SELF-PROPELLED MICRO-SWIMMERS IN A BRINKMAN FLUID

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ABSTRACT. We prove an existence, uniqueness, and regularity result for the motion of a self-propelled micro-swimmer in a particulate viscous medium, modeled as a Brinkman fluid. A suitable functional setting is introduced to solve the Brinkman system for the velocity field and the pressure of the fluid by variational techniques. The equations of motion are written by imposing a self-propulsion constraint, thus allowing the viscous forces and torques to be the only ones acting on the swimmer. From an infinite-dimensional control on the shape of the swimmer, a system of six ordinary differential equations for the spatial position and the orientation of the swimmer is obtained. This is dealt with standard techniques for ordinary differential equations, once the coefficients are proved to be measurable and bounded. The main result turns out to extend an analogous result previously obtained for the Stokes system.

Keywords: Brinkman equation, self-propelled motion, swimming, particulate media

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1. INTRODUCTION

Modeling the motion of living beings has stimulated scientists for many decades. The first attempts to study motion inside fluids date back to the pioneering works by Taylor [21] and Lighthill [16]. These papers and the 1977 paper by Purcell [18] point out that the description of motion in viscous fluids at low Reynolds number can involve some counter-intuitive facts. The low Reynolds number flow approximation is particularly efficient for microorganisms, while lager bodies or animals exploit more the inertial forces rather than the viscous ones. The recent literature has been populated by new and more refined results, both theoretical and experimental, in the two limit regimes. Concerning the viscous one, on which we concentrate in this paper, we recall that approximated theories, such as slender body approximation [3, 13], resistive force theory [10], and also others [19, 15], have been developed, and a number of biological experiments has been run to understand swimming strategies.

In a recent paper by S. Jung [11], the motion of *Caenorhabditis elegans* is observed in different environments: this nematode usually swims in saturated soil, and its behavior was studied in different saturation conditions as well as in a viscous fluid without solid particles. It must be noticed that the locomotion strategy of *C. elegans* is not completely understood, as it is shown by the many studies on this nematode in different conditions; nevertheless it has been taken as a model system to approach the study of many biological problems [24]. A satisfactory attempt to understand its locomotion dates back to [23], where the experiment was conducted in an environment close to the one in which *C. elegans* usually lives, yet the wet phase in which the particles are usually immersed was neglected. Other and more recent experiments have been run on agar composites [12, 14], and they could give a hint on the swimming strategies of *C. elegans*, showing that it moves more efficiently in a particulate medium rather than in a viscous fluid without particles [11].

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The aim of this paper is to provide a theoretical framework for the motion of a body in a particulate medium. Following the approach proposed in [11, III.C], we model the particulate medium surrounding the swimmer as a Brinkman fluid. We show that the framework we proposed in [5] also applies to the case of a Brinkman problem in an exterior domain. We prove the existence, uniqueness, and regularity of the solution to the equations of motion for a body swimming in such an environment, thus generalizing the result previously obtained for the Stokes system. The novelty in this work is that we are able to show that the hypotheses needed to solve the equations of motion for a swimmer in a Brinkman fluid are satisfied. These are the measurability and boundedness of the coefficients of the ordinary differential equations which govern the spatial position of the swimmer. Techniques from Calculus of Variations and results in the theory of Ordinary Differential Equations are used to achieve these results.

We shall define *swimming* the ability of an organism to propel itself in a fluid by changing its shape. The *self-propulsion* constraint is assumed: there are no other forces acting on the swimmer but the viscous interaction between the fluid and the swimmer itself. Also, we call *shape function* the map which describes the shape of the swimmer at any given time; the *position function* will describe its spatial position.

With these definitions in mind, the main result of this work, Theorem 4.6, proves that under the reasonable assumptions presented in Section 3 on the shape function a swimmer is able to advance in a particulate viscous fluid. It also shows that the significative shape functions that can provide net displacement are not simple rigid motions. Indeed, should the shape function, which is the one that the swimmer can control, be a rigid motion, then the resulting position function will turn out to be the inverse rigid motion, therefore implying no overall movement. As pointed out by Shapere and Wilczek [19], there must be a symmetry breaking for effective swimming to occur, thus avoiding the case of Purcell's Scallop Theorem [18]. In our case this is achieved by letting the shape vary in a rather non-trivial way, i.e., by allowing the control function be infinite-dimensional.

The paper is organized as follows. In Section 2 the functional setting for solving the Brinkman system in an exterior domain is presented. Consistent and general definition for the viscous force and torque and for the power expended during the swimming are given. In section 3 the kinematics setting is described and the equations of motion are obtained from the self-propulsion constraint on the swimmer. Moreover, regularity property for some of the coefficients of the equations of motion are proved. Eventually, in Section 4 the main theorem is stated and proved, once some technical results about the extension of boundary velocity fields are obtained. Finally, Section 5 provides some comments and hints on possible future directions.

2. Brinkman equation – functional setting

In this section we present some results about Brinkman equation. It was originally proposed in [4] to model a fluid flowing through a porous medium as a correction to Darcy's law by the addition of a diffusive term. A rigorous mathematical derivation from the Navier-Stokes equation via homogenization can be found in [1].

In a Lipschitz domain $\Omega\subset\mathbb{R}^3$, the Brinkman system reads

$$\begin{cases} \nu\Delta u-\alpha^2u=\nabla p & \text{in } \Omega,\\ \operatorname{div} u=0 & \text{in } \Omega,\\ u=U & \text{on } \partial\Omega,\\ u=0 & \text{at infinity.}\end{cases} \tag{2.1}$$

The positive constant α takes into account the permeability properties of the porous matrix and the viscosity of the fluid, the constant ν is an effective viscosity of the fluid, while the third equation in the system is the *no-slip* boundary condition. The condition $u=0$ at infinity is significant, and necessary, only when the domain Ω is unbounded. From now on, we will get rid of the effective viscosity, upon a redefinition of α, by setting $\nu=1$. A brief discussion on the constant ν can be found in Brinkman's paper [4].

In order to cast equation (2.1) in the weak form, we introduce the function spaces in which we will look for the weak solution. Define

$$\mathcal{X}(\Omega):=\{u\in H^1(\Omega;\mathbb{R}^3):\operatorname{div}u=0 \text{ in } \Omega\},\qquad \mathcal{X}_0(\Omega):=\{u\in H^1_0(\Omega;\mathbb{R}^3):\operatorname{div}u=0 \text{ in } \Omega\}.$$

Both $\mathcal{X}(\Omega)$ and $\mathcal{X}_0(\Omega)$ are equipped with the standard H^1 norm but we introduce this equivalent one

$$\|u\|^2_{\mathcal{X}(\Omega)} := \alpha^2 \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})},$$

the equivalence being a consequence of Korn's inequality.

