neq \boldsymbol{t_n}$, since $\boldsymbol{N}$ is related to $\boldsymbol{n}$ by the formula of the change of area, also known as Nanson's formula; see, e.g., Bonet and Wood, 2008).

# 3 Eshelby's Original Derivation of the Weak Form

Eshelby (1951) derived the weak form of the expression of the configurational force balance by means of a thought experiment subdivided in several steps. This form is weak as it is an integral equation expressing a *virtual work*. We note that, in this section, we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region $\mathcal{D}$ of the body as a functional on the manifold $\mathcal{M}$, the Eshelbian configuration space.

Eshelby (1951) considered a body $\mathcal{B}$, subjected to constraints and external loads, and in whose interior is located a *defect* of any kind: a point defect, a dislocation, an inclusion, or even a region in which the material properties are inhomogeneous. To fix ideas, we follow Eshelby's graphical example with a point defect, as shown in Figure 3. The left panel in Figure 3 shows what Eshelby called the *original* body, in which a region $\mathcal{D}$ (highlighted in dark grey), bounded by the smooth material surface $\Sigma = \partial\mathcal{D}$, is selected such that the defect is contained in $\mathcal{D}$. The right panel in Figure 3 represents a *replica* of the original body, in which a different region $\tilde{\mathcal{D}}$ (also highlighted in dark grey), bounded by the smooth material surface $\tilde{\Sigma} = \partial\tilde{\mathcal{D}}$, is selected so that the defect is contained in $\tilde{\mathcal{D}}$ (see also Kienzler and Herrmann, 2000). Since $\Sigma$ and $\tilde{\Sigma}$ are both smooth, it is always possible to find an Eshelbian configuration $\mathcal{Y}$ transforming $\mathcal{D}$ into $\tilde{\mathcal{D}}$, i.e., $\mathcal{Y}(\mathcal{D}) = \tilde{\mathcal{D}}$. Moreover, if $\Sigma$ and $\tilde{\Sigma}$ are "close enough", then $\tilde{\mathcal{D}}$ is obtainable from $\mathcal{D}$ through a perturbation of the form defined in Equation (21), whose domain restriction to $\mathcal{D}$ is

$$\mathcal{Y} : \mathcal{D} \to \mathcal{B} : X \mapsto \mathcal{Y}(X) = \mathcal{X}(X) + h\,\boldsymbol{U}(X), \tag{27}$$

where we recall that $h$ is a smallness parameter. Note that Eshelby (1951) chose $h\,\boldsymbol{U}$ to be a *uniform* material displacement field $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$ over $\mathcal{D}$. Eshelby's choice makes the procedure easier to illustrate and yields directly the *strong form* of the inclusion problem. Here, we derive the *weak form* first and then obtain the strong form by adding Eshelby's assumption, $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$, at the very end. However, it is helpful to keep the uniform displacement $-h\,\boldsymbol{U}_0$ in one's mind and, to this end, we chose to represent this uniform displacement in Figure 3, following Eshelby's original thought experiment.

We remark that, since the map $\mathcal{Y}$ of Equation (27) is Eshelbian, the body is undergoing *no* deformation, in the sense that it is *not* changing its shape, but only its configuration. Indeed, one chooses the surface $\Sigma$ enclosing the region $\mathcal{D}$ and the surface $\tilde{\Sigma}$ enclosing the region $\tilde{\mathcal{D}}$ *independently* and then finds a suitable $\mathcal{Y}$ mapping $\mathcal{D}$ into $\tilde{\mathcal{D}}$. Clearly, this mere fact does *not* displace the defect at all, but simply represents a different choice of enclosing surface. The displacement of the defect in the reference configuration actually takes place when we *replace* the region $\mathcal{D}$ in the original body with the region $\tilde{\mathcal{D}}$ cut from the replica body (which is straightforward in the case of a Eshelby's rigid displacement $-h\,\boldsymbol{U}_0$), where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are related by the *material transformation* $\mathcal{Y}$ described by (27). Note that, in this replacement, the defect is moved together with the region $\tilde{\mathcal{D}}$ (see point (iii) below).

Our goal is to determine the *variation in energy* accompanying this *change in reference configuration*. In order to achieve this, we perform the thought experiment proposed by Eshelby (1951, 1975) and described below.

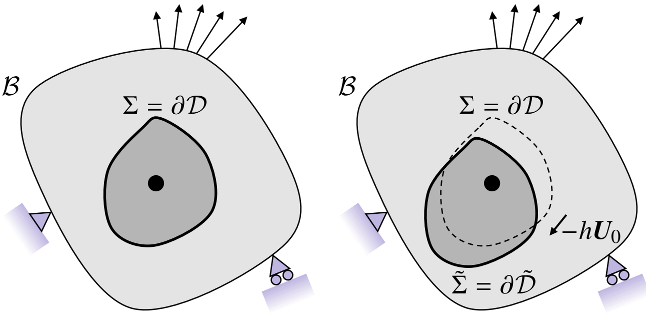


**Figure 3:** Determination of the force on a defect (the solid black circle). Left: original body, with the defect contained in a region $\mathcal{D}$, bounded by the smooth surface $\Sigma = \partial\mathcal{D}$. Right: replica body, with the defect contained in a different region $\tilde{\mathcal{D}}$, bounded by the smooth surface $\tilde{\Sigma} = \partial\tilde{\mathcal{D}}$. As in Eshelby's original scheme (Eshelby, 1975), here we depict the material displacement $h\,\boldsymbol{U}$ as being *uniform* over the material region $\mathcal{D}$ enclosed by the surface $\Sigma$, i.e., $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$ for every $X \in \mathcal{D}$.

(i) In the original body, cut out the material in the region $\mathcal{D}$. If the body is pre-stressed for any reason, then apply traction forces to the boundary $\Sigma = \partial\mathcal{D}$ of the cavity that has been created, in order to avoid relaxation.

(ii) Similarly, in the replica body, cut out the material in the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ and apply suitable tractions to the boundary $\tilde{\Sigma} = \partial\tilde{\mathcal{D}} = \partial[\mathcal{Y}(\mathcal{D})] \equiv \mathcal{Y}(\partial\mathcal{D})$ to prevent relaxation. Let us denote the total elastic energy $\mathcal{E}^{\text{el}}_{\mathcal{D}} : \mathcal{M} \to \mathbb{R}$ in $\mathcal{Y}(\mathcal{D})$ by

$$\mathcal{E}^{\text{el}}_{\mathcal{D}}(\mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} W = \int_{\mathcal{D}} \det(T\mathcal{Y})\, W \circ \mathcal{Y}, \tag{28}$$

where we used the theorem of the change of variables to transform the integral over the displaced region $\mathcal{Y}(\mathcal{D})$ into an integral over the original region $\mathcal{D}$. Similarly, in the original region, the total elastic energy would be

$$\mathcal{E}^{\text{el}}_{\mathcal{D}}(\mathcal{X}) = \int_{\mathcal{D}} W = \int_{\mathcal{D}} W \circ \mathcal{X}, \tag{29}$$

where we exploited the identity $\mathcal{X}(X) = X$ in writing $W = W \circ \mathcal{X}$. Therefore, the difference in energy due to the perturbation $\mathcal{Y}$ (i.e., due to the different selection of the surfaces $\tilde{\Sigma}$ and $\Sigma$) is

$$\mathcal{E}^{\text{el}}_{\mathcal{D}}(\mathcal{Y}) - \mathcal{E}^{\text{el}}_{\mathcal{D}}(\mathcal{X}) = \int_{\mathcal{D}} \det(T\mathcal{Y})\; W \circ \mathcal{Y} - \int_{\mathcal{D}} W \circ \mathcal{X} = \int_{\mathcal{D}} [\det(T\mathcal{Y})\; W \circ \mathcal{Y} - W \circ \mathcal{X}]. \tag{30}$$

By expressing the map $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$ (see Equation (21)), considering that, for $h \to 0$, $\det T\mathcal{Y} = 1 + h\,\mathrm{Div}\,\boldsymbol{U} + o(h)$ (see Equation (24)) and

$$W \circ \mathcal{Y} = W \circ (\mathcal{X} + h\,\boldsymbol{U}) = W \circ \mathcal{X} + h\,[(\mathrm{Grad}\,W) \circ \mathcal{X}]\,\boldsymbol{U} + o(h), \tag{31}$$

Equation (30) becomes

$$\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X}+h\,\boldsymbol{U})-\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X})=\int_{\mathcal{D}}\big[h\,(W\circ\mathcal{X})\,\mathrm{Div}\,\boldsymbol{U}+h\,[(\mathrm{Grad}\,W)\circ\mathcal{X}]\,\boldsymbol{U}+o(h)\big].\quad(32)$$

Now, we can divide both sides of Equation (32) by $h$ and take the limit for $h\longrightarrow 0$ so that, on the left-hand side, we have the *variational* Gâteaux derivative of $\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}$ with respect to the material displacement field $\boldsymbol{U}\in T_{\mathcal{X}}\mathcal{M}$, evaluated at the identity map $\mathcal{X}$, i.e.,

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X})=\lim_{h\to 0}\frac{\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X}+h\,\boldsymbol{U})-\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X})}{h}=\int_{\mathcal{D}}\big[(W\circ\mathcal{X})\,\mathrm{Div}\,\boldsymbol{U}+[(\mathrm{Grad}\,W)\circ\mathcal{X}]\,\boldsymbol{U}\big].\quad(33)$$

By using the identities $(\mathrm{Grad}\,W)\circ\mathcal{X}=\mathrm{Grad}\,W$ and $W\circ\mathcal{X}=W$, we can write

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X})=\int_{\mathcal{D}}\big[W\,\mathrm{Div}\,\boldsymbol{U}+[\mathrm{Grad}\,W]\,\boldsymbol{U}\big],\quad(34)$$

which, using by Leibniz' rule and the identity $\mathrm{Div}(W\,\boldsymbol{U})=\mathrm{Div}(W\,\boldsymbol{I}\,\boldsymbol{U})$ (where $\boldsymbol{I}$ is the material identity tensor), becomes

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X})=\int_{\mathcal{D}}\mathrm{Div}\big[W\,\boldsymbol{I}\,\boldsymbol{U}\big].\quad(35)$$

(iii) Before the deformation $\phi$ occurs, the region $\tilde{\mathcal{D}}=\mathcal{Y}(\mathcal{D})$ that had been isolated from the replica body could be *"transplanted"** into the cavity (resulting from the elimination of the original region $\mathcal{D}$) in the original body by simply applying the opposite displacement field $-h\,\boldsymbol{U}$. In Eshelby's choice of a uniform displacement, this would be the *rigid translation* $h\,\boldsymbol{U}_0$, as shown in Figure 4. This is as if the defect had been displaced of the amount $h\,\boldsymbol{U}_0$.

