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Eshelby's Inclusion Theory in the Light of Noether's Theorem

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We dedicate this work to the memory of our maestro Prof. Gaetano Giaquinta (Catania, Italy, 1945-2016), who first taught us Noether's Theorem and showed us its unifying beauty.

Abstract

In a variational setting describing the mechanics of a hyperelastic body with defects or inhomogeneities, we show how the application of Noether's theorem allows for obtaining the classical results by Eshelby. The framework is based on modern differential geometry. First, we present Eshelby's original derivation based on the cut-replace-weld thought experiment. Then, we show how Hamilton's standard variational procedure "with frozen coordinates", which Eshelby coupled with the evaluation of the gradient of the energy density, is shown to yield the strong form of Eshelby's problem. Finally, we demonstrate how Noether's theorem provides the weak form directly, thereby encompassing both procedures that Eshelby followed in his works. We also pursue a declaredly didactic intent, in that we attempt to provide a presentation that is as self-contained as possible, in a modern differential geometrical setting.

Keywords: Eshelby stress; energy-momentum tensor; configurational mechanics; inclusion; defect; Noether's theorem; variational principle

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1 Introduction

In a classical paper, Eshelby (1951) introduced the concept of *configurational force* as the 2 force required for a region containing a defect in a material body to undergo a material з virtual displacement. This idea led to the mechanical Maxwell energy-momentum tensor 4 that has been subsequently termed *Eshelby stress* in continuum mechanics (Maugin and 5 Trimarco, 1992). The procedure followed by Eshelby (1951) comprises a set of operations 6 in which the elastic energy in the interior of a region and the net work that the surface 7 tractions exert on the region are evaluated individually. In another work, Eshelby (1975) 8 used Hamilton's standard variational approach of field theory and found his energy-9 momentum tensor directly, using the components of the regular spatial displacement and 10 of the displacement gradient as the entities called *fields* in the jargon of field theory. In the 11 same paper, Eshelby (1975) also sketched the procedure for the case in which the *fields* 12 are the components of the configuration map, which is the common choice in modern 13 continuum mechanics. 14

Although initially conceived for a single inclusion or for a discrete set of inclusions, 15 Eshelby's theory naturally applies to *inhomogeneous materials* or materials with contin-16 uous distributions of defects. Epstein and Maugin (1990) obtained the Eshelby stress 17 using the concepts of material uniformity and material isomorphism introduced by Noll 18 (1967) for inhomogeneous materials. Gurtin (1995, 2000) reformulated and generalised 19 Eshelby's approach with the method of the varying control volumes and considered the 20 Eshelby stress as the appropriate stress of an *independent* material balance law. The Es-21 helby stress has been seen as the object capturing inhomogeneities and singularities (e.g., 22 Epstein and Maugin, 1990; Gurtin, 1995, 2000; Epstein and Maugin, 2000; Epstein and 23 Elżanowski, 2007; Verron et al., 2009; Weng and Wong, 2009; Maugin, 2011), or the driv-24 ing force of phenomena of material evolution such as plasticity and growth-remodelling 25 (e.g., Maugin and Epstein, 1998; Epstein and Maugin, 2000; Cermelli et al., 2001; Ep-26 stein, 2002; Imatani and Maugin, 2002; Grillo et al., 2003, 2005; Epstein, 2009, 2015; 27 Grillo et al., 2016, 2017; Hamedzadeh et al., 2019), or phase transitions, or evolution of 28 the interfaces among phases (e.g., Gurtin, 1986, 1993; Gurtin and Podio-Guidugli, 1996; 29 Fried and Gurtin, 1994, 2004). 30

In a didactic spirit, the aim of this work is to reproduce the results of Eshelby (1951, 31 1975) directly by means of the classical Noether's theorem (for a translation into English 32 of Noether's original 1918 paper, see Noether, 1971) for continuum systems, as presented 33 by Hill (1951). The derivation is made using the components of the configuration map as 34 the "fields" and those of the deformation gradient as the "gradients of the fields", while 35 an appropriate "topological" transformation represents the material virtual displacement 36 on the region containing the defect. We would like to emphasise that this work is 37 more than a mere rewrite of Eshelby's findings in a more modern notation. While the 38 relation between Eshelby's work and Noether's theorem has been highlighted in several 39 papers (e.g., Knowles and Sternberg, 1972; Eshelby, 1975; Fletcher, 1976; Edelen, 1981; 40 Golebiewska Herrmann, 1982; Olver, 1984a,b; Huang and Batra, 1996; Kienzler and 41 Herrmann, 2000; Maugin, 2011), to the best of our knowledge, no work in the literature 42 establishes an explicit relation between Eshelby's inclusion theory (and, specifically, the 43

44 procedure to deal with the presence of the inclusion; Eshelby, 1951, 1975) and Noether's
 45 theorem.

In Section 2, we introduce the notation and give some basic definitions. In particular, 46 we introduce standard and Eshelbian configurations and their variations, i.e., displace-47 ment fields. The setting is declaredly differential geometrical, although we avoid using 48 differentiable manifolds for simplicity. In Section 3, we review, with our notation and 49 within a suitable geometrical setting, Eshelby's original derivation (Eshelby, 1951) of 50 configurational forces. Similarly, in Section 4, we review Eshelby's variational deriva-51 tion (Eshelby, 1975). Finally, in Section 5, which is the core of the work, we introduce 52 Noether's theorem, and show how its application renders directly the results of both the 53 previous derivations. 54

55 2 Theoretical Background

In this section, we illustrate the notation that we employ and report some fundamental results relevant to this work. We generally use index-free notation but sometimes it is useful to show the corresponding expression in index notation. Therefore, we present most expressions in both notations. In index notation, the customary Einstein's summation convention for repeated indices is enforced throughout and a subscript preceded by a comma, as in $f_{,i}$, denotes partial differentiation with respect to its *i*-th argument.

62 2.1 General Notation and Basic Definitions

Here we review some basic definitions of continuum mechanics, in order to elucidate the 63 notation that we employ. The notation is essentially that of Truesdell and Noll (1965) 64 and Marsden and Hughes (1983), with some modifications (Federico, 2012; Federico 65 et al., 2016). We work in a simplified setting based on the use of affine spaces, whose 66 rigorous definition can be found, e.g., in the treatise by Epstein (2010). We could use 67 a presentation in terms of differentiable manifolds (Noll, 1967; Marsden and Hughes, 68 1983; Epstein, 2010; Segev, 2013), but using affine spaces avoids many of the intricacies 69 of higher-level differential geometry and makes the presentation more intuitive. 70

An affine space is a set S, called the point space, considered together with a vector 71 space \mathcal{V} , called the modelling space, and a mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{V}$: $(x, y) \mapsto y - x = u$. 72 This means that, at every point $x \in S$, it is possible to univocally attach the vector given 73 by u = y - x, for every point $y \in S$. The set of all vectors emanating from point x is 74 a vector space denoted $T_x S = \{ u \in \mathcal{V} : u = y - x, \text{ for all } y \in S \}$ and called *tangent* 75 space to S at x. In the differential geometrical definition, the tangent space $T_x S$ is the 76 set of the vectors that are each *tangent* at x to one of the infinite possible regular curves 77 $c: [a, b] \to S: s \mapsto c(s)$ such that $c(s_0) = x$, where $s_0 \in [a, b]$, i.e., the vectors (see 78 Figure 1) 79

$$\boldsymbol{u} = \lim_{h \to 0} \frac{c(s_0 + h) - c(s_0)}{h} = c'(s_0) \in T_x \mathcal{S}.$$
 (1)

For the case of an affine space S, this definition of tangent space T_xS coincides with that given by the expression u = y - x. Indeed, by varying the curve passing by x, we obtain all possible "tip points" y of the tangent vectors defined as u = y - x. The dual space of T_xS , i.e., the vector space of all linear maps $\varphi : T_xS \to \mathbb{R}$, is denoted T_x^*S and is called the *cotangent space* to S at x. The disjoint unions of all tangent and cotangent spaces are called *tangent bundle TS* and *cotangent bundle* T^*S , respectively.



Figure 1: Differential geometrical definition of tangent vector at a point $x \in S$. Left: the *secant* vector $c(s_0 + h) - c(s_0)$ passing by $x = c(s_0)$. Right: the tangent vector $u = c'(s_0)$ at $x = c(s_0)$, obtained as the limit of the secant.

Vector fields and covector fields (or fields of one-forms) on an open set $A \subseteq S$ are maps

$$\boldsymbol{u}: \mathcal{A} \subseteq \mathcal{S} \to T\mathcal{S}: \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) \in T_{\boldsymbol{x}}\mathcal{S}, \tag{2a}$$

$$\boldsymbol{\varphi}: \mathcal{A} \subseteq \mathcal{S} \to T^{\star}\mathcal{S}: x \mapsto \boldsymbol{\varphi}(x) \in T_x^{\star}\mathcal{S}, \tag{2b}$$

and tensor fields of higher order are defined analogously. Rather than speaking of
 contractions of vectors and covectors in a specific tangent and cotangent space, we can
 directly speak of the contractions of vector fields and covector fields in the tangent and
 cotangent bundle, and we denote the contraction by means of simple juxtaposition, i.e.,

$$\boldsymbol{\varphi} \, \boldsymbol{u} = \boldsymbol{u} \, \boldsymbol{\varphi} = \boldsymbol{\varphi}_a \, \boldsymbol{u}^a. \tag{3}$$

The physical space S is equipped with a metric tensor g, a symmetric and positive definite second-order tensor field defining the scalar product of two vector fields as

$$\boldsymbol{g}: T\mathcal{S} \times T\mathcal{S} \to \mathbb{R}: (\boldsymbol{u}, \boldsymbol{v}) \mapsto \langle \boldsymbol{u}, \boldsymbol{v} \rangle \equiv \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^a \, g_{ab} \, \boldsymbol{v}^b. \tag{4}$$

We assume use of the Levi-Civita connection, i.e., the covariant derivative associated with the metric tensor g via the Christoffel symbols given by (see, e.g., Marsden and Hughes, 1983)

$$\gamma_{bc}^{a} = \frac{1}{2} g^{ad} (g_{cd,b} + g_{bd,c} - g_{bc,d}), \tag{5}$$

⁹⁷ which are symmetric in their lower indices, i.e., $\gamma_{bc}^{a} = \gamma_{cb}^{a}$. The covariant derivative $\nabla_{u} v$ ⁹⁸ of the vector field v in the direction of the vector field u has the component expression

$$[\nabla_{\boldsymbol{u}}\boldsymbol{v}]^a \equiv v^a{}_{|b}\,\boldsymbol{u}^b = v^a{}_{,b}\,\boldsymbol{u}^b + \gamma^a_{bc}\,\boldsymbol{v}^c\,\boldsymbol{u}^b. \tag{6}$$

and defines the *gradient* grad v as the tensor field such that its definition as a linear map is $(\operatorname{grad} v)u \equiv \nabla_u v$, with components $[\operatorname{grad} v]^a{}_b = v^a{}_{|b}$. The covariant derivative and the gradient of a tensor field of arbitrary order are defined analogously.

Remark 1. A scalar is a tensor of order zero and thus we find more natural to use the 102 convention adopted by, e.g., Epstein (2010, see page 116) and to consider the gradient of 103 a scalar field f as the *covector* field (or one-form) grad f such that $(\operatorname{grad} f)(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} f$, 104 as for a tensor of any other order. Accordingly, the components of grad f are f_{a} . The 105 other possible convention is that adopted by Marsden and Hughes (1983, see page 69), 106 according to which the gradient of f is the vector field with components $g^{ab} f_{b}$. Note 107 that, in either case, since f is a tensor of order zero, the Christoffel symbols of the 108 connection are *not* involved in the gradient, which is thus connection-independent. There 109 are several advantages in defining the gradient as a covector. First, this definition is 110 *metric-independent*, whereas the vector definition clearly necessitates that a metric tensor 111 g be defined. Second, the covector definition accommodates the analytical mechanical 112 definition of force as a covector field: indeed, an integrable force is the negative of the 113 gradient of a potential energy and is thus consistently represented as a covector field. 114 Finally, with the covector definition of grad f, we have the remarkable chain of identities 115

$$\nabla f \equiv \operatorname{grad} f \equiv \mathrm{d} f \equiv \mathrm{D} f,\tag{7}$$

where d*f* is the *exterior derivative* of *f*, when seen as a zero-form (see, e.g., Epstein, 2010, page 116), and D*f* is the *Fréchet derivative* (or tangent map) of *f*, when seen as a point map from $A \subset S$ into \mathbb{R} .

In the following, the physical space S is identified with the affine space \mathbb{E}^3 , which is \mathbb{R}^3 considered both as the point space and as the modelling vector space.

2.2 Bodies, Configurations and the Deformation Gradient

¹²² In the simplified presentation that we adopt, a deformable continuous body \mathcal{B} is identified ¹²³ with one of its placements in the physical space \mathcal{S} , and this particular placement is called ¹²⁴ reference configuration. The body is assumed to be endowed with the material metric G, ¹²⁵ which induces the corresponding Levi-Civita connection, similarly to what seen for the ¹²⁶ spatial metric g.

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A configuration, or deformation, of the body is an embedding

$$\phi: \mathcal{B} \to \mathcal{S}: X \mapsto x = \phi(X), \tag{8}$$

i.e., a map such that its codomain-restriction $\phi : \mathcal{B} \to \phi(\mathcal{B})$ is a diffeomorphism, i.e., a continuos and differentiable map, which is invertible, with continuous and differentiable inverse $\Phi \equiv \phi^{-1} : \phi(\mathcal{B}) \to \mathcal{B}$. The configuration ϕ maps *material* points $X = (X^1, X^2, X^3)$ in the body \mathcal{B} into *spatial* points $x = (x^1, x^2, x^3)$ in \mathcal{S} , i.e., $\phi(X) = x$.

Since we are going to introduce another class of configurations, called *Eshelbian*, we shall refer to the standard definition of configuration given above as to a *conventional configuration*. The set of all *k*-times differentiable conventional configuration maps (with ¹³⁵ $k \in \mathbb{N}$ constitutes the *conventional* configuration space C of the body \mathcal{B} . Since S is an ¹³⁶ affine space, the space $C^k(\mathcal{B}, S)$ of the *k*-times differentiable maps from \mathcal{B} into S is ¹³⁷ an infinite-dimensional affine space. Thus, considering C as an open set in $C^k(\mathcal{B}, S)$ ¹³⁸ (Marsden and Hughes, 1983) makes C an infinite-dimensional trivial manifold. A tangent ¹³⁹ vector η in the functional tangent space $T_{\phi}C$ can be thought of as the tangent at ϕ to ¹⁴⁰ a curve of maps in C (i.e., a one-parameter family of maps in C), and is a vector field ¹⁴¹ *covering* the configuration ϕ , i.e.,

$$\eta: \mathcal{B} \to T\mathcal{S}: X \mapsto \eta(X) \in T_{\phi(X)}\mathcal{S} = T_x\mathcal{S}.$$
(9)

The vector field η is called a (conventional) *displacement field* (and, when compatible with the constraints, but not necessarily attained by the body, it is called a *virtual displacement*). Figure 2 shows the displacement $\eta(X) = \eta(\Phi(x))$ as a tangent vector at T_xS and an illustration of the configuration space with the displacement field η as a tangent vector at $T_{\phi}C$.



Figure 2: A conventional displacement field. Top: The displacement $\eta(X) = \eta(\Phi(x)) \in T_X S$ as a tangent vector attached at $x = \phi(X)$. Bottom: The displacement field η as a tangent vector attached at the configuration ϕ , which is a point in the configuration space C, here depicted as a surface, for the sake of an intuitive graphical representation.