The weak formulation of equation (2.1) is now given by

$$\begin{cases} \text{find } u \in \mathcal{X}(\Omega) \text{ such that} & u = U \text{ on } \partial\Omega, \\ 2\displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \,\mathrm{d}x + \alpha^2 \int_\Omega u\cdot w\,\mathrm{d}x = 0, & \text{for every } w \in \mathcal{X}_0(\Omega), \end{cases} \tag{2.2}$$

where the boundary velocity is a given function $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, the solution being the unique minimum in $\mathcal{X}(\Omega)$ of the strictly convex energy functional

$$\mathcal{E}(u) := 2\int_\Omega |\mathrm{E}u|^2\,\mathrm{d}x + \alpha^2\int_\Omega |u|^2\,\mathrm{d}x = \|u\|^2_{\mathcal{X}(\Omega)}.$$

Here and henceforth the symbol $\mathrm{E}u$ denotes the *symmetric gradient* of u, namely $\mathrm{E}u := \frac{1}{2}(\nabla u + (\nabla u)^T)$.

We call Ω an *exterior domain* with Lipschitz boundary if Ω is an unbounded, connected open set whose boundary $\partial\Omega$ is bounded and Lipschitz, see [5, Section 2]. If we consider the term $\alpha^2 u$ as a forcing term f in system (2.1), we can invoke a classical existence and uniqueness result, see, e.g., [6], [20], or [22].

Theorem 2.1. *Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Then the following results hold:*

(a) *Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with Lipschitz boundary. If*

$$\int_{\partial\Omega} U\cdot n\,\mathrm{d}S = 0, \tag{2.3}$$

there exists a unique solution u to problem (2.2)*. Moreover, there exists $p \in L^2(\Omega)$ such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

(b) *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then problem* (2.2) *has a solution. Moreover, there exists $p \in L^2_{\mathrm{loc}}(\Omega)$, with $p \in L^2(\Omega\cap\Sigma_\rho)$ for every $\rho > 0$, such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

The following density result is particularly useful when dealing with exterior domains.

Theorem 2.2 (Density [9])**.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then, the space $\{u \in C^\infty_c(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0$ in $\Omega\}$ is dense in $\mathcal{X}(\Omega)$ for the H^1 norm.* □

We now define some physically relevant quantities. The *stress tensor* associated with the velocity field u and the pressure p is given by

$$\sigma := -p\,\mathrm{I} + 2\mathrm{E}u. \tag{2.4}$$

The *viscous force* and *torque* are the resultant of the viscous forces and torques acting on the boundary $\partial\Omega$, respectively, and are given by

$$F := \int_{\partial\Omega} \sigma(x)n(x)\,\mathrm{d}S(x), \qquad M := \int_{\partial\Omega} x\times\sigma(x)n(x)\,\mathrm{d}S(x).$$

These definitions are valid under the condition that σn has a trace in $L^1(\partial\Omega;\mathbb{R}^3)$. Since, in general, this assumption is not fulfilled, we have to define the viscous force and torque in a different way, namely by introducing σn as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$. This will lead to a consistent definition of the *power* of the viscous force and torque. In order to do this, we introduce $\mathbb{M}^{3\times3}_{\mathrm{sym}}$, the space of 3×3 symmetric matrices, and recall that every $\sigma \in \mathbb{M}^{3\times3}_{\mathrm{sym}}$ can be orthogonally decomposed as $\sigma = \frac{1}{3}\operatorname{tr}\sigma\,\mathrm{I} + \sigma_D$ where the deviatoric part σ_D is traceless.

We are now ready to give the following

Definition 2.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary and let $\sigma \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ be such that $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$. The trace of σn, still denoted by σn, is defined as the unique element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ satisfying the equality*

$$\langle \sigma n, V\rangle_\Omega := \int_\Omega (\operatorname{div}\sigma)\cdot v\,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v\,\mathrm{d}x, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing between $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^3)$, and v is any function in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$.

If there is no risk of misunderstanding, the subscript Ω will be dropped whenever the domain of integration is clear. Notice that if σ is sufficiently smooth then integrating (2.5) by parts leads to the equality

$$\langle \sigma n, V\rangle_\Omega = \int_{\partial\Omega} \sigma n \cdot V \,\mathrm{d}S, \qquad \text{for every } V \in H^{1/2}(\partial\Omega;\mathbb{R}^3).$$

In the general case, the right-hand side of (2.5) is easily proved to be well defined, given the assumptions on σ. In fact, $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$ and $v \in L^2(\Omega;\mathbb{R}^3)$ make the first integral well defined, while the second one is also good since $\sigma : \mathrm{E}v = \sigma_D : \mathrm{E}v$, because of the symmetry of $\mathrm{E}v$, and both σ_D and $\mathrm{E}v$ belong to $L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Lastly, the definition is independent of the choice of $v \in \mathcal{X}(\Omega)$, since the right-hand side vanishes for every $v \in \mathcal{X}_0(\Omega)$: this follows from the very same computation for the more regular case, by the Density Theorem 2.2. It is easy to see that (2.5) defines a continuous linear functional on $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ by choosing $v \in \mathcal{X}(\Omega)$ an extension of V.

We now proceed in showing other useful properties of the duality pairing introduced in Definition 2.3. Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and let u be the solution to the Brinkman problem (2.2) with boundary datum U and let σ be the corresponding stress tensor. Since all the assumptions of Definition 2.3 are fulfilled, for any given $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ we have

$$\begin{aligned}\langle \sigma n, V\rangle &= \int_\Omega (\operatorname{div}\sigma)\cdot v \,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v \,\mathrm{d}x = \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x + \int_\Omega [-p\,\mathrm{I} : \mathrm{E}v + 2\mathrm{E}u : \mathrm{E}v]\,\mathrm{d}x \\ &= \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x - \int_\Omega p \operatorname{div} v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x \\ &= \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x,\end{aligned} \tag{2.6}$$

where v is an arbitrary element in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$. If we take, in particular, v to be the solution to problem (2.2) with boundary datum V, we recover the well known *reciprocity condition* (see, e.g., [8, Section 3-5])

$$\langle \sigma n, V\rangle = \langle \tau n, U\rangle,$$

with τ being the stress tensor associated with v. Moreover, by taking $U = V$ in (2.6) we obtain

$$\langle \sigma n, U\rangle = \alpha^2 \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})} = \|u\|^2_{\mathcal{X}(\Omega)}.$$

This equality allows us to show that the quadratic form $\langle \sigma n, U\rangle$ is positive definite: if $\langle \sigma n, U\rangle = 0$, then it follows that $u = 0$, and therefore $U = 0$.