However, after the deformation $\phi$ occurs, $\phi(\tilde{\mathcal{D}})=\phi(\mathcal{Y}(\mathcal{D}))$ from the replica and $\phi(\mathcal{D})$ from the original body are *different* in general, and thus $\phi(\tilde{\mathcal{D}})=\phi(\mathcal{Y}(\mathcal{D}))$ *may not* fit the cavity with deformed surface $\partial[\phi(\mathcal{D})]\equiv\phi(\partial\mathcal{D})=\phi(\Sigma)$ in the original body. Indeed, the points of the deformed surface $\partial[\phi(\mathcal{D})]\equiv\phi(\partial\mathcal{D})=\phi(\Sigma)$ in the original body and the points of the deformed surface $\partial[\phi(\mathcal{Y}(\mathcal{D}))]\equiv\phi(\partial(\mathcal{Y}(\mathcal{D})))=\phi(\partial\tilde{\mathcal{D}})=\phi(\tilde{\Sigma})$ in the replica body generally differ by the (conventional spatial) displacement

$$\phi(X+h\,\boldsymbol{U}(X))-\phi(X)=[\boldsymbol{F}(X)](h\,\boldsymbol{U}(X))+o(h),\quad(36)$$

which, recalling that $X+h\,\boldsymbol{U}(X)=\mathcal{Y}(X)$ and $X=\mathcal{X}(X)$, omitting the argument $X$ and using the linearity of $\boldsymbol{F}$, can be written as

$$\phi\circ\mathcal{Y}-\phi\circ\mathcal{X}=h\,\boldsymbol{F}\boldsymbol{U}+o(h).\quad(37)$$

In order to deform the surface $\phi(\Sigma)=\phi(\partial\mathcal{D})$ of the cavity in the original body in such a way that $\phi(\tilde{\mathcal{D}})$ from the replica body can exactly fit in it, we must *adjust* the

*We are borrowing the term "transplant" from Epstein and Maugin (2000) and Imatani and Maugin (2002), but with a more strictly "surgical" meaning.

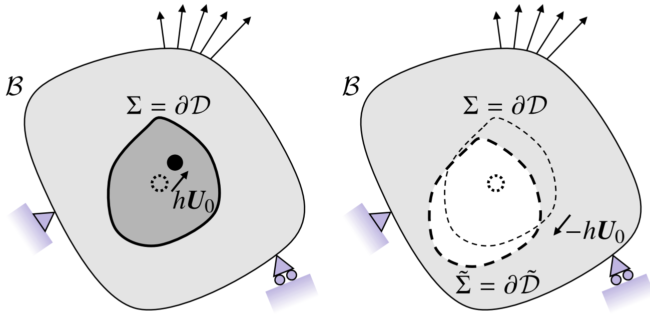


**Figure 4:** *Before* the deformation $\phi$ takes place, the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ could be transplanted from the replica (right panel) to the original body (left panel), into the cavity resulting from the removal of the original region $\mathcal{D}$, by simply applying the negative of the displacement $-h\,\boldsymbol{U}_0$. This procedure effectively displaces the defect by the amount $h\,\boldsymbol{U}_0$ in the original body. We remark that this no longer holds *after* deformation has taken place.

deformation. This can be achieved, in fact, by introducing a new deformation, $\bar{\phi}$, which, applied to $\mathcal{Y}(\mathcal{D}) = \tilde{\mathcal{D}}$, is such that the overall displacement is null, i.e.,

$$\bar{\phi}(\mathcal{Y}(X)) - \phi(X) = \mathbf{0}. \tag{38}$$

Since $\bar{\phi}$ has to adjust $\phi$ in order to eliminate the mismatch generated by the combined effect of $\mathcal{Y}$ and $\phi$ (note how the composition $\phi \circ \mathcal{Y}$ is, in fact, the mathematical representation of the "combined effect"), it is natural to define $\bar{\phi}$ as a perturbation of $\phi$. Hence, we set

$$\bar{\phi} = \phi + h\,\boldsymbol{\eta}, \tag{39}$$

where, without loss of generality, the same smallness parameter, $h$, is used as that defining $\mathcal{Y} = \mathcal{X} + h\boldsymbol{U}$. With the aid of (39), and in the limit $h \to 0$, Equation (38) becomes

$$\begin{aligned}
&\phi \circ (\mathcal{X} + h\,\boldsymbol{U}) + h\,\boldsymbol{\eta} \circ (\mathcal{X} + h\,\boldsymbol{U}) - \phi \circ \mathcal{X} \\
&= h\,\boldsymbol{F}\boldsymbol{U} + o(h) + h\,\boldsymbol{\eta} + h^2[\boldsymbol{\eta} \circ \mathcal{X}]\boldsymbol{U} + o(h^2) \\
&= h[\boldsymbol{F}\boldsymbol{U} + \boldsymbol{\eta}] + o(h) = \mathbf{0}.
\end{aligned} \tag{40}$$

At the lowest order, Equation (40) gives the condition sought for $\boldsymbol{\eta}$, i.e., that it has to compensate for $\boldsymbol{U}$, thereby yielding

$$\boldsymbol{F}\boldsymbol{U} + \boldsymbol{\eta} = \mathbf{0} \qquad \Rightarrow \qquad -h\,\boldsymbol{\eta} = h\,\boldsymbol{F}\boldsymbol{U}. \tag{41}$$

This interpretation of the displacement $\boldsymbol{\eta}$ is the core of Noether's Theorem, which will be addressed in Section 5.

The work necessary to adjust the deformation of $\mathcal{B} \setminus \mathcal{D}$ according to (39) is exerted by the first Piola-Kirchhoff surface traction $\boldsymbol{P}(-\boldsymbol{N}) = -\boldsymbol{P}\boldsymbol{N}$, where the minus sign comes from the fact that we regard $\boldsymbol{N}$ as the *outward* normal to the boundary $\Sigma = \partial\mathcal{D}$ of $\mathcal{D}$, which is *inward* with respect to the remainder $\mathcal{B} \setminus \mathcal{D}$ of the body. The integral of this work per unit referential area over the surface $\Sigma = \partial\mathcal{D}$ gives what Cermelli et al. (2001) called the "*net work*"

$$\begin{aligned} \mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{Y}) &= \int_{\partial\mathcal{D}} (-\boldsymbol{P}\boldsymbol{N})\,(-h\,\boldsymbol{\eta}) + o(h) \\ &= -h \int_{\partial\mathcal{D}} (\boldsymbol{P}\boldsymbol{N})\,(\boldsymbol{F}\boldsymbol{U}) + o(h) = -h \int_{\partial\mathcal{D}} [(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}]\,\boldsymbol{N} + o(h), \end{aligned} \tag{42}$$

where we rewrote the covector-vector contraction $(\boldsymbol{F}\boldsymbol{U})\,(\boldsymbol{P}\boldsymbol{N})$ by using the definition of transpose, i.e.,

$$\begin{aligned} (\boldsymbol{F}\boldsymbol{U})\,(\boldsymbol{P}\boldsymbol{N}) &= F^a{}_A\,U^A\,P_a{}^B\,N_B = (\boldsymbol{P}^T)^B{}_a\,F^a{}_A\,U^A\,N_B = [(\boldsymbol{F}^T\,\boldsymbol{P})^T]^B{}_A\,U^A\,N_B \\ &= [(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}]\,\boldsymbol{N}. \end{aligned} \tag{43}$$

Note that, for the sake of a lighter notation, we are writing $\boldsymbol{F}^T$ and $\boldsymbol{P}^T$ for $\boldsymbol{F}^T \circ \phi$ and $\boldsymbol{P}^T \circ \phi$. Rigorously speaking, the composition by $\phi$ would be necessary, since $\boldsymbol{F}^T$ and $\boldsymbol{P}^T$ are defined in the current configuration $\phi(\mathcal{B})$ (Marsden and Hughes, 1983). Since $\boldsymbol{N}$ is the *outward* normal to $\Sigma = \partial\mathcal{D}$, the net work (42) is the *negative* of the work that the Piola tractions $\boldsymbol{P}\boldsymbol{N}$ would exert over the displacement $-h\,\boldsymbol{\eta}$ of Equation (41) on the referential surface $\Sigma = \partial\mathcal{D}$, seen as the boundary of the referential region $\mathcal{D}$. This observation allows us to apply the divergence theorem to (42), which yields

$$\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{Y}) = -h \int_{\mathcal{D}} \mathrm{Div}\,[(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}] + o(h). \tag{44}$$

This can be made into an increment by expressing the map $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$, and considering that $\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{X}) = 0$, i.e.,

$$\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{X}) = -h \int_{\mathcal{D}} \mathrm{Div}\,[(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}] + o(h). \tag{45}$$

Now, dividing by $h$ and passing to the limit $h \to 0$, we obtain the functional directional derivative

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}})(\mathcal{X}) = \lim_{h\to 0} \frac{\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{X})}{h} = -\int_{\mathcal{D}} \mathrm{Div}\,[(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}]. \tag{46}$$

(iv) The deformed transformed region $\phi(\tilde{\mathcal{D}}) = \phi(\mathcal{Y}(\mathcal{D}))$ from the replica body can finally be exactly suited into the cavity left by the removal of $\mathcal{D}$ in the original body and we are able to weld together across the interface. We note that Eshelby (1975) needs to make considerations on the infinitesimals of order greater than $h$. In our approach, these are automatically taken care of (and eliminated) by the limit operation in Equation (46). To cite Eshelby (1975) verbatim, except for using our notation for the displacement, "We are now left with the system as it was to begin with, except that the defect has been shifted by $-h\,\boldsymbol{U} = h\,\boldsymbol{U}_0$, as required."

The associated variation in the total energy $\mathcal{E}_{\mathcal{D}} : \mathcal{M} \to \mathbb{R}$ of the system is obtained as $\mathcal{E}_{\mathcal{D}} = \mathcal{E}^{\mathrm{el}}_{\mathcal{D}} + \mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}$, i.e., by summing Equations (35) and (46), i.e.,

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big[W\,\boldsymbol{I}\,\boldsymbol{U}\big] - \int_{\mathcal{D}} \mathrm{Div}\,[(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}], \tag{47}$$

which can be written as

$$(\partial_{\boldsymbol{U}}\,\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big[\boldsymbol{\mathfrak{E}}^T\boldsymbol{U}\big] = \int_{\partial\mathcal{D}} (\boldsymbol{\mathfrak{E}}\boldsymbol{N})\,\boldsymbol{U}. \tag{48}$$

Equation (48) quantifies the variation in energy necessary to obtain a new reference configuration in which the defect is displaced in direction $\boldsymbol{U}$ with respect to the original one. In the context of the theory of defects, Eshelby (1951) called the tensor $\boldsymbol{\mathfrak{E}}$, with the expression

$$\boldsymbol{\mathfrak{E}} = W\,\boldsymbol{I}^T - \boldsymbol{F}^T\boldsymbol{P}, \qquad \mathfrak{E}_A{}^B = W\,\delta_A{}^B - F^a{}_A\,P_a{}^B, \tag{49}$$

the *Maxwell tensor of elasticity* and later (Eshelby, 1975) the *energy-momentum tensor*, in analogy with Maxwell's terminology from field theory. This analogy will be completely clear in Section 4. Later, Maugin and Trimarco (1992) gave $\boldsymbol{\mathfrak{E}}$ the name of *Eshelby stress* in his honour.

At the end of Eshelby's thought experiment, we have the expression in Equation (48), which can be thought of as the *virtual work* exerted by the Eshelby tractions $\boldsymbol{\mathfrak{E}}\boldsymbol{N}$ on the material displacement field $\boldsymbol{U}$ on the boundary $\partial\mathcal{D}$ of the region $\mathcal{D}$. Using Eshelby's assumption $\boldsymbol{U}(X) = -\boldsymbol{U}_0$ for every $X \in \mathcal{D}$, we can write Equation (48) as

$$(\partial_{-\boldsymbol{U}_0}\,\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\int_{\mathcal{D}} (\mathrm{Div}\,\boldsymbol{\mathfrak{E}})\,\boldsymbol{U}_0 = -\int_{\partial\mathcal{D}} (\boldsymbol{\mathfrak{E}}\boldsymbol{N})\,\boldsymbol{U}_0. \tag{50}$$

In order to obtain (in our notation) Equation (17) in the paper by Eshelby (1951), we use Cartesian coordinates, so that it is legitimate to rewrite the integral as

$$\boldsymbol{\mathfrak{F}}\,\boldsymbol{U}_0 = (\partial_{-\boldsymbol{U}_0}\,\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\left(\int_{\mathcal{D}} \mathrm{Div}\,\boldsymbol{\mathfrak{E}}\right)\boldsymbol{U}_0 = -\left(\int_{\partial\mathcal{D}} \boldsymbol{\mathfrak{E}}\,\boldsymbol{N}\right)\boldsymbol{U}_0, \tag{51}$$

where $\boldsymbol{\mathfrak{F}}$ was *defined* by Eshelby as the *total* inhomogeneity force, producing work over the uniform virtual displacement $\boldsymbol{U}_0$. We remark that the total inhomogeneity force $\boldsymbol{\mathfrak{F}}$ can only be defined in the case of Cartesian coordinates, which is the only particular case in which integration of a vector field makes sense (see warning at page 134 in the text by Marsden and Hughes, 1983).

