The Linear Quadratic Regulator for Periodic Hybrid Systems ⋆,⋆⋆

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Abstract

The main objective of this paper is to characterize feedback control laws that are optimal with respect to a quadratic cost functional in the framework of linear hybrid systems undergoing time-driven periodic jumps, namely the so-called hybrid Linear-Quadratic Regulator (LQR) problem. The optimal solution to the hybrid LQR problem is determined both in the case of finite–horizon and infinite–horizon optimal control problems by introducing a hybrid (periodic) extension of the classic Differential and Difference Riccati Equations, thus leading to the notion of Monodromy Riccati Equation. Interestingly, due to the periodic nature of the discrete-time events, the computation of the optimal feedback hinges upon the solution of a differential, rather than algebraic, Riccati equation also in the infinite–horizon case, hence yielding a time-varying, periodic control law. Necessary and sufficient conditions that ensure asymptotic stability of the closed–loop system are provided and discussed in detail in the case of infinite–horizon optimal control problems.

1 Introduction

The class of linear or nonlinear models described by *hybrid* dynamics has recently attracted increasing attention and, as a consequence, several studies have appeared in the literature concerning the extension or the adaptation of classic results on fundamental control problems to such a class of systems (Sun, 2006; Liberzon, 2012; Goebel et al., 2012). In particular, hybrid systems are characterized by the peculiar interplay between continuous–time evolution, according to the so-called *flow dynamics*, and discrete–time events, governed by the *jump dynamics*. It is then not surprising that the optimal control problem - which is among the most important tasks in control theory together with stabilization of a desired equilibrium point and output regulation - has immediately become the objective of intensive research effort in the context of hybrid systems, even before the introduction of the particularly useful formalism and results of Goebel et al. (2012), see *e.g.* D'Apice et al. (2003); Xu and Antsaklis (2004); Shaikh and Caines (2007); Yuan and Wu (2015). In particular, in Sussmann (1999), a version of the Pontryagin maximum (or, equivalently, minimum) principle for hybrid optimal control problems, under weak regularity conditions, is presented. On the other hand, in Cassandras et al. (2001), a class of optimal control problems is analyzed for hybrid systems, emphasizing the coupling between the time–driven and event–driven dynamics governing the switches.

In this work, the interest lies on a specially structured class of hybrid systems that has been recently widely studied (see Menini and Tornambe, 2000, 2002; Cox, Teel and Marconi, 2011; Cox, Marconi and Teel, 2011; Galeani et al., 2012; Carnevale, Galeani, Menini and Sassano, 2014b,a and references therein), namely that characterized by linear dynamics and by the presence of a clock variable, which satisfies a well–defined (periodic) dwell time between two consecutive jumps and which is available for feedback. Within the class of hybrid systems identified by the above features, the optimal control problem over infinite–horizon and with respect to a quadratic cost has been dealt with in Carnevale, Galeani

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and Sassano (2014), where only sufficient conditions ensuring that the optimal control stabilizes the hybrid system are given.

The main contribution of this paper is to provide a comprehensive characterization of the solution to the Linear-Quadratic Regulator problem in the presence of hybrid systems that exhibit periodic time-driven jumps, with the period and the initial timer *a priori* known. The optimal solution is constructed both in the finite–horizon as well as in the infinite–horizon cases. Moreover, the notion of Monodromy Riccati Equation is introduced and discussed. A preliminary version of this paper has appeared in Possieri and Teel (2016). With respect to Possieri and Teel (2016), here we provide the proofs of the main results, further insights and remarks and the derivation and use of the Monodromy (algebraic) Riccati equation.

The remainder of the paper is organized as follows: in Section 2, the considered class of hybrid systems is introduced and some preliminary results are stated. In Sections 3 and 4, the finite–horizon and infinite–horizon LQ optimal control problem are dealt with, respectively. In Sections 4.1 and 4.2, necessary and sufficient conditions guaranteeing that the infinite–horizon LQ optimal control stabilizes the closed loop systems are stated. Conclusion are drafted in Section 6.

2 Notation and Preliminaries

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ denote the set of real, integer, natural and complex numbers, respectively. Define $\mathbb{C}_g := \{s \in \mathbb{C} : |s| < 1\}$. Let $\lfloor a \rfloor$, be the largest integer lower than or equal to $a \in \mathbb{R}$. Given a symmetric, positive semi-definite matrix $M \in \mathbb{R}^{\nu\times\nu}$, let $|v|_M$, $v \in \mathbb{R}^\nu$, be the M–seminorm, i.e., $|v|_M := v'Mv$. Given $v \in \mathbb{C}^n$, let v^* (resp. v') be the complex conjugate (resp. transpose) of v.

Consider the hybrid system governed by the flow dynamics

$$\dot{\tau} = 1, \tag{1a}$$
$$\dot{x} = Ax + Bu_F, \tag{1b}$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and by the jump dynamics

$$\tau^+ = 0, \tag{1c}$$
$$x^+ = Ex + Fu_J, \tag{1d}$$

when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$, with state $x(t, k) \in \mathbb{R}^n$, flow input $u_F(t, k) \in \mathbb{R}^{m_1}$, jump input $u_J(k) \in \mathbb{R}^{m_2}$, and initial conditions $\tau(0, 0) = 0$, $x(0, 0) = x_0$ (the extension of the results of this paper to the case $\tau(0, 0) = \tau_0$, $\tau_0 \in [0, \tau_M]$, is discussed in the subsequent Corollaries 1 and 3). In (1), τ_M is a positive known constant that imposes a dwell–time constraint between two consecutive jumps. Each solution to (1) is then defined on the common *hybrid time domain*

$$\mathcal{T} := \{(t, k), t \in [k\tau_M, (k + 1)\tau_M], k \in \mathbb{N}\}, \tag{2}$$

which is therefore *a priori* fixed. Solutions to system (1) are *hybrid arcs*, *i.e.*, locally absolutely continuous functions mapping $(t, k) \in \mathcal{T}$ in the indicated set. By definition, $\dot{x}(t, k) = \frac{d}{dt}x(t, k)$ and $x^+(k\tau_M, k-1) = x(k\tau_M, k)$. Given $k \in \mathbb{Z}$, the shortcut $t_k := k\tau_M$ is used.

Let $Ee^{A\tau_M}$ be the *monodromy matrix* of system (1). As discussed in Carnevale et al. (2012a,b), system (1) is globally asymptotically stable if and only if all the eigenvalues of the monodromy matrix $Ee^{A\tau_M}$ (or, equivalently, of the matrix $e^{A\tau_M}E$) lie in the open unit circle in the complex plane, *i.e.* $\mathbb{C}_g$.

Let $\varphi(t, k, x_0, u_F, u_J)$ be the solution to system (1) at hybrid time $(t, k) \in \mathcal{T}$, with initial condition x_0, $\tau(0, 0) = 0$, and inputs $u_F(\cdot, \cdot)$, $u_J(\cdot)$. System (1) is *strongly reachable* if, for each $x \in \mathbb{R}^n$, there exist a finite $\kappa \in \mathbb{N}$ such that, for all $t \in (t_\kappa, t_{\kappa+1})$, there exist inputs $u_F(\cdot, \cdot)$ and $u_J(\cdot)$ such that $\varphi(t, \kappa, 0, u_F, u_J) = x$. On the other hand, system (1) (or, equivalently, the tuple (A, B, E, F, τ_M)) is *stabilizable* if for each $x_0 \in \mathbb{R}^n$ there exist $u_F(\cdot, \cdot)$, $u_J(\cdot)$ such that $\lim_{t+k\to\infty} x(t, k) = 0$. By Medina and Lawrence (2009, 2010); Carnevale, Galeani and Sassano (2013); Carnevale, Galeani and Sassano (2014); Carnevale et al. (2016); Possieri and Teel (2017); Carnevale et al. (2017), system (1) is stabilizable if and only if

$$\operatorname{rank}[\, Ee^{A\tau_M} - sI \;\; F \;\; R_{A,B} \,] = n, \quad \forall s \notin \mathbb{C}_g, \tag{3}$$

where $R_{A,B}$ is the *reachability matrix* associated to (1b),

$$R_{A,B} := [\, B \;\; AB \;\; \cdots \;\; A^{n-1}B \,].$$

On the other hand, let $y_F(t, k) \in \mathbb{R}^{q_1}$ and $y_J(k) \in \mathbb{R}^{q_2}$ be the measured outputs of system (1), defined as

$$y_F(t, k) := C_F x(t, k), \tag{4a}$$
$$y_J(k) := C_J x(t_k, k - 1). \tag{4b}$$

System (1) (or, equivalently, (A, E, C_F, C_J, τ_M)) is *detectable* if, for any initial condition $x_0 \in \mathbb{R}^n$, by using only measurements of the input functions $u_F(\cdot, \cdot)$, $u_J(\cdot)$ and of the outputs $y_F(\cdot, \cdot)$, $y_J(\cdot)$, it is possible to determine an estimate $\hat{x}(t, k)$ of $x(t, k)$ with the property that $\lim_{t+k\to\infty} \hat{x}(t, k) - x(t, k) = 0$ (or, equivalently since $(t, k) \in \mathcal{T}$, $\lim_{t\to\infty} \hat{x}(t, k) - x(t, k) = 0$ and $\lim_{k\to\infty} \hat{x}(t, k) - x(t, k) = 0$). System (1) is detectable if and only if, $\forall s \notin \mathbb{C}_g$,

$$\operatorname{rank}[\, (Ee^{A\tau_M})' - sI \;\; (C_J e^{A\tau_M})' \;\; O'_{A,C_F} \,]' = n, \tag{5}$$

where O_{A,C_F} is the *observability matrix* of (1b)–(4a),

$$O_{A,C_F} := [\, C_F' \ (C_F A)' \ \cdots \ (C_F A^{n-1})' \,]'.$$

On the other hand, system (1) is *observable* if, for any initial condition $x_0 \in \mathbb{R}^n$, there exists a hybrid time $(\theta, \kappa) \in \mathcal{T}$ such that, by using only measurements of the input functions $u_F(\cdot,\cdot)$, $u_J(\cdot)$ and of the outputs $y_F(\cdot,\cdot)$, $y_J(\cdot,)$ up to time (θ,κ), it is possible to determine x_0. By Possieri and Teel (2017), system (1) is observable if and only if (5) holds $\forall s \in \mathbb{C}$.