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The deformation gradient at point X is the *tangent map* of ϕ , i.e., the tensor

$$(T\phi)(X) = F(X) : T_X \mathcal{B} \to T_X \mathcal{S}, \tag{10}$$

with $x = \phi(X)$, expressing the Fréchet derivative of ϕ at X. Since the existence of the Fréchet derivative of ϕ implies the existence of its Gâteaux derivative (or directional derivative), F(X) can be defined through the limit

$$(\partial_{M}\phi)(X) := \lim_{h \to 0} \frac{\phi(X + hM) - \phi(X)}{h} = [(T\phi)(X)]M = [F(X)]M,$$
(11)

and the Gâteaux derivative $\partial_M \phi(X)$ of ϕ with respect to *any* tangent vector $M \in T_X \mathcal{B}$ equals the Fréchet derivative F(X)M, which is linear in M. In components, Equation (11) reads

$$(\underline{\partial}_{\boldsymbol{M}}\phi)^{a}(\underline{X}) \equiv (\underline{T}\phi)^{a}{}_{\underline{B}}(\underline{X})\underline{M}^{\underline{B}} \equiv \underline{F}^{a}{}_{\underline{B}}(\underline{X})\underline{M}^{\underline{B}} \equiv \phi^{a}{}_{\underline{B}}(\underline{X})\underline{M}^{\underline{B}},$$
(12)

where we recall that the comma denotes partial differentiation. Note that F(X) is a twopoint tensor as it has the *domain leg* in $T_X \mathcal{B}$ and the *codomain leg* in $T_X \mathcal{S}$. As a tensor field, the deformation gradient is

$$\boldsymbol{F}: \mathcal{B} \to T\mathcal{S} \otimes T^{\star}\mathcal{B}. \tag{13}$$

The deformation gradient F pushes-forward material vector fields M with components 157 M^A into spatial vector fields $\phi_* M = (F \circ \Phi) (M \circ \Phi)$ with components $(F^a{}_A \circ \Phi) (M^A \circ \Phi)$. 158 The inverse F^{-1} pulls-back spatial vector fields **m** with components m^a into material vec-159 tor fields $\phi^* \mathbf{m} = (\mathbf{F}^{-1} \circ \phi)(\mathbf{m} \circ \phi)$ with components $((\mathbf{F}^{-1})^A_a \circ \phi)(\mathbf{m}^a \circ \phi)$. The 160 transpose F^T pulls-back spatial covector fields π with components π_a into material cov-161 ector fields $\phi^* \pi = (F^T \circ \phi) (\pi \circ \phi)$ with components $((F^T)_A{}^a \circ \phi) (\pi_a \circ \phi) = F^a{}_A (\pi_a \circ \phi)$. 162 The inverse transpose F^{-T} pushes-forward material covector fields Π with compo-163 nents Π_A into spatial covector fields $\phi_*\Pi = (F^{-T} \circ \Phi)(\Pi \circ \Phi)$ with components 164 $((\boldsymbol{F}^{-T})_a{}^A \circ \Phi) (\Pi_A \circ \Phi) = (\boldsymbol{F}^{-1})^A{}_a (\Pi_A \circ \Phi).$ 165

The determinant $J = \det F$ has the meaning of *volume ratio*, in the spirit of the theorem of the change of variables applied to the transformation from the spatial region $\phi(\mathcal{R}) \subset S$ to the corresponding material region $\mathcal{R} \subset \mathcal{B}$.

2.3 Eshelbian Configurations and Their Tangent Maps

Grillo et al. (2003) introduced the concept of *admissible reference configuration set* of a body as the set of all reference configurations obtained by applying a diffeomorphism to the reference configuration \mathcal{B} representing the body (which has some similarities with the idea of boundary reparametresations introduced by Gurtin, 1995). Here, we make use of this concept in a slightly different way.

An *Eshelbian configuration* \mathcal{Y} is a diffeomorphism on the body \mathcal{B} . Since we define the 175 body \mathcal{B} as a trivial manifold, i.e., an open subset of the physical space \mathcal{S} , the codomain 176 of an Eshelbian configuration \mathcal{Y} should be the whole space \mathcal{S} and the image would be 177 an open set $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B}) \subset \mathcal{S}$. However, if the body \mathcal{B} were a non-trivial manifold, the 178 image $\hat{B} = \mathcal{Y}(B)$ would be another non-trivial manifold. To keep the notation as general 179 as possible, we prefer to avoid declaring S as the codomain of \mathcal{Y} . Rather, we consider all 180 admissible diffeomorphisms \mathcal{Y} , each with its image $\hat{\mathcal{B}}$, and we obtain the collection of all 181 admissible reference configurations $\hat{\mathcal{B}}$, which clearly also contains $\hat{\mathcal{B}}$ itself (see also Grillo 182

et al., 2003). Then, we consider the union $\mathcal{N} = \bigcup_{\mathcal{Y}} \tilde{\mathcal{B}}$ of these mutually diffeomorphic sets $\tilde{\mathcal{B}}$, and define the generic Eshelbian configuration as

$$\mathcal{Y}: \mathcal{B} \to \mathcal{N}: X \mapsto \tilde{X} = \mathcal{Y}(X), \tag{14}$$

which has the further notational advantage of not tying \mathcal{Y} to its specific image $\hat{\mathcal{B}}$.

Analogously to the case of a conventional configuration, the tangent map of an Eshelbian configuration at point X is the tensor

$$(T\mathcal{Y})(X): T_X\mathcal{B} \to T_{\tilde{X}}\mathcal{N},$$
 (15)

with $\tilde{X} = \mathcal{Y}(X)$. Again, $(T\mathcal{Y})(X)$ is the Fréchet derivative of \mathcal{Y} at X and, since \mathcal{Y} is a diffeomorphism, $(T\mathcal{Y})(X)$ can be computed by means of the Gâteaux derivative of \mathcal{Y} at X, i.e.,

$$(\partial_{\boldsymbol{M}}\mathcal{Y})(X) = \lim_{h \to 0} \frac{\mathcal{Y}(X + h\,\boldsymbol{M}) - \mathcal{Y}(X)}{h} = [(T\mathcal{Y})(X)]\boldsymbol{M}.$$
(16)

The *material identity map* is the particular case of Eshelbian configuration obtained by considering that $\mathcal{B} \subset \mathcal{N}$, and is defined as

$$\mathfrak{X}: \mathcal{B} \to \mathcal{B}: X \mapsto X = \mathfrak{X}(X), \tag{17}$$

¹⁹³ with the component representation

$$\mathfrak{X}^A: \mathcal{B} \to \mathbb{R}: X \mapsto X^A = \mathfrak{X}^A(X) \equiv \mathfrak{X}^A(X^1, X^2, X^3).$$
(18)

Its tangent map is clearly the (material) identity tensor in $T\mathcal{B}$, i.e.,

$$T\mathfrak{X} = \mathbf{I} : T\mathcal{B} \to T\mathcal{B}, \qquad (T\mathfrak{X})^{A}{}_{B} = \mathfrak{X}^{A}{}_{,B} = \delta^{A}{}_{B}.$$
 (19)

Also in the case of Eshelbian configurations, we can exploit the affine structure of S: since all sets $\tilde{\mathcal{B}}$ are open subsets of S, also $\mathcal{N} = \bigcup_{\mathcal{Y}} \tilde{\mathcal{B}} \subseteq S$ is an open set, and thus we can define the space of all Eshelbian configurations as an open subset \mathcal{M} of the infinite-dimensional affine space $C^k(\mathcal{B}, \mathcal{N})$, which makes \mathcal{M} an infinite-dimensional trivial manifold.

Remark 2. In our setting, in which the physical space S is an affine space and a body B199 is a subset of S, the distinction between a conventional configuration $\phi: \mathcal{B} \to \mathcal{S}$ and an 200 Eshelbian configuration $\mathcal{Y}: \mathcal{B} \to \mathcal{N}$ seems to fade out, because $\mathcal{N} = \bigcup_{\mathcal{Y}} \tilde{\mathcal{B}} \subseteq \mathcal{S}$. However 201 this is not the case, as will become clear from the explanation given in Section 3 (see also 202 Figures 3 and 4). Moreover, when \mathcal{B} is a general manifold, the distinction is fundamental. 203 In this case, while a conventional configuration ϕ remains an embedding of \mathcal{B} in \mathcal{S} , i.e., 204 it gives \mathcal{B} a placement $\phi(\mathcal{B}) \subset \mathcal{S}$, an Eshelbian configuration transforms the manifold \mathcal{B} 205 into *a different* manifold $\hat{\mathcal{B}}$. 206

A tangent vector
$$U \in T_{\mathcal{X}} \mathcal{M}$$
 is a vector field

$$\boldsymbol{U}: \boldsymbol{\mathcal{B}} \to T\boldsymbol{\mathcal{B}}: \boldsymbol{X} \mapsto \boldsymbol{U}(\boldsymbol{X}) \in T_{\boldsymbol{X}}\boldsymbol{\mathcal{B}},\tag{20}$$

and is called a *material displacement* field. When an Eshelbian configuration $\mathcal{Y} : \mathcal{B} \to \mathcal{N}$, is defined as a *perturbation* of the material identity \mathcal{X} , i.e.,

$$\mathcal{Y}(X) = \mathcal{X}(X) + h \, \boldsymbol{U}(X) = X + h \, \boldsymbol{U}(X), \quad \mathcal{Y}^{A}(X) = \mathcal{X}^{A}(X) + h \, \boldsymbol{U}^{A}(X) = X^{A} + h \, \boldsymbol{U}^{A}(X),$$
(21)

where $h \in \mathbb{R}$ is a smallness parameter and $U \in T_{\mathcal{X}}\mathcal{M}$, it is called an "infinitesimal transformation of the coordinates", in the language of field theory. Omitting the argument X, we can write

$$\mathcal{Y} = \mathcal{X} + h \mathbf{U}, \qquad \mathcal{Y}^A = \mathcal{X}^A + h U^A. \tag{22}$$

The tangent map of \mathcal{Y} in Equation (22) is expressed by

$$T\mathcal{Y} = T\mathcal{X} + h \operatorname{Grad} \boldsymbol{U} = \boldsymbol{I} + h \operatorname{Grad} \boldsymbol{U}, \quad (T\mathcal{Y})^{A}{}_{B} = (T\mathcal{X})^{A}{}_{B} + h U^{A}{}_{|B} = \delta^{A}{}_{B} + h U^{A}{}_{|B},$$
(23)

where I is the material identity tensor and Grad U, with components $U^{A}_{|B}$, is the gradient (or covariant derivative) of U. For $h \to 0$, the Jacobian determinant of $T\mathcal{Y}$ is

$$det(T\mathcal{Y}) = det(\boldsymbol{I} + h \operatorname{Grad} \boldsymbol{U}) = 1 + h \operatorname{Tr} (\operatorname{Grad} \boldsymbol{U}) + o(h)$$
$$= 1 + h \operatorname{Div} \boldsymbol{U} + o(h) = 1 + h U^{A}_{|A} + o(h).$$
(24)

216 2.4 Conventions on Forces and Stresses

As mentioned in Remark 1, in the analytical mechanics / field theory approach, followed by, e.g., Hill (1951) and Eshelby (1975), forces are regarded as *covector* fields, acting on velocity or displacement vector fields. Thus, the contraction of a force with a velocity or displacement is given precisely by (3). Consequently, the first leg of the stress (the "force leg") is a *covector*, while the second leg (the "area leg") is a *vector*. Indeed, in the expression of Cauchy's theorem, the traction vectors relative to the spatial and material elements of area are given by

$$\boldsymbol{t_n} = \boldsymbol{\sigma} \, \boldsymbol{n}, \quad \boldsymbol{t_N} = \boldsymbol{PN}, \qquad (\boldsymbol{t_n})_a = \sigma_a{}^b \, \boldsymbol{n_b}, \quad (\boldsymbol{t_N})_a = P_a{}^b \, N_B. \tag{25}$$

n

In Equation (25), *n* is the normal covector to a surface element at the spatial point $x = \phi(X)$ in the current configuration, *N* is the normal covector to the corresponding surface element at the material point *X* in the reference configuration and the first Piola-Kirchhoff stress is related to Cauchy stress by means of the backward Piola transformation

$$\boldsymbol{P} = J(\boldsymbol{\sigma} \circ \boldsymbol{\phi}) \, \boldsymbol{F}^{-T}, \qquad \boldsymbol{P}_a{}^B = J(\boldsymbol{\sigma}_a{}^b \circ \boldsymbol{\phi}) \, (\boldsymbol{F}^{-T})_b{}^B. \tag{26}$$

Equations (25) and (26) show that the tractions t_n and t_N are indeed *covectors* if the Cauchy stress σ and the first Piola-Kirchhoff stress P, respectively, are treated as "mixed" tensors (we remark that $t_N \neq t_n$, since N is related to n by the formula of the change of area, also known as Nanson's formula; see, e.g., Bonet and Wood, 2008).

3 Eshelby's Original Derivation of the Weak Form

Eshelby (1951) derived the weak form of the expression of the configurational force balance by means of a thought experiment subdivided in several steps. This form is weak as it is an integral equation expressing a *virtual work*. We note that, in this section, we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region \mathcal{D} of the body as a functional on the manifold \mathcal{M} , the Eshelbian configuration space.

Eshelby (1951) considered a body \mathcal{B} , subjected to constraints and external loads, and 238 in whose interior is located a *defect* of any kind: a point defect, a dislocation, an inclusion, 239 or even a region in which the material properties are inhomogeneous. To fix ideas, we 240 follow Eshelby's graphical example with a point defect, as shown in Figure 3. The left 241 panel in Figure 3 shows what Eshelby called the *original* body, in which a region \mathcal{D} 242 (highlighted in dark grey), bounded by the smooth material surface $\Sigma = \partial \mathcal{D}$, is selected 243 such that the defect is contained in \mathcal{D} . The right panel in Figure 3 represents a *replica* of 244 the original body, in which a different region \mathcal{D} (also highlighted in dark grey), bounded 245 by the smooth material surface $\hat{\Sigma} = \partial \hat{D}$, is selected so that the defect is contained in \hat{D} (see 246 also Kienzler and Herrmann, 2000). Since Σ and $\tilde{\Sigma}$ are both smooth, it is always possible 247 to find an Eshelbian configuration \mathcal{Y} transforming \mathcal{D} into \mathcal{D} , i.e., $\mathcal{Y}(\mathcal{D}) = \mathcal{D}$. Moreover, 248 if Σ and $\tilde{\Sigma}$ are "close enough", then $\tilde{\mathcal{D}}$ is obtainable from \mathcal{D} through a perturbation of the 249 form defined in Equation (21), whose domain restriction to \mathcal{D} is 250

$$\mathcal{Y}: \mathcal{D} \to \mathcal{B}: X \mapsto \mathcal{Y}(X) = \mathcal{X}(X) + h U(X), \tag{27}$$

where we recall that *h* is a smallness parameter. Note that Eshelby (1951) chose hUto be a *uniform* material displacement field $hU(X) = -hU_0$ over \mathcal{D} . Eshelby's choice makes the procedure easier to illustrate and yields directly the *strong form* of the inclusion problem. Here, we derive the *weak form* first and then obtain the strong form by adding Eshelby's assumption, $hU(X) = -hU_0$, at the very end. However, it is helpful to keep the uniform displacement $-hU_0$ in one's mind and, to this end, we chose to represent this uniform displacement in Figure 3, following Eshelby's original thought experiment.

We remark that, since the map \mathcal{Y} of Equation (27) is Eshelbian, the body is undergoing 258 *no* deformation, in the sense that it is *not* changing its shape, but only its configuration. 259 Indeed, one chooses the surface Σ enclosing the region \mathcal{D} and the surface $\hat{\Sigma}$ enclosing 260 the region \mathcal{D} independently and then finds a suitable \mathcal{Y} mapping \mathcal{D} into \mathcal{D} . Clearly, this 261 mere fact does *not* displace the defect at all, but simply represents a different choice of 262 enclosing surface. The displacement of the defect in the reference configuration actually 263 takes place when we *replace* the region \mathcal{D} in the original body with the region \mathcal{D} cut from 264 the replica body (which is straightforward in the case of a Eshelby's rigid displacement 265 $-h U_0$, where \mathcal{D} and \mathcal{D} are related by the *material transformation* \mathcal{Y} described by (27). 266 Note that, in this replacement, the defect is moved together with the region $\tilde{\mathcal{D}}$ (see point 267 (iii) below). 268

²⁶⁹ Our goal is to determine the *variation in energy* accompanying this *change in reference* ²⁷⁰ *configuration*. In order to achieve this, we perform the thought experiment proposed by ²⁷¹ Eshelby (1951, 1975) and described below.