# 4 Eshelby's Variational Derivation of the Strong Form

In his seminal paper, Eshelby (1975) used a variational approach and wrote the Euler-Lagrange equations for a generic system with a potential energy depending – in the language of classical field theory – on fields, "gradients" of fields and coordinates. In this quite general framework, Elasticity can be seen as a particular case. Here, we follow Eshelby's derivation (Eshelby, 1975) step by step, using our notation and adding our comments. Then, we shall show how this specialises to the case of large- and small-deformation Elasticity. The only difference with Eshelby's procedure is that, whenever we look at the variational problem as an elasticity problem, our fields are the components of the configuration map, rather than the components of the displacement. Note that, in contrast with Section 3, here we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region $\mathcal{D}$ of the body as a functional on the manifold $\mathcal{C}$, the conventional configuration space.

Let us assume a potential energy density $W$, defined per unit referential volume, given by

$$W(X) = \hat{W}(\phi(X), \boldsymbol{F}(X), X), \tag{52}$$

where $\phi$ is a collection of scalar fields (in the case of continuum mechanics, the configuration map, with components $\phi^a$), $\boldsymbol{F}$ is the collection of the gradients of the fields (in our case, the deformation gradient, with components $F^a{}_A = \phi^a{}_{,A}$), and $X$ is the collection of the independent variables (in our case, the material coordinates $X^A$). Note that we *distinguish* between the *scalar field* $W$ (function of the coordinates $X^A$) and the *associated constitutive function* $\hat{W}$ (function of the fields $\phi^a$, the gradients $F^a{}_A = \phi^a{}_{,A}$ and the coordinates $X^A$). By using the material identity map $\mathcal{X}$ of Equation (17) (such that $X = \mathcal{X}(X)$, in components, $X^A = \mathcal{X}^A(X)$)), the potential energy can be rewritten in the form

$$W(X) = \hat{W}(\phi(X), \boldsymbol{F}(X), \mathcal{X}(X)) = [\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})](X). \tag{53}$$

Thus, by dropping the argument $X$ on the far left and the far right sides, we have

$$W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{54}$$

In order to find the Euler-Lagrange equations associated with $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, we need to consider the total energy $\mathcal{E}_{\mathcal{B}} : \mathcal{C} \to \mathbb{R}$ over the whole body $\mathcal{B}$, given by

$$\mathcal{E}_{\mathcal{B}}(\phi) = \int_{\mathcal{B}} W = \int_{\mathcal{B}} \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{55}$$

and calculate its variation with respect to a conventional displacement $\boldsymbol{\eta}$, which is given by the Gâteaux derivative

$$\begin{aligned}(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) &= \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{B}}(\phi + h\,\boldsymbol{\eta}) - \mathcal{E}_{\mathcal{B}}(\phi)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \int_{\mathcal{B}} \left[\hat{W} \circ (\phi + h\,\boldsymbol{\eta}, \boldsymbol{F} + h\,\mathrm{Grad}\,\boldsymbol{\eta}, \mathcal{X}) - \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right],\end{aligned} \tag{56}$$

with $\boldsymbol{\eta}$ chosen in a suitable subset of $T_\phi\mathcal{C} \cap C^1(\mathcal{B}, T\mathcal{S})$, as will be clarified later in this section. In the jargon of field theory, this is called a "variation on the fields, with frozen coordinates", i.e., we are going to calculate the integral on the *fixed* domain $\mathcal{B}$. The transformation on the configuration map $\phi$ (the "fields" $\phi^a$) is given by

$$\phi \mapsto \bar{\phi} = \phi + h\,\boldsymbol{\eta}, \tag{57a}$$
$$\phi^a \mapsto \bar{\phi}^a = \phi^a + h\,\eta^a, \tag{57b}$$

and the transformation on the tangent map $T\phi = \boldsymbol{F}$ (the "gradients" $F^a{}_A = \phi^a{}_{,A}$) is

$$T\phi = \boldsymbol{F} \mapsto T\bar{\phi} = \bar{\boldsymbol{F}} = T(\phi + h\,\boldsymbol{\eta}) = \boldsymbol{F} + h\,\mathrm{Grad}\,\boldsymbol{\eta}, \tag{58a}$$
$$\phi^a{}_{,A} = F^a{}_A \mapsto \bar{F}^a{}_A = \bar{\phi}^a{}_{,A} = \phi^a{}_{,A} + h\,\eta^a{}_{|A} = F^a{}_A + h\,\eta^a{}_{|A}, \tag{58b}$$

where $\mathrm{Grad}\,\boldsymbol{\eta}$, with components $\eta^a{}_{|A}$, is the covariant derivative of the displacement $\boldsymbol{\eta}$.

We follow the standard derivation by expanding the argument of the integral as

$$\begin{aligned}\hat{W} \circ (\phi + h\,\boldsymbol{\eta}, \boldsymbol{F} + h\,\mathrm{Grad}\,\boldsymbol{\eta}, \mathcal{X}) - \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \\ = \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; h\,\eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; h\,\eta^a{}_{|A} + o(h),\end{aligned} \tag{59}$$

substituting in (56) and performing the limit, which results in

$$\begin{aligned}(\partial_{\boldsymbol{\eta}}\mathcal{E}_\mathcal{B})(\phi) &= \int_\mathcal{B} \left[\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \eta^a{}_{|A}\right] \\ &= \int_\mathcal{B} \big[-f_a\,\eta^a + P_a{}^A\,\eta^a{}_{|A}\big] = \int_\mathcal{B} \big[-\boldsymbol{f}\,\boldsymbol{\eta} + \boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{\eta}\big],\end{aligned} \tag{60}$$

where $\boldsymbol{f}$ and $\boldsymbol{P}$ are given by

$$f_a = -\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \boldsymbol{f} = -\frac{\partial \hat{W}}{\partial \phi} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \tag{61a}$$
$$P_a{}^A = \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \boldsymbol{P} = \frac{\partial \hat{W}}{\partial \boldsymbol{F}} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{61b}$$

In the case of elasticity in continuum mechanics, when the potential is given as the sum of an elastic potential and a potential of the external body forces, i.e.,

$$\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \hat{W}_{\mathrm{el}} \circ (\boldsymbol{F}, \mathcal{X}) + \hat{W}_{\mathrm{ext}} \circ (\phi, \mathcal{X}), \tag{62}$$

the covector field $\boldsymbol{f}$ and the tensor field $\boldsymbol{P}$ take the meaning of external body force per unit volume and first Piola-Kirchhoff stress, respectively. Now, considering that

$$\boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{\eta} = P_a{}^A\,\eta^a{}_{|A} = (P_a{}^A\,\eta^a)_{|A} - P_a{}^A{}_{|A}\,\eta^a = \mathrm{Div}(\boldsymbol{P}^T\boldsymbol{\eta}) - (\mathrm{Div}\,\boldsymbol{P})\,\boldsymbol{\eta}, \tag{63}$$

the variation becomes

$$\begin{aligned}(\partial_{\boldsymbol{\eta}}\mathcal{E}_\mathcal{B})(\phi) &= \int_\mathcal{B} \big[-\boldsymbol{f}\,\boldsymbol{\eta} + \mathrm{Div}(\boldsymbol{\eta}\,\boldsymbol{P}) - (\mathrm{Div}\,\boldsymbol{P})\,\boldsymbol{\eta}\big] \\ &= -\int_\mathcal{B} (\boldsymbol{f} + \mathrm{Div}\,\boldsymbol{P})\,\boldsymbol{\eta} + \int_\mathcal{B} \mathrm{Div}(\boldsymbol{\eta}\,\boldsymbol{P})\end{aligned} \tag{64}$$

and, by applying Gauss' divergence theorem,

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = -\int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P})\, \boldsymbol{\eta} + \int_{\partial\mathcal{B}} (\boldsymbol{P}\boldsymbol{N})\, \boldsymbol{\eta}, \tag{65}$$

where $\boldsymbol{N}$ is the normal to the boundary $\partial\mathcal{B}$ and $(\boldsymbol{P}\boldsymbol{N})\,\boldsymbol{\eta} = \boldsymbol{\eta}\,(\boldsymbol{P}\boldsymbol{N})$.

We now look for a configuration $\phi$ at which $\mathcal{E}_{\mathcal{B}}(\phi)$ is stationary. For this purpose, we impose the condition $(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = 0$, in which $\phi$ is unknown, and we study it under the restriction that $\boldsymbol{\eta}$ vanish on $\partial\mathcal{B}$ (Hill, 1951). This choice annihilates the surface integral on the right-hand-side of (65), so that the stationarity condition becomes

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = -\int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P})\, \boldsymbol{\eta} = 0, \qquad \boldsymbol{\eta} \in \mathcal{V}, \tag{66}$$

where $\mathcal{V} := \{\boldsymbol{\eta} \in T_\phi\mathcal{C} \cap C^1(\mathcal{B}, T\mathcal{S}) :\ \boldsymbol{\eta}(X) = \boldsymbol{0},\ \forall\, X \in \partial\mathcal{B}\}$. We require now that (66) be satisfied for all $\boldsymbol{\eta} \in \mathcal{V}$, which leads to the Euler-Lagrange equations

$$\boldsymbol{f} + \operatorname{Div} \boldsymbol{P} = \boldsymbol{0}, \qquad f_a + P_a{}^A{}_{|A} = 0. \tag{67}$$

If the external body forces acting on $\mathcal{B}$ are only those given by $\boldsymbol{f}$, which admit the potential density $\hat{W}_{\text{ext}} \circ (\phi, \mathcal{X})$, Equation (67) represents, in continuum mechanics, the Lagrangian (static) equilibrium equations, i.e., spatial equations described in terms of the material coordinates. If $\phi$ is a solution to (67), and the boundary of $\mathcal{B}$ can be written as the disjoint union of a Dirichlet part and a Neumann part, i.e., $\partial\mathcal{B} = \partial_D\mathcal{B} \sqcup \partial_N\mathcal{B}$, then the variation $(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi)$ in Equation (65) becomes

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = \int_{\partial\mathcal{B}} (\boldsymbol{P}\,\boldsymbol{N})\, \boldsymbol{\eta} = \int_{\partial_N\mathcal{B}} (\boldsymbol{P}\,\boldsymbol{N})\, \boldsymbol{\eta}, \tag{68}$$

where the surface integral is restricted to the Neumann boundary, $\partial_N\mathcal{B}$, because the displacement $\boldsymbol{\eta}$, although being arbitrary, has to vanish on the Dirichlet boundary, $\partial_D\mathcal{B}$. In this case, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ requires the vanishing of the surface integral on the far right-hand-side of Equation (65). This can be obtained if $\partial_N\mathcal{B}$ is a set of null measure, or if no contact forces are applied onto $\partial_N\mathcal{B}$. On the contrary, when contact forces are present, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ must be corrected by requiring that $(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi)$ be balanced by the work performed by the contact forces on $\boldsymbol{\eta}$. This result follows from the extended Hamilton's Principle (dell'Isola and Placidi, 2011).