We are now in a position to define the viscous force and torque in a rigorous way, by means of the duality product introduced in Definition 2.3.

Definition 2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary, let $u \in \mathcal{X}(\Omega)$ be the solution to the Brinkman problem* (2.2) *with boundary datum $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, let σ be the corresponding stress tensor defined by* (2.4)*, and let $\sigma n \in H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ be the trace on $\partial\Omega$ defined according to* (2.5)*. The* viscous force *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $F \in \mathbb{R}^3$ such that*

$$F\cdot V = \langle \sigma n, V\rangle \quad \textit{for every } V \in \mathbb{R}^3. \tag{2.7}$$

The torque *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $M \in \mathbb{R}^3$ such that*

$$M\cdot\omega = \langle \sigma n, W_\omega\rangle \quad \textit{for every } \omega \in \mathbb{R}^3, \tag{2.8}$$

where $W_\omega(x) := \omega \times x$ is the velocity field generated by the angular velocity ω.

Notice that this definition allows us to define two different physical quantities by means of the same mathematical object, namely the duality pairing defined in (2.5).

3. Kinematics and the equations of motion

In this section we describe the kinematics of the swimmer. The *motion* of a swimmer is described by a map $t \mapsto \varphi_t$, where, for every fixed t, the *state* φ_t is an orientation preserving bijective C^2 map from the *reference* configuration $A \subset \mathbb{R}^3$ into the *current* configuration $A_t \subset \mathbb{R}^3$. Given a distinguished point $x_0 \in A$, for every fixed t, we consider the following factorization

$$\varphi_t = r_t \circ s_t\,, \tag{3.1}$$

where the *position* function r_t is a rigid deformation and the *shape* function s_t is such that

$$s_t(x_0) = x_0 \quad \text{and} \quad \nabla s_t(x_0) \text{ is symmetric.} \tag{3.2}$$

We allow the map $t \mapsto s_t$ to be chosen in a suitable class of admissible shape changes and use it as a *control* to achieve propulsion as a consequence of the viscous reaction of the fluid. By contrast, $t \mapsto r_t$ is *a priori* unknown and it must be determined by imposing that the resulting $\varphi_t = r_t \circ s_t$ satisfies the equations of motion.

Since, as it is clear, the kinematics of the swimmer does not depend on the fluid the swimmer is surrounded by, we can adopt the same setting as in [5]. For the reader's convenience, we recall the results proved there, and refer the reader to the above mentioned paper and the references therein for a more detailed exposition.

The reference configuration of the swimmer $A \subset \mathbb{R}^3$ is a bounded connected open set of class C^2. The time-dependent deformation of A from the point of view of an external observer is described by a function $\varphi_t : \overline{A} \to \mathbb{R}^3$ with the following properties:

$$\varphi_t \in C^2(\overline{A};\mathbb{R}^3), \qquad \varphi_t \text{ is injective}, \qquad \det \nabla\varphi_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{3.3}$$

for every t; here and henceforth ∇ denotes the gradient with respect to the space variable. Under these hypotheses, $A_t := \varphi_t(A)$ is a bounded connected open set of class C^2 and

$$\text{the inverse} \quad \varphi_t^{-1} : \overline{A}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{A}_t;\mathbb{R}^3).$$

We also assume that

$$\text{the sets } \mathbb{R}^3 \setminus \overline{A}_t \text{ are connected for all } t \in [0,T]. \tag{3.4}$$

This assumption is technical and is made in order to prevent change of topology in the swimmer and in the surrounding fluid.

Concerning the regularity in time, we require that

$$\text{the map } t \mapsto \varphi_t \text{ belongs to } \mathrm{Lip}([0,T];C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T];C^2(\overline{A};\mathbb{R}^3)).$$

This condition implies that for almost every t there exists $\dot\varphi_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{\varphi_{t+h} - \varphi_t}{h} \to \dot\varphi_t\,, \quad \text{uniformly on } \overline{A} \text{ as } h \to 0.$$

From this, the Eulerian velocity on the boundary ∂A_t, defined by

$$U_t := \dot\varphi_t \circ \varphi_t^{-1}$$

belongs to $\mathrm{Lip}(\partial A_t;\mathbb{R}^3)$ with Lipschitz constant independent of t.

We now introduce the description of the kinematics from the point of view of the swimmer. Let $x_0 \in A$ be a distinguished point and let us look for a factorization of φ_t of the form (3.1). The function $s_t : A \to \mathbb{R}^3$ satisfies properties (3.2), in view of which it can be interpreted as a pure shape change from the point of view of an observer inertial with x_0, and the rigid motion $r_t : \mathbb{R}^3 \to \mathbb{R}^3$ is written in the form

$$r_t(z) = y_t + R_t z, \tag{3.5}$$

with $y_t \in \mathbb{R}^3$ and $R_t \in \mathrm{SO}(3)$, the set of orthogonal matrices with positive determinant. This allows us to say that the deformation φ_t, from the point of view of an external observer, is decomposed into a shape change followed by a rigid motion.

From (3.1), (3.3), and (3.5), the following properties of s_t can be inferred: for every t,

$$s_t \in C^2(\overline{A};\mathbb{R}^3), \qquad s_t \text{ is injective}, \qquad \det \nabla s_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{3.6a}$$

$$\text{the inverse} \quad s_t^{-1} : \overline{B}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{B}_t;\mathbb{R}^3), \tag{3.6b}$$

where $B_t := s_t(A)$, see Fig. 1. Note that (3.6b) is a consequence of (3.6a). Note also that B_t is a bounded connected open set of class C^2 and that $s_t(B_t) = A_t$ and $s_t(\partial B_t) = \partial A_t$.