Figure 3: Determination of the force on a defect (the solid black circle). Left: original body, with the defect contained in a region \mathcal{D} , bounded by the smooth surface $\Sigma = \partial \mathcal{D}$. Right: replica body, with the defect contained in a different region $\tilde{\mathcal{D}}$, bounded by the smooth surface $\tilde{\Sigma} = \partial \tilde{\mathcal{D}}$. As in Eshelby's original scheme (Eshelby, 1975), here we depict the material displacement h U as being *uniform* over the material region \mathcal{D} enclosed by the surface Σ , i.e., $h U(X) = -h U_0$ for every $X \in \mathcal{D}$.

- (i) In the original body, cut out the material in the region \mathcal{D} . If the body is pre-stressed for any reason, then apply traction forces to the boundary $\Sigma = \partial \mathcal{D}$ of the cavity that has been created, in order to avoid relaxation.
- (ii) Similarly, in the replica body, cut out the material in the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ and apply suitable tractions to the boundary $\tilde{\Sigma} = \partial \tilde{\mathcal{D}} = \partial [\mathcal{Y}(\mathcal{D})] \equiv \mathcal{Y}(\partial \mathcal{D})$ to prevent relaxation. Let us denote the total elastic energy $\mathcal{E}_{\mathcal{D}}^{el} : \mathcal{M} \to \mathbb{R}$ in $\mathcal{Y}(\mathcal{D})$ by

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} W = \int_{\mathcal{D}} \det(T\mathcal{Y}) W \circ \mathcal{Y}, \tag{28}$$

where we used the theorem of the change of variables to transform the integral over the displaced region $\mathcal{Y}(\mathcal{D})$ into an integral over the original region \mathcal{D} . Similarly, in the original region, the total elastic energy would be

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathfrak{X}) = \int_{\mathcal{D}} W = \int_{\mathcal{D}} W \circ \mathfrak{X},\tag{29}$$

where we exploited the identity $\mathcal{X}(X) = X$ in writing $W = W \circ \mathcal{X}$. Therefore, the difference in energy due to the perturbation \mathcal{Y} (i.e., due to the different selection of the surfaces $\tilde{\Sigma}$ and Σ) is

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathcal{Y}) - \mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathcal{X}) = \int_{\mathcal{D}} \det(T\mathcal{Y}) \ W \circ \mathcal{Y} - \int_{\mathcal{D}} W \circ \mathcal{X} = \int_{\mathcal{D}} [\det(T\mathcal{Y}) \ W \circ \mathcal{Y} - W \circ \mathcal{X}].$$
(30)

By expressing the map \mathcal{Y} as $\mathcal{Y} = \mathcal{X} + h U$ (see Equation (21)), considering that, for $h \to 0$, det $T\mathcal{Y} = 1 + h \operatorname{Div} U + o(h)$ (see Equation (24)) and

$$W \circ \mathcal{Y} = W \circ (\mathcal{X} + h \mathbf{U}) = W \circ \mathcal{X} + h [(\operatorname{Grad} W) \circ \mathcal{X}] \mathbf{U} + o(h),$$
(31)

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Equation (30) becomes

$$\mathcal{E}_{\mathcal{D}}^{\text{el}}(\mathcal{X} + h \boldsymbol{U}) - \mathcal{E}_{\mathcal{D}}^{\text{el}}(\mathcal{X}) = \int_{\mathcal{D}} \left[h \left(W \circ \mathcal{X} \right) \text{Div} \, \boldsymbol{U} + h \left[(\text{Grad } W) \circ \mathcal{X} \right] \boldsymbol{U} + o(h) \right]. \tag{32}$$

Now, we can divide both sides of Equation (32) by *h* and take the limit for $h \to 0$ so that, on the left-hand side, we have the *variational* Gâteaux derivative of $\mathcal{E}_{\mathcal{D}}^{el}$ with respect to the material displacement field $U \in T_{\mathcal{X}}\mathcal{M}$, evaluated at the identity map \mathcal{X} , i.e.,

$$(\partial_{U}\mathcal{E}_{\mathcal{D}}^{\mathrm{el}})(\mathcal{X}) = \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathcal{X} + h\mathbf{U}) - \mathcal{E}_{\mathcal{D}}^{\mathrm{el}}(\mathcal{X})}{h} = \int_{\mathcal{D}} \left[(W \circ \mathcal{X}) \operatorname{Div} \mathbf{U} + \left[(\operatorname{Grad} W) \circ \mathcal{X} \right] \mathbf{U} \right].$$
(33)

By using the identities (Grad W) $\circ \mathfrak{X} =$ Grad W and $W \circ \mathfrak{X} = W$, we can write

$$(\partial_{\boldsymbol{U}} \mathcal{E}_{\mathcal{D}}^{\text{el}})(\mathcal{X}) = \int_{\mathcal{D}} \left[\boldsymbol{W} \operatorname{Div} \boldsymbol{U} + \left[\operatorname{Grad} \boldsymbol{W} \right] \boldsymbol{U} \right], \tag{34}$$

which, using by Leibniz' rule and the identity Div(W U) = Div(W I U) (where I is the material identity tensor), becomes

$$(\partial_{\boldsymbol{U}} \mathcal{E}_{\mathcal{D}}^{\text{el}})(\mathcal{X}) = \int_{\mathcal{D}} \text{Div}[\boldsymbol{W} \boldsymbol{I} \boldsymbol{U}].$$
(35)

(iii) Before the deformation ϕ occurs, the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ that had been isolated from the replica body could be "*transplanted*"* into the cavity (resulting from the elimination of the original region \mathcal{D}) in the original body by simply applying the opposite displacement field -h U. In Eshelby's choice of a uniform displacement, this would be the *rigid translation* $h U_0$, as shown in Figure 4. This is as if the defect had been displaced of the amount $h U_0$.

However, after the deformation ϕ occurs, $\phi(\tilde{D}) = \phi(\mathcal{Y}(D))$ from the replica and $\phi(D)$ from the original body are *different* in general, and thus $\phi(\tilde{D}) = \phi(\mathcal{Y}(D))$ may not fit the cavity with deformed surface $\partial[\phi(D)] \equiv \phi(\partial D) = \phi(\Sigma)$ in the original body. Indeed, the points of the deformed surface $\partial[\phi(D)] \equiv \phi(\partial D) = \phi(\Sigma)$ in the original body and the points of the deformed surface $\partial[\phi(\mathcal{Y}(D)]] \equiv \phi(\partial(\mathcal{Y}(D))) = \phi(\partial\tilde{D}) = \phi(\partial\tilde{D}) = \phi(\tilde{\Sigma})$ in the replica body generally differ by the (conventional spatial) displacement

$$\phi(X + h U(X)) - \phi(X) = [F(X)](h U(X)) + o(h), \tag{36}$$

which, recalling that $X + h U(X) = \mathcal{Y}(X)$ and $X = \mathcal{X}(X)$, omitting the argument Xand using the linearity of F, can be written as

$$\phi \circ \mathcal{Y} - \phi \circ \mathcal{X} = h F U + o(h). \tag{37}$$

In order to deform the surface $\phi(\Sigma) = \phi(\partial D)$ of the cavity in the original body in such a way that $\phi(\tilde{D})$ from the replica body can exactly fit in it, we must *adjust* the

^{*}We are borrowing the term "transplant" from Epstein and Maugin (2000) and Imatani and Maugin (2002), but with a more strictly "surgical" meaning.



Figure 4: *Before* the deformation ϕ takes place, the region $\tilde{D} = \mathcal{Y}(D)$ could be transplanted from the replica (right panel) to the original body (left panel), into the cavity resulting from the removal of the original region D, by simply applying the negative of the displacement $-h U_0$. This procedure effectively displaces the defect by the amount $h U_0$ in the original body. We remark that this no longer holds *after* deformation has taken place.

deformation. This can be achieved, in fact, by introducing a new deformation, $\bar{\phi}$, which, applied to $\mathcal{Y}(\mathcal{D}) = \tilde{\mathcal{D}}$, is such that the overall displacement is null, i.e.,

$$\bar{\phi}(\mathcal{Y}(X)) - \phi(X) = \mathbf{0}. \tag{38}$$

Since $\bar{\phi}$ has to adjust ϕ in order to eliminate the mismatch generated by the combined effect of \mathcal{Y} and ϕ (note how the composition $\phi \circ \mathcal{Y}$ is, in fact, the mathematical representation of the "combined effect"), it is natural to define $\bar{\phi}$ as a perturbation of ϕ . Hence, we set

$$\bar{\phi} = \phi + h \,\eta,\tag{39}$$

where, without loss of generality, the same smallness parameter, *h*, is used as that defining $\mathcal{Y} = \mathcal{X} + hU$. With the aid of (39), and in the limit $h \to 0$, Equation (38) becomes

$$\phi \circ (\mathfrak{X} + h \mathbf{U}) + h \boldsymbol{\eta} \circ (\mathfrak{X} + h \mathbf{U}) - \phi \circ \mathfrak{X}$$

= $h \mathbf{F} \mathbf{U} + o(h) + h \boldsymbol{\eta} + h^2 [\boldsymbol{\eta} \circ \mathfrak{X}] \mathbf{U} + o(h^2)$
= $h [\mathbf{F} \mathbf{U} + \boldsymbol{\eta}] + o(h) = \mathbf{0}.$ (40)

At the lowest order, Equation (40) gives the condition sought for η , i.e., that it has to compensate for U, thereby yielding

$$FU + \eta = 0 \qquad \Rightarrow \qquad -h\eta = hFU. \tag{41}$$

This interpretation of the displacement η is the core of Noether's Theorem, which will be addressed in Section 5.

The work necessary to adjust the deformation of $\mathcal{B} \setminus \mathcal{D}$ according to (39) is exerted by the first Piola-Kirchhoff surface traction P(-N) = -PN, where the minus sign 325 326 327

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comes from the fact that we regard *N* as the *outward* normal to the boundary $\Sigma = \partial D$ of D, which is *inward* with respect to the remainder $B \setminus D$ of the body. The integral of this work per unit referential area over the surface $\Sigma = \partial D$ gives what Cermelli et al. (2001) called the "*net work*"

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{Y}) = \int_{\partial \mathcal{D}} (-PN) (-h \eta) + o(h)$$

= $-h \int_{\partial \mathcal{D}} (PN) (FU) + o(h) = -h \int_{\partial \mathcal{D}} [(F^T P)^T U] N + o(h),$ (42)

where we rewrote the covector-vector contraction (FU)(PN) by using the definition of transpose, i.e.,

$$(\boldsymbol{F}\boldsymbol{U})(\boldsymbol{P}\boldsymbol{N}) = F^{a}{}_{A} U^{A} P_{a}{}^{B} N_{B} = (\boldsymbol{P}^{T})^{B}{}_{a} F^{a}{}_{A} U^{A} N_{B} = [(\boldsymbol{F}^{T} \boldsymbol{P})^{T}]^{B}{}_{A} U^{A} N_{B}$$
$$= [(\boldsymbol{F}^{T} \boldsymbol{P})^{T} \boldsymbol{U}] \boldsymbol{N}.$$
(43)

Note that, for the sake of a lighter notation, we are writing F^T and P^T for $F^T \circ \phi$ 331 and $P^T \circ \phi$. Rigorously speaking, the composition by ϕ would be necessary, since 332 F^T and P^T are defined in the current configuration $\phi(\mathcal{B})$ (Marsden and Hughes, 333 1983). Since N is the *outward* normal to $\Sigma = \partial D$, the net work (42) is the *negative* 334 of the work that the Piola tractions **P**N would exert over the displacement $-h\eta$ 335 of Equation (41) on the referential surface $\Sigma = \partial \mathcal{D}$, seen as the boundary of the 336 referential region \mathcal{D} . This observation allows us to apply the divergence theorem to 337 (42), which yields 338

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{Y}) = -h \int_{\mathcal{D}} \operatorname{Div}\left[(\boldsymbol{F}^{T} \boldsymbol{P})^{T} \boldsymbol{U} \right] + o(h).$$
(44)

This can be made into an increment by expressing the map \mathcal{Y} as $\mathcal{Y} = \mathcal{X} + h U$, and considering that $\mathcal{E}_{\mathcal{D}}^{nw}(\mathcal{X}) = 0$, i.e.,

$$\mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{X} + h \boldsymbol{U}) - \mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{X}) = -h \int_{\mathcal{D}} \mathrm{Div}\left[(\boldsymbol{F}^{T} \boldsymbol{P})^{T} \boldsymbol{U}\right] + o(h).$$
(45)

Now, dividing by *h* and passing to the limit $h \rightarrow 0$, we obtain the functional directional derivative

$$(\partial_{\boldsymbol{U}}\mathcal{E}_{\mathcal{D}}^{\mathrm{nw}})(\mathcal{X}) = \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{X} + h\boldsymbol{U}) - \mathcal{E}_{\mathcal{D}}^{\mathrm{nw}}(\mathcal{X})}{h} = -\int_{\mathcal{D}} \operatorname{Div}\left[(\boldsymbol{F}^{T}\boldsymbol{P})^{T}\boldsymbol{U}\right].$$
(46)

(iv) The deformed transformed region $\phi(\tilde{D}) = \phi(\mathcal{Y}(D))$ from the replica body can finally be exactly suited into the cavity left by the removal of D in the original body and we are able to weld together across the interface. We note that Eshelby (1975) needs to make considerations on the infinitesimals of order greater than *h*. In our approach, these are automatically taken care of (and eliminated) by the limit operation in Equation (46). To cite Eshelby (1975) verbatim, except for using our notation for the displacement, "We are now left with the system as it was to begin with, except that the defect has been shifted by $-h U = h U_0$, as required."

The associated variation in the total energy $\mathcal{E}_{\mathcal{D}} : \mathcal{M} \to \mathbb{R}$ of the system is obtained as $\mathcal{E}_{\mathcal{D}} = \mathcal{E}_{\mathcal{D}}^{\text{el}} + \mathcal{E}_{\mathcal{D}}^{\text{nw}}$, i.e., by summing Equations (35) and (46), i.e.,

$$(\partial_{\boldsymbol{U}} \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \operatorname{Div} \left[\boldsymbol{W} \boldsymbol{I} \boldsymbol{U} \right] - \int_{\mathcal{D}} \operatorname{Div} \left[(\boldsymbol{F}^{T} \boldsymbol{P})^{T} \boldsymbol{U} \right], \tag{47}$$

which can be written as

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$$(\partial_U \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \operatorname{Div} \left[\mathfrak{E}^T U \right] = \int_{\partial \mathcal{D}} (\mathfrak{E}N) U.$$
(48)

Equation (48) quantifies the variation in energy necessary to obtain a new reference configuration in which the defect is displaced in direction U with respect to the original one. In the context of the theory of defects, Eshelby (1951) called the tensor \mathfrak{E} , with the expression

$$\boldsymbol{\mathfrak{G}} = W \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}, \qquad \boldsymbol{\mathfrak{G}}_A{}^B = W \,\delta_A{}^B - F^a{}_A P_a{}^B, \tag{49}$$

the *Maxwell tensor of elasticity* and later (Eshelby, 1975) the *energy-momentum tensor*, in analogy with Maxwell's terminology from field theory. This analogy will be completely clear in Section 4. Later, Maugin and Trimarco (1992) gave **©** the name of *Eshelby stress* in his honour.