If Equation (65) is referred to a set $\mathcal{D} \subset \mathcal{B}$, and is evaluated for a configuration $\phi$ solving (67), the volume integral vanishes by virtue of the Euler-Lagrange equations, while internal contact forces are exchanged through $\partial\mathcal{D}$. In this case, $\boldsymbol{\eta}$ is not required to vanish on $\partial\mathcal{D}$, and the variational procedure leads to

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{D}})(\phi) = \int_{\partial\mathcal{D}} (\boldsymbol{P}\boldsymbol{N})\, \boldsymbol{\eta}, \tag{69}$$

thereby returning the virtual work exerted by the contact forces acting on $\partial\mathcal{D}$.

Let us now assume that $\phi$ satisfies the Euler-Lagrange equations (67), and let us take the material gradient $\mathrm{Grad}\, W$ of the energy density $W$, i.e., the partial derivatives of $W$ with respect to $X^B$,

$$\begin{aligned} W_{,B} &= \left[\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right]_{,B} \\ &= \left[\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right] \phi^a{}_{,B} + \left[\frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right] F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^B} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \\ &= -f_a\, F^a{}_B + P_a{}^A\, F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^B} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \end{aligned} \tag{70}$$

where we used the definitions of the components of the deformation gradient, $F^a{}_A = \phi^a{}_{,A}$, of the body force and the first Piola-Kirchhoff stress, and $F^a{}_{A|B}$ are the components of the third-order two-point tensor $\mathrm{Grad}\,\boldsymbol{F}$. The last term in Equation (70) is usually called "explicit" gradient of the field $W$ and denoted $(\partial W/\partial X^B)|_{\mathrm{expl}}$ in the literature (e.g., Eshelby, 1975; Epstein and Maugin, 1990), whereas we regard it as the collection of the partial derivatives of the constitutive function $\hat{W}$ with respect to $\mathcal{X}^B$ (which, we recall, are the functions such that $\mathcal{X}^B(X) = X^B$). The negative of the "explicit" gradient defines the *material inhomogeneity force* or *configurational force*

$$\mathfrak{F} = -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \mathfrak{F}_A = -\frac{\partial \hat{W}}{\partial \mathcal{X}^A} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{71}$$

Substituting the expressions of the Lagrangian force $\boldsymbol{f}$, the Piola-Kirchhoff stress $\boldsymbol{P}$, and the configurational force $\mathfrak{F}$ into Equation (70), we obtain

$$\mathrm{Grad}\, W = -\boldsymbol{F}^T \boldsymbol{f} + \boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{F} - \mathfrak{F}, \tag{72}$$

where the double contraction ":" in the second term is of the two legs of $\boldsymbol{P}$ with the first two legs of $\mathrm{Grad}\,\boldsymbol{F}$. By invoking the symmetry of the Christoffel symbols $\Gamma^A_{BC}$ associated with the Levi-Civita Connection induced by the material metric $\boldsymbol{G}$, so that $F^a{}_{A|B} = F^a{}_{B|A}$, we work out the second term on the right-hand-side of (72) in components, i.e.,

$$P_a{}^A\, F^a{}_{A|B} = P_a{}^A\, F^a{}_{B|A} = (P_a{}^A\, F^a{}_B)_{|A} - P_a{}^A{}_{|A}\, F^a{}_B, \tag{73}$$

which, in component-free notation, reads

$$\boldsymbol{P} : \mathrm{Grad}\boldsymbol{F} = \mathrm{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \boldsymbol{F}^T \mathrm{Div}\boldsymbol{P}. \tag{74}$$

By substituting this result into (72), we obtain

$$\begin{aligned} \mathrm{Grad}\, W &= -\boldsymbol{F}^T \boldsymbol{f} + \mathrm{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \boldsymbol{F}^T \mathrm{Div}\boldsymbol{P} - \mathfrak{F} \\ &= -\boldsymbol{F}^T [\boldsymbol{f} + \mathrm{Div}\boldsymbol{P}] + \mathrm{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \mathfrak{F}. \end{aligned} \tag{75}$$

Moreover, using the Euler-Lagrange equation (67) yields

$$\mathrm{Grad}\, W = \mathrm{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \mathfrak{F}. \tag{76}$$

Finally, by virtue of the identity $\operatorname{Grad} W = \operatorname{Div}\left(W\boldsymbol{I}^T\right)$, where $\boldsymbol{I}$ is the material identity tensor, Equation (76) becomes

$$\mathfrak{F} + \operatorname{Div}\mathfrak{E} = \mathbf{0}, \qquad \mathfrak{F}_A + \mathfrak{E}_A{}^B{}_{|B} = 0, \tag{77}$$

where $\mathfrak{E}$ is the Eshelby stress defined as in Equation (49).

Similarly to other field theories, like Electromagnetism or General Relativity, the tensor $\mathfrak{E}$ defined in (49) plays the role of the ("spatial" part of the) energy-momentum tensor of the theory under study. However, we emphasise that, while $\mathfrak{E}$ has been obtained with the aid of a variational argument in the present framework, more general approaches exist, in which $\mathfrak{E}$ is introduced as a primary dynamical quantity (Gurtin, 1995). Equation (77) is called *material equilibrium equation* or *configurational equilibrium equation* (Gurtin, 1995), by analogy with the equilibrium equation (67) described by the Euler-Lagrange equations.

According to Equation (71), if the body $\mathcal{B}$ is homogeneous, then we have

$$\mathfrak{F}_A(X) = -\left[\frac{\partial \hat{W}}{\partial \mathcal{X}^A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right](X) = 0, \qquad \forall X \in \mathcal{B}, \tag{78}$$

and Equation (77) implies the vanishing of the divergence of the Eshelby stress. On the contrary, if there is *any* inhomogeneity in $\mathcal{D}$ (i.e., the derivative $\partial\hat{W}/\partial\mathcal{X}^A$ is non-vanishing), this will be captured by the integral of the traction forces $\mathfrak{E}\,\boldsymbol{N}$ of the Eshelby stress over the boundary $\partial\mathcal{D}$.

We now show that Equation (48) yields the *weak formulation* of the *strong form* described in Equation (77). This is easy to see by referring to Equation (51), which we obtained from Equation (48) (or Equation (50)) by working in Cartesian coordinates and using Eshelby's displacement $\boldsymbol{U} = -\boldsymbol{U}_0$, constant over $\mathcal{D}$. Indeed, by solving the material equilibrium equation (77) for $\mathfrak{F}$, using Cartesian coordinates, integrating over $\mathcal{D}$, applying Gauss' theorem and contracting both sides with $\boldsymbol{U}_0$, we obtain the *total configurational force* on the region $\mathcal{D}$ as the covector $\mathscr{F}$ such that

$$\mathscr{F}\,\boldsymbol{U}_0 = -\left(\int_{\mathcal{D}} \operatorname{Div}\mathfrak{E}\right)\boldsymbol{U}_0 = -\left(\int_{\partial\mathcal{D}} \mathfrak{E}\,\boldsymbol{N}\right)\boldsymbol{U}_0 = \left(\int_{\mathcal{D}} \mathfrak{F}\right)\boldsymbol{U}_0, \tag{79}$$

i.e., $\mathscr{F}$ is the integral of the inhomogeneity force density $\mathfrak{F}$, as we see by comparing with Equation (51). Note that, if the body $\mathcal{D}$ is homogeneous, Equations (77) and (78) imply the vanishing of the divergence of the Eshelby stress, and therefore the vanishing of the volume integral and the equivalent surface integral on the right-hand-side of Equation (79).

# 5 Derivation of the Weak Form with Noether Theorem

In Noether's Theorem, we need to contemporarily transform the domain and perform a variation on the arguments of the Lagrangian. In the jargon of classical field theory, these are called a *transformation of the coordinates* (material coordinates, in our case) and a *variation of the fields*, respectively. Together, these give the *total variation*. We have already shown the transformation of the material coordinates in Section 2.3 and the variation on the fields in Section 4 and we turn now to the total variation. Then, we apply Noether's theorem to *directly* obtain Eshelby's results. In the application of Noether's Theorem, we define the total energy $\mathcal{E}_{\mathcal{D}}$ of a region $\mathcal{D}$ as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$.

## 5.1 Total Variation

In the language of field theory, the *total variation* is obtained by evaluating the *variation of the fields at frozen coordinates* given in (57) and (58) at the transformed points $\tilde{X} = \mathcal{Y}(X)$, where $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U} : \mathcal{B} \to \tilde{\mathcal{B}}$ is the *infinitesimal transformation of the coordinates* defined in (21), with $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$. In order to avoid confusion, some care must be exercised.

We recall that the manifold $\mathcal{C}$ is the configuration space of the body $\mathcal{B}$, a configuration $\phi$ is an element of $\mathcal{C}$ and a displacement field $\boldsymbol{\eta}$ is a tangent vector of $T_\phi\mathcal{C}$. Let us denote by $\tilde{\mathcal{C}}$ the configuration space of the "perturbed" body $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B})$, to which the points $\tilde{X} = \mathcal{Y}(X)$ belong. Consider the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ and the *restriction* of the configuration $\phi$ and the displacement field $\boldsymbol{\eta}$ defined in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ (see Figure 5). In this restriction, it is legitimate to evaluate $\phi$ and $\boldsymbol{\eta}$ at $\tilde{X}$.

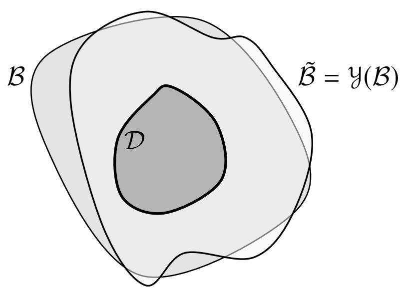


**Figure 5:** A domain $\mathcal{D}$ (dark grey) in the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ between the body $\mathcal{B}$ (solid grey) and the perturbed body $\tilde{\mathcal{B}}$ (transparent grey).