Remark 1. The framework considered in this paper has few similarities with the so-called *lifting approach* (see Heemels et al., 2016), hence a comparison is in order. By employing the framework proposed in Bamieh et al. (1991); Chen and Francis (2012), a lifting procedure can be applied to periodic continuous-time systems to derive equivalent discrete-time models. These model involve, however, infinite-dimensional operators between the (lifted) inputs and the state of the plant. On the other hand, the approach proposed here just involves finite-dimensional matrix multiplications leading to a simpler representation of the process being analyzed. △

3 Finite–Horizon LQ optimal control

Let x be a solution to system (2), with continuous-time input u_F and discrete-time input u_J. Define the cost function

$$J_{x_0}(u_F,u_J) = \int_0^T (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\,dt + \\ + \sum_{k=1}^{K} (|x(t_k,k-1)|_{Q_J} + |u_J(k)|_{R_J}) + |x(T,K)|_Z, \quad (6)$$

with $T \in \mathbb{R}$, $T>0$, $K = \lfloor T/\tau_M \rfloor$, Q_F, Q_J, Z are positive semidefinite symmetric matrices in $\mathbb{R}^{n\times n}$, R_F and R_J are positive definite symmetric (hence invertible) matrices in $\mathbb{R}^{m_1\times m_1}$ and $\mathbb{R}^{m_2\times m_2}$, respectively. Note that, the cost function J in (6) is the extension of classical cost functions for continuous–time and discrete–time linear systems. In fact, we are assuming that the matrices Q_F, Q_J and Z are positive semidefinite, whence the cost function J penalizes the transient and final state deviation, both for continuous time and for discrete time. On the other hand, the matrices R_F and R_J are assumed to be positive definite; therefore, the cost function J penalizes also the control effort. Consider the following problem.

Problem 1. Let system (1) and the cost function (6) be given. Find, if any, $u^\star = (u_F^\star, u_J^\star)$ that minimizes J from initial condition $(0, x_0)$, $x_0 \in \mathbb{R}^n$, of (1). ■

For each $0 \leqslant \theta \leqslant T$ and $0 \leqslant \kappa \leqslant K$, with $\kappa \in \mathbb{N}$ being such that $(\theta,\kappa) \in \mathcal{T}$, define the *cost–to–go function* from state x at $(\theta,\kappa) \in \mathcal{T}$ by

$$V(\theta,\kappa,x) := \inf_{u(\cdot,\cdot)} \Bigg\{ \int_\theta^T (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\,dt \\ + \sum_{k=\kappa+1}^{K} (|x(t_k,k-1)|_{Q_J} + |u_J(k)|_{R_J}) + |x(T,K)|_Z \Bigg\},$$

with $x(t,k)$ initialized at $x(\theta,\kappa) = x$ and satisfying the dynamics given in (1). Computing the cost–to–go $V(0,0,x_0)$ corresponds to obtaining a solution to Problem 1. Thus, we consider two different cases: $K \neq T/\tau_M$ and $K = T/\tau_M$. The subsequent Theorem 1 gives a solution to Problem 1 for both cases, while Corollary 1 extends it to any initial condition $(\tau_0, x_0) \in [0,\tau_M]\times\mathbb{R}^n$.

3.1 Case $K \neq T/\tau_M$

In this section, we assume that $K \neq T/\tau_M$. The value of the cost–to–go function at (T,K) is

$$V(T,K,x) = |x|_Z.$$

Let $t \in [t_K, T]$, which is a non-empty interval since $K \neq T/\tau_M$. Computing the value of the cost–to–go function $V(t,K,x)$ corresponds to finding a solution to:

$$V(t,K,x) = \inf_{u(\cdot,K)} \left\{ \int_t^T (|x|_{Q_F} + |u|_{R_F}) + |x(T,K)|_Z \right\}, \quad (7)$$

subject to the dynamics given in (1b). By Dorato et al. (1994), since R_F is positive definite, there exists a unique solution to the problem given in (7) that is

$$u_F^\star(t,K) = -R_F^{-1}B'P(t,K)x,$$

where $P(t,K)$ is the (symmetric) solution to

$$-\dot{P} = A'P + PA + Q_F - PBR_F^{-1}B'P, \quad (8)$$

with final condition $P(T,K) = Z$. By Theorem 3.14 of Athans and Falb (2007), the ordinary differential equation (8) admits a unique solution $P(t,K)$ such that $P(T,K) = Z$, for all $t \in [t_K, T]$. In addition, the value of the cost–to–go function V for any $t \in [t_K,T]$ is given by

$$V(t,K,x) = |x|_{P(t,K)}. \quad (9)$$

By the structure of J, the cost–to–go function is greater than or equal to zero. Therefore, by (8) and (9), $P(t,K)$ is symmetric and positive semidefinite for any $t \in [t_K,T]$.

Remark 2. It is possible to relax the hypothesis on the matrix R_F, dealing with matrices that are positive semidefinite. Different techniques have been proposed in the literature to solve such a problem (see, for instance, Saberi and Sannuti, 1987; Kalaimani et al., 2013 and references therein). In particular, in Hautus and Silverman (1983); Willems et al. (1986) it is shown that, by including distributions in the allowable controls, there still exists a solution to the LQ problem and that the regular part of the optimal LQ control can still be written as a state feedback (Ferrante and Ntogramatzidis, 2016). Furthermore, the results given in Cobb (1983) and Bender and Laub (1987b) could be used to deal with systems in descriptor form. △

In (9), an expression valid also for the cost–to–go function $V(t_K, K, x)$ is obtained. Hence, the two cases $K \neq T/\tau_M$ and $K = T/\tau_M$ can be now considered jointly.

3.2 Case $K = T/\tau_M$

The cost–to–go function V for (t_K, K) is given by

$$V(t_K, K, x) = |x|_{\bar{Z}},$$

with $\bar{Z} = Z$, if $K = T/\tau_M$, or, by the discussion in the previous section, $\bar{Z} = P(t_K, K)$, if $K \neq T/\tau_M$. Note that the matrix $\bar{Z}$ is symmetric and positive semidefinite in both cases. By the Dynamic Programming Algorithm (Bertsekas, 1995), the function $V(t_K, K-1, x)$ can be computed by solving the following problem:

$$V(t_K, K-1, x) = \inf_u \left\{ |x|_{Q_J} + |u|_{R_J} + |Ex + Fu|_{\bar{Z}} \right\}.$$

By Anderson and Moore (2007), since R_J is positive definite, there exists a unique solution to this problem, given by

$$u_J^\star(K) = -(R_J + F'\bar{Z}F)^{-1} F'\bar{Z}Ex,$$

and the cost–to–go function is given by

$$V(t_K, K-1, x) = |x|_{P(t_K, K-1)}, \tag{10}$$

where $P(t_K, K-1)$ is the (symmetric) solution to

$$P(t_K, K-1) = \\ Q_J + E'\bar{Z}E - E'\bar{Z}F(R_J + F'\bar{Z}F)^{-1}F'\bar{Z}E. \tag{11}$$

Since R_J is positive definite and $\bar{Z}$ is positive semidefinite, the matrix $(R_J + F'\bar{Z}F)$ is positive definite, whence invertible. Moreover, by (10), the matrix $P(t_K, K-1)$ is symmetric and positive semidefinite, because V is greater than or equal to zero.

Remark 3. It is possible to relax the hypothesis on the matrix R_J, dealing with matrices that are positive semidefinite. In Ferrante and Ntogramatzidis (2013, 2014), resorting to the so–called Constrained Generalized Discrete Algebraic Riccati equation, a solution to the discrete–time optimization problem is given for matrices R_J that are positive semidefinite. Furthermore, the results given in Bender and Laub (1987a) could be used to deal with systems in descriptor form. △

Let $t \in [t_{K-1}, t_K]$ and consider the cost–to–go function $V(t, K-1, x)$. Computing the value of such a function for all $t \in [t_{K-1}, t_K]$ corresponds to finding a solution to:

$$\begin{aligned} &V(t, K-1, x) \\ &\quad = \inf_{u_F(\cdot, K-1)} \left\{ \int_t^{t_K} \left(|x|_{Q_F} + |u_F|_{R_F} \right) dt \right. \\ &\qquad\qquad \left. + |x(t_K, K-1)|_{P(t_K, K-1)} \right\}, \end{aligned} \tag{12}$$

subject to the dynamics in (1b). Such a problem is wholly similar to the one given in (7). Hence, there exists a unique solution to the problem given in (12), that is

$$u_F^\star(t, K-1) = -R_F^{-1}B'P(t, K-1)x,$$

where $P(t, K-1)$ is the unique symmetric and positive semidefinite solution to (8), with the final condition $P(t_K, K-1)$ computed in (11). Additionally, for any $t \in [t_{K-1}, t_K]$, the value of the cost–to–go function V is

$$V(t, K-1, x) = |x|_{P(t, K-1)}. \tag{13}$$

Note that, by (13), the value of the cost–to–go function at (t_{K-1}, K) is $V(t_{K-1}, K-1, x) = |x|_{P(t_{K-1}, K-1)}$. Therefore, by repeating the procedure given in this section backwards, one can obtain the cost–to–go $V(0, 0, x)$, that is the lower bound to the cost function J given in (6). Additionally, this procedure provides a feedback control input $u^\star$ that achieves such a minimum. Hence, $u^\star$ is a solution to Problem 1. The next theorem and corollary formalize the results given in Sections 3.1 and 3.2.

Theorem 1. *Let system* (1) *and the cost function* (6) *be given. There is a unique solution to Problem 1, given by*

$$u_F^\star(t, k) = -R_F^{-1}B'P(t, k)x, \tag{14a}$$
$$u_J^\star(k) = -(R_J + F'P(t_k, k)F)^{-1}F'P(t_k, k)Ex, \tag{14b}$$

where $P(t, k)$ is the solution to the hybrid system described by the flow dynamics

$$-\dot{\tau} = 1, \tag{15a}$$
$$-\dot{P} = A'P + PA + Q_F - PBR_F^{-1}B'P, \tag{15b}$$

when $(\tau, P) \in [0, \tau_M] \times \mathbb{R}^{n \times n}$, and by the jumps dynamics

$$\tau^+ = 0, \tag{15c}$$
$$P = Q_J + E'P^+E - E'P^+F\Lambda^{-1}F'P^+E, \tag{15d}$$

when $(\tau, P) \in \{\tau_M\} \times \mathbb{R}^{n \times n}$, where $\Lambda = R_J + F'P^+F$, with terminal conditions $\tau(T, K) = T - K\tau_M$, $P(T, K) = Z$. Furthermore, letting x_0 be the initial condition of the system (1), *the minimum of J is given by*

$$J^\star = |x_0|_{P(0,0)}. \qquad \blacksquare$$

Note that, as usual, (15) need to be solved backwards in time starting from the given boundary condition at (T,K), and then the possible singularity of E does not create any problem in computing P from P^+ in (15d) (whereas going forward in time, the computation of P^+ from P might be impossible for singular E).

