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***Abstract*—This paper investigates the efficiency of a perturbative approach for the statistical assessment of differential transmission lines affected by random parameters. Within the polynomial chaos framework, the novel technique reformulates the so-called stochastic Galerkin method in a decoupled and iterative fashion. Instead of solving the classical, augmented and fully coupled transmission line equations, the new approach iteratively solves multiple uncoupled line equations with nominal per-unit-length parameters and suitable equivalent distributed sources accounting for their variability. The methodology is applied to a system of up to four PCB differential pairs. A computational advantage is observed against the classical stochastic Galerkin method for large problems in terms of number of random parameters and/or conductors.**

***Index Terms*—Differential signaling, perturbation technique, polynomial chaos, statistical analysis, stochastic Galerkin method, transmission line modeling.**

I. Introduction

Differential signaling is widely adopted in modern high-speed electronic systems for the mitigation of detrimental effects such as crosstalk and radiated emission/susceptibility. A symmetric and perfectly balanced differential line (DL) limits the signal transmission to the differential mode (DM), generating no common mode (CM) noise. However, manufacturing variability may in practice break the symmetry of the line geometry, thus giving rise to DM-to-CM conversion and degrading signal integrity and electromagnetic compatibility performance.

Some recent papers investigated mode conversion from a theoretical viewpoint in various deterministic scenarios [1]–[7]. Nevertheless, a statistical assessment is extremely beneficial, as the source of such asymmetry is often stochastic in nature.

In recent literature, the method of generalized polynomial chaos expansion (PCE) has been extensively used for stochastic signal integrity analysis [8]– [10]. It was shown that using the PCE, in particular in conjunction with the stochastic Galerkin method (SGM), allows outperforming the blind and brute-force Monte Carlo (MC) method. This is true at least for a moderate number of random variables (RVs), whereas the efficiency rapidly decreases as the number of parameters increases, as a consequence of the so-called "curse of dimensionality" [11]. Indeed, the SGM projects the stochastic telegraphers' equations onto the polynomial basis functions, and results in equivalent, deterministic telegraphers' equations of larger size in the unknown PCE coefficients. The system size scales rather inconveniently with the number of RVs, thus limiting the applicability to problems with a small to moderate number of parameters.

A preliminary approach for mitigating this issue was presented in [12], in which an iterative and decoupled reformulation of the SGM was introduced that builds upon a perturbation approach. In this previous work, only a proof of principle was presented based on a single DL, showing the feasibility of this alternative approach. In the present paper, the technique is applied to more challenging examples, and the scaling of the efficiency with respect to the number of parameters is assessed. It is thus confirmed that the proposed perturbative reformulation outperforms the classical SGM for larger problems. As a test case, a structure with up to four uniform PCB differential lines and 15 RVs is considered.

II. Classical and Perturbative SGM Formulations

In this section, the perturbative reformulation of the SGM, originally presented in [12] for a single DL, is extended to the general case. The starting point for the discussion are the well-known frequency-domain telegraphers' equations for a multiconductor transmission line (MTL) with N signal conductors, which read [13]

$$\frac{d}{dz}\mathbf{V}(z,\omega) = -\mathbf{Z}(\omega)\mathbf{I}(z,\omega) \tag{1a}$$

$$\frac{d}{dz}\mathbf{I}(z,\omega) = -\mathbf{Y}(\omega)\mathbf{V}(z,\omega), \tag{1b}$$

where ω is the angular frequency, $\mathbf{Z}, \mathbf{Y} \in \mathbb{R}^{N\times N}$ are the per-unit-length (p.u.l.) impedance and admittance matrices, whereas vectors $\mathbf{V}, \mathbf{I} \in \mathbb{R}^{N}$ collect voltages and currents along the conductors. In the following, the variable ω is omitted for brevity of notaiton.

If the MTL is affected by a certain number d of RVs, embedded in vector $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_d]$, the impedance and admittance matrices, as well as the voltage and current vectors, are inherently $\boldsymbol{\xi}$-dependent and hence stochastic. In the framework of generalized polynomial chaos, stochastic quantities are expressed as PCEs [8], e.g.,

$$\mathbf{Z}(\boldsymbol{\xi}) \approx \sum_{k=0}^{K} \mathbf{Z}_k \varphi_k(\boldsymbol{\xi}) \tag{2a}$$

$$\mathbf{V}(z, \boldsymbol{\xi}) \approx \sum_{k=0}^{K} \mathbf{V}_k(z) \varphi_k(\boldsymbol{\xi}), \tag{2b}$$

where the impedance coefficients $\mathbf{Z}_k$ are determined based on the known variability of the geometry, while the voltage coefficients $\mathbf{V}_k$ are to be determined. Analogous relations hold for the admittance matrix and the vector of currents. From the PCEs (2), statistical information is readily obtained by post-processing. The number of coefficients is related to the number of RVs d and polynomial order p by $K+1 = (p+d)!/(p!d!)$.