We now define the total variation $\mathcal{C} \to \tilde{\mathcal{C}} : \phi \mapsto \bar{\phi}$ by evaluating the *variations of the fields at frozen coordinates* of Equations (57) and (58) at $\tilde{X} \in \tilde{\mathcal{B}} \cap \mathcal{B}$, i.e., we define

$$\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X}), \qquad \bar{\phi}^a(\tilde{X}) = \phi^a(\tilde{X}) + h\,\eta^a(\tilde{X}), \tag{80}$$

$$\bar{\boldsymbol{F}}(\tilde{X}) = \boldsymbol{F}(\tilde{X}) + h\,(\mathrm{Grad}\,\boldsymbol{\eta})(\tilde{X}), \qquad \bar{F}^a{}_A(\tilde{X}) = F^a{}_A(\tilde{X}) + h\,\eta^a{}_{|A}(\tilde{X}), \tag{81}$$

where $h$ is, with no loss of generality, the same smallness parameter as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$. To obtain the final form of the total variation, we substitute the transformation (21) of the coordinates into the variations on the configuration (80) and on the tangent (81) of the configuration, respectively, and use Taylor expansion. For the configuration, we have

$$
\begin{aligned}
\bar{\phi}(\tilde{X}) &= \phi(X + h\,\boldsymbol{U}(X)) + h\,\boldsymbol{\eta}(X + h\,\boldsymbol{U}(X)) \\
&= \phi(X) + h\,\boldsymbol{F}(X)\,\boldsymbol{U}(X) + h\,\boldsymbol{\eta}(X) + o(h), \qquad (82a)\\
\bar{\phi}^a(\tilde{X}) &= \phi^a(X + h\,\boldsymbol{U}(X)) + h\,\eta^a(X + h\,\boldsymbol{U}(X)) \\
&= \phi^a(X) + h\,F^a{}_B(X)\,U^B(X) + h\,\eta^a(X) + o(h), \qquad (82b)
\end{aligned}
$$

from which, using $\bar{\phi}(\tilde{X}) = \bar{\phi}(\mathcal{Y}(X)) = (\bar{\phi} \circ \mathcal{Y})(X)$ and omitting the argument $X$, we have

$$
\bar{\phi} \circ \mathcal{Y} = \phi + h\,(\boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U}) + o(h) = \phi + h\,\boldsymbol{w} + o(h), \qquad (83a)
$$

$$
\bar{\phi}^a \circ \mathcal{Y} = \phi^a + h\,(\eta^a + F^a{}_B\,U^B) + o(h) = \phi^a + h\,w^a + o(h), \qquad (83b)
$$

where

$$
\boldsymbol{w} = \boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U}, \qquad w^a = \eta^a + F^a{}_B\,U^B. \qquad (84)
$$

For the tangent map, we have

$$
\begin{aligned}
\bar{\boldsymbol{F}}(\tilde{X}) &= \boldsymbol{F}(X + h\,\boldsymbol{U}(X)) + h\,(\mathrm{Grad}\,\boldsymbol{\eta})(X + h\,\boldsymbol{U}(X)) \\
&= \boldsymbol{F}(X) + h\,(\mathrm{Grad}\,\boldsymbol{F})(X)\,\boldsymbol{U}(X) + h\,(\mathrm{Grad}\,\boldsymbol{\eta})(X) + o(h), \qquad (85a)\\
\bar{F}^a{}_A(\tilde{X}) &= F^a{}_A(X + h\,\boldsymbol{U}(X)) + h\,\eta^a{}_{|A}(X + h\,\boldsymbol{U}(X)) \\
&= F^a{}_A(X) + h\,F^a{}_{A|B}(X)\,U^B(X) + h\,\eta^a{}_{|A}(X) + o(h), \qquad (85b)
\end{aligned}
$$

and thus,

$$
\bar{\boldsymbol{F}} \circ \mathcal{Y} = \boldsymbol{F} + h\,(\mathrm{Grad}\,\boldsymbol{\eta} + (\mathrm{Grad}\,\boldsymbol{F})\,\boldsymbol{U}) + o(h) = \boldsymbol{F} + h\,\boldsymbol{Y} + o(h), \qquad (86a)
$$

$$
\bar{F}^a{}_A \circ \mathcal{Y} = F^a{}_A + h\,(\eta^a{}_{|A} + F^a{}_{A|B}\,U^B) + o(h) = F^a{}_A + h\,Y^a{}_A + o(h), \qquad (86b)
$$

where

$$
\boldsymbol{Y} = \mathrm{Grad}\,\boldsymbol{\eta} + (\mathrm{Grad}\,\boldsymbol{F})\,\boldsymbol{U}, \qquad Y^a{}_A = \eta^a{}_{|A} + F^a{}_{A|B}\,U^B. \qquad (87)
$$

## 5.2 Variation of the Total Energy

Since we are working in the static case, we replace the action functional and the Lagrangian density with the total energy functional $\mathcal{E}$ and the potential energy density $W$. The total energy in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ is a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$, i.e.,

$$
\mathcal{E}_{\mathcal{D}} : \mathcal{C} \times \mathcal{M} \to \mathbb{R} : (\phi, \mathcal{Y}) \mapsto \mathcal{E}_{\mathcal{D}}(\phi, \mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad (88)
$$

where the integration domain $\mathcal{Y}(\mathcal{D})$ must belong to the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$. We now consider the coordinate transformation $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$, where $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$ is a tangent vector at the identity $\mathcal{X}$, and the field transformation is $\bar{\phi} = \phi + h\,\boldsymbol{\eta}$, where $\boldsymbol{\eta} \in T_{\phi}\mathcal{C}$ is a tangent vector at the configuration $\phi$. The variation of the energy is given by the directional derivative

$$
\begin{aligned}
(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) &= \lim_{h\to 0}\frac{\mathcal{E}_{\mathcal{D}}(\bar{\phi},\mathcal{Y})-\mathcal{E}_{\mathcal{D}}(\phi,\mathcal{X})}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_{\mathcal{Y}(\mathcal{D})}\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})-\int_{\mathcal{D}}\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\right],
\end{aligned}
\tag{89}
$$

evaluated at the conventional configuration $\phi$ and Eshelbian configuration $\mathcal{X}$, with respect to the pair of tangent vectors $(\boldsymbol{\eta},\boldsymbol{U})\in T_{(\phi,\mathcal{X})}(\mathcal{C}\times\mathcal{M})$ in the product manifold $\mathcal{C}\times\mathcal{M}$. Note also that, in the second integral, we used $\mathcal{X}(\mathcal{D})=\mathcal{D}$.

Application of the theorem of the change of variables on the first integral in (89) yields

$$
\int_{\mathcal{Y}(\mathcal{D})}\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X}) = \left[\int_{\mathcal{D}}(1+h\,\mathrm{Div}\,\boldsymbol{U})\,\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y}\right]+o(h),
\tag{90}
$$

where the determinant $\det(T\mathcal{Y})=1+h\,\mathrm{Div}\,\boldsymbol{U}+o(h)$ follows from Equation (24). We now notice that

$$
\begin{aligned}
\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y} &= \hat{W}\circ(\bar{\phi}\circ\mathcal{Y},\bar{\boldsymbol{F}}\circ\mathcal{Y},\mathcal{X}\circ\mathcal{Y})\\
&= \hat{W}\circ(\phi+h\,\boldsymbol{w}+o(h),\boldsymbol{F}+h\,\boldsymbol{Y}+o(h),\mathcal{X}+h\,\boldsymbol{U}),
\end{aligned}
\tag{91}
$$

where we made use of the total variations (83) and (86), as well as of the identity $\mathcal{X}\circ\mathcal{Y}=\mathcal{Y}=\mathcal{X}+h\,\boldsymbol{U}$. Now, we expand in Taylor series up to the first order, and obtain

$$
\begin{aligned}
\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y} = \hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X}) &+ \frac{\partial\hat{W}}{\partial\phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,w^a\\
&+\frac{\partial\hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,Y^a{}_A\\
&+\frac{\partial\hat{W}}{\partial\mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,U^B+o(h).
\end{aligned}
\tag{92}
$$

Using Equations (90), (91) and (92) in the variation of the energy (89), we have

$$
\begin{aligned}
(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) &= \lim_{h\to 0}\frac{1}{h}\left[\int_{\mathcal{D}}h\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\;U^B{}_{|B}+\right.\right.\\
&\left.\left.+\frac{\partial\hat{W}}{\partial\phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;w^a+\frac{\partial\hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;Y^a{}_A+\frac{\partial\hat{W}}{\partial\mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\;U^B\right)+o(h)\right].
\end{aligned}
\tag{93}
$$

The smallness parameter cancels out and the term $o(h)$ disappears in the limit $h\to 0$. Thus, we write

$$
\begin{aligned}
(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}}&\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\;U^B{}_{|B}+\right.\\
&\left.+\frac{\partial\hat{W}}{\partial\phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;w^a+\frac{\partial\hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;Y^a{}_A+\frac{\partial\hat{W}}{\partial\mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\;U^B\right),
\end{aligned}
\tag{94}
$$

and we use the explicit expressions (84) and (87) of the total variations $\boldsymbol{w}$ and $\boldsymbol{Y}$:

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \Bigg( \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; U^B{}_{|B} + \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; (\eta^a + F^a{}_B\, U^B) + \\ + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; (\eta^a{}_{|A} + F^a{}_{A|B}\, U^B) + \frac{\partial \hat{W}}{\partial \mathcal{X}^B} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; U^B \Bigg). \quad (95)$$

Since

$$(\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}))_{,B} = \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; F^a{}_B + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \delta^A{}_B, \quad (96)$$

we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \Bigg( \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; U^B{}_{|B} + (\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}))_{,B}\, U^B + \\ + \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \eta^a{}_{|A} \Bigg). \quad (97)$$

Using Leibniz' rule in the first two terms and in the last two terms and separating the integrals, we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \left[ \left( \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; U^B \right)_{|B} + \left( \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\; \eta^a \right)_{|A} \right] \\ + \int_{\mathcal{D}} \left[ \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \left( \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \right)_{|A} \right] \eta^a. \quad (98)$$

Now we use the definitions (61), which, in the context of continuum mechanics, give the body force $\boldsymbol{f}$ and the first Piola-Kirchhoff stress $\boldsymbol{P}$, use $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and change index $A$ into $B$ in the first integral. So, we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \left[ \left( W\, U^B \right)_{|B} + \left( \eta^a\, P_a{}^B \right)_{|B} \right] - \int_{\mathcal{D}} \left( f_a + P_a{}^A{}_{|A} \right) \eta^a, \quad (99)$$

which corresponds to Equation (17) in the paper by Hill (1951). In the first integral, we use $U^B = U^A\; \delta_A{}^B$ in the first term and the definition (83) of the total variation $\boldsymbol{w}$ to eliminate $\eta^a = w^a - F^a{}_A\, U^A$ in the second term, and then we split the first integral into two, to obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \left[ U^A \left( W\, \delta_A{}^B - F^a{}_A\, P_a{}^B \right) \right]_{|B} + \\ + \int_{\mathcal{D}} \left( w^a\, P_a{}^B \right)_{|B} - \int_{\mathcal{D}} \left( f_a + P_a{}^A{}_{|A} \right) \eta^a, \quad (100)$$

where we recognise the Eshelby stress $\mathfrak{E}_A{}^B = W\, \delta_A{}^B - F^a{}_A\, P_a{}^B$ defined in Equation (49). Finally, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \left( U^A\; \mathfrak{E}_A{}^B \right)_{|B} + \int_{\mathcal{D}} \left( w^a\, P_a{}^B \right)_{|B} - \int_{\mathcal{D}} \left( f_a + P_a{}^A{}_{|A} \right) \eta^a, \quad (101)$$

which, in component-free formalism, reads

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \operatorname{Div}\big(\boldsymbol{\mathfrak{C}}^T\boldsymbol{U}\big) + \int_{\mathcal{D}} \operatorname{Div}\big(\boldsymbol{P}^T\boldsymbol{w}\big) - \int_{\mathcal{D}} (\boldsymbol{f} + \operatorname{Div}\boldsymbol{P})\,\boldsymbol{\eta}. \tag{102}$$

If the variation (102) is evaluated for a configuration $\phi$ solving the Euler-Lagrange equations (67), we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \operatorname{Div}\big(\boldsymbol{\mathfrak{C}}^T\boldsymbol{U}\big) + \int_{\mathcal{D}} \operatorname{Div}\big(\boldsymbol{P}^T\boldsymbol{w}\big), \tag{103}$$

where the first two integrals contain the contributions to the *Noether current density* $\boldsymbol{\mathfrak{C}}^T\boldsymbol{U} + \boldsymbol{P}^T\boldsymbol{w}$. The extension of the result (103) to the case of the presence of non-integrable body forces $\boldsymbol{f}$ is treated in Appendix A.