At the end of Eshelby's thought experiment, we have the expression in Equation (48), which can be thought of as the *virtual work* exerted by the Eshelby tractions $\mathfrak{G}N$ on the material displacement field U on the boundary $\partial \mathcal{D}$ of the region \mathcal{D} . Using Eshelby's assumption $U(X) = -U_0$ for every $X \in \mathcal{D}$, we can write Equation (48) as

$$(\partial_{-\boldsymbol{U}_0} \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\int_{\mathcal{D}} (\text{Div}\,\boldsymbol{\mathfrak{G}})\,\boldsymbol{U}_0 = -\int_{\partial\mathcal{D}} (\boldsymbol{\mathfrak{G}}N)\,\boldsymbol{U}_0.$$
(50)

In order to obtain (in our notation) Equation (17) in the paper by Eshelby (1951), we use
 Cartesian coordinates, so that it is legitimate to rewrite the integral as

$$\mathcal{F}\boldsymbol{U}_{0} = (\partial_{-\boldsymbol{U}_{0}}\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\left(\int_{\mathcal{D}} \operatorname{Div}\boldsymbol{\mathfrak{E}}\right)\boldsymbol{U}_{0} = -\left(\int_{\partial\mathcal{D}}\boldsymbol{\mathfrak{E}}\,\boldsymbol{N}\right)\boldsymbol{U}_{0},\tag{51}$$

where \mathcal{F} was *defined* by Eshelby as the *total* inhomogeneity force, producing work over the uniform virtual displacement U_0 . We remark that the total inhomogeneity force \mathcal{F} can only be defined in the case of Cartesian coordinates, which is the only particular case in which integration of a vector field makes sense (see warning at page 134 in the text by Marsden and Hughes, 1983).

4 Eshelby's Variational Derivation of the Strong Form

In his seminal paper, Eshelby (1975) used a variational approach and wrote the Euler-375 Lagrange equations for a generic system with a potential energy depending - in the 376 language of classical field theory - on fields, "gradients" of fields and coordinates. In 377 this quite general framework, Elasticity can be seen as a particular case. Here, we follow 378 Eshelby's derivation (Eshelby, 1975) step by step, using our notation and adding our 379 comments. Then, we shall show how this specialises to the case of large- and small-380 deformation Elasticity. The only difference with Eshelby's procedure is that, whenever 381 we look at the variational problem as an elasticity problem, our fields are the components 382 of the configuration map, rather than the components of the displacement. Note that, in 383 contrast with Section 3, here we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region \mathcal{D} of the body as 384 a functional on the manifold C, the conventional configuration space. 385

Let us assume a potential energy density *W*, defined per unit referential volume, given by

$$W(X) = \hat{W}(\phi(X), \boldsymbol{F}(X), X), \tag{52}$$

where ϕ is a collection of scalar fields (in the case of continuum mechanics, the con-388 figuration map, with components ϕ^a), F is the collection of the gradients of the fields 389 (in our case, the deformation gradient, with components $F^a{}_A = \phi^a{}_A$), and X is the 390 collection of the independent variables (in our case, the material coordinates X^A). Note 391 that we *distinguish* between the *scalar field* W (function of the coordinates X^A) and the 392 associated constitutive function \hat{W} (function of the fields ϕ^a , the gradients $F^a{}_A = \phi^a{}_A$ 393 and the coordinates X^A). By using the material identity map \mathfrak{X} of Equation (17) (such 394 that $X = \mathcal{X}(X)$, in components, $X^A = \mathcal{X}^A(X)$), the potential energy can be rewritten in 395 the form 396

$$W(X) = \hat{W}(\phi(X), F(X), \mathcal{X}(X)) = [\hat{W} \circ (\phi, F, \mathcal{X})](X).$$
(53)

Thus, by dropping the argument X on the far left and the far right sides, we have

$$W = \hat{W} \circ (\phi, F, \mathcal{X}). \tag{54}$$

In order to find the Euler-Lagrange equations associated with $W = \hat{W} \circ (\phi, F, \mathcal{X})$, we need to consider the total energy $\mathcal{E}_{\mathcal{B}} : \mathcal{C} \to \mathbb{R}$ over the whole body \mathcal{B} , given by

$$\mathcal{E}_{\mathcal{B}}(\phi) = \int_{\mathcal{B}} W = \int_{\mathcal{B}} \hat{W} \circ (\phi, \boldsymbol{F}, \boldsymbol{\mathcal{X}}),$$
(55)

and calculate its variation with respect to a conventional displacement η , which is given by the Gâteaux derivative

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{B}}(\phi + h \boldsymbol{\eta}) - \mathcal{E}_{\mathcal{B}}(\phi)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} \int_{\mathcal{B}} \left[\hat{W} \circ (\phi + h \boldsymbol{\eta}, \boldsymbol{F} + h \operatorname{Grad} \boldsymbol{\eta}, \boldsymbol{\mathcal{X}}) - \hat{W} \circ (\phi, \boldsymbol{F}, \boldsymbol{\mathcal{X}}) \right],$$
(56)

with η chosen in a suitable subset of $T_{\phi}C \cap C^1(\mathcal{B}, TS)$, as will be clarified later in this section. In the jargon of field theory, this is called a "variation on the fields, with frozen coordinates", i.e., we are going to calculate the integral on the *fixed* domain \mathcal{B} . The transformation on the configuration map ϕ (the "fields" ϕ^a) is given by

$$\phi \mapsto \bar{\phi} = \phi + h \, \eta, \tag{57a}$$

$$\phi^a \mapsto \bar{\phi}^a = \phi^a + h \eta^a, \tag{57b}$$

and the transformation on the tangent map $T\phi = F$ (the "gradients" $F^a{}_A = \phi^a{}_A$) is

$$T\phi = F \mapsto T\bar{\phi} = \bar{F} = T(\phi + h\eta) = F + h\operatorname{Grad}\eta,$$
(58a)

$$\phi^{a}_{,A} = F^{a}_{\ A} \mapsto \bar{F}^{a}_{\ A} = \bar{\phi}^{a}_{\ A} = \phi^{a}_{\ A} + h \eta^{a}_{\ |A} = F^{a}_{\ A} + h \eta^{a}_{\ |A}, \tag{58b}$$

where Grad η , with components $\eta^a|_A$, is the covariant derivative of the displacement η .

408 We follow the standard derivation by expanding the argument of the integral as

$$\hat{W} \circ (\phi + h \eta, F + h \operatorname{Grad} \eta, \mathfrak{X}) - \hat{W} \circ (\phi, F, \mathfrak{X}) =$$

$$= \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, F, \mathfrak{X}) h \eta^{a} + \frac{\partial \hat{W}}{\partial F^{a}{}_{A}} \circ (\phi, F, \mathfrak{X}) h \eta^{a}{}_{|A} + o(h), \quad (59)$$

⁴⁰⁹ substituting in (56) and performing the limit, which results in

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = \int_{\mathcal{B}} \left[\frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) \eta^{a} + \frac{\partial \hat{W}}{\partial F^{a}{}_{A}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) \eta^{a}{}_{|A} \right]$$
$$= \int_{\mathcal{B}} \left[-f_{a} \eta^{a} + P_{a}{}^{A} \eta^{a}{}_{|A} \right] = \int_{\mathcal{B}} \left[-f \boldsymbol{\eta} + \boldsymbol{P} : \operatorname{Grad} \boldsymbol{\eta} \right], \qquad (60)$$

410 where f and P are given by

$$f_a = -\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, F, \mathcal{X}), \qquad \qquad f = -\frac{\partial \hat{W}}{\partial \phi} \circ (\phi, F, \mathcal{X}) \tag{61a}$$

$$P_a{}^A = \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, F, \mathcal{X}), \qquad \qquad P = \frac{\partial \hat{W}}{\partial F} \circ (\phi, F, \mathcal{X}). \tag{61b}$$

In the case of elasticity in continuum mechanics, when the potential is given as the sum of an elastic potential and a potential of the external body forces, i.e.,

$$\hat{W} \circ (\phi, F, \mathcal{X}) = \hat{W}_{el} \circ (F, \mathcal{X}) + \hat{W}_{ext} \circ (\phi, \mathcal{X}),$$
(62)

the covector field f and the tensor field P take the meaning of external body force per unit volume and first Piola-Kirchhoff stress, respectively. Now, considering that

$$\boldsymbol{P}: \operatorname{Grad} \boldsymbol{\eta} = P_a{}^A \eta^a{}_{|A} = (P_a{}^A \eta^a)_{|A} - P_a{}^A{}_{|A} \eta^a = \operatorname{Div}(\boldsymbol{P}^T \boldsymbol{\eta}) - (\operatorname{Div} \boldsymbol{P}) \boldsymbol{\eta}, \quad (63)$$

415 the variation becomes

$$(\partial_{\eta} \mathcal{E}_{\mathcal{B}})(\phi) = \int_{\mathcal{B}} \left[-f \eta + \operatorname{Div}(\eta P) - (\operatorname{Div} P) \eta \right]$$
$$= -\int_{\mathcal{B}} (f + \operatorname{Div} P) \eta + \int_{\mathcal{B}} \operatorname{Div}(\eta P)$$
(64)

and, by applying Gauss' divergence theorem,

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = -\int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P}) \,\boldsymbol{\eta} + \int_{\partial \mathcal{B}} (\boldsymbol{P}N) \,\boldsymbol{\eta}, \tag{65}$$

where *N* is the normal to the boundary $\partial \mathcal{B}$ and $(PN)\eta = \eta (PN)$.

We now look for a configuration ϕ at which $\mathcal{E}_{\mathcal{B}}(\phi)$ is stationary. For this purpose, we impose the condition $(\partial_{\eta}\mathcal{E}_{\mathcal{B}})(\phi) = 0$, in which ϕ is unknown, and we study it under the restriction that η vanish on $\partial \mathcal{B}$ (Hill, 1951). This choice annihilates the surface integral on the right-hand-side of (65), so that the stationarity condition becomes

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\boldsymbol{\phi}) = -\int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P}) \,\boldsymbol{\eta} = 0, \qquad \boldsymbol{\eta} \in \mathcal{V}, \tag{66}$$

where $\mathcal{V} := \{ \eta \in T_{\phi}\mathcal{C} \cap C^{1}(\mathcal{B}, T\mathcal{S}) : \eta(X) = \mathbf{0}, \forall X \in \partial \mathcal{B} \}$. We require now that (66) be satisfied for all $\eta \in \mathcal{V}$, which leads to the Euler-Lagrange equations

$$f + \text{Div } P = 0, \qquad f_a + P_a{}^A{}_{|A} = 0.$$
 (67)

If the external body forces acting on \mathcal{B} are only those given by f, which admit the potential density $\hat{W}_{\text{ext}} \circ (\phi, \mathfrak{X})$, Equation (67) represents, in continuum mechanics, the Lagrangian (static) equilibrium equations, i.e., spatial equations described in terms of the material coordinates. If ϕ is a solution to (67), and the boundary of \mathcal{B} can be written as the disjoint union of a Dirichlet part and a Neumann part, i.e., $\partial \mathcal{B} = \partial_D \mathcal{B} \sqcup \partial_N \mathcal{B}$, then the variation $(\partial_{\eta} \mathcal{E}_{\mathcal{B}})(\phi)$ in Equation (65) becomes

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = \int_{\partial \mathcal{B}} (\boldsymbol{P} N) \, \boldsymbol{\eta} = \int_{\partial_{N} \mathcal{B}} (\boldsymbol{P} N) \, \boldsymbol{\eta}, \tag{68}$$

where the surface integral is restricted to the Neumann boundary, $\partial_N \mathcal{B}$, because the 430 displacement η , although being arbitrary, has to vanish on the Dirichlet boundary, $\partial_D \mathcal{B}$. 431 In this case, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ requires the vanishing of the surface integral 432 on the far right-hand-side of Equation (65). This can be obtained if $\partial_N \mathcal{B}$ is a set of null 433 measure, or if no contact forces are applied onto $\partial_N \mathcal{B}$. On the contrary, when contact 434 forces are present, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ must be corrected by requiring that 435 $(\partial_n \mathcal{E}_{\mathcal{B}})(\phi)$ be balanced by the work performed by the contact forces on η . This result 436 follows from the extended Hamilton's Principle (dell'Isola and Placidi, 2011). 437

If Equation (65) is referred to a set $\mathcal{D} \subset \mathcal{B}$, and is evaluated for a configuration ϕ solving (67), the volume integral vanishes by virtue of the Euler-Lagrange equations, while internal contact forces are exchanged through $\partial \mathcal{D}$. In this case, η is not required to vanish on $\partial \mathcal{D}$, and the variational procedure leads to

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{D}})(\phi) = \int_{\partial \mathcal{D}} (\boldsymbol{P}N) \,\boldsymbol{\eta},\tag{69}$$

thereby returning the virtual work exerted by the contact forces acting on ∂D .

Let us now assume that ϕ satisfies the Euler-Lagrange equations (67), and let us take the material gradient Grad W of the energy density W, i.e., the partial derivatives of W with respect to X^B ,

$$W_{,B} = \left[\hat{W} \circ (\phi, F, \mathfrak{X})\right]_{,B}$$

$$= \left[\frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, F, \mathfrak{X})\right] \phi^{a}_{,B} + \left[\frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\phi, F, \mathfrak{X})\right] F^{a}_{A|B} + \frac{\partial \hat{W}}{\partial \mathfrak{X}^{B}} \circ (\phi, F, \mathfrak{X})$$

$$= -f_{a} F^{a}_{\ B} + P_{a}^{\ A} F^{a}_{\ A|B} + \frac{\partial \hat{W}}{\partial \mathfrak{X}^{B}} \circ (\phi, F, \mathfrak{X}), \tag{70}$$

where we used the definitions of the components of the deformation gradient, $F^a{}_A = \phi^a{}_A$, 446 of the body force and the first Piola-Kirchhoff stress, and $F^{a}{}_{A|B}$ are the components of 447 the third-order two-point tensor Grad F. The last term in Equation (70) is usually called 448 "explicit" gradient of the field W and denoted $(\partial W/\partial X^B)|_{expl}$ in the literature (e.g., 449 Eshelby, 1975; Epstein and Maugin, 1990), whereas we regard it as the collection of the 450 partial derivatives of the constitutive function \hat{W} with respect to \mathfrak{X}^{B} (which, we recall, 451 are the functions such that $\chi^B(X) = X^B$). The negative of the "explicit" gradient defines 452 the material inhomogeneity force or configurational force 453

$$\mathfrak{F} = -\frac{\partial \hat{W}}{\partial \mathfrak{X}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}), \qquad \mathfrak{F}_A = -\frac{\partial \hat{W}}{\partial \mathfrak{X}^A} \circ (\phi, \boldsymbol{F}, \mathfrak{X}). \tag{71}$$

Substituting the expressions of the Lagrangian force f, the Piola-Kirchhoff stress P, and the configurational force \mathfrak{F} into Equation (70), we obtain

Grad
$$W = -F^T f + P$$
: Grad $F - \mathfrak{F}$, (72)

where the double contraction ":" in the second term is of the two legs of P with the first two legs of Grad F. By invoking the symmetry of the Christoffel symbols Γ_{BC}^{A} associated with the Levi-Civita Connection induced by the material metric G, so that $F^{a}{}_{A|B} = F^{a}{}_{B|A}$, we work out the second term on the right-hand-side of (72) in components, i.e.,

$$P_a{}^A F^a{}_{A|B} = P_a{}^A F^a{}_{B|A} = (P_a{}^A F^a{}_B)_{|A} - P_a{}^A{}_{|A} F^a{}_B,$$
(73)

⁴⁶⁰ which, in component-free notation, reads

$$\boldsymbol{P}: \operatorname{Grad} \boldsymbol{F} = \operatorname{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \boldsymbol{F}^T \operatorname{Div} \boldsymbol{P}.$$
(74)

⁴⁶¹ By substituting this result into (72), we obtain

Grad
$$W = -F^T f + \text{Div}(F^T P) - F^T \text{Div}P - \mathfrak{F}$$

$$= -F^T [f + \text{Div}P] + \text{Div}(F^T P) - \mathfrak{F}.$$
(75)

⁴⁶² Moreover, using the Euler-Lagrange equation (67) yields

$$\operatorname{Grad} W = \operatorname{Div}(\boldsymbol{F}^T \boldsymbol{P}) - \boldsymbol{\mathfrak{F}}.$$
(76)

Finally, by virtue of the identity Grad $W = \text{Div}(WI^T)$, where I is the material identity tensor, Equation (76) becomes

$$\mathfrak{F} + \operatorname{Div} \mathfrak{E} = \mathbf{0}, \qquad \mathfrak{F}_A + \mathfrak{E}_A{}^B{}_{|B} = \mathbf{0}, \tag{77}$$

where \mathfrak{E} is the Eshelby stress defined as in Equation (49).