Replacing the PCEs (2) into (1) and performing a Galerkin projection, yields the following deterministic though augmented MTL-like equations in the PCE coefficients:

$$\frac{d}{dz} \widetilde{\mathbf{V}}(z) = -\widetilde{\mathbf{Z}} \widetilde{\mathbf{I}}(z) \tag{3a}$$

$$\frac{d}{dz} \widetilde{\mathbf{I}}(z) = -\widetilde{\mathbf{Y}} \widetilde{\mathbf{V}}(z), \tag{3b}$$

with $\widetilde{\mathbf{V}} = [\mathbf{V}_0, \ldots, \mathbf{V}_K]^T$ and $\widetilde{\mathbf{I}} = [\mathbf{I}_0, \ldots, \mathbf{I}_K]^T$, whereas $\widetilde{\mathbf{Z}}$ and $\widetilde{\mathbf{Y}}$ are block matrices constructed using a suitable combination of the coefficients $\mathbf{Z}_k$ and $\mathbf{Y}_k$. The size of equations (3) is $N(K+1)$ and rapidly becomes intractable when the number of RVs and/or expansion order is increased, due to the coupled and dense nature of the involved matrices.

Nonetheless, it is possible to factor out the block-diagonal components $\mathbf{Z}_0, \mathbf{Y}_0$, corresponding to the average of the impedance and admittance matrices. This allows rewriting (3) as

$$\frac{d}{dz} \widetilde{\mathbf{V}}(z) = -(\mathbb{I} \otimes \mathbf{Z}_0) \widetilde{\mathbf{I}}(z) + (\mathbb{I} \otimes \mathbf{Z}_0 - \widetilde{\mathbf{Z}}) \widetilde{\mathbf{I}}(z) \tag{4a}$$

$$\frac{d}{dz} \widetilde{\mathbf{I}}(z) = -(\mathbb{I} \otimes \mathbf{Y}_0) \widetilde{\mathbf{V}}(z) + (\mathbb{I} \otimes \mathbf{Y}_0 - \widetilde{\mathbf{Y}}) \widetilde{\mathbf{V}}(z), \tag{4b}$$

where $\mathbb{I}$ is the identity matrix $\in \mathbb{R}^{(K+1)\times(K+1)}$, and $\otimes$ denotes the Kronecker product.

Equations (4) are formally equivalent to (3). However, it is now noted that the first matrix term in the r.h.s. of (4) i) has a block-diagonal structure and ii) is much larger (in norm) than the second one. The second point can be understood by considering that the PCE is typically dominated by the zero-order coefficient (average value). These two considerations allow relaxing (4) and further recasting them as a set of uncoupled and iterative subproblems:

$$\frac{d}{dz} \mathbf{V}_k^{(m)}(z) = -\mathbf{Z}_0 \mathbf{I}_k^{(m)}(z) + \left[\mathbb{I} \otimes \mathbf{Z}_0 - \widetilde{\mathbf{Z}} \right]_{k*} \mathbf{I}_k^{(m-1)}(z) \tag{5a}$$

$$\frac{d}{dz} \mathbf{I}_k^{(m)}(z) = -\mathbf{Y}_0 \mathbf{V}_k^{(m)}(z) + \left[\mathbb{I} \otimes \mathbf{Y}_0 - \widetilde{\mathbf{Y}} \right]_{k*} \mathbf{V}_k^{(m-1)}(z), \tag{5b}$$

where m is the iteration index, while the notation $[\,\cdot\,]_{k*}$ denotes the kth row matrix block.

Equations (5) correspond to the telegraphers' equations of an MTL with distributed sources [13], which require integration along the line length. At every iteration, each PCE coefficient $\mathbf{V}_k$ and $\mathbf{I}_k$ is solved independently for $k = 0, \ldots, K$. This is possible because the distributed sources only depend on the solution at the previous iteration, and they are therefore explicitly known at a given step. The procedure is terminated based on the convergence of the solution in relative difference.

Despite the integration of the distributed sources, the decoupled nature of (5) allows for a faster solution w.r.t. the original coupled problem (3), even when a relatively high number of iterations is required. Indeed, the computational cost of the new solution scales linearly with the number of PCE coefficients. The assessment of the efficiency scaling is the main subject of the present paper and is discussed in the next section.