## 5.3 Eshelby's Results and Conservation of Noether's Current

The variational procedure followed in Section 5.2 was conducted by introducing the one-parameter families of transformations $\mathcal{Y}(X) = X + h\,\boldsymbol{U} = \tilde{X}$ and $\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X})$, which allowed to compute the Gâteaux derivative of total energy $\mathcal{E}_{\mathcal{D}}$ along the pair of directions $(\boldsymbol{\eta},\boldsymbol{U})$. Transformations of this kind are said to be *symmetries* if they do not alter the numerical value of $\mathcal{E}_{\mathcal{D}}$, i.e., if it holds true that $\mathcal{E}_{\mathcal{D}}(\bar{\phi},\mathcal{Y}) = \mathcal{E}_{\mathcal{D}}(\phi,\mathcal{X})$ for sufficiently small values of $h$. Following an argument reported by Hill (1951), a condition ensuring the compliance with this equality and the form-invariance of the Euler-Lagrange equations is obtained by means of what in field theory is called a *divergence transformation* (Hill, 1951; Maugin, 1993). For the case of an infinitesimal symmetry transformation, the divergence transformation reads

$$\int_{\mathcal{D}} (1 + h\operatorname{Div}\boldsymbol{U})\,\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y} = \int_{\mathcal{D}} \big[\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X}) + h\operatorname{Div}\boldsymbol{\Omega}\big], \tag{104}$$

where $\boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}}\circ\mathcal{X}$ is a vector field to be determined. Note that, in order to leave the Euler-Lagrange equations (67) invariant, $\hat{\boldsymbol{\Omega}}$ must *not* depend on $\boldsymbol{F}$ (Hill, 1951). By dividing Equation (104) by $h$ and taking the limit for $h \to 0$, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) - \int_{\mathcal{D}} \operatorname{Div}\boldsymbol{\Omega} = \int_{\mathcal{D}} \big[\operatorname{Div}\big(\boldsymbol{\mathfrak{C}}^T\boldsymbol{U}\big) + \operatorname{Div}\big(\boldsymbol{P}^T\boldsymbol{w}\big) - \operatorname{Div}\boldsymbol{\Omega}\big] = 0. \tag{105}$$

According to this result, to a given pair $\boldsymbol{U}$ and $\boldsymbol{w}$ there corresponds the conservation law

$$\operatorname{Div}\big(\boldsymbol{\mathfrak{C}}^T\boldsymbol{U}\big) + \operatorname{Div}\big(\boldsymbol{P}^T\boldsymbol{w}\big) - \operatorname{Div}\boldsymbol{\Omega} = 0, \tag{106}$$

which allows to determine $\boldsymbol{\Omega}$. In several circumstances of interest, such as the one related to the conservation of momentum or angular momentum, one can take $\boldsymbol{\Omega}$ to be zero from the outset and look for transformations $\boldsymbol{U}$ and $\boldsymbol{w}$ leading to conservation laws of the form

$$\operatorname{Div}\big(\boldsymbol{\mathfrak{C}}^T\boldsymbol{U}\big) + \operatorname{Div}\big(\boldsymbol{P}^T\boldsymbol{w}\big) = 0. \tag{107}$$

In the remainder of our work, we specialise to this case in order to retrieve Eshelby's result in the light of Noether's theorem. Some remarks on divergence transformations are reported in Appendix B.

Eshelby (1975) imposed $\boldsymbol{\eta} = -\boldsymbol{F}\,\boldsymbol{U}$, i.e., that the conventional displacement $\boldsymbol{\eta}$ be equal to the negative of the push-forward of the material displacement $\boldsymbol{U}$, as shown in Equation (41), in order to preserve compatibility. This condition, in turn, imposes the vanishing of the total variation, i.e., $\boldsymbol{w} = \boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U} = \boldsymbol{0}$. With this hypothesis, the integral of $\mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big)$ in Equation (103) vanishes identically and the variation reduces to

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big). \tag{108}$$

which coincides with the result shown in Equation (48).

Now we can exploit Noether's theorem to obtain Eshelby's final result. Noether's Theorem states that

> *For every continuous symmetry under which the integral $\mathcal{E}_{\mathcal{D}}$ is invariant, there is a conserved current density.*

In this case, the Noether current density is $\mathfrak{E}^T \boldsymbol{U}$. For it to be conserved, the divergence $\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big)$ has to vanish and, in fact, a direct computation, in which the configurational force balance (77) is used, yields the condition

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} + (\mathrm{Div}\mathfrak{E})\,\boldsymbol{U} = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} - \mathfrak{F}\,\boldsymbol{U} = 0. \tag{109}$$

Equation (109) is known as *Noetherian identity* (Podio-Guidugli, 2001), and places restrictions on the class of transformations $\boldsymbol{U}$ that comply with the requirement $\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = 0$, which can thus be said to be *symmetry transformations*. Indeed, a field $\boldsymbol{U}$ is a symmetry transformation (i.e., it leaves $\mathcal{E}_{\mathcal{D}}$ invariant) if, and only if, it satisfies (109) (for a similar result in a different context, see also Grillo et al., 2003, 2019). Looking at (109), we notice that, when the inhomogeneity force, $\mathfrak{F}$, vanishes identically i.e., when the body is *materially homogeneous* and, thus, the energy density $\hat{W}$ *does not* depend on the material points, the Noetherian identity reduces to

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} = 0. \tag{110}$$

This result implies that *any* arbitrary uniform displacement field $\boldsymbol{U}$, for which $\mathrm{Grad}\,\boldsymbol{U} = \boldsymbol{0}$, annihilates the divergence of the Noether current density and is, thus, a symmetry transformation. A body endowed with this property is said to enjoy the symmetry of *material homogeneity*. We notice, however, that, when $\mathfrak{F}$ is not null, $\boldsymbol{U}$ may no longer be uniform. This means that $\mathfrak{F}$ *breaks* the symmetry of material homogeneity and a new class of transformations $\boldsymbol{U}$ has to be determined.

We also note that, under the hypothesis of homogeneous material, Equation (108) implies the vanishing of the divergence of $\mathfrak{E}^T \boldsymbol{U}$, and not of $\mathfrak{E}$. In order to obtain the vanishing of the divergence of the Eshelby stress $\mathfrak{E}$, we implement the last of Eshelby's hypotheses, namely the fact that the material displacement $\boldsymbol{U}$ is uniform on $\mathcal{D}$ and given by $\boldsymbol{U}(X) = -\boldsymbol{U}_0$, for every $X \in \mathcal{D}$. This implies that in the integral of $\mathrm{Div}(\mathfrak{E}^T \boldsymbol{U})$ in Equation (111), the displacement $\boldsymbol{U} = -\boldsymbol{U}_0$ can be brought out of the divergence, i.e.,

$$(\partial_{(\boldsymbol{\eta},-\boldsymbol{U}_0)}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = -\int_{\mathcal{D}} (\mathrm{Div}\,\mathfrak{E})\,\boldsymbol{U}_0. \tag{111}$$

which coincides with Equation (50) obtained using Eshelby's original procedure. Now, the vanishing of the variation due to the homogeneity of the material implies the vanishing of $\mathrm{Div}\,\mathfrak{E}$, as in the strong form (77) considered with condition (78).

# 6 Summary

In this work we systematically reviewed the two procedures proposed by Eshelby to study the effect of inhomogeneity in an elastic body, in the differential geometric picture of continuum mechanics. The first procedure (Eshelby, 1951) involves the classical cutting-replacing-welding operations and is mathematically represented by defining the energy as a functional on the manifold $\mathcal{M}$ of the *Eshelbian configurations* $\mathcal{Y}$ (which transform the domain $\mathcal{D}$ containing the inclusion/defect), and performing a variation *on the coordinates*, i.e., a variational derivative made with respect to a *material* displacement field $\boldsymbol{U}$, seen as a variation of the identity Eshelbian configuration $\mathcal{X}$. The second procedure (Eshelby, 1975) follows Hamilton's principle of stationary action. Accordingly, the energy is defined as a functional on the manifold $\mathcal{C}$ of the *conventional configurations* $\phi$, and a variation is performed *on the fields*, i.e., a variational derivative is calculated with respect to a *spatial* displacement, seen as a variation of the configuration map $\phi$.

The natural manner to unify the two procedures is the use of *Noether's Theorem*, in which a variation on both fields and coordinates (total variation) is used. Indeed, to obtain this result, we defined the energy as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$ of the conventional configurations $\phi$ and the Eshelbian configurations $\mathcal{Y}$, and performed a variational derivative with respect to the pair $(\boldsymbol{\eta}, \boldsymbol{U})$, which is a variation with respect to the pair $(\phi, \mathcal{X})$. While certainly no additional proof was needed to demonstrate the beauty and generality of Noether's Theorem, we find that it is insightful to look at Eshelby's theory of defects from the point of view of Noether's conservation laws.

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# Appendix

# A Monogenic and Polygenic Forces

The variational setting adopted in our work serves as a basis for the employment of Noether's Theorem (see Section 5), which, for first order theories, is generally enunciated for a Lagrangian density function depending on "fields and gradients of the fields". Hence, the expression of the energy density used so far, i.e., $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, is meant to replicate, up to the sign, the standard functional dependence of a generic Lagrangian density function, for which Noether's Theorem is formulated. In principle, however, *neither* the introduction of the Eshelby stress tensor *nor* that of the configurational force density require any variational framework. Indeed, as clearly shown by Gurtin (1995), the existence of these quantities stands on its own, and it necessitates neither the hypothesis of hyperelastic material nor the assumption of body forces descending from a generalised potential density. The Eshelby stress tensor, for instance, is defined also for a generic Cauchy elastic material (for a definition of Cauchy elastic materials, see, e.g., Ogden, 1984), for which the first Piola-Kirchhoff stress tensor, $\boldsymbol{P}$, cannot be determined by differentiating the body's free energy density with respect to its deformation gradient tensor. In this respect, we recall Gurtin's words: "*My derivation of Eshelby's relation is accomplished without recourse to constitutive equations or to a variational principle*" (Gurtin, 1995). Yet, what is referred to as "Eshelby stress tensor" and "configurational force density" within a given theory may well depend on whether or not the body is hyperelastic and the body forces admit a potential.

To focus on the consequences of the existence of such a potential, we consider first a hyperelastic and inhomogeneous material with energy density $W^{\text{el}} := \check{W}^{\text{el}} \circ (\boldsymbol{F}, \mathcal{X})$, and subjected to body forces for which no integrability hypothesis is made. Then, following Gurtin's approach (Gurtin, 1995), the following configurational force balance applies

$$\operatorname{Div} \mathfrak{E}^{\text{el}} + \mathfrak{F}^{\text{el}} = \boldsymbol{0}, \tag{112}$$

where $\mathfrak{E}^{\text{el}} := W^{\text{el}} \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}$ is the Eshelby stress tensor obtained by using $W^{\text{el}}$ as free energy density, and $\mathfrak{F}^{\text{el}}$ is the configurational force density satisfying Equation (112). Note that, for the sake of a lighter notation, we write $\boldsymbol{F}^T$ in lieu of $\boldsymbol{F}^T \circ \phi$ throughout this section.