Similarly to other field theories, like Electromagnetism or General Relativity, the ten-466 sor & defined in (49) plays the role of the ("spatial" part of the) energy-momentum tensor 467 of the theory under study. However, we emphasise that, while & has been obtained with 468 the aid of a variational argument in the present framework, more general approaches exist, 469 in which **&** is introduced as a primary dynamical quantity (Gurtin, 1995). Equation (77) 470 is called material equilibrium equation or configurational equilibrium equation (Gurtin, 471 1995), by analogy with the equilibrium equation (67) described by the Euler-Lagrange 472 equations. 473

According to Equation (71), if the body \mathcal{B} is homogeneous, then we have

$$\mathfrak{F}_{A}(X) = -\left[\frac{\partial \hat{W}}{\partial \mathfrak{X}^{A}} \circ (\phi, \boldsymbol{F}, \mathfrak{X})\right](X) = 0, \qquad \forall X \in \mathcal{B},$$
(78)

and Equation (77) implies the vanishing of the divergence of the Eshelby stress. On the contrary, if there is *any* inhomogeneity in \mathcal{D} (i.e., the derivative $\partial \hat{W} / \partial \mathcal{X}^A$ is nonvanishing), this will be captured by the integral of the traction forces **C** *N* of the Eshelby stress over the boundary $\partial \mathcal{D}$.

We now show that Equation (48) yields the *weak formulation* of the *strong form* described in Equation (77). This is easy to see by referring to Equation (51), which we obtained from Equation (48) (or Equation (50)) by working in Cartesian coordinates and using Eshelby's displacement $U = -U_0$, constant over \mathcal{D} . Indeed, by solving the material equilibrium equation (77) for \mathfrak{F} , using Cartesian coordinates, integrating over \mathcal{D} , applying Gauss' theorem and contracting both sides with U_0 , we obtain the *total configurational force* on the region \mathcal{D} as the covector \mathcal{F} such that

$$\mathcal{F}\boldsymbol{U}_{0} = -\left(\int_{\mathcal{D}} \operatorname{Div}\boldsymbol{\mathfrak{E}}\right)\boldsymbol{U}_{0} = -\left(\int_{\partial \mathcal{D}}\boldsymbol{\mathfrak{E}}N\right)\boldsymbol{U}_{0} = \left(\int_{\mathcal{D}}\boldsymbol{\mathfrak{F}}\right)\boldsymbol{U}_{0},\tag{79}$$

⁴⁸⁶ i.e., \mathcal{F} is the integral of the inhomogeneity force density \mathfrak{F} , as we see by comparing ⁴⁸⁷ with Equation (51). Note that, if the body \mathcal{D} is homogeneous, Equations (77) and (78) ⁴⁸⁸ imply the vanishing of the divergence of the Eshelby stress, and therefore the vanishing ⁴⁸⁹ of the volume integral and the equivalent surface integral on the right-hand-side of ⁴⁹⁰ Equation (79).

5 Derivation of the Weak Form with Noether Theorem

⁴⁹² In Noether's Theorem, we need to contemporarily transform the domain and perform a ⁴⁹³ variation on the arguments of the Lagrangian. In the jargon of classical field theory, these are called a *transformation of the coordinates* (material coordinates, in our case) and a *variation of the fields*, respectively. Together, these give the *total variation*. We have already shown the transformation of the material coordinates in Section 2.3 and the variation on the fields in Section 4 and we turn now to the total variation. Then, we apply Noether's theorem to *directly* obtain Eshelby's results. In the application of Noether's Theorem, we define the total energy $\mathcal{E}_{\mathcal{D}}$ of a region \mathcal{D} as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$.

501 5.1 Total Variation

In the language of field theory, the *total variation* is obtained by evaluating the *variation of the fields at frozen coordinates* given in (57) and (58) at the transformed points $\tilde{X} = \mathcal{Y}(X)$, where $\mathcal{Y} = \mathcal{X} + h U : \mathcal{B} \to \tilde{\mathcal{B}}$ is the *infinitesimal transformation of the coordinates* defined in (21), with $U \in T_{\mathcal{X}} \mathcal{M}$. In order to avoid confusion, some care must be exercised.

We recall that the manifold C is the configuration space of the body \mathcal{B} , a configuration ϕ is an element of C and a displacement field η is a tangent vector of $T_{\phi}C$. Let us denote by \tilde{C} the configuration space of the "perturbed" body $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B})$, to which the points $\tilde{X} = \mathcal{Y}(X)$ belong. Consider the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ and the *restriction* of the configuration ϕ and the displacement field η defined in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ (see Figure 5). In this restriction, it is legitimate to evaluate ϕ and η at \tilde{X} .



Figure 5: A domain \mathcal{D} (dark grey) in the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ between the body \mathcal{B} (solid grey) and the perturbed body $\tilde{\mathcal{B}}$ (transparent grey).

We now define the total variation $\mathcal{C} \to \tilde{\mathcal{C}} : \phi \mapsto \bar{\phi}$ by evaluating the *variations of the fields at frozen coordinates* of Equations (57) and (58) at $\tilde{X} \in \tilde{\mathcal{B}} \cap \mathcal{B}$, i.e., we define

$$\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h \eta(\tilde{X}), \qquad \qquad \bar{\phi}^a(\tilde{X}) = \phi^a(\tilde{X}) + h \eta^a(\tilde{X}), \qquad (80)$$

$$\bar{F}(\tilde{X}) = F(\tilde{X}) + h (\operatorname{Grad} \eta)(\tilde{X}), \qquad \bar{F}^a{}_A(\tilde{X}) = F^a{}_A(\tilde{X}) + h \eta^a{}_{|A}(\tilde{X}), \qquad (81)$$

where *h* is, with no loss of generality, the same smallness parameter as $\mathcal{Y} = \mathcal{X} + h U$. To obtain the final form of the total variation, we substitute the transformation (21) of the coordinates into the variations on the configuration (80) and on the tangent (81) of the ⁵¹⁷ configuration, respectively, and use Taylor expansion. For the configuration, we have

$$\bar{\phi}(\tilde{X}) = \phi(X + h U(X)) + h \eta(X + h U(X))$$

$$= \phi(X) + h F(X) U(X) + h \eta(X) + o(h), \qquad (82a)$$

$$\bar{\phi}^{a}(\tilde{X}) = \phi^{a}(X + h U(X)) + h \eta^{a}(X + h U(X))$$

$$= \phi^{a}(X) + h F^{a}{}_{B}(X) U^{B}(X) + h \eta^{a}(X) + o(h),$$
(82b)

from which, using $\bar{\phi}(\tilde{X}) = \bar{\phi}(\mathcal{Y}(X)) = (\bar{\phi} \circ \mathcal{Y})(X)$ and omitting the argument X, we have

$$\bar{\phi} \circ \mathcal{Y} = \phi + h\left(\boldsymbol{\eta} + \boldsymbol{F} \boldsymbol{U}\right) + o(h) = \phi + h \boldsymbol{w} + o(h), \tag{83a}$$

$$\bar{\phi}^{a} \circ \mathcal{Y} = \phi^{a} + h \left(\eta^{a} + F^{a}{}_{B} U^{B} \right) + o(h) = \phi^{a} + h w^{a} + o(h), \tag{83b}$$

519 where

$$\boldsymbol{w} = \boldsymbol{\eta} + \boldsymbol{F} \boldsymbol{U}, \qquad \boldsymbol{w}^{a} = \boldsymbol{\eta}^{a} + \boldsymbol{F}^{a}{}_{B} \boldsymbol{U}^{B}.$$
(84)

⁵²⁰ For the tangent map, we have

$$F(\tilde{X}) = F(X + h U(X)) + h (\operatorname{Grad} \eta)(X + h U(X))$$

= $F(X) + h (\operatorname{Grad} F)(X) U(X) + h (\operatorname{Grad} \eta)(X) + o(h),$ (85a)
 $\bar{F}^{a}{}_{A}(\tilde{X}) = F^{a}{}_{A}(X + h U(X)) + h \eta^{a}{}_{|A}(X + h U(X))$
= $F^{a}{}_{A}(X) + h F^{a}{}_{A|B}(X) U^{B}(X) + h \eta^{a}{}_{|A}(X) + o(h),$ (85b)

521 and thus,

$$\overline{F} \circ \mathcal{Y} = F + h (\operatorname{Grad} \eta + (\operatorname{Grad} F) U) + o(h) = F + h Y + o(h),$$
(86a)

$$\bar{F}^{a}{}_{A} \circ \mathcal{Y} = F^{a}{}_{A} + h\left(\eta^{a}{}_{|A} + F^{a}{}_{A|B} U^{B}\right) + o(h) = F^{a}{}_{A} + hY^{a}{}_{A} + o(h),$$
(86b)

522 where

$$\boldsymbol{Y} = \operatorname{Grad} \boldsymbol{\eta} + (\operatorname{Grad} \boldsymbol{F}) \boldsymbol{U}, \qquad \boldsymbol{Y}^{a}{}_{A} = \boldsymbol{\eta}^{a}{}_{|A} + \boldsymbol{F}^{a}{}_{A|B} \boldsymbol{U}^{B}.$$
(87)

523 5.2 Variation of the Total Energy

Since we are working in the static case, we replace the action functional and the Lagrangian density with the total energy functional \mathcal{E} and the potential energy density W. The total energy in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ is a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$, i.e.,

$$\mathcal{E}_{\mathcal{D}}: \mathcal{C} \times \mathcal{M} \to \mathbb{R}: (\phi, \mathcal{Y}) \mapsto \mathcal{E}_{\mathcal{D}}(\phi, \mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} \hat{W} \circ (\phi, F, \mathcal{X}),$$
(88)

where the integration domain $\mathcal{Y}(\mathcal{D})$ must belong to the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$. We now consider the coordinate transformation $\mathcal{Y} = \mathcal{X} + h U$, where $U \in T_{\mathcal{X}} \mathcal{M}$ is a tangent vector at the identity \mathcal{X} , and the field transformation is $\bar{\phi} = \phi + h \eta$, where $\eta \in T_{\phi} \mathcal{C}$ is a tangent vector at the configuration ϕ . The variation of the energy is given by the directional derivative

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathfrak{X}}) = \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{D}}(\bar{\boldsymbol{\phi}},\boldsymbol{\mathfrak{Y}}) - \mathcal{E}_{\mathcal{D}}(\boldsymbol{\phi},\boldsymbol{\mathfrak{X}})}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} \left[\int_{\boldsymbol{\mathfrak{Y}}(\mathcal{D})} \hat{W} \circ (\bar{\boldsymbol{\phi}},\bar{\boldsymbol{F}},\boldsymbol{\mathfrak{X}}) - \int_{\mathcal{D}} \hat{W} \circ (\boldsymbol{\phi},\boldsymbol{F},\boldsymbol{\mathfrak{X}}) \right], \qquad (89)$$

evaluated at the conventional configuration ϕ and Eshelbian configuration \mathcal{X} , with respect to the pair of tangent vectors $(\boldsymbol{\eta}, \boldsymbol{U}) \in T_{(\phi, \mathcal{X})}(\mathcal{C} \times \mathcal{M})$ in the product manifold $\mathcal{C} \times \mathcal{M}$. Note also that, in the second integral, we used $\mathcal{X}(\mathcal{D}) = \mathcal{D}$.

Application of the theorem of the change of variables on the first integral in (89) yields

535

$$\int_{\mathcal{Y}(\mathcal{D})} \hat{W} \circ (\bar{\phi}, \bar{F}, \mathcal{X}) = \left[\int_{\mathcal{D}} (1 + h \operatorname{Div} \boldsymbol{U}) \, \hat{W} \circ (\bar{\phi}, \bar{F}, \mathcal{X}) \circ \mathcal{Y} \right] + o(h), \tag{90}$$

where the determinant $det(T\mathcal{Y}) = 1 + h \operatorname{Div} U + o(h)$ follows from Equation (24). We now notice that

$$\hat{W} \circ (\bar{\phi}, \bar{F}, \mathfrak{X}) \circ \mathfrak{Y} = \hat{W} \circ (\bar{\phi} \circ \mathfrak{Y}, \bar{F} \circ \mathfrak{Y}, \mathfrak{X} \circ \mathfrak{Y})$$
$$= \hat{W} \circ (\phi + h w + o(h), F + h Y + o(h), \mathfrak{X} + h U), \tag{91}$$

where we made use of the total variations (83) and (86), as well as of the identity $\chi \circ \mathcal{Y} = \mathcal{Y} = \mathcal{X} + h U$. Now, we expand in Taylor series up to the first order, and obtain

$$\hat{W} \circ (\bar{\phi}, \bar{F}, \mathfrak{X}) \circ \mathcal{Y} = \hat{W} \circ (\phi, F, \mathfrak{X}) + \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, F, \mathfrak{X}) h w^{a} + \frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\phi, F, \mathfrak{X}) h Y^{a}_{A} + \frac{\partial \hat{W}}{\partial \mathfrak{X}^{B}} \circ (\phi, F, \mathfrak{X}) h U^{B} + o(h).$$
(92)

⁵⁴⁰ Using Equations (90), (91) and (92) in the variation of the energy (89), we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathfrak{X}) = \lim_{h \to 0} \frac{1}{h} \left[\int_{\mathcal{D}} h \left(\hat{W} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) U^{B}_{|B} + \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) w^{a} + \frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) Y^{a}_{A} + \frac{\partial \hat{W}}{\partial \mathfrak{X}^{B}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) U^{B} \right] + o(h) \right].$$
(93)

The smallness parameter cancels out and the term o(h) disappears in the limit $h \rightarrow 0$. Thus, we write

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left(\hat{W} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. \boldsymbol{U}^{B} \right|_{B} + \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. \boldsymbol{w}^{a} + \frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. \boldsymbol{Y}^{a}_{A} + \frac{\partial \hat{W}}{\partial \mathcal{X}^{B}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. \boldsymbol{U}^{B} \right) \right\}, \quad (94)$$

and we use the explicit expressions (84) and (87) of the total variations w and Y:

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left(\hat{W} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. U^{B}_{|B} + \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left(\eta^{a} + F^{a}_{|B} \left. U^{B} \right) + \frac{\partial \hat{W}}{\partial F^{a}_{|A}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left(\eta^{a}_{|A} + F^{a}_{|A|B} \left. U^{B} \right) + \frac{\partial \hat{W}}{\partial \mathcal{X}^{B}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. U^{B} \right) \right).$$
(95)

544 Since

$$(\hat{W} \circ (\phi, \boldsymbol{F}, \mathfrak{X}))_{,B} = \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) F^{a}{}_{B} + \frac{\partial \hat{W}}{\partial F^{a}{}_{A}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) F^{a}{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathfrak{X}^{A}} \circ (\phi, \boldsymbol{F}, \mathfrak{X}) \delta^{A}{}_{B},$$
(96)

545 we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left(\hat{W} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. U^{B}_{|B} + (\hat{W} \circ (\phi,\boldsymbol{F},\mathcal{X}))_{,B} \left. U^{B} + \frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \right. \eta^{a} + \frac{\partial \hat{W}}{\partial F^{a}_{|A}} \circ (\phi,\boldsymbol{F},\mathcal{X}) \left. \eta^{a}_{|A} \right) \right|_{A} \right).$$
(97)

⁵⁴⁶ Using Leibniz' rule in the first two terms and in the last two terms and separating the ⁵⁴⁷ integrals, we have

$$\begin{aligned} (\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathfrak{X}}) &= \int_{\mathcal{D}} \left[\left(\hat{W} \circ (\boldsymbol{\phi},\boldsymbol{F},\boldsymbol{\mathfrak{X}}) \ \boldsymbol{U}^{B} \right)_{|B} + \left(\frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\boldsymbol{\phi},\boldsymbol{F},\boldsymbol{\mathfrak{X}}) \ \boldsymbol{\eta}^{a} \right)_{|A} \right] \\ &+ \int_{\mathcal{D}} \left[\frac{\partial \hat{W}}{\partial \phi^{a}} \circ (\boldsymbol{\phi},\boldsymbol{F},\boldsymbol{\mathfrak{X}}) - \left(\frac{\partial \hat{W}}{\partial F^{a}_{A}} \circ (\boldsymbol{\phi},\boldsymbol{F},\boldsymbol{\mathfrak{X}}) \right)_{|A} \right] \boldsymbol{\eta}^{a}. \end{aligned} \tag{98}$$