Corollary 1. *Let system* (1) *be given. Consider the cost*

$$\bar{J}_{\theta,\kappa} = \int_{\theta}^{T} (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\, dt + \\ + \sum_{k=\kappa+1}^{K} (|x(t_k,k-1)|_{Q_J} + |u_J(k)|_{R_J}) + |x(T,K)|_Z,$$

with $(\theta,\kappa) \in \mathcal{T}$. *The control input* $u^\star = (u_F^\star, u_J^\star)$ *given in* (14) *minimizes the cost function* $\bar{J}_{\theta,\kappa}$ *for any* $(\tau_0, x_0) \in [0,\tau_M] \times \mathbb{R}^n$ *such that* $\tau(\theta,\kappa) = \tau_0 := \theta - \kappa\tau_M$, $x(\theta,\kappa) = x_0$. *Moreover, the minimum of the cost function* $\bar{J}_{\theta,\kappa}$ *is*

$$\bar{J}^\star_{\theta,\kappa} = |x_0|_{P(\theta,\kappa)}. \qquad \blacksquare$$

3.3 The monodromy Riccati equation

Following the same idea employed in Possieri and Sassano (2018), the Hybrid Riccati equation (15) can be interpreted in a monodromy sense, thus obtaining a *monodromy Riccati equation* (briefly, *MRE*). Namely, define

$$\begin{bmatrix} \psi_{1,1}(t) & \psi_{1,2}(t) \\ \psi_{2,1}(t) & \psi_{2,2}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} -A & BR^{-1}B' \\ Q & A' \end{bmatrix} t \right),$$

and let

$$\Phi(\tau, \bar{Z}) = (\psi_{2,1}(\tau) + \psi_{2,2}(\tau)\bar{Z})(\psi_{1,1}(\tau) + \psi_{1,2}(\tau)\bar{Z})^{-1}.$$

By classical results about LQ optimal control, the flow dynamics of system (15) are such that $\Phi(t_{\mathrm{f}} - t_0, \bar{Z})$ is the solution to system (15b) at time t_0 with final condition $P(t_{\mathrm{f}}) = \bar{Z}$. Thus, by using a construction wholly similar to the one employed in Possieri and Sassano (2018), letting $P_k := P(t_k, k-1)$, one has that

$$P_{k-1} = Q_J + E'\Phi(\tau_M, P_k)E - E'\Phi(\tau_M, P_k)F \cdot \\ \cdot (R_J + F'\Phi(\tau_M, P_k)F)^{-1} F'\Phi(\tau_M, P_k)E, \quad (16)$$

where $P_K = Z$ if $T = t_{K+1}$, or $P_K = Q_J + E'\Phi(T - K\tau_M, Z)E - E'\Phi(T - K\tau_M, Z)F(R_J + F'\Phi(T - K\tau_M, Z)F)^{-1}F'\Phi(T - K\tau_M, Z)E$, otherwise.

Thus, consider the following theorem which guarantees existence of solutions to the MRE (16).

Theorem 2. *For each positive semidefinite* $Z \in \mathbb{R}^{n\times n}$, *each* $(T,K) \in \mathcal{T}$, *and each* $k \in \mathbb{N}$, $k \leqslant K$, *there exists a unique solution to the MRE* (16). $\blacksquare$

Proof. Since system (1) is linear, the solution $x(t,j)$ corresponding to $u_F = 0$ and $u_J = 0$ satisfies

$$|x(t,k)| \leqslant \varrho^k \exp(\lambda(t - t_k))|x_0|,$$

for some $\varrho, \lambda \in \mathbb{R}$ and for all $(t,k) \in \mathcal{T}$ (see Carnevale et al., 2012b). By Corollary 1, this implies that, for each $x_0 \in \mathbb{R}^n$, each positive semidefinite $Z \in \mathbb{R}^{n\times n}$, each $(T,K) \in \mathcal{T}$, and each $k \in \mathbb{N}$, $k \leqslant K$, one has

$$|x_0|_{P(t_k,k-1)} = \bar{J}^\star_{t_k,k-1} \leqslant c\, \frac{\varrho^{2(k+1)} - 1}{\varrho^2 - 1} |x_0|^2,$$

where

$$c = \frac{1}{2\lambda}|Q_F|(e^{2\lambda\tau_M} - 1) + |Q_F|e^{2\lambda\tau_M}, \qquad (17)$$

and hence the solution to the MRE (16), which by construction satisfies $P_k = P(t_k, k-1)$, does not blow up in finite time. Uniqueness of the solution P_k follows by classical arguments about discrete-time LQ optimal control and by Theorem 1. $\square$

3.4 Comparison with LQ optimization over time scales

In Satco and Turcu (2013), a method to study hybrid systems with the analysis carried out on a time scale is given. Let $\mathbb{T}$ be a *time scale*, i.e., a nonempty closed subset of $\mathbb{R}$ (Hilscher and Zeidan, 2004). Define the operators $\varsigma : \mathbb{T} \to \mathbb{T}$,

$$\varsigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and $\varpi : \mathbb{T} \to \mathbb{T}$,

$$\varpi(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Thus, $t \in \mathbb{T}$ is *right–dense*, *left–dense*, *right–scattered*, and *left–scattered* if $\varsigma(t) = t$, $\varpi(t) = t$, $\varsigma(t) > t$, and $\varpi(t) < t$, respectively. The Δ–derivative of $f : \mathbb{T} \to \mathbb{R}$ is

$$f^\Delta := \lim_{s \to t, s \in \mathbb{T}\setminus\{\varsigma(t)\}} \frac{f(\varsigma(t)) - f(s)}{\varsigma(t) - s}.$$

Note that, if t is right–dense, $f^\Delta(t) = \dot{f}(t)$ is the usual time derivative, while if t is right–scattered and $\varsigma(t) = t+1$, then $f^\Delta = f(t+1) - f(t)$. A property is said to hold Δ–almost everywhere on $\mathbb{T}$ if it holds for every $t \in \mathbb{T}$, except for t in some Δ–measurable subset of $\mathbb{T}$ of measure 0. Let $\Phi : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. A solution $x : \mathbb{T} \to \mathbb{R}^n$ to

$$x^\Delta(t) = \Phi(t, x(t)) \qquad (18)$$

is a function x that is absolutely continuous and (18) holds Δ-almost everywhere. In Hilscher and Zeidan (2012), a solution to the LQR problem with respect to system (18), where $\Psi(t, x(t)) = A(t)x + B(t)u$, with

$A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{T} \to \mathbb{R}^{n\times m}$ being piecewise right dense-continuous functions (Hilscher and Zeidan, 2012), is given in terms of a time scale Riccati equation. Theorem 1 and Corollary 1 translate the results given in Hilscher and Zeidan (2012) for dynamical systems over time scales to the hybrid setting considered in this paper. Namely, following Satco and Turcu (2013), define the time scale $\mathbb{T} := \bigcup_{k\in\mathbb{N}}[k + k\tau_M, k + (k+1)\tau_M]$, and

$$\Psi(t,x) := \begin{cases} Ax + Bu_F, & \text{if } \varsigma(t) \leqslant t, \\ Ex + Fu_J, & \text{if } \varsigma(t) > t, \end{cases} \tag{19}$$

the dynamics given in (15) reads as the time scale Riccati equation (namely, equation (R) in Hilscher and Zeidan, 2012) on the hybrid setting of this paper. Moreover, Theorem 1 and Corollary 1 can be extended to deal with unbounded hybrid domains, as detailed in the subsequent Section 4.

4 Infinite–horizon LQ optimal control

Let x be a solution to system (1), with continuous-time input u_F and discrete-time input u_J. Consider the cost function

$$J_\infty = \int_0^\infty (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\, dt + \\ + \sum_{k=1}^{\infty} (|x(t_k, k-1)|_{Q_J} + |u_J(k)|_{R_J}), \tag{20}$$

where the matrices Q_F, R_F, Q_J, R_J satisfy the same assumptions stated for (6). The main objective of this section is formalized in the following problem.

Problem 2. Let the system (1) and the cost function (20) be given. Find, if any, a control input $u^\star_\infty = (u^\star_{F,\infty}, u^\star_{J,\infty})$ that minimizes the cost J_∞ from initial condition $(0, x_0)$, $x_0 \in \mathbb{R}^n$, of (1). ■

Consider the following assumption.

Assumption 1. The system (1) is stabilizable. ○

As shown in the remainder of this section, Assumption 1 is sufficient (but not necessary) to guarantee the existence of a solution to Problem 2 (see the subsequent Example 1). However, Assumption 1 is clearly necessary to guarantee that the control input $u^\star_\infty$ that minimizes the cost function J_∞ stabilizes the system (the stabilization properties of a solution $u^\star_\infty$ to Problem 2 are investigated in subsequent Section 4.1).

Example 1. Consider the hybrid system (1) and the cost function (20), with $R_F = 1$, $R_J = 1$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ Q_F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.1 & 0 \\ 0 & 2 \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ Q_J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This hybrid system is not stabilizable. However, there is a solution to Problem 2, that is $u^\star_{F,\infty} = 0$, $u^\star_{J,\infty} = 0$. △

Let $P_T(t,k)$ be the solution to the Hybrid Riccati Equation (15), such that $P_T(T,K) = Z := 0$. The next lemma characterizes $P_T(\sigma, 0)$ for $T \to \infty$ and $\sigma \in [0, \tau_M]$.