III. Application to Multiple DLs

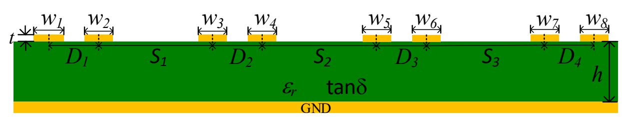

Fig. 1. Test structure with multiple differential pairs. Cross-sectional view of the case study with $N = 8$ conductors (4 DLs).

In order to generate different test cases of increasing complexity, a modular PCB structure consisting of up to four differential pairs is considered, as depicted in Fig. 1. The RVs that are possibly taken into consideration are the trace widths (nominal value of $w_i = 0.6$ mm, $i = 1, \ldots, N$) and their separations (nominal values of $D_i = 1.1$ mm, $i = 1, \ldots, N/2$, and $S_i = 2.2$ mm, $i = 1, \ldots, N/2 - 1$), all independent and Gaussian with a relative standard deviation of 10%. The other geometrical and material parameters are as follows: $h = 1.425$ mm, $t = 0.035$ mm, $\varepsilon_r = 4.4$, $\tan\delta = 0.001$. The traces are made of copper with a conductivity of 58 MS/m. Eight scenarios are investigated, as summarized in Table I: the number of differential pairs is increased from one to four. For each structure, two cases are considered, one with variability in the trace widths only, and one with variability in both the trace widths and their separations. This allows generating a relevant collection of test cases, with a number of RVs ranging from 2 to 15, and complexity in terms of matrix size ranging from 12 to 1088.

Statistical estimates of the mixed-mode S-parameters are evaluated by means of the perturbative reformulation of the SGM outlined in Section II, and compared versus the predictions obtained by the MC method and classical SGM. For all test cases, a second-order PCE is used. As an example, Fig. 2 shows the mean value and standard deviation of $|S_{c8d1}|$ for the configuration with four DLs and eight RVs. An excellent accuracy is established. Similar accuracy is observed also for other S-parameters and test cases. It is important to remark

TABLE I
TEST CASES AND COMPLEXITY

Number of conductors N	Number of RVs and SGM problem size			
	Case 1		Case 2	
	d	$N(K+1)$	d	$N(K+1)$
2 (1 DL)	2	12	3	20
4 (2 DLs)	4	60	7	144
6 (3 DLs)	6	168	11	468
8 (4 DLs)	8	360	15	1088

that the difference w.r.t. the classical SGM can be arbitrarily reduced by considering a sufficiently high number of iterations. For the considered test cases, the iterations required to achieve 0.1% convergence range from 3 up to a maximum of 19, depending on the frequency point.

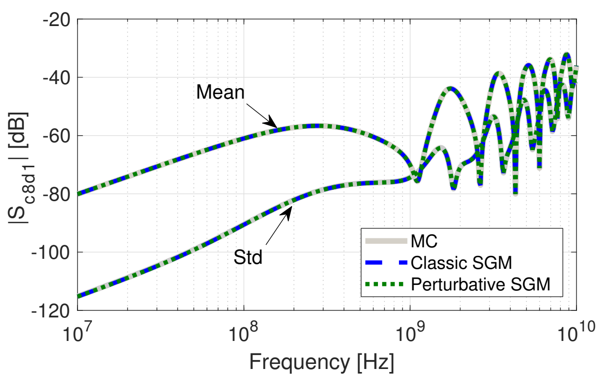

Fig. 2. Statistical estimates (average and standard deviation) of $|S_{c8d1}|$ in the configuration with four DLs and eight RVs. The results are computed with MC (solid gray line), classical SGM (dashed black line), and perturbative SGM (dotted green line).

Fig. 3 compares the speed-ups obtained by the classical and perturbative SGM formulations against the standard MC simulation (10000 samples). The comparison clearly shows that the proposed perturbative SGM outperforms the classical SGM for problems with eight RVs or more, *and* remains more efficient than MC even for the problem with 15 RVs. This is quite remarkable, since the perturbative SGM requires the evaluation and integration of the distributed sources along the line, as opposed to MC and the classical SGM.

IV. Conclusion

In this paper, the efficiency of the perturbative reformulation of the SGM for MTLs is investigated for problems with increasing complexity in terms of equation size. The perturbative SGM solves the originally coupled SGM problem in an iterative and uncoupled manner, thus achieving a significant efficiency improvement. Several PCB configurations with multiple DLs are used as test cases. It is shown that the perturbative SGM outperforms the classical SGM for problems with more than eight RVs, while still beating MC up to at least 15 RVs. Future work will extend the advocated approach to the analysis of nonuniform MTLs.