To identify $\mathfrak{F}^{\text{el}}$ from Equation (112), we compute explicitly the divergence of $\mathfrak{E}^{\text{el}}$, while recalling the equilibrium equation $\operatorname{Div} \boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}$. Thus, we find

$$\mathfrak{F}^{\text{el}} = -\operatorname{Div} \mathfrak{E}^{\text{el}} = -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f}, \tag{113}$$

thereby reaching the conclusion that $\mathfrak{F}^{\text{el}}$ consists of the sum of two contributions, denoted by

$$\mathfrak{F}^{\text{el,inh}} := -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}), \tag{114a}$$

$$\mathfrak{F}^{\text{el,b}} := -\boldsymbol{F}^T \boldsymbol{f}, \tag{114b}$$

and ascribable to the inhomogeneity of the material and to the presence of the body force $\boldsymbol{f}$, respectively. We emphasise that Equations (113), (114a) and (114b) are true *regardless* of any prescription on the integrability of $\boldsymbol{f}$. Still, without loss of generality, we may assume the splitting $\boldsymbol{f} = \boldsymbol{f}^{\mathrm{p}} + \boldsymbol{f}^{\mathrm{m}}$, where $\boldsymbol{f}^{\mathrm{m}}$ is assumed to admit the generalised energy potential density $W^{\mathrm{m}} = \check{W}^{\mathrm{m}} \circ (\phi, \mathcal{X})$, such that

$$\boldsymbol{f}^{\mathrm{m}} = -\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \mathcal{X}). \tag{115}$$

In the terminology of Lanczos (1970, page 30), $\boldsymbol{f}^{\mathrm{p}}$ is said to be "*polygenic*", whereas $\boldsymbol{f}^{\mathrm{m}}$ is referred to as a "*monogenic*" force density, because it is "*generated by a single scalar function*", i.e., $\check{W}^{\mathrm{m}}$.

The splitting $\boldsymbol{f} = \boldsymbol{f}^{\mathrm{p}} + \boldsymbol{f}^{\mathrm{m}}$ and Equation (115) permit to rewrite $\boldsymbol{\mathfrak{F}}^{\mathrm{el}}$ as

$$\begin{aligned}\boldsymbol{\mathfrak{F}}^{\mathrm{el}} &= -\frac{\partial \check{W}^{\mathrm{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^{T} \boldsymbol{f} \\ &= -\frac{\partial \check{W}^{\mathrm{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) + \boldsymbol{F}^{T} \left[\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \mathcal{X})\right] - \boldsymbol{F}^{T} \boldsymbol{f}^{\mathrm{p}},\end{aligned} \tag{116}$$

and, since it holds true that

$$\mathrm{Grad}\, W^{\mathrm{m}} = \boldsymbol{F}^{T} \left[\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \mathcal{X})\right] + \frac{\partial \check{W}^{\mathrm{m}}}{\partial \mathcal{X}} \circ (\phi, \mathcal{X}), \tag{117}$$

the force density $\boldsymbol{\mathfrak{F}}^{\mathrm{el}}$ takes on the expression

$$\boldsymbol{\mathfrak{F}}^{\mathrm{el}} = -\frac{\partial \check{W}^{\mathrm{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \frac{\partial \check{W}^{\mathrm{m}}}{\partial \mathcal{X}} \circ (\phi, \mathcal{X}) + \mathrm{Grad}\, W^{\mathrm{m}} - \boldsymbol{F}^{T} \boldsymbol{f}^{\mathrm{p}}. \tag{118}$$

Moreover, by exploiting the identity $\mathrm{Grad}\, W^{\mathrm{m}} = \mathrm{Div}(W^{\mathrm{m}} \boldsymbol{I}^{T})$, setting

$$W^{\mathrm{el}} = \check{W}^{\mathrm{el}} \circ (\boldsymbol{F}, \mathcal{X}) = \hat{W}^{\mathrm{el}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \text{with } \frac{\partial \hat{W}^{\mathrm{el}}}{\partial \phi} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \boldsymbol{0}, \tag{119a}$$

$$W^{\mathrm{m}} = \check{W}^{\mathrm{m}} \circ (\phi, \mathcal{X}) = \hat{W}^{\mathrm{m}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \text{with } \frac{\partial \hat{W}^{\mathrm{m}}}{\partial \boldsymbol{F}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \boldsymbol{0}, \tag{119b}$$

and defining the overall energy density, $\hat{W} := \hat{W}^{\mathrm{el}} + \hat{W}^{\mathrm{m}}$, we obtain

$$\boldsymbol{\mathfrak{F}}^{\mathrm{el}} = -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \mathrm{Div}(W^{\mathrm{m}} \boldsymbol{I}^{T}) - \boldsymbol{F}^{T} \boldsymbol{f}^{\mathrm{p}}. \tag{120}$$

Finally, substituting this result into Equation (112) yields

$$\mathrm{Div}\left(W^{\mathrm{el}} \boldsymbol{I}^{T} - \boldsymbol{F}^{T} \boldsymbol{P}\right) - \frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \mathrm{Div}(W^{\mathrm{m}} \boldsymbol{I}^{T}) - \boldsymbol{F}^{T} \boldsymbol{f}^{\mathrm{p}} = \boldsymbol{0}, \tag{121}$$

which can be recast in the form

$$\mathrm{Div}\left(W \boldsymbol{I}^{T} - \boldsymbol{F}^{T} \boldsymbol{P}\right) - \frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^{T} \boldsymbol{f}^{\mathrm{p}} = \boldsymbol{0}. \tag{122}$$

We recognise that the term under divergence in Equation (122) is the Eshelby stress tensor used in our work, i.e., $\mathfrak{E} = W\boldsymbol{I}^T - \boldsymbol{F}^T\boldsymbol{P}$, which is constructed with the energy density $W$. Accordingly, the corresponding configurational force is given by

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}} = \mathfrak{F}^{\mathrm{el}} - \operatorname{Grad} W^{\mathrm{m}}, \tag{123}$$

so that Equation (122) returns the configurational force balance $\operatorname{Div}\mathfrak{E} + \mathfrak{F} = \boldsymbol{0}$. In the absence of polygenic forces, i.e., for $\boldsymbol{f}^{\mathrm{p}} = \boldsymbol{0}$, the form of the configurational force balance is maintained up to the re-definition of $\mathfrak{F}$, which reduces to

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{124}$$

a result stating that the inhomogeneity force $\mathfrak{F}$ acquires the meaning of an *effective* force accounting for two contributions: the inhomogeneities of the material featuring in the body's hyperelastic behaviour and, thus, represented by $W^{\mathrm{el}}$, and the inhomogeneities of the energy density $W^{\mathrm{m}}$, which describes the interaction of the body with its surrounding world (e.g., via the mass density).

# B Divergence Transformation

Let us consider a field theoretical framework and analyse a static problem, described by the Lagrangian density function $\mathcal{L} = \hat{\mathcal{L}} \circ (\varphi, \operatorname{Grad}\varphi, \mathcal{X})$, in which $\varphi$ is a scalar field (the generalisation to the situation in which $\varphi$ is a collection of $N$ scalar fields is straightforward). We emphasise that $\varphi$ is *not* the deformation here, but only a generic scalar field, as it could be the case for temperature or for the scalar potential in Electromagnetism. Consequently, the evaluation $\varphi(X)$, with $X \in \mathcal{B}$, only represents the value taken by $\varphi$ at $X$, i.e., it is not the embedding of the material point $X$ into the three-dimensional Euclidean space. Within this setting, the quantity $\operatorname{Grad}\varphi$ need not be the "material gradient" of $\varphi$. Still, we maintain the notation introduced so far in our work in order not to generate confusion.

After renaming $\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}_{\mathrm{old}}$, we express the *divergence transformation* as (Hill, 1951)

$$\hat{\mathcal{L}}_{\mathrm{new}} \circ (\varphi, \operatorname{Grad}\varphi, \mathcal{X}) = \hat{\mathcal{L}}_{\mathrm{old}} \circ (\varphi, \operatorname{Grad}\varphi, \mathcal{X}) + \operatorname{Div}\boldsymbol{\Omega}, \tag{125}$$

where $\boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}} \circ (\varphi, \mathcal{X})$ is an arbitrary vector field. Moreover, we notice that the vector-valued function $\hat{\boldsymbol{\Omega}}$ has to be independent of $\operatorname{Grad}\varphi$.

A first direct consequence of (125) is that the overall Lagrangian† associated with the

†In a more general – yet conceptually equivalent – framework, we should speak of action functional, rather than "overall Lagrangian", with the former being defined as the time integral of the latter over a given (bounded) time interval. However, since all the quantities introduced in the present work are independent of time because of the hypothesis of static problem, the action and the "overall Lagrangian" are defined up to a multiplicative constant representing the width of the given time interval. For this reason, the formulation used in our work is totally equivalent to the general one.

body transforms from

$$L^{\text{old}}_{\mathcal{B}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{old}} \circ (\varphi, \text{Grad}\varphi, \mathcal{X})] \tag{126}$$

into

$$L^{\text{new}}_{\mathcal{B}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{new}} \circ (\varphi, \text{Grad}\,\varphi, \mathcal{X})], \tag{127}$$

where $L^{\text{old}}_{\mathcal{B}}(\varphi)$ and $L^{\text{new}}_{\mathcal{B}}(\varphi)$ differ from each other by the boundary term $\int_{\partial\mathcal{B}} \boldsymbol{\Omega}\boldsymbol{N}$, i.e.,

$$L^{\text{new}}_{\mathcal{B}}(\varphi) = L^{\text{old}}_{\mathcal{B}}(\varphi) + \int_{\partial\mathcal{B}} [\hat{\boldsymbol{\Omega}} \circ (\varphi, \mathcal{X})]\boldsymbol{N}. \tag{128}$$

Since the variational procedure yielding the stationarity conditions for $L^{\text{old}}_{\mathcal{B}}(\varphi)$ and $L^{\text{new}}_{\mathcal{B}}(\varphi)$ requires the fields $\varphi$ and $\bar{\varphi} = \varphi + h\,\eta$ to coincide with each other on $\partial\mathcal{B}$ (indeed, $\eta$ is chosen such that it vanishes on $\partial\mathcal{B}$), a field $\varphi$ for which $L^{\text{old}}_{\mathcal{B}}(\varphi)$ is stationary makes $L^{\text{new}}_{\mathcal{B}}(\varphi)$ stationary too. Moreover, such a field has to satisfy the same set of Euler-Lagrange equations. Indeed, upon recalling the expression of the covariant divergence of $\boldsymbol{\Omega}$, i.e.,

$$\begin{aligned} \text{Div}\,\boldsymbol{\Omega} &= \Omega^A{}_{,A} + \Gamma^A_{BA}\Omega^B \\ &= \left[\frac{\partial\hat{\Omega}^A}{\partial\varphi} \circ (\varphi, \mathcal{X})\right]\varphi_{,A} + \frac{\partial\hat{\Omega}^A}{\partial\mathcal{X}^A} \circ (\varphi, \mathcal{X}) + \Gamma^A_{BA}[\hat{\Omega}^B \circ (\varphi, \mathcal{X})], \end{aligned} \tag{129}$$

and substituting (129) into (125), we find that another consequence of Equation (125) is given by the identities

$$\begin{aligned} \frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi} \circ (\dots) = \frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi} \circ (\dots) &+ \left[\frac{\partial^2\hat{\Omega}^A}{\partial\varphi^2} \circ (\varphi, \mathcal{X})\right]\varphi_{,A} \\ &+ \frac{\partial^2\hat{\Omega}^A}{\partial\mathcal{X}^A\partial\varphi} \circ (\varphi, \mathcal{X}) + \Gamma^A_{BA}\left[\frac{\partial\hat{\Omega}^B}{\partial\varphi} \circ (\varphi, \mathcal{X})\right], \end{aligned} \tag{130a}$$

$$\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}} \circ (\dots) = \frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}} \circ (\dots) + \frac{\partial\hat{\Omega}^B}{\partial\varphi} \circ (\varphi, \mathcal{X}), \tag{130b}$$

$$\begin{aligned} \left[\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}} \circ (\dots)\right]_{|B} = \left[\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}} \circ (\dots)\right]_{|B} &+ \left[\frac{\partial^2\hat{\Omega}^B}{\partial\varphi^2} \circ (\varphi, \mathcal{X})\right]\varphi_{,B} \\ &+ \frac{\partial^2\hat{\Omega}^B}{\partial\varphi\partial\mathcal{X}^B} \circ (\varphi, \mathcal{X}) + \Gamma^B_{DB}\left[\frac{\partial\hat{\Omega}^D}{\partial\varphi} \circ (\varphi, \mathcal{X})\right], \end{aligned} \tag{130c}$$

which imply the invariance of the Euler-Lagrange equations under the transformation (125), i.e.,

$$\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi} \circ (\dots) - \left(\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}} \circ (\dots)\right)_{|B} = \frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi} \circ (\dots) - \left(\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}} \circ (\dots)\right)_{|B} = 0. \tag{131a}$$