Lemma 1. *Let Assumption 1 hold. There is a symmetric and positive semidefinite $P_\infty(\sigma)$, $\sigma \in [0,\tau_M]$ such that*

$$P_\infty(\sigma) = \lim_{T\to\infty} P_T(\sigma, 0). \qquad ■$$

Proof. Let σ be a constant, $\sigma \in [0,\tau_M]$, and let $x_\sigma = x(\sigma, 0)$. If system (1) is stabilizable, then, by Proposition 1 of Carnevale, Galeani and Sassano (2014), there exists a *time invariant dynamic linear state feedback* Σ such that the eigenvalues of the monodromy matrix of the closed loop system lie inside the open unit circle in the complex plane. By Carnevale et al. (2012a,b), if the eigenvalues of the monodromy matrix have modulus less than 1, the origin is a globally exponentially stable equilibrium, i.e., there exist constants $\alpha, \lambda \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$ such that $|x(t,k)| \leqslant \alpha e^{-\lambda(t-\sigma)}|x_\sigma|$. Additionally, one has that $|x|_M \leqslant |M||x(t,k)|^2$ (Meyer, 2000). Therefore, if system (1) is stabilizable, then there exists a time–invariant dynamic linear state feedback Σ such that, letting $\bar{x}(t,k)$, $\bar{u} = (\bar{u}_F, \bar{u}_J)$ be the solution to the closed–loop system, there exist constants $\alpha, \beta_F, \beta_J, \lambda \in \mathbb{R}$, $\alpha, \beta_F, \beta_J, \lambda > 0$ such that

$$\begin{aligned} &\int_\sigma^\infty (|\bar{x}(t,k)|_{Q_F} + |\bar{u}_F(t,k)|_{R_F})\, dt \\ &\qquad + \sum_{k=1}^{\infty} (|\bar{x}(t_k,k-1)|_{Q_J} + |\bar{u}_J(k)|_{R_J}) \\ &\leqslant \int_\sigma^\infty \alpha^2(|Q_F| + \beta_F|R_F|)|x_\sigma|^2 e^{-2\lambda(t-\sigma)} dt \\ &\quad + \sum_{k=1}^{\infty} \alpha^2(|Q_J| + \beta_J|R_J|)|x_\sigma|^2 e^{-2\lambda(k\tau_M - \sigma)} \\ &\hspace{10em} =: a(x_\sigma) < \infty. \end{aligned}$$

Note that, Σ needs not be a solution to Problem 1, i.e., $x'_\sigma P_T(\sigma,0)x_\sigma \leqslant \int_\sigma^T (|\bar{x}(t,k)|_{Q_F} + |\bar{u}_F(t,k)|_{R_F}) dt + \sum_{k=1}^{K}(|\bar{x}(t_k,k)|_{Q_J} + |\bar{u}_J(k)|_{R_J}) \leqslant a(x_\sigma)$.

Let $T_1, T_2 \in \mathbb{R}$, with $T_2 > T_1 > \sigma$. Let $K_1 = \lfloor T_1/\tau_M \rfloor$ and $K_2 = \lfloor T_2/\tau_M \rfloor$. Let $P_{T_1}(t,k)$ and $P_{T_2}(t,k)$ be the solution to (15) with $P_{T_1}(T_1,K_1) = 0$ and $P_{T_2}(T_2,K_2) = 0$, respectively. Let $K_{F,i}(t,k) = -R_F^{-1}B'P_{T_i}(t,k)$ and $K_{J,i}(k) = -(R_J + F'P_{T_i}(t_k,k)F)^{-1}F'P_{T_i}(t_k,k)E$, $i = 1,2$. Let $x_{T_i}(t,k)$ be the solution to system (1) with control inputs $u_{F,T_i}(k,t) = K_{F,i}(t,k)x_{T_i}(t,k)$, $u_{J,T_i}(k) = K_{j,i}(k)x_{T_i}(t_k,k)$, $i = 1,2$. By Corollary 1, the control inputs $u_{F,T_1}(k,t)$, $u_{J,T_1}(k)$ minimize the cost function $J_1 = \int_\sigma^{T_1} (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\, dt +$

$\sum_{k=1}^{K_1}(|x(t_k,k-1)|_{Q_J}+|u_J(k)|_{R_J})+|x(T_1,K_1)|_Z$, and the minimum of J_1 is given by $x'_\sigma P_{T_1}(\sigma,0)x_\sigma$. Note that control inputs $u_{F,T_2}(k,t)$, $u_{J,T_2}(k)$ are generically not optimal for the cost function J_1. Therefore, since $|x(t,k)|_{Q_F}, |u_F(t,k)|_{R_F}, |x(t_k,k-1)|_{Q_J}, |u_J(k)|_{R_J}$ are greater than or equal to zero and $T_2 > T_1$, one has that

$$
\begin{aligned}
|x_\sigma|_{P_{T_1}(\sigma,0)} &\leqslant \int_\sigma^{T_1}(|x_{T_2}(t,k)|_{Q_F}+|u_{F,T_2}(t,k)|_{R_F})dt \\
&\quad + \sum_{k=1}^{K_1}(|x_{T_2}(t_k,k-1)|_{Q_J}+|u_{J,T_2}(k)|_{R_J}) \\
&\leqslant \int_\sigma^{T_2}(|x_{T_2}(t,k)|_{Q_F}+|u_{F,T_2}(t,k)|_{R_F})dt \\
&+\sum_{k=1}^{K_2}(|x_{T_2}(t_k,k-1)|_{Q_J}+|u_{J,T_2}(k)|_{R_J}) = |x_\sigma|_{P_{T_2}(\sigma,0)}.
\end{aligned}
$$

Hence, since the sequence of solutions $P_T(\sigma,0)$ parameterized by the terminal time T is upper–bounded (due to the fact that $|x_\sigma|_{P_T(\sigma,0)} \leqslant a(x_\sigma)$) and non–decreasing, there exists a finite limit for $T\to\infty$ of $|x_\sigma|_{P_T(\sigma,0)}$, for any x_σ. Therefore, there exists a finite limit for each entry of $P_T(\sigma,0)$, for $T\to\infty$ and $P_\infty(\sigma)$ is symmetric. Moreover, since $\lim_{T\to\infty} x'_\sigma P_T(\sigma,0)x_\sigma$ exists and is greater than or equal to zero for each $x_\sigma\in\mathbb{R}^n$, then $P_\infty(\sigma)$ is positive semidefinite. □

Corollary 2. *Let Assumption 1 hold. The matrix $P_\infty(\cdot)$ satisfies*

$$
\begin{aligned}
\frac{d}{d\sigma}P_\infty(\sigma) &= -A'P_\infty(\sigma)-P_\infty(\sigma)A-Q_F+ \\
&\quad + P_\infty(\sigma)BR_F^{-1}B'P_\infty(\sigma), \qquad (21a)
\end{aligned}
$$

for all $\sigma\in[0,\tau_M]$, and

$$
\begin{aligned}
&P_\infty(\tau_M) = Q_J + E'P_\infty(0)E+ \\
&- E'P_\infty(0)F(R_J+F'P_\infty(0)F)^{-1}F'P_\infty(0)E. \qquad (21b)
\end{aligned}
$$

■

Proof. The proof follows directly from the hybrid system (15) and from the fact that, by Lemma 1, there exists $P_\infty(\sigma)=\lim_{T\to\infty}P_T(\sigma,0)$, for all $\sigma\in[0,\tau_M]$. □

Remark 4. Due to the fact that infinite-horizon LQ optimal control for LTI systems yields algebraic Riccati equations with constant solutions, one may wonder whether it is reasonable to expect to potentially find a constant solution to the differential equation (21a) together with the two-point boundary conditions (21b). Unfortunately, this is impossible in general since this would imply solving, with respect to the constant matrix P_C, the coupled algebraic Riccati equations

$$
\begin{aligned}
0 &= Q_F + P_CA + A'P_C - P_CBR_F^{-1}B'P_C, \qquad (22a)\\
P_C &= Q_J + E'P_CE \\
&\quad - EP_CF(R_J+F'P_CF)^{-1}F'P_CE, \qquad (22b)
\end{aligned}
$$

which is generically unfeasible since the symmetric matrix P_C contains only $\frac{n(n+1)}{2}$ scalar unknowns, whereas the symmetric matrix equations (22) contain $n(n+1)$ independent scalar equations. On the other hand, as usual in ordinary differential equations with *two point* (or *split*) *boundary conditions* arising in optimal control, the differential Riccati equation (21a) needs $\frac{n(n+1)}{2}$ boundary conditions, which are given by (21b) (which contain $\frac{n(n+1)}{2}$ scalar equations relating the values of $P_\infty(\cdot)$ at two different time instants, hence the name "two boundary point"). Nonetheless, the structure of the boundary conditions (21b) entails that finding the solution of interest of (21a) seems not to be a trivial task; the discussion in Section 4.2 will provide an answer to this question as well. △

The following theorem gives a solution to Problem 2.

Theorem 3. *Let the system* (1) *and the cost function* (20) *be given and let Assumption 1 hold. Then there exists a solution to Problem 2 for any $x_0\in\mathbb{R}^n$, given by*

$$
\begin{aligned}
u^\star_{F,\infty}(\tau,x) &= -R_F^{-1}B'P_\infty(\tau)x, \qquad (23a)\\
u^\star_{J,\infty}(x) &= -(R_J+F'P_\infty(0)F)^{-1}F'P_\infty(0)Ex, \qquad (23b)
\end{aligned}
$$

where $P_\infty(\tau)=\lim_{T\to\infty}P_T(\tau,0)$. Additionally, the minimum of the cost function J_∞ in (20) *is given by*

$$J^\star_\infty = |x_0|_{P_\infty(0)}.$$

■

Proof. Let $x_\infty(t,k)$ be the solution to the closed–loop system (1) with the control inputs given in (23). Consider the cost function

$$
\begin{aligned}
\bar{J}_K(u_\infty) &= \int_0^{t_K}(|x_\infty(t,k)|_{Q_F}+|u_{F,\infty}(t,k)|_{R_F})dt \\
&\quad + \sum_{k=1}^{K}(|x_\infty(t_k,k-1)|_{Q_J}+|u_{J,\infty}(k)|_{R_J}).
\end{aligned}
$$

Since the control input given in (23) is not necessarily a solution to Problem 1 with respect to $\bar{J}_K$, one has that $x'_0P_{t_K}(0,0)x_0 \leqslant \bar{J}_K(u_\infty)$. Note that, by (21a) and (9), $\int_0^{\tau_M}(|x_\infty(t,k)|_{Q_F}+|u_{F,\infty}(t,k)|_{R_F})dt = x'_0P_\infty(0)x_0 - x'(\tau_M,0)P_\infty(\tau_M)x(\tau_M,0)$. On the other hand, by (21b) and (10), $|x_\infty(\tau_M,0)|_{Q_J}+|u_{J,\infty}(0)|_{R_J} = x'(\tau_M,0)P_\infty(\tau_M)x(\tau_M,0) - x'(\tau_M,1)P_\infty(0)x(\tau_M,1)$. Therefore, by considering that the system (1) and the cost function (20) are time–invariant, one has that $\bar{J}_K = x'_0P_\infty(0)x_0 - x'(t_K,K)P_\infty(0)x(t_K,K) \leqslant x'_0P_\infty(0)x_0$, because, by Lemma 1, $P_\infty(0)$ is positive semidefinite. Therefore, one has that $\lim_{K\to\infty} x_0' P_{t_K}(0,0)x_0 \leqslant \lim_{K\to\infty} \bar{J}_K \leqslant x_0' P_\infty(0)x_0$, whence, by considering that, by Lemma 1, $\lim_{K\to\infty} x_0' P_{t_K}(0,0)x_0 = x_0' P_\infty(0)x_0$, one has that $\lim_{K\to\infty} \bar{J}_K = |x_0|_{P_\infty(0)}$. Hence, we proved that the input $u^\star$ given in (23) leads to a value for the cost function $J_\infty(u_\infty) = |x_0|_{P_\infty(0)}$. We still need to prove that this is the minimum of the cost J_∞ given in (20).