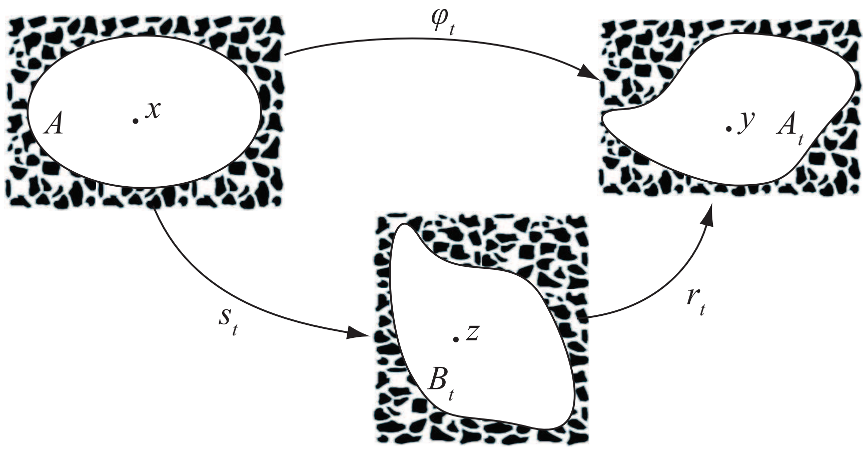

FIGURE 1. Notation for the kinematics.

Moreover, since A is bounded and s_t is continuous, there exists a ball Σ_ρ centered at 0 with radius ρ such that

$$A \subset\subset \Sigma_{\rho-1} \qquad \text{and} \qquad B_t \subset\subset \Sigma_{\rho-1}\,.$$

Lastly, (3.4) implies that

$$\text{the sets } \Sigma_\rho \setminus \overline{B}_t \text{ are connected for all } t \in [0,T]. \tag{3.7}$$

By means of the Polar Decomposition Theorem and the factorization (3.1), it is possible to give explicit formulae for R_t and y_t that clearly show that the maps $t \mapsto R_t$ and $t \mapsto y_t$ are Lipschitz continuous. Since $s_t = r_t^{-1} \circ \varphi_t$,

$$\text{the map } t \mapsto s_t \text{ belongs to } \mathrm{Lip}([0,T]; C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T]; C^2(\overline{A};\mathbb{R}^3)), \tag{3.8}$$

The third property in (3.6a) and (3.8) imply that $\|s_t^{-1}\|_{C^2(\overline{B}_t;\mathbb{R}^3)} \leqslant C < +\infty$, with C independent of t. Moreover, condition (3.8) yields the existence of $\dot{s}_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{s_{t+h} - s_t}{h} \to \dot{s}_t\,, \quad \text{uniformly on } \overline{A}, \text{ as } h \to 0.$$

Other properties of s_t that are worth mentioning, and whose full derivation can be found in [5, Section 3] are:

$$\begin{aligned}
&\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; H^{1/2}(\partial A;\mathbb{R}^3)),\\
&\mathrm{Lip}(\dot{s}_t) \leqslant L, \text{ with } L \text{ independent of } t,\\
&\text{for any fixed } x \in \overline{A}, \text{ the map } t \mapsto \dot{s}_t(x) \text{ is measurable.}
\end{aligned}$$

To conclude the description of the kinematics of the swimmer, we give the form of the boundary velocity on the intermediate configuration B_t. It turns out that, if we define $V_t(z) := R_t^T U_t(r_t(z))$ and $W_t(z) := \dot{s}_t(s_t^{-1}(z))$, for every $z \in \partial B_t$, an elementary computation shows that for almost every $t \in [0,T]$

$$V_t(z) = R_t^T \dot{y}_t + R_t^T \dot{R}_t z + W_t(z) \quad \text{for every } z \in \partial B_t\,.$$

We proceed now to the description of the motion of the swimmer. The motion $t \mapsto \varphi_t$ determines for almost every $t \in [0,T]$ the Eulerian velocity U_t through the formula

$$U_t(y) := \dot{\varphi}_t(\varphi_t^{-1}(y)) \quad \text{for almost every } y \in \partial A_t\,.$$

Notice that $U_t \in H^{1/2}(\partial A_t;\mathbb{R}^3)$ for almost every $t \in [0,T]$. By applying Theorem 2.1 (b) with $\Omega = A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$ and, for almost every $t \in [0,T]$, we obtain a unique solution u_t to the problem

$$\begin{cases} \text{find } u_t \in \mathcal{X}(A_t^{\mathrm{ext}}) \text{ such that} & u_t = U_t \text{ on } \partial A_t\,,\\ 2\displaystyle\int_{A_t^{\mathrm{ext}}} \mathrm{E}u_t : \mathrm{E}w \,\mathrm{d}y + \alpha^2 \int_{A_t^{\mathrm{ext}}} u_t \cdot w \,\mathrm{d}y = 0 & \text{for every } w \in \mathcal{X}_0(A_t^{\mathrm{ext}}). \end{cases} \tag{3.9}$$

Let F_{A_t,U_t} and M_{A_t,U_t} be the viscous force and torque determined by the velocity field U_t according to (2.7) and (2.8). By neglecting inertia and imposing the self-propulsion constraint, the equations of motion reduce to the vanishing of the viscous force and torque, i.e.,

$$F_{A_t,U_t} = 0 \quad \text{and} \quad M_{A_t,U_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{3.10}$$

By assuming that φ_t is factorized as $\varphi_t = r_t \circ s_t$, where r_t is a rigid motion as in (3.5) and $t \mapsto s_t$ is a prescribed shape function, our aim is to find $t \mapsto r_t$ so that the equations of motion (3.10) are satisfied. To this extent, we present Theorem 3.1 below, whose result is that (3.10) is equivalent to a system of ordinary differential equations where the unknown functions are the translation $t \mapsto y_t$ and the rotation $t \mapsto R_t$ of the map $t \mapsto r_t$.

The coefficients of these differential equations are defined starting from the intermediate configuration described by the sets $B_t = s_t(A)$ introduced before and the 3×3 matrices K_t, C_t, J_t, depending only on the geometry of B_t, whose entries are defined by

$$(K_t)_{ij} := \langle \sigma[e_j]n, e_i \rangle_{B_t^{\text{ext}}}, \tag{3.11a}$$

$$(C_t)_{ij} := \langle \sigma[e_j]n, e_i \times z \rangle_{B_t^{\text{ext}}}, \tag{3.11b}$$

$$(J_t)_{ij} := \langle \sigma[e_j \times z]n, e_i \times z \rangle_{B_t^{\text{ext}}}, \tag{3.11c}$$

where $B_t^{\text{ext}} := \mathbb{R}^3 \setminus \overline{B}_t$, the duality product is given in Definition 2.3 by formula (2.5), and $\sigma[W]$ denotes the stress tensor associated with the outer Brinkman problem in B_t^{ext} with boundary datum W. The notation $\sigma[W]$ is chosen to emphasize the linear dependence of σ on W. Formula (2.6) shows that K_t and J_t are symmetric. The matrix

$$\begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}$$

is often called in the literature *grand resistance matrix*, and is symmetric and invertible. It originally arises in the case of a Stokes system [8], but the adaptation to the Brinkman system is straightforward: it only shares the structure with the original one, while the values of the entries are computed with a different formula, namely (2.6). Let

$$\begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} := \begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}^{-1} \tag{3.12}$$

be its inverse. For almost every $t \in [0,T]$, we defined $W_t = \dot{s}_t \circ s_t^{-1}$, and let F_t^{sh} and M_t^{sh} be the viscous force and torque on ∂B_t determined by the boundary velocity field W_t. The components of F_t^{sh} and M_t^{sh} are given, according to (2.7) and (2.8), by

$$(F_t^{\text{sh}})_i = \langle \sigma[W_t]n, e_i \rangle_{B_t^{\text{ext}}}, \tag{3.13a}$$

$$(M_t^{\text{sh}})_i = \langle \sigma[W_t]n, e_i \times z \rangle_{B_t^{\text{ext}}}. \tag{3.13b}$$