Now we use the definitions (61), which, in the context of continuum mechanics, give the body force f and the first Piola-Kirchhoff stress P, use $W = \hat{W} \circ (\phi, F, \mathcal{X})$ and change index A into B in the first integral. So, we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathcal{X}}) = \int_{\mathcal{D}} \left[\left(W \ \boldsymbol{U}^{B} \right)_{|B} + \left(\eta^{a} \ \boldsymbol{P}_{a}{}^{B} \right)_{|B} \right] - \int_{\mathcal{D}} \left(f_{a} + \boldsymbol{P}_{a}{}^{A}{}_{|A} \right) \eta^{a}, \tag{99}$$

which corresponds to Equation (17) in the paper by Hill (1951). In the first integral, we use $U^B = U^A \delta_A{}^B$ in the first term and the definition (83) of the total variation w to eliminate $\eta^a = w^a - F^a{}_A U^A$ in the second term, and then we split the first integral into two, to obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathfrak{X}}) = \int_{\mathcal{D}} \left[U^{A} \left(W \,\delta_{A}{}^{B} - F^{a}{}_{A} P_{a}{}^{B} \right) \right]_{|B} + \int_{\mathcal{D}} \left(w^{a} P_{a}{}^{B} \right)_{|B} - \int_{\mathcal{D}} \left(f_{a} + P_{a}{}^{A}{}_{|A} \right) \eta^{a}, \tag{100}$$

where we recognise the Eshelby stress $\mathfrak{E}_A{}^B = W \, \delta_A{}^B - F^a{}_A P_a{}^B$ defined in Equation (49). Finally, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\mathcal{X}) = \int_{\mathcal{D}} \left(U^{A} \mathfrak{E}_{A}{}^{B} \right)_{|B} + \int_{\mathcal{D}} \left(w^{a} P_{a}{}^{B} \right)_{|B} - \int_{\mathcal{D}} \left(f_{a} + P_{a}{}^{A}{}_{|A} \right) \eta^{a}, \quad (101)$$

⁵⁵⁷ which, in component-free formalism, reads

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathcal{X}}) = \int_{\mathcal{D}} \operatorname{Div}(\boldsymbol{\mathfrak{G}}^{T}\boldsymbol{U}) + \int_{\mathcal{D}} \operatorname{Div}(\boldsymbol{P}^{T}\boldsymbol{w}) - \int_{\mathcal{D}} (\boldsymbol{f} + \operatorname{Div}\boldsymbol{P})\boldsymbol{\eta}.$$
(102)

⁵⁵⁸ If the variation (102) is evaluated for a configuration ϕ solving the Euler-Lagrange ⁵⁵⁹ equations (67), we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathcal{X}}) = \int_{\mathcal{D}} \operatorname{Div}(\boldsymbol{\mathfrak{G}}^{T}\boldsymbol{U}) + \int_{\mathcal{D}} \operatorname{Div}(\boldsymbol{P}^{T}\boldsymbol{w}), \tag{103}$$

where the first two integrals contain the contributions to the *Noether current density* ${\bf \mathfrak{G}}^T {\bf U} + {\bf P}^T {\bf w}$. The extension of the result (103) to the case of the presence of nonintegrable body forces f is treated in Appendix A.

563 5.3 Eshelby's Results and Conservation of Noether's Current

The variational procedure followed in Section 5.2 was conducted by introducing the one-564 parameter families of transformations $\mathcal{Y}(X) = X + h U = \tilde{X}$ and $\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h \eta(\tilde{X})$, 565 which allowed to compute the Gâteaux derivative of total energy $\mathcal{E}_{\mathcal{D}}$ along the pair of 566 directions (η , U). Transformations of this kind are said to be symmetries if they do not alter 567 the numerical value of $\mathcal{E}_{\mathcal{D}}$, i.e., if it holds true that $\mathcal{E}_{\mathcal{D}}(\bar{\phi}, \mathcal{Y}) = \mathcal{E}_{\mathcal{D}}(\phi, \mathcal{X})$ for sufficiently 568 small values of h. Following an argument reported by Hill (1951), a condition ensuring 569 the compliance with this equality and the form-invariance of the Euler-Lagrange equations 570 is obtained by means of what in field theory is called a *divergence transformation* (Hill, 571 1951; Maugin, 1993). For the case of an infinitesimal symmetry transformation, the 572 divergence transformation reads 573

$$\int_{\mathcal{D}} (1 + h \operatorname{Div} \boldsymbol{U}) \, \hat{W} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) \circ \mathcal{Y} = \int_{\mathcal{D}} \left[\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + h \operatorname{Div} \boldsymbol{\Omega} \right], \tag{104}$$

where $\Omega = \hat{\Omega} \circ \mathcal{X}$ is a vector field to be determined. Note that, in order to leave the Euler-Lagrange equations (67) invariant, $\hat{\Omega}$ must *not* depend on *F* (Hill, 1951). By dividing Equation (104) by *h* and taking the limit for $h \to 0$, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathcal{X}}) - \int_{\mathcal{D}} \operatorname{Div} \boldsymbol{\Omega} = \int_{\mathcal{D}} \left[\operatorname{Div} (\boldsymbol{\mathfrak{G}}^{T}\boldsymbol{U}) + \operatorname{Div} (\boldsymbol{P}^{T}\boldsymbol{w}) - \operatorname{Div} \boldsymbol{\Omega} \right] = 0.$$
(105)

According to this result, to a given pair U and w there corresponds the conservation law

$$\operatorname{Div}(\mathfrak{E}^{T}\boldsymbol{U}) + \operatorname{Div}(\boldsymbol{P}^{T}\boldsymbol{w}) - \operatorname{Div}\boldsymbol{\Omega} = 0, \qquad (106)$$

which allows to determine Ω . In several circumstances of interest, such as the one related to the conservation of momentum or angular momentum, one can take Ω to be zero from the outset and look for transformations U and w leading to conservation laws of the form

$$\operatorname{Div}(\mathfrak{E}^T \boldsymbol{U}) + \operatorname{Div}(\boldsymbol{P}^T \boldsymbol{w}) = 0.$$
(107)

In the remainder of our work, we specialise to this case in order to retrieve Eshelby's result in the light of Noether's theorem. Some remarks on divergence transformations are reported in Appendix B.

Eshelby (1975) imposed $\eta = -F U$, i.e., that the conventional displacement η be equal to the negative of the push-forward of the material displacement U, as shown in Equation (41), in order to preserve compatibility. This condition, in turn, imposes the vanishing of the total variation, i.e., $w = \eta + F U = 0$. With this hypothesis, the integral of Div $(P^T w)$ in Equation (103) vanishes identically and the variation reduces to

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\boldsymbol{\mathfrak{X}}) = \int_{\mathcal{D}} \operatorname{Div}\big(\boldsymbol{\mathfrak{G}}^{T}\boldsymbol{U}\big).$$
(108)

⁵⁸⁹ which coincides with the result shown in Equation (48).

Now we can exploit Noether's theorem to obtain Eshelby's final result. Noether's Theorem states that

For every continuous symmetry under which the integral $\mathcal{E}_{\mathcal{D}}$ is invariant, there is a conserved current density.

In this case, the Noether current density is $\mathfrak{E}^T U$. For it to be conserved, the divergence Div $(\mathfrak{E}^T U)$ has to vanish and, in fact, a direct computation, in which the configurational force balance (77) is used, yields the condition

$$\operatorname{Div}(\mathfrak{G}^T U) = \mathfrak{G}: \operatorname{Grad} U + (\operatorname{Div}\mathfrak{G}) U = \mathfrak{G}: \operatorname{Grad} U - \mathfrak{F} U = 0.$$
(109)

Equation (109) is known as *Noetherian identity* (Podio-Guidugli, 2001), and places restric-597 tions on the class of transformations U that comply with the requirement $\text{Div}(\mathbf{\mathfrak{G}}^T U) = 0$, 598 which can thus be said to be symmetry transformations. Indeed, a field U is a symmetry 599 transformation (i.e., it leaves $\mathcal{E}_{\mathcal{D}}$ invariant) if, and only if, it satisfies (109) (for a similar 600 result in a different context, see also Grillo et al., 2003, 2019). Looking at (109), we 601 notice that, when the inhomogeneity force, \mathfrak{F} , vanishes identically i.e., when the body is 602 *materially homogeneous* and, thus, the energy density \hat{W} does not depend on the material 603 points, the Noetherian identity reduces to 604

$$\operatorname{Div}(\mathfrak{G}^T U) = \mathfrak{G} : \operatorname{Grad} U = 0.$$
(110)

This result implies that *any* arbitrary uniform displacement field U, for which Grad U =**0**, annihilates the divergence of the Noether current density and is, thus, a symmetry transformation. A body endowed with this property is said to enjoy the symmetry of *material homogeneity*. We notice, however, that, when \mathfrak{F} is not null, U may no longer be uniform. This means that \mathfrak{F} *breaks* the symmetry of material homogeneity and a new class of transformations U has to be determined.

⁶¹¹ We also note that, under the hypothesis of homogeneous material, Equation (108) ⁶¹² implies the vanishing of the divergence of $\mathfrak{E}^T U$, and not of \mathfrak{E} . In order to obtain the ⁶¹³ vanishing of the divergence of the Eshelby stress \mathfrak{E} , we implement the last of Eshelby's ⁶¹⁴ hypotheses, namely the fact that the material displacement U is uniform on \mathcal{D} and given by $U(X) = -U_0$, for every $X \in \mathcal{D}$. This implies that in the integral of $\text{Div}(\mathfrak{G}^T U)$ in Equation (111), the displacement $U = -U_0$ can be brought out of the divergence, i.e.,

$$(\partial_{(\boldsymbol{\eta},-\boldsymbol{U}_0)}\mathcal{E}_{\mathcal{D}})(\boldsymbol{\phi},\mathfrak{X}) = -\int_{\mathcal{D}} (\operatorname{Div} \boldsymbol{\mathfrak{E}}) \boldsymbol{U}_0.$$
(111)

which coincides with Equation (50) obtained using Eshelby's original procedure. Now, the vanishing of the variation due to the homogeneity of the material implies the vanishing of Div **C**, as in the strong form (77) considered with condition (78).

620 6 Summary

In this work we systematically reviewed the two procedures proposed by Eshelby to study 621 the effect of inhomogeneity in an elastic body, in the differential geometric picture of 622 continuum mechanics. The first procedure (Eshelby, 1951) involves the classical cutting-623 replacing-welding operations and is mathematically represented by defining the energy as 624 a functional on the manifold \mathcal{M} of the *Eshelbian configurations* \mathcal{Y} (which transform the 625 domain \mathcal{D} containing the inclusion/defect), and performing a variation on the coordinates, 626 i.e., a variational derivative made with respect to a *material* displacement field U, seen 627 as a variation of the identity Eshelbian configuration \mathfrak{X} . The second procedure (Eshelby, 628 1975) follows Hamilton's principle of stationary action. Accordingly, the energy is 629 defined as a functional on the manifold C of the *conventional configurations* ϕ , and a 630 variation is performed on the fields, i.e., a variational derivative is calculated with respect 631 to a *spatial* displacement, seen as a variation of the configuration map ϕ . 632

The natural manner to unify the two procedures is the use of *Noether's Theorem*, 633 in which a variation on both fields and coordinates (total variation) is used. Indeed, to 634 obtain this result, we defined the energy as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$ 635 of the conventional configurations ϕ and the Eshelbian configurations \mathcal{Y} , and performed 636 a variational derivative with respect to the pair (η, U) , which is a variation with respect to 637 the pair (ϕ, \mathcal{X}) . While certainly no additional proof was needed to demonstrate the beauty 638 and generality of Noether's Theorem, we find that it is insightful to look at Eshelby's 639 theory of defects from the point of view of Noether's conservation laws. 640

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647 Appendix

A Monogenic and Polygenic Forces

The variational setting adopted in our work serves as a basis for the employment of 649 Noether's Theorem (see Section 5), which, for first order theories, is generally enunciated 650 for a Lagrangian density function depending on "fields and gradients of the fields". 651 Hence, the expression of the energy density used so far, i.e., $W = \hat{W} \circ (\phi, F, \mathcal{X})$, is meant 652 to replicate, up to the sign, the standard functional dependence of a generic Lagrangian 653 density function, for which Noether's Theorem is formulated. In principle, however, 654 *neither* the introduction of the Eshelby stress tensor *nor* that of the configurational force 655 density require any variational framework. Indeed, as clearly shown by Gurtin (1995), the 656 existence of these quantities stands on its own, and it necessitates neither the hypothesis 657 of hyperelastic material nor the assumption of body forces descending from a generalised 658 potential density. The Eshelby stress tensor, for instance, is defined also for a generic 659 Cauchy elastic material (for a definition of Cauchy elastic materials, see, e.g., Ogden, 660 1984), for which the first Piola-Kirchhoff stress tensor, P, cannot be determined by 661 differentiating the body's free energy density with respect to its deformation gradient 662 tensor. In this respect, we recall Gurtin's words: "My derivation of Eshelby's relation 663 is accomplished without recourse to constitutive equations or to a variational principle" 664 (Gurtin, 1995). Yet, what is referred to as "Eshelby stress tensor" and "configurational 665 force density" within a given theory may well depend on whether or not the body is 666 hyperelastic and the body forces admit a potential. 667

To focus on the consequences of the existence of such a potential, we consider first a hyperelastic and inhomogeneous material with energy density $W^{\text{el}} := \check{W}^{\text{el}} \circ (F, \mathfrak{X})$, and subjected to body forces for which no integrability hypothesis is made. Then, following Gurtin's approach (Gurtin, 1995), the following configurational force balance applies

$$\operatorname{Div} \mathfrak{E}^{\operatorname{el}} + \mathfrak{F}^{\operatorname{el}} = \mathbf{0}, \tag{112}$$

where $\mathfrak{E}^{\text{el}} := W^{\text{el}} I^T - F^T P$ is the Eshelby stress tensor obtained by using W^{el} as free energy density, and \mathfrak{F}^{el} is the configurational force density satisfying Equation (112). Note that, for the sake of a lighter notation, we write F^T in lieu of $F^T \circ \phi$ throughout this section.

To identify \mathfrak{F}^{el} from Equation (112), we compute explicitly the divergence of \mathfrak{E}^{el} , while recalling the equilibrium equation Div P + f = 0. Thus, we find

$$\boldsymbol{\mathfrak{F}}^{\text{el}} = -\text{Div}\,\boldsymbol{\mathfrak{G}}^{\text{el}} = -\frac{\partial \check{W}^{\text{el}}}{\partial \boldsymbol{\mathfrak{X}}} \circ (\boldsymbol{F}, \boldsymbol{\mathfrak{X}}) - \boldsymbol{F}^{T}\boldsymbol{f}, \qquad (113)$$

thereby reaching the conclusion that \mathfrak{F}^{el} consists of the sum of two contributions, denoted by

$$\boldsymbol{\mathfrak{F}}^{\text{el,inh}} := -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}), \qquad (114a)$$

$$\mathfrak{F}^{\mathrm{el},\mathrm{b}} := -\boldsymbol{F}^T \boldsymbol{f},\tag{114b}$$

and ascribable to the inhomogeneity of the material and to the presence of the body force *f*, respectively. We emphasise that Equations (113), (114a) and (114b) are true *regardless* of any prescription on the integrability of *f*. Still, without loss of generality, we may assume the splitting $f = f^p + f^m$, where f^m is assumed to admit the generalised energy potential density $W^m = \check{W}^m \circ (\phi, \mathfrak{X})$, such that

$$\boldsymbol{f}^{\mathrm{m}} = -\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \boldsymbol{\mathcal{X}}). \tag{115}$$

In the terminology of Lanczos (1970, page 30), f^{p} is said to be "*polygenic*", whereas f^{m} is referred to as a "*monogenic*" force density, because it is "*generated by a single scalar function*", i.e., \check{W}^{m} .