Assume that there exists $\tilde{u}$ that leads to a cost $J_\infty(\tilde{u}) < |x_0|_{P_\infty(0)}$. This implies that there exists a $\bar{T}$ such that $\int_0^{\bar{T}} (|\tilde{x}(t,k)|_{Q_F} + |\tilde{u}_F(t,k)|_{R_F})dt + \sum_{k=1}^{\bar{K}} (|\tilde{x}(t_k,k-1)|_{Q_J} + |\tilde{u}_J(k)|_{R_J}) < x_0' P_{\bar{T}}(0,0)x_0$, where $\bar{K} = \lfloor \bar{T}/\tau_M \rfloor$. But, by Theorem 1, $x_0' P_{\bar{T}}(0,0)x_0$ is the minimum of the left–hand term in the previous equation, leading to a contradiction. Therefore, the control input $u^\star$ given in (23) is a solution to Problem 2. $\square$

Remark 5. As in the classic results about LQ regulation, the optimal control solving the hybrid optimal LQ problem turns out to be a linear state feedback. Nonetheless, as written in (23), such feedback appears to be dependent on the timer state variable τ, *even if* the x-dynamics and the cost functional are time-invariant and even if an infinite-horizon overall problem is considered; however, a feedback depending on the timer variable during flow is perhaps expected, considering that optimization between each pair of jumps occurs on a finite interval, and then classic LQ theory over finite intervals is known to yield time-varying feedback laws. Moreover, the fact that the property of optimality for the underlying plant is achieved in closed loop with a state feedback depending on the timer variable is not surprising since for certain stabilizable plants in such a class also stability may be achieved via state feedback *necessarily* containing components depending on τ. For instance, consider the system (1) with data

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{24a}$$

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F = 0, \tag{24b}$$

$$E = \begin{bmatrix} 0 & 0 \\ E_1 & E_2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}. \tag{24c}$$

The system is neither flow-reachable nor jump reachable, but it is clearly hybrid reachable since

$$e^{AT} = \begin{bmatrix} \cos(T) & \sin(T) & 0 & 0 \\ -\sin(T) & \cos(T) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25a}$$

$$Ee^{AT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos(T) & \sin(T) & \alpha & 0 \\ -\sin(T) & \cos(T) & 0 & \alpha \end{bmatrix}, \tag{25b}$$

$$\operatorname{im}(R_{A,B}) = \operatorname{im} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{25c}$$

so that it is immediate to check that

$$\operatorname{rank} \begin{bmatrix} Ee^{A\tau_M} - sI & F & R_{A,B} \end{bmatrix} = 4, \quad \forall s \in \mathbb{C}. \tag{26}$$

As a consequence, all the closed loop eigenvalues of the monodromy matrix can be assigned by using a suitable *hybrid dynamic* linear state feedback controller. However, it is easily shown that no *static*, time-invariant, linear state feedback controller can stabilize the plant. In fact, let

$$u = Kx, \; K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \tag{27a}$$

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & A_2 \end{bmatrix}, \tag{27b}$$

$$\bar{A}_1 = A_1 + B_1 K_1, \; \bar{A}_{12} = B_1 K_2, \tag{27c}$$

$$e^{\bar{A}T} = \begin{bmatrix} e^{\bar{A}_1 T} & M \\ 0 & I_2 \end{bmatrix}, \tag{27d}$$

$$M = \int_0^T e^{\bar{A}_1(T-\tau)} B_1 K_2 d\tau = \bar{B}_1 K_2, \tag{27e}$$

$$Ee^{\bar{A}T} = \begin{bmatrix} 0 & 0 \\ e^{\bar{A}_1 T} & \alpha I_2 + \bar{B}_1 K_2 \end{bmatrix}. \tag{27f}$$

Note that since $\operatorname{rank}(\bar{B}_1 K_2) \leq \operatorname{rank}(\bar{B}_1) = 1$, only one of the eigenvalues of $\alpha I_2 + \bar{B}_1 K_2$ can be different from α (in fact, $[\,1\;\;0\,]$ is a left eigenvector of $\alpha I_2 + \bar{B}_1 K_2$ relative to the eigenvalue α, whatever choice of K_2 is made). Hence, the monodromy matrix dynamics always contain two eigenvalues at zero and always contain one eigenvalue at α. Such conclusions make perhaps more interesting the discussion in the subsequent Section 4.2, where an alternative LTI (dynamic) controller implementing the same input as (23) is provided and discussed. $\triangle$

Remark 6. Consider the hybrid autonomous system

$$\dot{\tau} = 1, \tag{28a}$$
$$\dot{x} = Ax, \tag{28b}$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and

$$\tau^+ = 0, \tag{28c}$$
$$x^+ = Ex, \tag{28d}$$

when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$. By Carnevale et al. (2012b), such a system is globally asymptotically stable if and only if the matrix $e^{A\tau_M}E$ is Schur. Nonetheless, there need not exist positive definite matrices W_F, W_J and X such that

$$A^\top X + XA + W_F = 0,$$
$$E^\top XE - X + W_J = 0,$$

due to the fact that there need not exist a common Lyapunov function for (28b) and (28d) that is independent of the timer state variable τ. Nonetheless, if $e^{A\tau_M}E$ is Schur, then there is a Lyapunov function for system (28). Namely, letting W be any positive definite matrix, letting X be the positive definite, symmetric solution to

$$E^\top e^{A^\top \tau_M} X e^{A\tau_M} E - X + W = 0,$$

whose existence is guaranteed by the fact that $e^{A\tau_M}E$ is Schur, it can be easily derived that the function

$$\Omega(\tau, x) := e^{-\gamma\tau} x^\top e^{A^\top(\tau_M - \tau)} X e^{A(\tau_M-\tau)} x,$$

where $\gamma > 0$ is a sufficiently small constant, is a Lyapunov function for system (28). Note that such a function, as the optimal control law (23), depends on the timer state variable τ (see also Example 3.21 of Goebel et al., 2012). △

The following corollary extends the results of Theorem 3 to any initial condition $(\tau_0, x_0) \in [0, \tau_M] \times \mathbb{R}^n$ of (1).

Corollary 3. *Let the assumptions of Theorem 3 hold. Consider the cost function*

$$\hat{J}_{\theta,\kappa} = \int_\theta^\infty \left(|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F}\right) dt + \sum_{k=\kappa+1}^{\infty} \left(|x(t_k, k-1)|_{Q_J} + |u_J(k)|_{R_J}\right),$$

subject to the dynamics given in (1)*, with initial condition* $(\tau_0, x_0) \in [0, \tau_M] \times \mathbb{R}^n$*,* $\tau(\theta, \kappa) = \tau_0 := \theta - \kappa\tau_M$*,* $x(\theta,\kappa) = x_0$*,* $(\theta, \kappa) \in \mathcal{T}$*. The control input given in* (23) *minimizes* $\hat{J}_{\theta,\kappa}$ *and the minimum of such a function is given by*

$$\hat{J}^\star_{\theta,\kappa} = |x_0|_{P_\infty(\tau_0)}. \qquad \blacksquare$$

4.1 Stabilization properties of the LQ infinite–horizon optimal control

As in standard LQ control theory (Dorato et al., 1994), the optimal solution of Problem 2 is not necessarily stabilizing. Hence, consider the following problem.

Problem 3. Let the system (1) and the cost function (20) be given. Find, if any, a control input $u^\star_\infty = (u^\star_{F,\infty}, u^\star_{J,\infty})$ that is a solution to Problem 2 and that stabilizes system (1). ■

The following lemma characterizes the dynamical behavior of system (1) when the control input $u^\star_\infty$ is applied.

Lemma 2. *Let the assumptions of Theorem 3 hold. Let*

$$K_F(\tau) := -R_F^{-1} B' P_\infty(\tau), \tag{29a}$$
$$K_J := -(R_J + F' P_\infty(0) F)^{-1} F' P_\infty(0) E, \tag{29b}$$

where $P_\infty(\cdot)$ *is the solution to* (21)*. Let* $x(t,k)$ *be the solution to the closed–loop system with flow dynamics*

$$\dot{\tau} = 1, \tag{30a}$$
$$\dot{x} = (A + BK_F(\tau))x, \tag{30b}$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$*, and jump dynamics*

$$\tau^+ = 0, \tag{30c}$$
$$x^+ = (E + FK_J)x, \tag{30d}$$

when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$*, with* $\tau(0,0) = 0$*,* $x(0,0) = x_0$*,* $x_0 \in \mathbb{R}^n$*. The following statements are equivalent.*

(i) The initial condition x_0 *is such that* $P_\infty(0)x_0 = 0$*.*
(ii) The solution $x(t,k)$ *is such that*

$$Q_F x(t,k) = 0, \tag{31a}$$
$$K_F(\tau(t,k))x(t,k) = 0, \tag{31b}$$
$$Q_J x(t_k, k-1) = 0, \tag{31c}$$
$$K_J x(t_k, k-1) = 0, \tag{31d}$$

for all $t \in [t_k, t_{k+1})$*,* $k \in \mathbb{N}$*.*
(iii) The solution $x(t,k)$ *is* $x(t,k) = e^{A(t-t_k)}(Ee^{A\tau_M})^k x_0$ *and* $P_\infty(\tau(t,k))x(t,k) = 0$*, for all* $(t,k) \in \mathcal{T}$*.* ■

Proof. Consider the function

$$L(t,k) := x'(t,k) P_\infty(\tau(t,k)) x(t,k).$$

Since the matrix $P_\infty(\sigma)$ is symmetric and positive semidefinite for every $\sigma \in [0, \tau_M]$ (see Corollary 3), $L(t,k) = 0$ if and only if $P_\infty(\tau(t,k))x(t,k) = 0$.