Consider now the linear operator $\mathcal{A} : \mathbb{R}^3 \to \mathbb{M}^{3\times 3}$ that associates to every $\omega \in \mathbb{R}^3$ the only skew-symmetric matrix $\mathcal{A}(\omega)$ such that $\mathcal{A}(\omega)z = \omega \times z$; therefore, ω is the axial vector of $\mathcal{A}(\omega)$. Finally, we define a vector b_t and a matrix Ω_t according to

$$b_t := H_t F_t^{\text{sh}} + D_t^T M_t^{\text{sh}}, \qquad \Omega_t := \mathcal{A}(D_t F_t^{\text{sh}} + L_t M_t^{\text{sh}}), \tag{3.14}$$

which depend on s_t and, most importantly on $\dot{s}_t$, via (3.13) and the definition of W_t.

Theorem 3.1. *Assume that the shape function $t \mapsto s_t$ satisfies* (3.6) *and* (3.8)*, and that the position function $t \mapsto r_t$ satisfies* (3.5) *and is Lipschitz continuous with respect to time. Then the following conditions are equivalent:*

(i) *the deformation function $t \mapsto \varphi_t := r_t \circ s_t$ satisfies the equations of motion* (3.10)*;*
(ii) *the functions $t \mapsto y_t$ and $t \mapsto R_t$ satisfy the system*

$$\dot{y}_t = R_t b_t, \qquad \dot{R}_t = R_t \Omega_t, \quad \textit{for almost every } t \in [0,T], \tag{3.15}$$

where b_t and Ω_t are defined in (3.14). □

The proof was given in [5] and need not be modified, so we skip it. It is developed by setting the problem in the intermediate configuration B_t, assuming the point of view of the coordinate system of the shape functions. Changing the variables according to $y = r_t(z)$, $z \in B_t^{\text{ext}}$, the velocity field $v_t(z) := R_t^T u_t(r_t(z))$ is the solution to the problem

$$\begin{cases} \text{find } v_t \in \mathcal{X}(B_t^{\text{ext}}) \text{ such that} & v_t = V_t \text{ on } \partial B_t\,, \\ 2\displaystyle\int_{B_t^{\text{ext}}} \mathrm{E}v_t : \mathrm{E}w \,\mathrm{d}z + \alpha^2 \int_{B_t^{\text{ext}}} v_t \cdot w \,\mathrm{d}z = 0, & \text{for every } w \in \mathcal{X}_0(B_t^{\text{ext}}), \end{cases} \tag{3.16}$$

where $V_t(z) = R_t^T U_t(r_t(z))$, see Fig. 2.

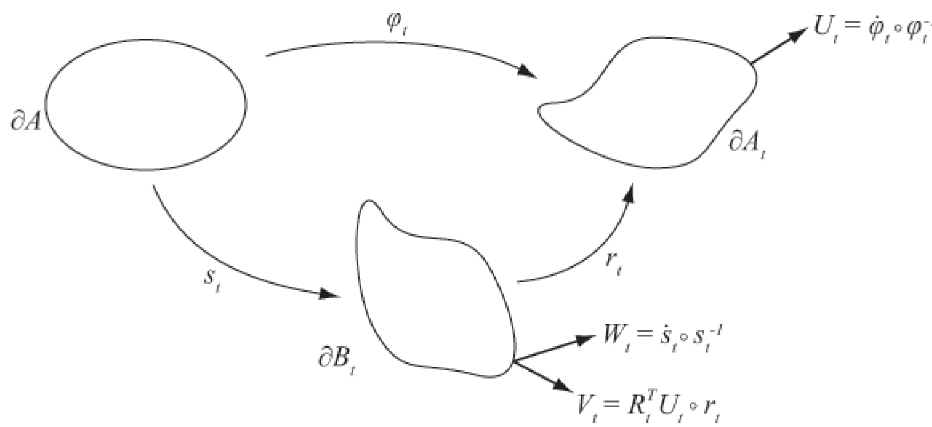

FIGURE 2. Notation for the boundary velocities (we neglect here the surrounding particulate medium).

Denote by F_{B_t,V_t} and M_{B_t,V_t} the viscous force and torque on ∂B_t determined by the velocity field v_t according to (2.7) and (2.8), with $\Omega = B_t^{\text{ext}}$. A straightforward computation yields $F_{B_t,V_t} = R_t^T F_{A_t,U_t}$ and $M_{B_t,V_t} = R_t^T M_{A_t,U_t}$, so that the equations of motion (3.10) reduce to

$$F_{B_t,V_t} = 0 \quad \text{and} \quad M_{B_t,V_t} = 0 \quad \text{for almost every } t \in [0,T].$$

Again by a simple manipulation we obtain the following form of the equations of motion

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\text{sh}} \\ M_t^{\text{sh}} \end{bmatrix} \quad \text{for almost every } t \in [0,T],$$

which read, by means of (3.14), as (3.15).

Now, the standard theory of ordinary differential equations with possibly discontinuous coefficients [7] ensures that the Cauchy problem for (3.15) has one and only one Lipschitz solution $t \mapsto R_t$, $t \mapsto y_t$, provided that the functions $t \mapsto \Omega_t$ and $t \mapsto b_t$ are measurable and bounded. By (3.12) and (3.14), this happens when the functions

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \quad t \mapsto F_t^{\text{sh}}, \quad t \mapsto M_t^{\text{sh}} \tag{3.17}$$

are measurable and bounded. The continuity of the first three functions will be proved in the last part of this section. The proof of the measurability and boundedness of the last two functions in (3.17) requires some technical tools that will be developed in Section 4.