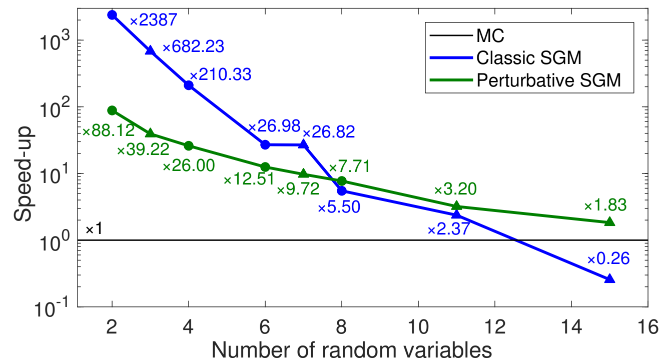

Fig. 3. Speed-up of the classical and perturbative SGM implementations w.r.t. MC analysis. Markers ● and ▲ refer to cases 1 and 2 in Table I, respectively.

References

[1] F. Grassi, G. Spadacini, and S. A. Pignari, “The concept of weak imbalance and its role in the emissions and immunity of differential lines,” *IEEE Trans. Electromagn. Compat.*, vol. 55, no. 6, pp. 1346–1349, Dec. 2013.

[2] Y. Kami, F. Xiao, and K. Murano, “Mode-port-network approach to analyze power-line emc problems for plc,” in *Proc. 20th Int. Zurich Symp. Electromagn. Compat.*, Zurich, Switzerland, Jan 2009, pp. 9–12.

[3] A. Sugiura and Y. Kami, “Generation and propagation of common-mode currents in a balanced two-conductor line,” *IEEE Trans. Electromagn. Compat.*, vol. 54, no. 2, pp. 466–473, Apr. 2012.

[4] K. Sejima, Y. Toyota, K. Iokibe, L. R. Koga, and T. Watanabe, “Experimental model validation of mode-conversion sources introduced to modal equivalent circuit,” in *Proc. IEEE 2012 Int. Symp. on Electromagn. Compat.*, Pittsburgh, PA, USA, Aug 2012, pp. 492–497.

[5] F. Grassi, Y. Yang, X. Wu, G. Spadacini, and S. A. Pignari, “On mode conversion in geometrically unbalanced differential lines and its analogy with crosstalk,” *IEEE Trans. Electromagn. Compat.*, vol. 57, no. 2, pp. 283–291, Apr. 2015.

[6] F. Grassi, P. Manfredi, X. Liu, J. Sun, X. Wu, D. Vande Ginste, and S. A. Pignari, “Effects of undesired asymmetries and nonuniformities in differential lines,” *IEEE Trans. Electromagn. Compat.*, vol. 59, no. 5, pp. 1613–1624, Oct. 2017.

[7] X. Wu, F. Grassi, P. Manfredi, and D. Vande Ginste, “Perturbative analysis of differential-to-common mode conversion in asymmetric nonuniform interconnects,” *IEEE Trans. Electromagn. Compat.*, vol. 60, no. 1, pp. 7–15, Feb. 2018.

[8] P. Manfredi, D. Vande Ginste, I. S. Stievano, D. De Zutter, and F. G. Canavero, “Stochastic transmission line analysis via polynomial chaos methods: an overview,” *IEEE Electromagn. Compat. Mag.*, vol. 6, no. 3, pp. 77–84, 2017.

[9] M. Ahadi and S. Roy, “Sparse linear regression (spliner) approach for efficient multidimensional uncertainty quantification of high-speed circuits,” *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 35, no. 10, pp. 1640–1652, Oct. 2016.

[10] M. R. Rufuie, E. Gad, M. Nakhla, and R. Achar, “Generalized hermite polynomial chaos for variability analysis of macromodels embedded in nonlinear circuits,” *IEEE Trans. Compon. Packag. Manuf. Technol.*, vol. 4, no. 4, pp. 673–684, Apr. 2014.

[11] J. Bai, G. Zhang, D. Wang, A. P. Duffy, and L. Wang, “Performance comparison of the SGM and the SCM in EMC simulation,” *IEEE Trans. Electromagn. Compat.*, vol. 58, no. 6, pp. 1739–1746, Dec. 2016.

[12] X. Wu, F. Grassi, P. Manfredi, and D. Vande Ginste, “Perturbative statistical assessment of PCB differential interconnects,” in *Signal and Power Integrity (SPI), 2018 IEEE 22nd Workshop on*. Brest, France: IEEE, May 2018, pp. 1–4.

[13] C. R. Paul, *Analysis of multiconductor transmission lines*. John Wiley & Sons, 2008.