By (21a) and Lemma 1, one has that

$$\dot{L}(t,k) = -x'(t,k) Q_F x(t,k) - x'(t,k) K_F'(\tau(t,k)) R_F K_F(\tau(t,k)) x(t,k).$$

By considering that Q_F is positive semidefinite and R_F is positive definite, $\dot{L}(t,k) \leqslant 0$, $\forall t \in [t_k, t_{k+1}]$, and $L(t,k) = c_k$, $c_k \in \mathbb{R}$, $\forall t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, if and only if (31a) and (31b) hold for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. On the other hand, by (21b) and Lemma 1, it results that

$$L(t_k,k) = x'(t_k,k)P_\infty(0)x(t_k,k) = \\ x'(t_k,k-1)(P_\infty(\tau_M) - (Q_J + K_J'R_JK_J))x(t_k,k-1).$$

Therefore, since Q_J is positive semidefinite and R_J is positive definite, $L(t_k,k) \leqslant L(t_k,k-1)$ and $L(t_k,k-1) = L(t_k,k)$ if and only if (31c) and (31d) hold, $\forall k \in \mathbb{N}$.

Thus, $L(t,k)$ is nonincreasing, lower and upper bounded (indeed, $0 \leqslant L(t,k) \leqslant |x(0,0)|_{P_\infty(0)}$). Therefore, $\lim_{t+k\to\infty} L(t,k) = c$, $c \in \mathbb{R}$, $c \geqslant 0$. Additionally, $L(t,k) = c$, $c \in \mathbb{R}$, $c \geqslant 0$, for all $(t,k) \in \mathcal{T}$ if and only if (31) holds.

(i) $\Longrightarrow$ (ii) If x_0 is such that $P_\infty(0)x_0 = 0$, then $L(t,k) = 0$, for all $(t,k) \in \mathcal{T}$, because L is nonincreasing and lower bounded. Hence, (31) holds.

(ii) $\Longrightarrow$ (iii) If (31) holds, then the hybrid arc $x(t,k)$ solution to (30) is a solution to (1) with $u_F = 0$ and $u_J = 0$. Since

$$\begin{aligned} x_0'P_\infty(0)x_0 &= J_\infty^\star \\ &= \int_0^\infty (|x(t,k)|_{Q_F} + |u_F(t,k)|_{R_F})\, dt \\ &\qquad + \sum_{k=1}^{\infty} (|x(t_k,k-1)|_{Q_J} + |u_J(k)|_{R_J}) \end{aligned}$$

if (31) holds, then $P_\infty(0)x_0 = 0$. Thus, by considering that if x_0 is such that $P_\infty(0)x_0 = 0$, then $L(t,k) = 0$, for all $(t,k) \in \mathcal{T}$, one has that $P(\tau(t,k))x(t,k) = 0$, for all $(t,k) \in \mathcal{T}$.

(iii) $\Longrightarrow$ (i) Since $P_\infty(\tau(t,k))x(t,k) = 0$ for any t and k, $P_\infty(0)x_0 = 0$. □

The following theorem specifies additional conditions on the cost function J_∞ under which the state feedback (23) solution to Problem 2 stabilizes system (1).

Theorem 4. *Let the system* (1) *and the cost function* (20) *be given. Let Assumption 1 hold. If the tuple* (A, E, Q_J, Q_F, τ_M) *is detectable, then the control input* (23) *is such that the closed–loop system is asymptotically stable.* ■

Proof. Assume that there exists $x_0 \in \mathbb{R}^n$ such that, letting $x(t,k)$ be the solution to (30) such that $x(0,0) = x_0$, $\lim_{t+k\to\infty} x(t,k) \neq 0$, i.e., the control input (23) does not guarantee convergence of the trajectories of system (1). Note that although this statement contradicts only attractivity, since the closed-loop system given by (1) and (23) is linear, it also contradicts asymptotic stability. Define the change of coordinates $\hat{x} = T^{-1}x$ that is such that

$$\hat{P}_\infty(0) = TP_\infty(0)T^{-1} = \begin{bmatrix} \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\lambda_i \in \Lambda(P_\infty(0))$, $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$, $i = 1, \ldots, \ell$, where ℓ is the number of eigenvalues of $P_\infty(0)$ different from 0. Hence, let $\hat{x}_0 = T^{-1}x_0 = [\,\hat{x}_{0,o}' \ \hat{x}_{0,i}'\,]'^\top$, with $\hat{x}_{0,o} \in \mathbb{R}^\ell$, $\hat{x}_{0,i} \in \mathbb{R}^{n-\ell}$. Since system (30) is linear, one has that $x(t,k) = x_o(t,k) + x_i(t,k)$, where $x_i(t,k)$ is the solution to (30) with $x_i(0,0) = T[\,0\ \hat{x}_{0,i}'\,]'$, while $x_o(t,k)$ is the solution to (30) with $x_o(0,0) = T[\,\hat{x}_{0,o}'\ 0\,]'$.

Consider the hybrid arc $x_i(t,k)$ and assume that $\lim_{t+k\to\infty} x_i(t,k) \neq 0$. Since $P_\infty(0)x_i(0,0) = 0$, by Lemma 2, (31) holds for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, i.e., $Q_F x_i(t,k) = 0$, $\forall (t,k) \in \mathcal{T}$, and $x_i(t,k) = e^{A\tau(t,k)}(Ee^{A\tau_M})^k T[\,0\ \hat{x}_{0,i}'\,]'$. This is in contradiction with (A, E, Q_J, Q_F, τ_M) being detectable. Hence, if (A, E, Q_J, Q_F, τ_M) is detectable and $x_i(t,k)$ is such that $P_\infty(0)x_i(t,k) = 0$, then $\lim_{t+k\to\infty} x_i(t,k) = 0$.

Consider $x_o(t,k)$. Note that if $\hat{x}_{0,o} \neq 0$, then

$$P_\infty(0)x_o(0,0) \neq 0.$$

Since $P_\infty(0)x_o(0,0) \neq 0$, then (31) does not hold. Define

$$L(t,k) := x'(t,k)P_\infty(\tau(t,k))x(t,k).$$

Since (31) does not hold, $L(t_k,k) > L(t_{k+1},k+1)$ (see the proof of Lemma 2). Let $\hat{x}_o(t_k,k) = T^{-1}x_o(t_k,k) = [\,\hat{x}_{o,1}'(t_k,k)\ \hat{x}_{o,2}'(t_k,k)\,]'$. By considering that

$$L(t_k,k) = \hat{x}_{o,1}'(t_k,k)\mathrm{diag}(\lambda_1, \ldots, \lambda_\ell)\hat{x}_{o,1}(t_k,k),$$

and that $L(t_k,k)$ is monotonically decreasing and lower bounded, $\lim_{k\to\infty} \hat{x}_{o,1}(t_k,k) = 0$. On the other hand, since $\hat{x}_{o,2}'(t_k,k)$ is such that $\hat{P}_\infty(0)\hat{x}_{o,2}'(t_k,k) = 0$, (A, E, Q_J, Q_F, τ_M) is detectable, and the hybrid arc $x(t,k)$, solution to system (30) with $x(0,0) = \bar{x}$, equals $\breve{x}(t - t_\kappa, k - \kappa)$, where $\breve{x}(t,k)$ is the solution to system (30) with $\breve{x}(t_\kappa,\kappa) = \bar{x}$, $\kappa \in \mathbb{N}$, one has that $\lim_{k\to\infty} \hat{x}_{o,2}'(t_k,k) = 0$. Therefore, since $\lim_{k\to\infty} x_o(t_k,k) = 0$, one has that $\lim_{t+k\to\infty} x_o(t,k) = 0$.

In fact, letting

$$H := \begin{bmatrix} A & -BR_F^{-1}B' \\ -Q_F & -A' \end{bmatrix},$$

$$\psi(\tau) := [\,I\ 0\,]e^{H\tau}[\,I\ P_\infty(0)\,]'$$

by Carnevale, Galeani and Sassano (2014), one has that $x_o(t,k) = \psi(\tau(t,k))x_o(t_k,k)$. Hence, letting $M = \max_{\sigma\in[0,\tau_M]} |\phi(\sigma)| < \infty$, $|x_o(t,k)| \leqslant M|x_o(t_k,k)|$.

Therefore, since

$$\lim_{t+k\to\infty} x_i(t,k) = 0,$$
$$\lim_{t+k\to\infty} x_o(t,k) = 0,$$

one has that $\lim_{t+k\to\infty} x(t,k) = 0$, i.e., the control input (23) stabilizes system (1). □

The following theorem shows that the additional conditions on the cost function J_∞ stated in Theorem 4 are, in fact, necessary to guarantee that the optimal control $u^\star_\infty$, solution to Problem 2, stabilizes system (1).

Theorem 5. *Let the system* (1) *and the cost function* (20) *be given. Let Assumption 1 hold and let $u^\star = (u^\star_F, u^\star_J)$ be a solution to Problem 2. Then $u^\star$ stabilizes system* (1) *only if (A, E, Q_J, Q_F, τ_M) is detectable.* ■

Proof. Assume that (A, E, Q_J, Q_F, τ_M) is not detectable. Hence, $\exists\lambda \in \Lambda(Ee^{A\tau_M})$, $\lambda \notin \mathbb{C}_g$, such that there exists a vector $w \in \mathbb{C}^n$ such that

$$Ee^{A\tau_M}w = \lambda w, \tag{32a}$$
$$Q_Je^{A\tau_M}w = 0, \tag{32b}$$
$$\begin{bmatrix} Q_F & A'Q_F & \cdots & (A')^{n-1}Q_F\end{bmatrix}' w = 0. \tag{32c}$$

Consider the following two cases

(a) $\lambda \in \mathbb{R}$. Since $Ee^{A\tau_M} \in \mathbb{R}^{n\times n}$, the vector w is in $\mathbb{R}^n$. Let $x(t,k)$ be the solution to system (1) with $u_F = 0$, $u_J = 0$, such that $x(0,0) = w$. Hence $x(t,k) = e^{A\tau(t,k)}(Ee^{A\tau_M})^k w = e^{A\tau(t,k)}\lambda^k w$. Thus, by (32c), one has that $Q_Fx(t,k) = 0$, while, by (32b), it results that $Q_Jx(t_k,k-1) = 0$. Therefore, the control inputs $u_F = 0$, $u_J = 0$ are such $J_\infty = 0$, i.e., $u_F = 0$, $u_J = 0$ is a solution to Problem 2.