We need the following notion of set convergence: given a sequence of sets $(S_k)_k$, we say that S_k converge to S_∞, $S_k \to S_\infty$, if for every $\varepsilon > 0$ there exists m such that for every $k \geqslant m$

$$S_\infty^{-\varepsilon} \subset S_k \subset S_\infty^{+\varepsilon}, \tag{3.18}$$

where $S_\infty^{-\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, \mathbb{R}^3 \setminus S_\infty) \geqslant \varepsilon\}$ and $S_\infty^{+\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, S_\infty) \leqslant \varepsilon\}$. The next lemma states a continuity property of the set-valued function $t \mapsto B_t$.

Lemma 3.2 ([5])**.** *Let s_t satisfy* (3.8)*. Then if $t \to t_\infty$ the sets B_t converge to the set B_{t_∞} in the sense of* (3.18)*.* □

Theorem 3.3. *Let w_t be the solution to the exterior Brinkman problem* (2.2) *on B_t^{ext} with boundary datum W on ∂B_t, where W can be either a constant vector $a \in \mathbb{R}^3$ or the rotation $W_\omega := \omega \times z$, with $\omega \in \mathbb{R}^3$. Define $\widetilde{w}_t$ to be the extension*

$$\widetilde{w}_t := \begin{cases} W & \text{on } B_t\,, \\ w_t & \text{on } B_t^{\mathrm{ext}}, \end{cases} \tag{3.19}$$

Assume that $t \mapsto s_t$ satisfies (3.8). *Then the map $t \mapsto \widetilde{w}_t$ is continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$.*

Proof. Let $(t_k)_k \subset [0,T]$ be a sequence that converges to $t_\infty \in [0,T]$. Lemma 3.2 ensures the convergence of the sets B_{t_k} to B_{t_∞} in the sense of (3.18).

Since w_{t_k} are solutions to Brinkman problems, we have the bound $2\int_{B_{t_k}^{\mathrm{ext}}} |\mathrm{E}w_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{B_{t_k}^{\mathrm{ext}}} |w_{t_k}|^2\,\mathrm{d}z \leqslant C$, which, in turn, implies that

$$2\int_{\mathbb{R}^3} |\mathrm{E}\widetilde{w}_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} |\widetilde{w}_{t_k}|^2\,\mathrm{d}z \leqslant C.$$

Therefore, $\widetilde{w}_t$ admits a subsequence that converges weakly to a function $w^* \in \mathcal{X}(\mathbb{R}^3)$. By the convergence of the B_{t_k}, it is easy to see that $w^* = W$ on B_{t_∞}. We now prove that $w^*|_{B_{t_\infty}^{\mathrm{ext}}}$ solves the exterior Brinkman problem on B_{t_∞}. Too see it, consider a test function $\varphi \in C_c^\infty(B_{t_\infty}^{\mathrm{ext}};\mathbb{R}^3)$. For k large enough, $\varphi \in C_c^\infty(B_{t_k}^{\mathrm{ext}};\mathbb{R}^3)$, so that

$$2\int_{\mathrm{spt}\,\varphi} \mathrm{E}w_{t_k} : \mathrm{E}\varphi\,\mathrm{d}z + \alpha^2 \int_{\mathrm{spt}\,\varphi} w_{t_k}\cdot\varphi\,\mathrm{d}z = 0.$$

This equality passes to the limit as $k \to \infty$, showing that $w^*|_{B_{t_\infty}^{\mathrm{ext}}}$ is a solution to the Brinkman problem at t_∞. Therefore, $w^* = \widetilde{w}_{t_\infty}$, and we have proved that $t \mapsto w_t$ is strongly continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$. □

We can now prove the following continuity result for the elements of the grand resistance matrix by means of Theorem 3.3.

Proposition 3.4. *Assume that s_t satisfies* (3.6) *and* (3.8). *Then the functions*

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{3.20}$$

and consequently $t \mapsto H_t$, $t \mapsto D_t$, $t \mapsto L_t$, are continuous.

Proof. Formulae (3.11) and (2.6) provide us with an explicit form for the elements of the grand resistance matrix

$$(K_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t^j : \mathrm{E}v_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t^j \cdot v_t^i\,\mathrm{d}z, \tag{3.21a}$$

$$(C_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t^j : \mathrm{E}\hat{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t^j \cdot \hat{v}_t^i\,\mathrm{d}z, \tag{3.21b}$$

$$(J_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}\hat{v}_t^j : \mathrm{E}\hat{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} \hat{v}_t^j \cdot \hat{v}_t^i\,\mathrm{d}z, \tag{3.21c}$$

where v_t^i and $\hat{v}_t^i$ are the functions defined in (3.19) with $W = e_i$ and $W = e_i \times z$, respectively. We prove the result for K_t only, since the others are similar. We write

$$(K_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}\widetilde{v}_t^j : \mathrm{E}\widetilde{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} \widetilde{v}_t^j \cdot \widetilde{v}_t^i\,\mathrm{d}z - \alpha^2 \int_{B_t} e_j \cdot e_i\,\mathrm{d}z,$$

where $\widetilde{v}_t^i$ and $\widetilde{v}_t^j$ are the extensions considered in (3.19). By Theorem 3.3, the first two integrals are continuous with respect to t. The continuity of the last integral is guaranteed by Lemma 3.2. □

The proof of the measurability and boundedness of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ is a delicate issue. The difficulty arises from the fact that both the domains B_t and the boundary data $W_t = \dot{s}_t \circ s_t^{-1}$ depend on time. Moreover, since it is meaningful and interesting to consider boundary values W_t that might be discontinuous with respect to t, we cannot expect the functions $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ to be continuous.

To prove the measurability we start from an integral representation of F_t^{sh} and M_t^{sh}, similar to (3.21). As $\int_{\partial B_t} W_t \cdot n\,\mathrm{d}S$ is not necessarily zero, we will not be able to compute integrals over the whole space $\mathbb{R}^3$, so we will have to work in the complement of an open ball $\Sigma_\varepsilon^0 \subset\subset B_t$. Since, in general, this inclusion holds only locally in time, we first fix $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$ and select $\delta > 0$ and $\varepsilon > 0$ so that the open ball $\Sigma_\varepsilon^0 := \Sigma_\varepsilon(z^0)$ of radius ε centered at z^0 satisfies

$$\Sigma_\varepsilon^0 \subset\subset B_t\,, \quad \text{for all } t \in I_\delta(t_0) := [0,T] \cap (t_0-\delta, t_0+\delta). \tag{3.22}$$

This is possible thanks to the continuity properties of $t \mapsto s_t$ listed in the first part of this section.