The splitting $f = f^{p} + f^{m}$ and Equation (115) permit to rewrite \mathfrak{F}^{el} as

$$\mathbf{\mathfrak{F}}^{\text{el}} = -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\mathbf{F}, \mathcal{X}) - \mathbf{F}^{T} \mathbf{f}$$
$$= -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\mathbf{F}, \mathcal{X}) + \mathbf{F}^{T} \left[\frac{\partial \check{W}^{\text{m}}}{\partial \phi} \circ (\phi, \mathcal{X}) \right] - \mathbf{F}^{T} \mathbf{f}^{\text{p}}, \tag{116}$$

and, since it holds true that

Grad
$$W^{\mathrm{m}} = \mathbf{F}^{T} \left[\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \mathfrak{X}) \right] + \frac{\partial \check{W}^{\mathrm{m}}}{\partial \mathfrak{X}} \circ (\phi, \mathfrak{X}),$$
 (117)

the force density $\boldsymbol{\mathfrak{F}}^{el}$ takes on the expression

$$\mathfrak{F}^{\text{el}} = -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \frac{\partial \check{W}^{\text{m}}}{\partial \mathcal{X}} \circ (\phi, \mathcal{X}) + \text{Grad} \, W^{\text{m}} - \boldsymbol{F}^{T} \boldsymbol{f}^{\text{p}}.$$
(118)

⁶⁹¹ Moreover, by exploiting the identity Grad $W^m = \text{Div}(W^m I^T)$, setting

$$W^{\text{el}} = \check{W}^{\text{el}} \circ (F, \mathcal{X}) = \hat{W}^{\text{el}} \circ (\phi, F, \mathcal{X}), \quad \text{with } \frac{\partial \hat{W}^{\text{el}}}{\partial \phi} \circ (\phi, F, \mathcal{X}) = \mathbf{0}, \quad (119a)$$

$$W^{\rm m} = \check{W}^{\rm m} \circ (\phi, \mathfrak{X}) = \hat{W}^{\rm m} \circ (\phi, F, \mathfrak{X}), \qquad \text{with } \frac{\partial \hat{W}^{\rm m}}{\partial F} \circ (\phi, F, \mathfrak{X}) = \mathbf{0}, \qquad (119b)$$

and defining the overall energy density, $\hat{W} := \hat{W}^{el} + \hat{W}^{m}$, we obtain

$$\mathfrak{F}^{\text{el}} = -\frac{\partial \hat{W}}{\partial \chi} \circ (\phi, \boldsymbol{F}, \chi) + \text{Div}(W^{\text{m}}\boldsymbol{I}^{T}) - \boldsymbol{F}^{T}\boldsymbol{f}^{\text{p}}.$$
(120)

⁶⁹³ Finally, substituting this result into Equation (112) yields

$$\operatorname{Div}\left(W^{\mathrm{el}}\boldsymbol{I}^{T}-\boldsymbol{F}^{T}\boldsymbol{P}\right)-\frac{\partial\hat{W}}{\partial\boldsymbol{X}}\circ\left(\phi,\boldsymbol{F},\boldsymbol{X}\right)+\operatorname{Div}(W^{\mathrm{m}}\boldsymbol{I}^{T})-\boldsymbol{F}^{T}\boldsymbol{f}^{\mathrm{p}}=\boldsymbol{0},\qquad(121)$$

⁶⁹⁴ which can be recast in the form

Div
$$(WI^T - F^T P) - \frac{\partial \hat{W}}{\partial \chi} \circ (\phi, F, \chi) - F^T f^p = 0.$$
 (122)

We recognise that the term under divergence in Equation (122) is the Eshelby stress tensor used in our work, i.e., $\mathfrak{E} = WI^T - F^T P$, which is constructed with the energy density *W*. Accordingly, the corresponding configurational force is given by

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \chi} \circ (\phi, F, \chi) - F^T f^p = \mathfrak{F}^{el} - \operatorname{Grad} W^m, \qquad (123)$$

⁶⁹⁸ so that Equation (122) returns the configurational force balance Div $\mathfrak{E} + \mathfrak{F} = \mathbf{0}$. In the ⁶⁹⁹ absence of polygenic forces, i.e., for $f^p = \mathbf{0}$, the form of the configurational force balance ⁷⁰⁰ is maintained up to the re-definition of \mathfrak{F} , which reduces to

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{124}$$

⁷⁰¹ a result stating that the inhomogeneity force \mathfrak{F} acquires the meaning of an *effective* force ⁷⁰² accounting for two contributions: the inhomogeneities of the material featuring in the ⁷⁰³ body's hyperelastic behaviour and, thus, represented by W^{el} , and the inhomogeneities of ⁷⁰⁴ the energy density W^{m} , which describes the interaction of the body with its surrounding ⁷⁰⁵ world (e.g., via the mass density).

B Divergence Transformation

Let us consider a field theoretical framework and analyse a static problem, described by 707 the Lagrangian density function $\mathcal{L} = \hat{\mathcal{L}} \circ (\varphi, \operatorname{Grad} \varphi, \mathfrak{X})$, in which φ is a scalar field (the 708 generalisation to the situation in which φ is a collection of N scalar fields is straightfor-709 ward). We emphasise that φ is *not* the deformation here, but only a generic scalar field, 710 as it could be the case for temperature or for the scalar potential in Electromagnetism. 711 Consequently, the evaluation $\varphi(X)$, with $X \in \mathcal{B}$, only represents the value taken by φ at X, 712 i.e., it is not the embedding of the material point X into the three-dimensional Euclidean 713 space. Within this setting, the quantity Grad φ need not be the "material gradient" of 714 φ . Still, we maintain the notation introduced so far in our work in order not to generate 715 confusion. 716

After renaming $\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}_{old}$, we express the *divergence transformation* as (Hill, 1951)

$$\hat{\mathcal{L}}_{\text{new}} \circ (\varphi, \text{Grad}\,\varphi, \mathfrak{X}) = \hat{\mathcal{L}}_{\text{old}} \circ (\varphi, \text{Grad}\,\varphi, \mathfrak{X}) + \text{Div}\,\mathbf{\Omega}, \tag{125}$$

where $\Omega = \hat{\Omega} \circ (\varphi, \chi)$ is an arbitrary vector field. Moreover, we notice that the vectorvalued function $\hat{\Omega}$ has to be independent of Grad φ .

720

A first direct consequence of (125) is that the overall Lagrangian^{\dagger} associated with the

 $^{^{\}dagger}$ In a more general – yet conceptually equivalent – framework, we should speak of action functional, rather than "overall Lagrangian", with the former being defined as the time integral of the latter over a given (bounded) time interval. However, since all the quantities introduced in the present work are independent of time because of the hypothesis of static problem, the action and the "overall Lagrangian" are defined up to a multiplicative constant representing the width of the given time interval. For this reason, the formulation used in our work is totally equivalent to the general one.

721 body transforms from

$$L_{\mathcal{B}}^{\text{old}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{old}} \circ (\varphi, \operatorname{Grad}\varphi, \mathfrak{X})]$$
(126)

722 into

$$L_{\mathcal{B}}^{\text{new}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{new}} \circ (\varphi, \text{Grad}\,\varphi, \mathcal{X})], \qquad (127)$$

where $L_{\mathcal{B}}^{\text{old}}(\varphi)$ and $L_{\mathcal{B}}^{\text{new}}(\varphi)$ differ from each other by the boundary term $\int_{\partial \mathcal{B}} \Omega N$, i.e.,

$$L_{\mathcal{B}}^{\text{new}}(\varphi) = L_{\mathcal{B}}^{\text{old}}(\varphi) + \int_{\partial \mathcal{B}} [\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})] N.$$
(128)

Since the variational procedure yielding the stationarity conditions for $L_{\mathcal{B}}^{\text{old}}(\varphi)$ and $L_{\mathcal{B}}^{\text{new}}(\varphi)$ requires the fields φ and $\bar{\varphi} = \varphi + h \eta$ to coincide with each other on $\partial \mathcal{B}$ (indeed, η is chosen such that it vanishes on $\partial \mathcal{B}$), a field φ for which $L_{\mathcal{B}}^{\text{old}}(\varphi)$ is stationary makes $L_{\mathcal{B}}^{\text{new}}(\varphi)$ stationary too. Moreover, such a field has to satisfy the same set of Euler-Lagrange equations. Indeed, upon recalling the expression of the covariant divergence of Ω , i.e.,

Div
$$\Omega = \Omega^{A}_{,A} + \Gamma^{A}_{BA} \Omega^{B}$$

= $\left[\frac{\partial \hat{\Omega}^{A}}{\partial \varphi} \circ (\varphi, \chi)\right] \varphi_{,A} + \frac{\partial \hat{\Omega}^{A}}{\partial \chi^{A}} \circ (\varphi, \chi) + \Gamma^{A}_{BA} [\hat{\Omega}^{B} \circ (\varphi, \chi)],$ (129)

and substituting (129) into (125), we find that another consequence of Equation (125) is
 given by the identities

$$\frac{\partial \hat{\mathcal{L}}_{\text{new}}}{\partial \varphi} \circ (\ldots) = \frac{\partial \hat{\mathcal{L}}_{\text{old}}}{\partial \varphi} \circ (\ldots) + \left[\frac{\partial^2 \hat{\Omega}^A}{\partial \varphi^2} \circ (\varphi, \mathfrak{X}) \right] \varphi_{,A} + \frac{\partial^2 \hat{\Omega}^A}{\partial \mathfrak{X}^A \partial \varphi} \circ (\varphi, \mathfrak{X}) + \Gamma^A_{BA} \left[\frac{\partial \hat{\Omega}^B}{\partial \varphi} \circ (\varphi, \mathfrak{X}) \right], \quad (130a)$$

$$\frac{\partial \hat{\mathcal{L}}_{\text{new}}}{\partial \varphi_{,B}} \circ (\ldots) = \frac{\partial \hat{\mathcal{L}}_{\text{old}}}{\partial \varphi_{,B}} \circ (\ldots) + \frac{\partial \hat{\Omega}^{B}}{\partial \varphi} \circ (\varphi, \mathcal{X}),$$
(130b)

$$\begin{bmatrix} \frac{\partial \hat{\mathcal{L}}_{\text{new}}}{\partial \varphi_{,B}} \circ (\dots) \end{bmatrix}_{|B} = \begin{bmatrix} \frac{\partial \hat{\mathcal{L}}_{\text{old}}}{\partial \varphi_{,B}} \circ (\dots) \end{bmatrix}_{|B} + \begin{bmatrix} \frac{\partial^2 \hat{\Omega}^B}{\partial \varphi^2} \circ (\varphi, \mathfrak{X}) \end{bmatrix} \varphi_{,B} \\ + \frac{\partial^2 \hat{\Omega}^B}{\partial \varphi \partial \mathfrak{X}^B} \circ (\varphi, \mathfrak{X}) + \Gamma^B_{DB} \begin{bmatrix} \frac{\partial \hat{\Omega}^D}{\partial \varphi} \circ (\varphi, \mathfrak{X}) \end{bmatrix},$$
(130c)

which imply the invariance of the Euler-Lagrange equations under the transformation (125), i.e.,

$$\frac{\partial \hat{\mathcal{L}}_{\text{old}}}{\partial \varphi} \circ (\dots) - \left(\frac{\partial \hat{\mathcal{L}}_{\text{old}}}{\partial \varphi_{,B}} \circ (\dots) \right)_{|B} = \frac{\partial \hat{\mathcal{L}}_{\text{new}}}{\partial \varphi} \circ (\dots) - \left(\frac{\partial \hat{\mathcal{L}}_{\text{new}}}{\partial \varphi_{,B}} \circ (\dots) \right)_{|B} = 0. \quad (131a)$$

We emphasise that this result holds true because $\text{Div}[\hat{\Omega} \circ (\varphi, \mathfrak{X})]$ solves identically the Euler-Lagrange equations, i.e.,

$$\frac{\partial}{\partial \varphi} \operatorname{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathfrak{X})] - \operatorname{Div}\left(\frac{\partial}{\partial \operatorname{Grad} \varphi} \operatorname{Div}[\hat{\mathbf{\Omega}} \circ (\varphi, \mathfrak{X})]\right) = 0.$$
(132)

⁷³⁶ If φ is a collection of *N* independent scalar fields, Equation (132) becomes a system of *N* ⁷³⁷ scalar equations, i.e., in components,

$$\frac{\partial}{\partial \varphi^{\mu}} \operatorname{Div}[\hat{\Omega} \circ (\varphi, \mathfrak{X})] - \left(\frac{\partial}{\partial \varphi^{\mu}{}_{,A}} \operatorname{Div}[\hat{\Omega} \circ (\varphi, \mathfrak{X})]\right)_{|A} = 0, \quad \mu = 1, \dots, N.$$
(133)

738 However, the quantity

$$\frac{\partial}{\partial \varphi^{\mu}{}_{,A}} \operatorname{Div}[\hat{\Omega} \circ (\varphi, \mathfrak{X})], \qquad \mu = 1, \dots, N, \quad A = 1, 2, 3, \tag{134}$$

⁷³⁹ is not, in general, the component of a tensor field. Indeed, if it were, for example for ⁷⁴⁰ N = 3, the covariant divergence constituting the second term on the left-hand-side of ⁷⁴¹ Equation (133) would require to differentiate the tensors $e^{\mu} \otimes E_A$ of a suitable tensor ⁷⁴² basis, thereby yielding a term, obtained by differentiating e^{μ} , that does not cancel with ⁷⁴³ the first summand of Equation (133). Hence, Equation (133) would not be satisfied.

The situation just depicted occurs when the "fields" of the triplet $(\varphi^1, \varphi^2, \varphi^3)$ acquire the meaning of the components of the deformation, an object that has the mathematical meaning of an embedding and, thus, that is not truly identifiable with a collection of genuine scalar fields. Indeed, when $(\varphi^1, \varphi^2, \varphi^3)$ is replaced by (ϕ^1, ϕ^2, ϕ^3) , the corresponding "gradient" is none other than **F** and, more importantly, the quantity in (134) becomes (with $a \in \{1, 2, 3\}$ and $A \in \{1, 2, 3\}$)

$$\frac{\partial}{\partial \phi^{a}{}_{,A}} \operatorname{Div}[\hat{\mathbf{\Omega}} \circ (\phi, \mathfrak{X})] = \frac{\partial}{\partial F^{a}{}_{A}} \operatorname{Div}[\hat{\mathbf{\Omega}} \circ (\phi, \mathfrak{X})],$$
(135)

which takes on the meaning of a fictitious first Piola-Kirchhoff stress tensor. The consequence of this result is that the covariant divergence of the right-hand-side of Equation (135) does not cancel with $\partial \text{Div}[\hat{\Omega} \circ (\phi, \chi)]/\partial \phi^a$. This leads us to the conclusion, already stated by Maugin (1993, see page 100), that $\hat{\Omega}$ should depend "*at most*" on *X* "*and not on the fields*".