(b) $\lambda \notin \mathbb{R}$. Define $w_a, w_b \in \mathbb{R}^n$ such that $w = w_a + \iota w_b$, where ι is the imaginary unit. Since $\lambda = \varrho e^{\iota\theta}$, $\varrho,\theta \in \mathbb{R}$, is an eigenvalue of $Ee^{A\tau_M} \in \mathbb{R}^{n\times n}$, then $\lambda^* \in \Lambda(Ee^{A\tau_M})$. Additionally, by (32b) and (32c), one has that $Q_Je^{A\tau_M}w^* = 0$ and $[\,Q_F\ A'Q_F\ \cdots\ (A')^{n-1}Q_F\,]'w^* = 0$. Hence, one has that

$$\begin{aligned} Q_Je^{A\tau_M}w_a &= 0,\\ \begin{bmatrix} Q_F & A'Q_F & \cdots & (A')^{n-1}Q_F\end{bmatrix}' w_a &= 0,\\ Q_Je^{A\tau_M}w_b &= 0,\\ \begin{bmatrix} Q_F & A'Q_F & \cdots & (A')^{n-1}Q_F\end{bmatrix}' w_b &= 0. \end{aligned} \tag{33}$$

Let $x(t,k)$ be the solution to system (1) with $u_F = 0$, $u_J = 0$, such that $x(0,0) = c_1w_a + c_2w_b$, $c_1, c_2 \in \mathbb{R}$. One has that $x(t,k) = e^{A\tau(t,k)}(\bar{c}_1\varrho^k\cos(\theta k + \phi_1)w_a + \bar{c}_2\varrho^k\cos(\theta k + \phi_2)w_b)$, $\bar{c}_1, \bar{c}_2, \phi_1, \phi_2 \in \mathbb{R}$. Hence, by (33), one has that $Q_Fx(t,k) = 0$, and $Q_Jx(t_k, k-1) = 0$. Therefore, the control inputs $u_F = 0$, $u_J = 0$ are such $J_\infty = 0$, i.e., $u_F = 0$, $u_J = 0$ is a solution to Problem 2.

Note that, in both cases (a) and (b), $\lambda \notin \mathbb{C}_g$. Therefore, by Lemma 1 of Carnevale et al. (2012b), $u^\star$ does not stabilize system (1). □

Note that the control input $u^\star$ given in (23) is independent of k and hence of the initial condition for the timer $\tau(0,0) = \tau_0$, $\tau_0 \in [0,\tau_M]$. Therefore, under the assumptions of Theorem 4, system (30) is globally asymptotically stable also for any initial condition $(\tau_0, x_0) \in [0,\tau_M]\times\mathbb{R}^n$ of (1).

Lemma 3. *Let the system* (1) *and the cost function* (20) *be given. Let Assumption 1 hold. If the tuple (A, E, Q_J, Q_F, τ_M) is observable, then the matrix $P_\infty(0)$ is positive definite.* ■

Proof. Let $x_\infty(t,k)$, $u_\infty(t,k)$ be the solution to the closed–loop system (1) with the control inputs given in (23). Consider the cost function

$$\bar{J}_n = \int_0^{t_n} (|x_\infty(t,k)|_{Q_F} + |u_{F,\infty}(t,k)|_{R_F})dt + \sum_{k=1}^{n} (|x_\infty(t_k,k-1)|_{Q_J} + |u_{J,\infty}(k)|_{R_J}).$$

By letting $\bar{x}$ be the initial condition of system (1), by Corollary 1, one has that $\bar{J}_n = \bar{x}'P_\infty(0)\bar{x} - x'_\infty(t_n,n)\,P_\infty(0)x_\infty(t_n,n)$. By Corollary 3, the matrix $P_\infty(0)$ is positive semidefinite. Assume that there exists $\check{x} \in \mathbb{R}^n$ such that $P_\infty(0)\check{x} = 0$. By considering that Q_F, Q_J, R_F, R_J, and $P_\infty(0)$ are positive semidefinite matrices, by Horn and Johnson (2012), there exist matrices $\bar{C}_F$, $\bar{C}_J$, $\bar{R}_F$, $\bar{R}_J$, and $\bar{P}$ with the property that $Q_F = \bar{C}_F'\bar{C}_F$, $Q_J = \bar{C}_J'\bar{C}_J$, $R_F = \bar{R}_F'\bar{R}_F$, $R_J = \bar{R}_J'\bar{R}_J$, $\bar{P}'\bar{P} = \bar{P}_\infty(0)$, respectively. Hence, we have that $\int_0^{t_n}(x'_\infty(t,k)\bar{C}_F'\bar{C}_Fx_\infty(t,k) + u'_{F,\infty}(t,k)\bar{R}_F'\bar{R}_Fu_{F,\infty}(t,k))dt + \sum_{k=1}^n(x'_\infty(t_k,k-1)\bar{C}_J'\bar{C}_Jx_\infty(t_k,k-1) + u'_{J,\infty}(k)\bar{R}_F'\bar{R}_Fu_{J,\infty}(t,k)) + x'(t_n,n)\bar{P}'\bar{P}x(t_n,n) = 0$. Hence, since R_F and R_J are positive definite matrices, this implies $u_{F,\infty} = 0$ and $u_{J,\infty} = 0$. Therefore, letting $\bar{x}(t,k)$ be a solution to

$$\begin{aligned} \dot{\bar{x}}(t,k) &= A\bar{x}(t,k),\\ \bar{x}^+(t_k,k) &= E\bar{x}(t_k,k), \end{aligned} \tag{34}$$

with $\bar{x}(0,0) = \bar{x}$, $\bar{x} \neq 0$, it must necessarily be that

$$\begin{aligned} \bar{C}_F \bar{x}(t,k) &= 0, \\ \bar{C}_J \bar{x}(t_k, k-1) &= 0, \end{aligned} \tag{35}$$

for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, $k \leqslant n$. Therefore, the following relations hold for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, $k \leqslant n$,

$$Q_F e^{A\tau(t,k)} (E e^{A\tau_M})^k \breve{x} = 0, \tag{36a}$$

$$Q_J e^{A\tau_M} (E e^{A\tau_M})^k \breve{x} = 0, \tag{36b}$$

which implies that the tuple (A, E, Q_J, Q_F, τ_M) is not observable, that is a contradiction. Hence, since $P_\infty(0)$ is positive semidefinite (by Theorem 3) and nonsingular, $P_\infty(0)$ is positive definite. $\square$

4.2 The monodromy algebraic Riccati equation

Following a construction wholly similar to the one given in Section 3.3, it can be easily verified that the solution $P_\infty(\tau_M) =: \bar{P}$ of the two-point boundary value problem (21a), if any, satisfies the following *monodromy algebraic Riccati equation* (briefly, *MARE*)

$$\begin{aligned} \bar{P} = Q_J + E'\Phi(\tau_M, \bar{P})E - E'\Phi(\tau_M, \bar{P})F \cdot \\ \cdot (R_J + F'\Phi(\tau_M, \bar{P})F)^{-1} F'\Phi(\tau_M, \bar{P})E. \end{aligned} \tag{37}$$

By construction and by Lemma 1, if system (1) is stabilizable, then a solution to the MARE (37) exists and can be determined either by computing numerically the solution to the MRE (16) for $Z = 0$, $K \to \infty$ and $k = 0$ or by directly determining the solution to (37). Once such a positive semidefinite solution $\bar{P}$ has been determined, the solution $P_\infty(\sigma)$ of the two-point boundary value problem (21a) is given by

$$P(\sigma) = U(\sigma) V^{-1}(\sigma),$$

where $U(\cdot)$ and $V(\cdot)$ are the solution to (Liberzon, 2011)

$$\begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix},$$

with final condition $U(\tau_M) = I$ and $V(\tau_M) = \bar{P}$.

As anticipated above, while the result in Theorem 3 provides an optimal stabilizing linear state feedback solution, a time-invariant, although potentially dynamic, implementation would clearly be more desirable.

Theorem 6. *Let $P_\infty(\cdot)$ be as defined in Corollary 2. The LTI hybrid controller given by*

$$\dot{p} = -A'p - Q_F x, \tag{38a}$$

$$p^+ = P_\infty(0)(E + FK_J)x, \tag{38b}$$

$$u^*_{F,\infty}(t,k) = -R_F^{-1} B' p(t,k), \tag{38c}$$

$$u^*_{J,\infty}(k) = K_J x(t_k, k), \tag{38d}$$

with $p(0,0) = P_\infty(0)x(0,0)$ and K_J as in (29b), provides the same optimal control state feedback (and performance) as the one specified in (23). ■

Proof. Exactly as in the classic (non-hybrid) derivation of LQ control from the Pontryagin Minimum Principle (PMP), defining the Hamiltonian function $H(x,p,u)$ for the problem of interest as

$$H(x,p,u) = x'Q_F x + u'R_F u + p'(Ax + Bu),$$

the control law during flows, which is uniquely identified by minimizing the strictly convex (in u) function $H(x,p,u)$, can be expressed as the linear time invariant feedback from the costate given by (38c), provided that the costate evolution is governed by the dynamics

$$\dot{p} = -\nabla_x H(x,p,u),$$

and the initial state for each flow interval is suitably initialized according to (38b), which implements the relation that $p(t,k) = \nabla_x J^*(x,t,k)$ (that is, the classic result that the costate provides sensitivity information about the value function, namely its gradient at the current event (x,t,k)). In particular, such relation at jump times becomes $p((k+1)\tau_M, k+1) = P_\infty(0)x((k+1)\tau_M, k+1)$ so that substituting $x((k+1)\tau_M, k+1) = Ex((k+1)\tau_M, k) + FK_J x((k+1)\tau_M, k)$ relation (38b) holds. $\square$

State p in (38) is actually a well-known variable in classic optimal control, namely the *costate*, given by

$$p(t,k) := P_\infty(\tau(t,k))x(t,k). \tag{39}$$

In the Hamiltonian formulation, the closed–loop optimal state and costate evolution is described by the equations:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR_F^{-1}B' \\ -Q_F & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{40a}$$

$$\begin{bmatrix} I & FR_J^{-1}F' \\ 0 & E' \end{bmatrix} \begin{bmatrix} x^+ \\ p^+ \end{bmatrix} = \begin{bmatrix} E & 0 \\ -Q_J & I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{40b}$$

where, as in the remainder of the paper, (40a) holds for all $(t,k) \in \mathcal{T}$, whereas (40b) holds for $t = (k +$

$1)\tau_M$. When $\det(E) \neq 0$, (40b) is readily solved. On the other hand, when $\det(E) = 0$, neither of the matrices in (40b) is invertible , so that neither $[\,(x^+)'\ (p^+)'\,]'$ nor $[\,x'\ p'\,]'$ can be explicitly derived in terms of the other, and then a different solution technique must be used. By a nontrivial extension of classic arguments (along the path provided *e.g.* in Pappas et al., 1980, Sec. II), the following result can be proven, making completely constructive the previous results (in practice, providing a way to compute $P_\infty(0)$).