Next we consider the solution w_t to the problem

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma_\varepsilon^{0,\text{ext}})} : w \in \mathcal{X}(\Sigma_\varepsilon^{0,\text{ext}}),\ w = W_t \text{ on } \partial B_t\,,\ \text{and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma_\varepsilon^0\right\}$$

In order for the flux condition (2.3) to be fulfilled by w_t on $\partial B_t \cup \partial\Sigma_\varepsilon^0$, we choose

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S.$$

Finally, putting together (3.13) and (2.6), we obtain the following explicit integral representation of F_t^{sh} and M_t^{sh}

$$(F_t^{\text{sh}})_i = 2\int_{\Sigma_\varepsilon^{0,\text{ext}}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z + \alpha^2\int_{\Sigma_\varepsilon^{0,\text{ext}}} w_t\cdot v_t^i\,\mathrm{d}z - \alpha^2\int_{Q_{\varepsilon,t}} w_t\cdot v_t^i\,\mathrm{d}z$$

$$(M_t^{\text{sh}})_i = 2\int_{\Sigma_\varepsilon^{0,\text{ext}}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z + \alpha^2\int_{\Sigma_\varepsilon^{0,\text{ext}}} w_t\cdot \hat{v}_t^i\,\mathrm{d}z - \alpha^2\int_{Q_{\varepsilon,t}} w_t\cdot \hat{v}_t^i\,\mathrm{d}z$$

where v_t^i and $\hat{v}_t^i$ have been defined in the proof of Proposition 3.4 and $Q_{\varepsilon,t} := B_t \setminus \overline{\Sigma}_\varepsilon^0$. We deduce from Theorem 3.3 and Lemma 3.2 that the functions $t \mapsto v_t^i$ and $t \mapsto \hat{v}_t^i$ are continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\text{ext}})$. Therefore, the measurability and boundedness of $t \mapsto F_t^{\text{sh}}$ and $t \mapsto M_t^{\text{sh}}$ will be proved once $t \mapsto w_t$ is proved to be measurable. We first show that $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\text{ext}})$ and eventually we will prove that the function $t \mapsto \int_{Q_{\varepsilon,t}} w_t\,\mathrm{d}z$ is continuous with respect to time. These two results are proved in the next Section.

4. Extensions of boundary data and main result

In order to prove the main result, some work is still to be done to prove the regularity property of the coefficients of the equations of motion (3.15). To this aim, results concerning the extension of boundary data are needed to be able to use standard variational techniques to solve the relevant minimum problem of Theorem 4.4. The following result has been proved in [5].

Proposition 4.1 (Solenoidal extension operators). *Assume that s_t satisfies* (3.6) *and* (3.8), *and let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$. Let $\delta > 0$ and $\varepsilon > 0$ be such that* (3.22) *holds true. Then there exists a uniformly bounded family $(\mathcal{T}_t)_{t\in I_\delta(t_0)}$ of continuous linear operators*

$$\mathcal{T}_t\colon H^{1/2}(\partial A;\mathbb{R}^3) \to \mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma}_\varepsilon^0)$$

such that

(i) *for all $t \in I_\delta(t_0)$ and for all $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$,*

$$\mathcal{T}_t(\Phi) = \Phi\circ s_t^{-1} \quad \text{on } \partial B_t\,,$$
$$\mathcal{T}_t(\Phi) = \lambda_t\frac{z}{|z|^3} \quad \text{on } \partial\Sigma_\rho\,,$$

(ii) *for every $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$ the map $t \mapsto \mathcal{T}_t(\Phi)$ is continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma}_\varepsilon^0)$.*

In particular, the following estimate holds

$$\|\mathcal{T}_t(\Phi)\|_{H^1(\Sigma_\rho\setminus\overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)} \leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \tag{4.1}$$

where the constant C is independent of t and Φ. □

Proposition 4.2. *Assume that s_t satisfies* (3.6), (3.7), *and* (3.8). *Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (3.22). *Suppose, in addition, that for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi^{t_0}_t : \Sigma_\rho \to \Sigma_\rho$ coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Psi^{t_0}_t = s_{t_0} \circ s_t^{-1}$ on B_t. Let the map $t \mapsto \Phi_t$ belong to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. Let w_t be the solution to the problem*

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)} : \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon),\ w = \Phi_t \circ s_t^{-1} \text{ on } \partial B_t \text{ and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma^0_\varepsilon\right\}, \tag{4.2}$$

where $\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t}(\Phi_t \circ s_t^{-1})\cdot n \,\mathrm{d}S$. Then $t \mapsto w_t$ belongs to $C^0(I_\delta(t_0); \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon))$.

Proof. The proof can be easily adapted from that of [5, Proposition 6.1]; the following important estimate provides a uniform bound for the norms of the w_t's in $\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ that will also be useful in the proof of Proposition 4.3

$$\begin{aligned} 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_{t_k}|^2\,\mathrm{d}z &\leqslant 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|\mathrm{E}\psi_{t_k}|^2\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|\psi_{t_k}|^2\,\mathrm{d}z \\ &\leqslant \|\psi_{t_k}\|^2_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)} \leqslant C^2(\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|)^2 \leqslant (CM)^2, \end{aligned} \tag{4.3}$$

where $\psi_t \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is defined by

$$\psi_t := \begin{cases} \mathcal{T}_t(\Phi_t) & \text{in } \Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon \\ \lambda_t \dfrac{z}{|z|^3} & \text{in } \Sigma^{\mathrm{ext}}_\rho \end{cases}$$

and is the function provided by Proposition 4.1 and extended on $\Sigma^{\mathrm{ext}}_\rho$, C is the constant in (4.1), and $M > 0$ is a uniform upper bound of $\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|$, whose existence is guaranteed by the fact that $t \mapsto \Phi_t$ belongs to $L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. □

Proposition 4.3. *Under the hypotheses of Proposition 4.2, recalling that $Q_{\varepsilon,t} = B_t \setminus \overline{\Sigma}^0_\varepsilon$, the maps*

$$t \mapsto \int_{Q_{\varepsilon,t}} w_t\,\mathrm{d}z, \qquad t \mapsto \int_{Q_{\varepsilon,t}} z \times w_t\,\mathrm{d}z \tag{4.4}$$

where $t \mapsto w_t \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is the solution to the minimum problem (4.2) *as in Proposition 4.2, are continuous with respect to time in $I_\delta(t_0)$.*