Since we consider a static problem, for which the body's Lagrangian density function coincides with the negative of its total energy density, following Hill (1951), we introduce the functions $W_{\text{old}} = \hat{W}_{\text{old}} \circ (\phi, F, \mathcal{X})$ and $W_{\text{new}} = \hat{W}_{\text{new}} \circ (\phi, F, \mathcal{X})$, and we reformulate the transformation (125) as

$$-\hat{W}_{\text{new}} \circ (\phi, F, \mathcal{X}) = -\hat{W}_{\text{old}} \circ (\phi, F, \mathcal{X}) + \text{Div}\,\Omega, \tag{136}$$

with $\Omega \equiv \hat{\Omega} \circ \mathcal{X}$. For the reasons outlined above, the divergence transformation (136) is such that the overall energies $\mathcal{E}_{\mathcal{D}}^{\text{old}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, F, \mathcal{X})$ and $\mathcal{E}_{\mathcal{D}}^{\text{new}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\text{new}} \circ (\phi, F, \mathcal{X})$ ⁷⁶¹ (ϕ, F, \mathfrak{X}) are stationary for the same deformation ϕ , which thus satisfies the same Euler-⁷⁶² Lagrange equations. Indeed, since $\hat{\Omega}$ is independent of ϕ , it holds true that

$$\frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\ldots) = \frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\ldots), \tag{137a}$$

$$\frac{\partial \hat{W}_{\text{new}}}{\partial F^b{}_B} \circ (\dots) = \frac{\partial \hat{W}_{\text{old}}}{\partial F^b{}_B} \circ (\dots), \tag{137b}$$

$$\frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\dots) - \left(\frac{\partial \hat{W}_{\text{old}}}{\partial F^b{}_B} \circ (\dots)\right)_{|B} = \frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\dots) - \left(\frac{\partial \hat{W}_{\text{new}}}{\partial F^b{}_B} \circ (\dots)\right)_{|B} = 0. \quad (137c)$$

After proving this property, we superimpose the transformations $X \mapsto \tilde{X} = \mathcal{Y}(X) = X + h U$ and $\phi(X) \mapsto \bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h \eta(\tilde{X})$ to the divergence transformation (136), and we require the invariance of the overall energy under the resulting, global transformation (Hill, 1951). This yields the equality

$$\underbrace{\int_{\mathcal{D}} \{ [\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{F}, \mathcal{X})] \circ \mathcal{Y} \} \det(T\mathcal{Y})}_{\equiv \mathcal{E}_{\mathcal{D}}^{\text{new}}(\bar{\phi}, \mathcal{Y})} = \underbrace{\int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, F, \mathcal{X})}_{\equiv \mathcal{E}_{\mathcal{D}}^{\text{old}}(\phi, \mathcal{X})},$$
(138)

where $T\mathcal{Y}$ is the tangent map of \mathcal{Y} . By applying a "rescaled" divergence transformation to the left-hand-side of Equation (138), i.e.,

$$\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{F}, \mathcal{X}) = \hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{F}, \mathcal{X}) - \text{Div}(h \, \mathbf{\Omega}), \tag{139}$$

769 we obtain

$$\int_{\mathcal{D}} \{ [\hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{F}, \mathcal{X})] \circ \mathcal{Y} - \text{Div}(h \, \Omega) \circ \mathcal{Y} \} \det(T \mathcal{Y}) = \int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, F, \mathcal{X}) \,.$$
(140)

We remark that the smallness parameter h, which multiplies Ω in (139) and (140), has been introduced in order to make the divergence transformation infinitesimal, as is the case for the transformations on the material points and on the deformation.

By rearranging Equation (140), so as to separate the transformations on the material points and on the deformation from the divergence transformation, we find

$$\int_{\mathcal{D}} \{ [\hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{F}, \mathfrak{X}) \circ \mathfrak{Y}] \det(T \mathfrak{Y}) - \hat{W}_{\text{old}} \circ (\phi, F, \mathfrak{X}) \} = \int_{\mathcal{D}} [\text{Div}(h \, \Omega) \circ \mathfrak{Y}] \det(T \mathfrak{Y}) \,.$$
(141)

 $_{775}$ By using the result reported in (103), at the first order in *h*, Equation (141) becomes

$$\int_{\mathcal{D}} \operatorname{Div}[\mathfrak{G}^{T} \boldsymbol{U} + \boldsymbol{P}^{T} \boldsymbol{w}] = \int_{\mathcal{D}} \operatorname{Div} \boldsymbol{\Omega} \quad \Rightarrow \quad \int_{\mathcal{D}} \operatorname{Div}[\mathfrak{G}^{T} \boldsymbol{U} + \boldsymbol{P}^{T} \boldsymbol{w} - \boldsymbol{\Omega}] = 0, \quad (142)$$

thereby implying that Noether's current density is given by $\Im = \mathfrak{E}^T U + \mathbf{P}^T \mathbf{w} - \mathbf{\Omega}$ and that, after localisation, the conservation laws should be sought for in the form

$$\operatorname{Div}[\mathfrak{G}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = 0.$$
(143)

The choice of Ω depends on the type of conservation law and on the associated class of symmetry which one is interested in looking at.

Within the present context, Equation (143) constitutes the most general form of conservation law pertaining to Noether's current. This result, however, can be exploited in much deeper detail: indeed, granted the Euler-Lagrange equations f + Div P = 0, if, for a given choice of the fields U, w and Ω , (143) is satisfied as an identity, then a specific physical quantity is conserved and the fields are said to be *symmetries*.

For the problem under investigation, Equation (143) can be recast in the equivalent form (Hill, 1951; Grillo et al., 2003, 2019)

$$\operatorname{Div}[\mathfrak{G}^T U + \mathbf{P}^T w - \mathbf{\Omega}] = (\operatorname{Div}\mathfrak{G})U + \mathfrak{G} : \operatorname{Grad} U + (\operatorname{Div} \mathbf{P})w + \mathbf{P} : \operatorname{Grad} w - \operatorname{Div}\mathbf{\Omega}$$
$$= -\mathfrak{F} U + \mathfrak{G} : \operatorname{Grad} U - fw + \mathbf{P} : \operatorname{Grad} w - \operatorname{Div}\mathbf{\Omega} = 0. \quad (144)$$

⁷⁸⁷ If one is interested in looking at the conservation of linear momentum, one sets U = 0, $\Omega = 0$ and $w = w_0$, with w_0 being a uniform displacement field. In this case, ⁷⁸⁹ Equation (144) is not satisfied. Indeed, it occurs that

$$\operatorname{Div}[\mathbf{\mathfrak{G}}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = \operatorname{Div}[\boldsymbol{P}^T \boldsymbol{w}_0] = -\boldsymbol{f} \boldsymbol{w}_0 \neq 0, \quad (145)$$

which shows that linear momentum is not conserved because of the body forces f.

⁷⁹¹ On the same footing, the presence of the inhomogeneity force, \mathfrak{F} , spoils the conser-⁷⁹² vation of the pseudo-momentum (Maugin, 1993), and this is reflected by the fact that ⁷⁹³ uniform translations of material points, hereafter denoted by $U = U_0$, are not symmetry ⁷⁹⁴ transformations. This is encompassed by Equation (144) by setting w = 0 and $\Omega = 0$, ⁷⁹⁵ thereby obtaining

$$\operatorname{Div}[\mathfrak{G}^{T}\boldsymbol{U} + \boldsymbol{P}^{T}\boldsymbol{w} - \boldsymbol{\Omega}] = -\mathfrak{F}\boldsymbol{U}_{0} \neq 0.$$
(146)

⁷⁹⁶ In fact, Hill (1951) presents several examples, from which we largely took inspiration, ⁷⁹⁷ and, among those, he shows that the only case in which Ω should be taken different ⁷⁹⁸ from the null vector is the case in which velocity transformations are applied, a situation ⁷⁹⁹ referred to as the *centre-of-mass theorem*. I feel indebtly to my parents for

References

Bonet, J. and Wood, R. D. (2008). Nonlinear Continuum Mechanics for Finite Element
 Analysis (Second Edition). Cambridge University Press, Cambridge, UK.

⁸⁰³ Cermelli, P., Fried, E., and Sellers, S. (2001). Configurational stress, yield and flow
 ⁸⁰⁴ in rate-independent plasticity. *Proceedings of the Royal Society of London Series A*,
 ⁸⁰⁵ 457:1447–1467.

dell'Isola, F. and Placidi, L. (2011). Variational principles are a powerful tool also for
 formulating field theories. In dell'Isola, F. and Gavrilyuk, S., editors, *Variational Models and Methods in Solid and Fluid Mechanics*, pages 1–15. Springer-Verlag.

- Edelen, D. G. B. (1981). Aspects of variational arguments in the theory of elasticity: Fact
 and folklore. *International Journal of Solids and Structures*, 17:729–740.
- Epstein, M. (2002). The eshelby tensor and the theory of continuous distributions of
 inhomogeneities. *Mechanics Research Communications*, 29:501–506.
- Epstein, M. (2009). The split between remodelling and aging. *International Journal of Non-Linear Mechanics*, 44:604–609.
- Epstein, M. (2010). *The Geometrical Language of Continuum Mechanics*. Cambridge
 University Press, Cambridge, UK.
- Epstein, M. (2015). Mathematical characterization and identification of remodeling,
 growth, aging and morphogenesis. *Journal of the Mechanics and Physics of Solids*,
 84:72–84.
- Epstein, M. and Elżanowski, M. (2007). *Material Inhomogeneities and Their Evolution*.
 Springer, Berlin, Germany.
- Epstein, M. and Maugin, G. A. (1990). The energy momentum tensor and material uniformity in finite elasticity. *Acta Mechanica*, 83:127–133.
- Epstein, M. and Maugin, G. A. (2000). Thermomechanics of volumetric growth in
 uniform bodies. *International Journal of Plasticity*, 16:951–978.
- Eshelby, J. D. (1951). The force on an elastic singularity. *Philosophical Transactions of the Royal Society A*, 244A:87–112.
- Eshelby, J. D. (1975). The elastic energy-momentum tensor. *Journal of Elasticity*, 5:321–335.
- Federico, S. (2012). Covariant formulation of the tensor algebra of non-linear elasticity.
 International Journal of Non-Linear Mechanics, 47:273–284.
- Federico, S., Grillo, A., and Segev, R. (2016). Material description of fluxes in terms of
 differential forms. *Continuum Mechanics and Thermodynamics*, 28:379–390.
- Fletcher, D. C. (1976). Conservation laws in linear elastodynamics. *Archive for Rational Mechanics and Analysis*, 60:329–353.
- Fried, E. and Gurtin, M. E. (1994). Dynamic solid-solid transitions with phase characterized by an order parameter. *Physica D*, 72:287–308.
- Fried, E. and Gurtin, M. E. (2004). A unified treatment of evolving interfaces accounting
 for small deformations and atomic transport with emphasis on grain-boundaries and
 epitaxy. *Advances in Applied Mechanics*, 40:1–177.
- ⁸⁴¹ Golebiewska Herrmann, A. (1982). Material momentum tensor and path-independent
 ⁸⁴² integrals of fracture mechanics. *International Journal of Solids and Structures*, 18:319–
 ⁸⁴³ 326.

Grillo, A., Di Stefano, S., and Federico, S. (2019). Growth and remodelling from the
 perspective of Noether's theorem. *Mechanics Research Communications*, 97:89–95.

Grillo, A., Federico, S., Giaquinta, G., Herzog, W., and La Rosa, G. (2003). Restoration
 of the symmetries broken by reversible growth in hyperelastic bodies. *Theoretical and Applied Mechanics*, 30:311–331.

- Grillo, A., Prohl, R., and Wittum, G. (2016). A poroplastic model of structural re organisation in porous media of biomechanical interest. *Continuum Mechanics and Thermodynamics*, 28:579–601.
- Grillo, A., Prohl, R., and Wittum, G. (2017). A generalised algorithm for anelastic
 processes in elastoplasticity and biomechanics. *Mathematics and Mechanics of Solids*,
 22:502–527.
- Grillo, A., Zingali, G., Federico, S., Herzog, W., and Giaquinta, G. (2005). The role of
 material in homogeneities in biological growth. *Theoretical and Applied Mechanics*,
 32:21–38.
- Gurtin, M. E. (1986). Two-phase deformations of elastic solids. In *The Breadth and Depth of Continuum Mechanics*, pages 147–175. Springer, Berlin.
- Gurtin, M. E. (1993). The dynamics of solid-solid phase transitions 1. Coherent interfaces.
 Archive for Rational Mechanics and Analysis, 123:305–335.
- ⁸⁶² Gurtin, M. E. (1995). The nature of configurational forces. *Archive for Rational Mechanics* ⁸⁶³ and Analysis, 131:67–100.
- ⁸⁶⁴ Gurtin, M. E. (2000). Configurational Forces as Basic Concepts of Continuum Physics.
 ⁸⁶⁵ Springer.
- Gurtin, M. E. and Podio-Guidugli, P. (1996). On configurational inertial forces at a phase
 interface. *Journal of Elasticity*, 44:255–269.
- Hamedzadeh, A., Grillo, A., Epstein, M., and Federico, S. (2019). Remodelling of
 biological tissues with fibre recruitment and reorientation in the light of the theory of
 material uniformity. *Mechanics Research Communications*, 96:56–61.
- Hill, E. L. (1951). Hamilton's principle and the conservation theorems of mathematical
 physics. *Reviews of Modern Physics*, 23(3):253.
- Huang, Y.-N. and Batra, R. C. (1996). Energy-momentum tensors in nonsimple elastic dielectrics. *Journal of Elasticity*, 42:275–281.
- Imatani, S. and Maugin, G. A. (2002). A constitutive model for material growth and its
 application to three-dimensional finite element analysis. *Mechanics Research Communications*, 29:477–483.
- Kienzler, R. and Herrmann, G. (2000). *Mechanics in Material Space with Applications* to Defect and Fracture Mechanics. Springer-Verlag, Berlin, Germany, first edition.

- Knowles, J. K. and Sternberg, E. (1972). On a class of conservation laws in linearized 880 and finite elastostatics. Archive for Rational Mechanics and Analysis, 44:187–211. 881
- Lanczos, C. (1970). The Variational Principle of Mechanics. University of Toronto Press, 882 Toronto, Canada, fourth edition. 883
- Marsden, J. E. and Hughes, T. J. R. (1983). Mathematical Foundations of Elasticity. 884 Prentice-Hall, Englewood Cliff, NJ, USA. 885
- Maugin, G. A. (1993). *Material Inhomogeneities in Elasticity*. CRC Press, Boca Raton, 886 FL, USA. 887
- Maugin, G. A. (2011). Configurational Forces: Thermomechanics, Physics, Mathemat-888 ics, and Numerics. CRC Press, Boca Raton, FL, USA. 889
- Maugin, G. A. and Epstein, M. (1998). Geometrical material structure of elastoplasticity. 890 International Journal of Plasticity, 14:109–115. 891
- Maugin, G. A. and Trimarco, C. (1992). Pseudomomentum and material forces in 892 nonlinear elasticity: variational formulations and application to brittle fracture. Acta 893 Mechanica, 94:1-28. 894
- Noether, E. (1971). Invariant variation problems. Transport Theory and Statistical 895 Physics, 1:186-207. 896
- Noll, W. (1967). Materially uniform simple bodies with inhomogeneities. Archives for 897 Rational Mechanics and Analysis, 27:1–32. 898
- Ogden, R. W. (1984). Non-Linear Elastic Deformations. Ellis Horwood, New York, 899 USA. 900
- Olver, P. J. (1984a). Conservation laws in elasticity: I: general results. Archive for 901 Rational Mechanics and Analysis, 85:111-129. 902
- Olver, P. J. (1984b). Conservation laws in elasticity II: linear homogeneous isotropic 903 elastostatics. Archive for Rational Mechanics and Analysis, 85:131-160. 904
- Podio-Guidugli, P. (2001). Configurational balances via variational arguments. Interfaces 905 and Free Boundaries, 3:223–232. 906
- Segev, R. (2013). Notes on metric independent analysis of classical fields. Mathematical 907 Methods in the Applied Sciences, 36:497–566. 908
- Truesdell, C. and Noll, W. (1965). The Non-Linear Field Theories of Mechanics, volume 909 III of S. Flügge, ed., Encyclopedia of Physics. Springer-Verlag, Berlin, Germany. 910
- Verron, E., Aït-Bachir, M., and Castaing, P. (2009). Some new properties of the Eshelby 911 stress tensor. In IUTAM Symposium on Progress in the Theory and Numerics of 912 Configurational Mechanics, pages 27–35. Springer. 913
- Weng, G. J. and Wong, D. T. (2009). Thermodynamic driving force in ferroelectric crystals 914 with a rank-2 laminated domain pattern, and a study of enhanced electrostriction. 915 Journal of the Mechanics and Physics of Solids, 57:571–597.
- 916