We emphasise that this result holds true because $\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})]$ solves identically the Euler-Lagrange equations, i.e.,

$$\frac{\partial}{\partial \varphi}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})] - \mathrm{Div}\left(\frac{\partial}{\partial \mathrm{Grad}\,\varphi}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})]\right) = 0. \tag{132}$$

If $\varphi$ is a collection of $N$ independent scalar fields, Equation (132) becomes a system of $N$ scalar equations, i.e., in components,

$$\frac{\partial}{\partial \varphi^\mu}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})] - \left(\frac{\partial}{\partial {\varphi^\mu}_{,A}}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})]\right)_{|A} = 0, \quad \mu = 1, \ldots, N. \tag{133}$$

However, the quantity

$$\frac{\partial}{\partial {\varphi^\mu}_{,A}}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})], \qquad \mu = 1, \ldots, N, \quad A = 1, 2, 3, \tag{134}$$

is not, in general, the component of a tensor field. Indeed, if it were, for example for $N = 3$, the covariant divergence constituting the second term on the left-hand-side of Equation (133) would require to differentiate the tensors $\boldsymbol{e}^\mu \otimes \boldsymbol{E}_A$ of a suitable tensor basis, thereby yielding a term, obtained by differentiating $\boldsymbol{e}^\mu$, that does not cancel with the first summand of Equation (133). Hence, Equation (133) would not be satisfied.

The situation just depicted occurs when the "fields" of the triplet $(\varphi^1, \varphi^2, \varphi^3)$ acquire the meaning of the components of the deformation, an object that has the mathematical meaning of an embedding and, thus, that is not truly identifiable with a collection of genuine scalar fields. Indeed, when $(\varphi^1, \varphi^2, \varphi^3)$ is replaced by $(\phi^1, \phi^2, \phi^3)$, the corresponding "gradient" is none other than $\boldsymbol{F}$ and, more importantly, the quantity in (134) becomes (with $a \in \{1, 2, 3\}$ and $A \in \{1, 2, 3\}$)

$$\frac{\partial}{\partial {\phi^a}_{,A}}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\phi, \mathcal{X})] = \frac{\partial}{\partial {F^a}_A}\mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\phi, \mathcal{X})], \tag{135}$$

which takes on the meaning of a fictitious first Piola-Kirchhoff stress tensor. The consequence of this result is that the covariant divergence of the right-hand-side of Equation (135) does not cancel with $\partial \mathrm{Div}[\hat{\mathbf{\Omega}} \circ (\phi, \mathcal{X})]/\partial \phi^a$. This leads us to the conclusion, already stated by Maugin (1993, see page 100), that $\hat{\mathbf{\Omega}}$ should depend "***at most***" on $X$ "*and **not** on the fields*".

Since we consider a static problem, for which the body's Lagrangian density function coincides with the negative of its total energy density, following Hill (1951), we introduce the functions $W_{\mathrm{old}} = \hat{W}_{\mathrm{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and $W_{\mathrm{new}} = \hat{W}_{\mathrm{new}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, and we reformulate the transformation (125) as

$$-\hat{W}_{\mathrm{new}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = -\hat{W}_{\mathrm{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \mathrm{Div}\,\mathbf{\Omega}, \tag{136}$$

with $\mathbf{\Omega} \equiv \hat{\mathbf{\Omega}} \circ \mathcal{X}$. For the reasons outlined above, the divergence transformation (136) is such that the overall energies $\mathcal{E}^{\mathrm{old}}_{\mathcal{D}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\mathrm{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and $\mathcal{E}^{\mathrm{new}}_{\mathcal{D}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\mathrm{new}} \circ$

$(\phi, \boldsymbol{F}, \mathcal{X})$ are stationary for the same deformation $\phi$, which thus satisfies the same Euler-Lagrange equations. Indeed, since $\hat{\boldsymbol{\Omega}}$ is independent of $\phi$, it holds true that

$$\frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\ldots) = \frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\ldots), \tag{137a}$$

$$\frac{\partial \hat{W}_{\text{new}}}{\partial {F^b}_B} \circ (\ldots) = \frac{\partial \hat{W}_{\text{old}}}{\partial {F^b}_B} \circ (\ldots), \tag{137b}$$

$$\frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\ldots) - \left(\frac{\partial \hat{W}_{\text{old}}}{\partial {F^b}_B} \circ (\ldots)\right)_{|B} = \frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\ldots) - \left(\frac{\partial \hat{W}_{\text{new}}}{\partial {F^b}_B} \circ (\ldots)\right)_{|B} = 0. \tag{137c}$$

After proving this property, we superimpose the transformations $X \mapsto \tilde{X} = \mathcal{Y}(X) = X + h\,\boldsymbol{U}$ and $\phi(X) \mapsto \bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X})$ to the divergence transformation (136), and we require the invariance of the overall energy under the resulting, global transformation (Hill, 1951). This yields the equality

$$\underbrace{\int_{\mathcal{D}} \{[\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X})] \circ \mathcal{Y}\} \det(T\mathcal{Y})}_{\equiv \mathcal{E}^{\text{new}}_{\mathcal{D}}(\bar{\phi}, \mathcal{Y})} = \underbrace{\int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})}_{\equiv \mathcal{E}^{\text{old}}_{\mathcal{D}}(\phi, \mathcal{X})}, \tag{138}$$

where $T\mathcal{Y}$ is the tangent map of $\mathcal{Y}$. By applying a "rescaled" divergence transformation to the left-hand-side of Equation (138), i.e.,

$$\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) = \hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) - \mathrm{Div}(h\,\boldsymbol{\Omega}), \tag{139}$$

we obtain

$$\int_{\mathcal{D}} \{[\hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X})] \circ \mathcal{Y} - \mathrm{Div}(h\,\boldsymbol{\Omega}) \circ \mathcal{Y}\} \det(T\mathcal{Y}) = \int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{140}$$

We remark that the smallness parameter $h$, which multiplies $\boldsymbol{\Omega}$ in (139) and (140), has been introduced in order to make the divergence transformation infinitesimal, as is the case for the transformations on the material points and on the deformation.

By rearranging Equation (140), so as to separate the transformations on the material points and on the deformation from the divergence transformation, we find

$$\int_{\mathcal{D}} \{[\hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) \circ \mathcal{Y}] \det(T\mathcal{Y}) - \hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})\} = \int_{\mathcal{D}} [\mathrm{Div}(h\,\boldsymbol{\Omega}) \circ \mathcal{Y}] \det(T\mathcal{Y}). \tag{141}$$

By using the result reported in (103), at the first order in $h$, Equation (141) becomes

$$\int_{\mathcal{D}} \mathrm{Div}[\mathfrak{C}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w}] = \int_{\mathcal{D}} \mathrm{Div}\,\boldsymbol{\Omega} \;\; \Rightarrow \;\; \int_{\mathcal{D}} \mathrm{Div}[\mathfrak{C}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = 0, \tag{142}$$

thereby implying that Noether's current density is given by $\mathfrak{J} = \mathfrak{C}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}$ and that, after localisation, the conservation laws should be sought for in the form

$$\mathrm{Div}[\mathfrak{C}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = 0. \tag{143}$$

The choice of $\boldsymbol{\Omega}$ depends on the type of conservation law and on the associated class of symmetry which one is interested in looking at.

Within the present context, Equation (143) constitutes the most general form of conservation law pertaining to Noether's current. This result, however, can be exploited in much deeper detail: indeed, granted the Euler-Lagrange equations $\boldsymbol{f} + \mathrm{Div}\,\boldsymbol{P} = \boldsymbol{0}$, if, for a given choice of the fields $\boldsymbol{U}$, $\boldsymbol{w}$ and $\boldsymbol{\Omega}$, (143) is satisfied as an identity, then a specific physical quantity is conserved and the fields are said to be *symmetries*.

For the problem under investigation, Equation (143) can be recast in the equivalent form (Hill, 1951; Grillo et al., 2003, 2019)

$$\begin{aligned}\mathrm{Div}[\boldsymbol{\mathfrak{E}}^T\boldsymbol{U} + \boldsymbol{P}^T\boldsymbol{w} - \boldsymbol{\Omega}] &= (\mathrm{Div}\,\boldsymbol{\mathfrak{E}})\boldsymbol{U} + \boldsymbol{\mathfrak{E}} : \mathrm{Grad}\,\boldsymbol{U} + (\mathrm{Div}\,\boldsymbol{P})\boldsymbol{w} + \boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{w} - \mathrm{Div}\,\boldsymbol{\Omega} \\ &= -\boldsymbol{\mathfrak{F}}\,\boldsymbol{U} + \boldsymbol{\mathfrak{E}} : \mathrm{Grad}\,\boldsymbol{U} - \boldsymbol{f}\boldsymbol{w} + \boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{w} - \mathrm{Div}\,\boldsymbol{\Omega} = 0\,. \quad (144)\end{aligned}$$

If one is interested in looking at the conservation of linear momentum, one sets $\boldsymbol{U} = \boldsymbol{0}$, $\boldsymbol{\Omega} = \boldsymbol{0}$ and $\boldsymbol{w} = \boldsymbol{w}_0$, with $\boldsymbol{w}_0$ being a uniform displacement field. In this case, Equation (144) is not satisfied. Indeed, it occurs that

$$\mathrm{Div}[\boldsymbol{\mathfrak{E}}^T\boldsymbol{U} + \boldsymbol{P}^T\boldsymbol{w} - \boldsymbol{\Omega}] = \mathrm{Div}[\boldsymbol{P}^T\boldsymbol{w}_0] = -\boldsymbol{f}\boldsymbol{w}_0 \neq 0\,, \quad (145)$$

which shows that linear momentum is not conserved because of the body forces $\boldsymbol{f}$.

On the same footing, the presence of the inhomogeneity force, $\boldsymbol{\mathfrak{F}}$, spoils the conservation of the pseudo-momentum (Maugin, 1993), and this is reflected by the fact that uniform translations of material points, hereafter denoted by $\boldsymbol{U} = \boldsymbol{U}_0$, are not symmetry transformations. This is encompassed by Equation (144) by setting $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{\Omega} = \boldsymbol{0}$, thereby obtaining

$$\mathrm{Div}[\boldsymbol{\mathfrak{E}}^T\boldsymbol{U} + \boldsymbol{P}^T\boldsymbol{w} - \boldsymbol{\Omega}] = -\boldsymbol{\mathfrak{F}}\,\boldsymbol{U}_0 \neq 0\,. \quad (146)$$

In fact, Hill (1951) presents several examples, from which we largely took inspiration, and, among those, he shows that the only case in which $\boldsymbol{\Omega}$ should be taken different from the null vector is the case in which velocity transformations are applied, a situation referred to as the *centre-of-mass theorem*. I feel indebtly to my parents for

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