Proof. We check the continuity with the definition

$$\begin{aligned} \left|\int_{Q_{t+h}} w_{t+h}\,\mathrm{d}z - \int_{Q_t} w_t\,\mathrm{d}z\right| &= \left|\int_{Q_{t+h}}(w_{t+h}-w_t)\,\mathrm{d}z + \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} w_t(\chi_{Q_{t+h}}(z) - \chi_{Q_t}(z))\,\mathrm{d}z\right| \\ &\leqslant \left(\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_{t+h}-w_t|^2\,\mathrm{d}z\right)^{\frac12}|Q_{t+h}|^{\frac12} + \left(\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_t|^2\,\mathrm{d}z\right)^{\frac12}|Q_{t+h}\triangle Q_t|^{\frac12} \\ &\leqslant \|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}|Q_{t+h}|^{\frac12} + \|w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}|Q_{t+h}\triangle Q_t|^{\frac12} \\ &\leqslant |\Sigma_\rho|^{\frac12}\|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)} + CM\,|Q_{t+h}\triangle Q_t|^{\frac12} \xrightarrow{h\to 0} 0. \end{aligned}$$

Here, χ_Q denotes the characteristic function of the set Q, $\triangle$ is the symmetric difference operator, and CM is the uniform (with respect to t) upper bound coming from (4.3). The continuity for the second map is achieved in the same way. □

Proposition 4.2 and Proposition 4.3 combined together give the continuity of $t \mapsto F^{\mathrm{sh}}_t$ and $t \mapsto M^{\mathrm{sh}}_t$ with respect to time, in the case of regular boundary data $\Phi_t \circ s_t^{-1}$ on ∂B_t, where the map $t \mapsto \Phi_t$ belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. The next results will prove that when the boundary data on ∂B_t are given by $\dot{s}_t \circ s_t^{-1}$, then the maps $t \mapsto F^{\mathrm{sh}}_t$ and $t \mapsto M^{\mathrm{sh}}_t$ are measurable and bounded.

Theorem 4.4. *Assume that s_t satisfies* (3.6), (3.7), *and* (3.8). *Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (3.22). *Suppose, in addition, that for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi^{t_0}_t : \Sigma_\rho \to \Sigma_\rho$ coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Psi^{t_0}_t = s_{t_0} \circ s_t^{-1}$ on B_t. Let w_t be the solution to the problem*

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)} : w \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon),\ w = \dot{s}_t \circ s_t^{-1} \text{ on } \partial B_t,\ \text{and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma^0_\varepsilon\right\}.$$

Then the function $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. Moreover, also the functions (4.4) *considered in Proposition 4.3 are measurable and bounded in $I_\delta(t_0)$.*

Proof. It suffices to convolve the boundary datum with a suitable regularizing kernel and to apply Propositions 4.2 and 4.3. By passing to the limit, the continuity is lost but the functions turn out to be measurable and bounded. □

Proposition 3.4 and Theorem 4.4 give the regularity result for b_t and Ω_t in (3.14), as stated in the following result.

Theorem 4.5. *Assume that $t \mapsto s_t$ satisfies* (3.6)*,* (3.7)*, and* (3.8)*. Then the vector b_t and the matrix Ω_t in* (3.14) *are bounded and measurable with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then $t \mapsto (b_t\,,\Omega_t)$ belongs to $C^0([0,T];\mathbb{R}^3{\times}\mathbb{M}^{3\times3})$.* □

We are now in a position to state the existence, uniqueness, and regularity result for the equations of motion (3.15).

Theorem 4.6. *Assume that $t \mapsto s_t$ satisfies* (3.6)*,* (3.7)*, and* (3.8)*. Let $y^* \in \mathbb{R}^3$ and $R^* \in \mathrm{SO}(3)$. Then* (3.15) *has a unique absolutely continuous solution $t \mapsto (y_t\,,R_t)$ defined in $[0,T]$ with values in $\mathbb{R}^3{\times}\mathrm{SO}(3)$ such that $y_0 = y^*$ and $R_0 = R^*$. In other words, there exists a unique rigid motion $t \mapsto r_t(z) = y_t + R_t z$ such that the deformation function $t \mapsto \varphi_t = r_t \circ s_t$ satisfies the equations of motion* (3.10)*.*

Moreover this solution is Lipschitz continuous with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then the solution $t \mapsto (y_t\,,R_t)$ belongs to $C^1([0,T];\mathbb{R}^3{\times}\mathrm{SO}(3))$. □

Proof. The existence and uniqueness of the solution of the Cauchy problem for (3.15) follow immediately from Theorem 4.5, by standard results on ordinary differential equations with bounded measurable coefficients, see, e.g., [7, Theorem I.5.1]. The assertion concerning the deformation function $t \mapsto \varphi_t$ and the equation of motion (3.10) follows from the equivalence Theorem 3.1. The Lipschitz continuity of the solution follows from the boundedness of the right-hand sides of the equation in (3.15).

If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then Theorem 4.5 ensures that the coefficients of the equations in (3.15) are continuous with respect to t, and therefore the solutions are of class C^1. □

5. Conclusions and future work

We have shown that the framework for modeling the motion of a deformable body in a viscous fluid that we presented in [5] also fits in the case of a particulate system for which the Brinkman equation is assumed to model the fluid phase of the surrounding medium. A suitable functional setting has been developed, and the solution to the Brinkman system has been found by solving a minimum problem for the associated functional. Some extra terms appeared, with respect to the Stokes case, due to the presence of the $-\alpha^2 u$ term in the Brinkman system. Nonetheless, the corresponding integrals, depending on time both in the integrand function and in the domain of integration, have been proved to be continuous with respect to time, thus allowing the coefficients of the equations of motion to be regular enough.

Another noteworthy feature of our work is that the infinite-dimensional control $t \mapsto s_t$ is coupled with and determines a finite-dimensional function to describe the position of the swimmer. In previous works [18], [17], and [2], only swimmers with a finite number of shape parameters were dealt with. Here, we have been able to extend the study to the case of a more complex deformations.

In our model, we neglected the interactions between the solid particles and the swimmer, considering only the body-fluid phase viscous interaction. We think this is a reasonable approximation for using a simple model such as the Brinkman equation. Also, the mathematical model to describe the experiments in [11] is the same, and in that case the elastic and adhesive interactions between the nematode and the surrounding particles are neglected as well. Nevertheless, we think it can be interesting to develop more complex models to take into account also that kind of contact forces, and that could be the object of a future study.

Even though it has not been addressed in this work, we also expect our model to be able to predict, on the basis of an energy comparison, whether swimming in a particulate medium is more efficient than swimming in a plain viscous fluid; that would be an interesting theoretical check of the thesis advanced by Jung on the basis of his experimental results that *C. elegans* swims more efficiently in a particulate medium.

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