Theorem 7. *Assume that system* (1) *is stabilizable and the matrices* Q_F *and* Q_J *are such that the tuple* (A, E, Q_J, Q_F, τ_M) *is detectable. Let* $V = [\,V_1'\ V_2'\,]'$, $V_1, V_2 \in \mathbb{R}^{n\times n}$*, be a basis for the generalized eigenspace corresponding to eigenvalues* λ *with* $|\lambda| < 1$ *for the pair* $(H_{JL}e^{H_F\tau_M}, H_{JR})$ *where*

$$H_F := \begin{bmatrix} A & -BR_F^{-1}B' \\ -Q_F & -A' \end{bmatrix}, \tag{41a}$$

$$H_{JR} := \begin{bmatrix} I & FR_J^{-1}F' \\ 0 & E' \end{bmatrix}, \tag{41b}$$

$$H_{JL} := \begin{bmatrix} E & 0 \\ -Q_J & I \end{bmatrix}. \tag{41c}$$

Then, $P_\infty(0)$ *in* (38b) *is given by* $P_\infty(0) = V_2V_1^{-1}$. ■

Remark 7. The results given in this section have some connections with the ones given in Ferrante and Ntogramatzidis (2013) and in Ferrante et al. (2005). However, while the scope of Ferrante and Ntogramatzidis (2013) and Ferrante et al. (2005) is to determine the solution to an LQ optimization problem with affine constraints on the initial and the terminal state for discrete-time and continuous-time systems, respectively, the main objective of this section is to determine the solution to an LQ hybrid optimization problem. Furthermore, the numerical tools proposed in this section to determine a solution to Problem 2 differ from the ones given in Ferrante and Ntogramatzidis (2013) and Ferrante et al. (2005). In fact, while the approaches given in Ferrante et al. (2005) and in Ferrante and Ntogramatzidis (2013) rely on the solutions of an algebraic Riccati equation and of a Lyapunov equation and on the so-called extended symplectic pencil, respectively, it has been shown in Theorem 7 that the solution to Problem 2 can be determined by solving a generalized eigenvalue problem. △

5 A physically motivated example

Consider a disk of radius r, total mass m, and inertia $\mathcal{I}$, moving on an horizontal plane between two parallel walls, orthogonal to the plane of motion and infinitely massive. Let $l + 2r$, $l > 0$, be the distance between the two walls, let (x_c, y_c) be the coordinates of the center of mass of the disk, and let α denote the angular position of the disk (Fig. 1).

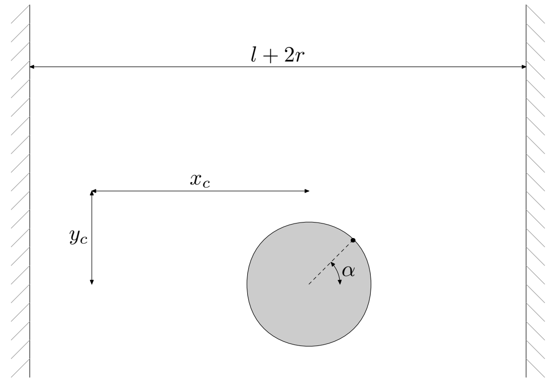

Fig. 1. A rotating disk bouncing between two walls.

Assume that all the impacts are elastic and occur with pre–impact conditions such that the infinitesimal interval in which the disk is in contact with the wall consists in a first interval of sliding followed by a second interval of rolling, i.e.,

$$|\dot{y}_c(t_k, k-1) + r\dot{\alpha}(t_k, k-1)| \leqslant 2\zeta\mu|\dot{x}_c(t_k, k-1)|, \tag{42}$$

where $\zeta = \frac{r^2 m}{\mathcal{I}}$ and μ is the coefficient of kinetic friction characterizing the infinitesimal sliding phase (Carnevale, Galeani and Menini, 2013). Assuming, additionally, that $|\dot{x}_c(t)| = |\dot{x}_c(0)| = v > 0$, a comprehensive state–space description of this mechanical system, with state $x = [\,y_c\ \dot{y}_c\ \alpha\ \dot{\alpha}\,]' \in \mathbb{R}^4$, is

$$\dot{\tau} = 1, \tag{43a}$$

$$\dot{x} = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{bmatrix} u_y, \tag{43b}$$

when $(\tau, x) \in [0, \frac{l}{v}] \times \mathbb{R}^4$, and

$$\tau^+ = 0, \tag{43c}$$

$$x^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\zeta^{-1} & 0 & -\zeta^{-1}r \\ 0 & 0 & 1 & 0 \\ 0 & -r^{-1}(1-\zeta^{-1}) & 0 & \zeta^{-1} \end{bmatrix} x, \tag{43d}$$

when $(\tau, x) \in \{\frac{l}{v}\} \times \mathbb{R}^4$, $\tau(0,0) = \frac{x_{c,0}}{v}$, and $x(0,0) = [\,y_{c,0}\ \dot{y}_{c,0}\ \alpha_0\ \dot{\alpha}_0\,]'$. It can be easily checked that system (43) satisfy (3), whence it is stabilizable. Consider the cost J_∞ given in (20). Note that the matrix E is not invertible and hence (40) cannot be used directly.

In order to satisfy the admissible motion condition (42), a reasonable choice for Q_J is

$$Q_J := \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & r^2 \end{bmatrix}, \tag{44}$$

so that the cost function J_∞ penalizes the term $\gamma\,|\dot{y}_c(t_k, k-1) + r\dot{\alpha}(t_k, k-1)|^2$. Thus, letting $\gamma = 10^2$, $v = 0.1\text{m/s}$, $l = 0.1\text{m}$, $r = 0.1\text{m}$, $m = 0.5\text{kg}$, $R_F = 1$, $R_J = 1$, and

$$Q_F := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we solved the MARE (37), that is, due to the matrix inversion, a system of polynomial equations in the entries of the matrix $\bar{P}$, obtaining

$$\bar{P} = \begin{bmatrix} 1.760 & -0.1081 & 0.07429 & 0.01087 \\ -0.1081 & 272.2 & -16.53 & -7.222 \\ 0.07429 & -16.53 & 2.592 & 1.653 \\ 0.01087 & -7.222 & 1.653 & 2.722 \end{bmatrix}.$$

Thus, the matrix $P_\infty(0)$ can be computed as $P_\infty(0) = U(0)V^{-1}(0)$, where

$$\begin{bmatrix} U(0) \\ V(0) \end{bmatrix} = \exp\left(- \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \tau_M \right) \begin{bmatrix} I \\ \bar{P} \end{bmatrix},$$

or, equivalently, as $P_\infty(0) = \Phi(\tau_M, \bar{P})$,

$$P_\infty(0) = \begin{bmatrix} 1.760 & 0.7293 & 0.07429 & 0.09459 \\ 0.7293 & 0.5887 & 0.02222 & 0.02586 \\ 0.07429 & 0.02222 & 2.592 & 3.308 \\ 0.09459 & 0.02586 & 3.308 & 6.888 \end{bmatrix}. \tag{45}$$

It is worth pointing out that, by Theorem 7, the same matrix can be determined by solving the generalized eigenvalue problem

$$H_{JL} e^{H_F \tau_M} z = \lambda H_{JR} z,$$

where H_{JL}, H_{JR}, and H_F are defined as in (41). Solving the equation $\det(H_{JL} e^{H_F \tau_M} - \lambda H_{JR}) = 0$, we find the finite eigenvalues 0, $0.016 - 0.225\imath$, $0.016 + 0.225\imath$, $0.321 - 4.423\imath$, $0.321 + 4.423\imath$, 0.561, 1.782. The eighth eigenvalue is an infinite one, in accordance with Section II-B of Pappas et al. (1980). The eigenvectors corresponding to the four eigenvalues in $\mathbb{C}_g$ can be put in a matrix

$$V = \left[\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right] = \left[\begin{array}{cccc} 0.018 & 0.018 & -0.45 & -0.44 \\ 0.020\imath & -0.020\imath & 0 & 0.86 \\ -0.10 + 0.36\imath & -0.10 - 0.36\imath & -0.056 & -0.011 \\ -0.06 - 0.20\imath & -0.06 + 0.20\imath & 0.024 & 0.011 \\ \hline 0.023 + 0.016\imath & 0.023 - 0.016\imath & -0.82 & -0.14 \\ 0.013 + 0.012\imath & 0.013 - 0.012\imath & -0.33 & 0.18 \\ -0.44 + 0.29\imath & -0.44 - 0.29\imath & -0.10 & 0 \\ -0.71 - 0.17\imath & -0.71 + 0.17\imath & -0.059 & 0.021 \end{array}\right].$$

It can be easily verified that the matrix $P_\infty(0)$ given in (45) matches with $V_2 V_1^{-1}$.

A numerical simulation of the behavior of system (43) in closed loop with the optimal controller (38) has been carried out assuming $x(0,0) = [\,-0.2\text{m}\ \ 0\text{m/s}\ \ 1\text{rad}\ \ 0\text{rad/s}\,]'$. Figure 2 depicts the results of such a simulation.

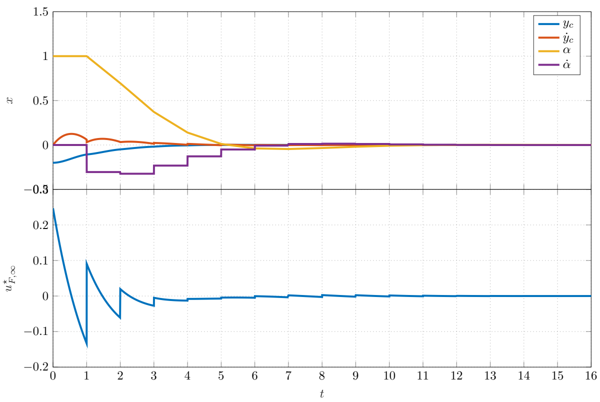

Fig. 2. Numerical simulation of system (43) in closed loop with the optimal controller (38).

As shown by such a figure, the optimal controller (38) stabilizes the closed-loop system due to the fact that system (43) is stabilizable and the matrices Q_F and Q_J are such that the tuple (A, E, Q_J, Q_F, τ_M) is detectable. Further, the admissible motion condition (42) is satisfied in the simulations with $\mu \geqslant 0.0522$.

6 Conclusion

This paper deals with the finite–horizon and infinite–horizon LQ optimal control problem for a class of hybrid system. It is shown that the solution to such problems can be obtained by an hybrid extension of the classical Differential and Difference Riccati Equations, solving such problems for the non–hybrid case. Necessary and sufficient conditions, ensuring that the optimal control stabilizes the hybrid system, are given.

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