Algorithms to compute the largest invariant set contained in an algebraic set for continuous-time and discrete-time nonlinear systems

Laura Menini, Corrado Possieri and Antonio Tornambè

***Abstract*—In this paper, some computational tools are proposed to determine the largest invariant set, with respect to either a continuous-time or a discrete-time system, that is contained in an algebraic set. In particular, it is shown that if the vector field governing the dynamics of the system is polynomial and the considered analytic set is a variety, then algorithms from algebraic geometry can be used to solve the considered problem. Examples of applications of the method (spanning from the characterization of the stability to the computation of the zero dynamics) are given all throughout the paper.**

***Index Terms*—Nonlinear systems, invariance, asymptotic stability.**

I. Introduction

The "second method" of Lyapunov, which makes use of a function that is monotonically decreasing along the solution of a systems, is one of the most widespread and literature pervasive tools to characterize the stability of nonlinear system [1], [2], [3], [4], [5], [6], [7]. In [8], [9], such a method has been generalized in order to possibly deal with monotonically non-increasing functions, by exploiting the concept of invariant set. Since these seminal works, the notion of invariant set has been extensively used in order to characterize the properties of nonlinear dynamical systems. For example, in [10], invariant sets are used to characterize the disturbances that let the output of a discrete-time, linear system vanish identically, in [11], controlled invariant sets are used to solve the disturbance decoupling problem, in [12], invariant sets are used to characterize some observability properties, and, in [13], (controlled) invariant sets are used to extend the notion of zero to nonlinear dynamical systems (see also [14] for a comprehensive review of the literature about the output zeroing problem and the zero dynamics of a system).

Due to the importance of invariant sets in both theoretical research and engineering practice (especially when dealing with stability of nonlinear dynamical system), a lot of research effort has been carried out to characterize such sets [15], [4], [16], [17], [18], [19], [20].

The main objective of this paper is to provide computational tools to determine the largest invariant set, with respect to either a continuous-time or a discrete-time nonlinear system, that is contained in an algebraic set. Such a problem is crucial in several control applications, as shown by the subsequent Theorems 1 and 6, that require the solution to such a problem to guarantee asymptotic stability of a given set. The goal of computing such an invariant set is pursued by showing that this set can be determined by solving a set of analytic equations. Furthermore, it is shown that, if the vector field governing the dynamics of the system is polynomial and the algebraic set is an affine variety, the largest invariant set with respect to the system that is contained in the affine variety can be determined through computationally efficient algorithms, which use tools borrowed from algebraic geometry that have already been used to solve control problems, such as observer design [21], [22], generation of algebraic certificates of (possibly, asymptotic) stability [23], solving the equations that arise in the harmonic balancing method [24], multi-objective optimal design of controllers for linear plants [25], checking the controllability of polynomial systems [26], and motion planning of mobile robots [27], [28]. Examples are given all throughout the paper to illustrate and corroborate the theoretical results. Applications of the proposed methods to several control problems are reported as, *e.g.*, stability analysis of nonlinear systems, test of the zero-state observability property, solution to the output zeroing problem, computation of the zero dynamics of a nonlinear plant.

The results given in this paper are related to the ones concerning the output zeroing of nonlinear autonomous systems [29], [30], [31]. The main difference between the results given in this paper and the ones available in the literature is that herein algorithmic procedures (ready to be implemented in CAS software, such as `Macaulay2` [32]) are proposed to determine the largest invariant set that is contained in an algebraic set with respect to either a continuous-time or a discrete-time analytic (possibly, polynomial) system.

Organization of the paper: the notation used in this paper is introduced in Section II, together with some preliminary results. The problem of determining the largest invariant set, which is contained in an analytic set, with respect to either a continuous-time or a discrete-time system is addressed in Sections III and IV, respectively. Examples of application of the given methods are reported in Section V. Conclusions are given in Section VI.

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II. Notation and preliminaries

Given a *multi-index* $\alpha = [\ \alpha_1 \ \ \cdots \ \ \alpha_n\ ]^\top$, $\alpha_i \in \mathbb{Z}_{\geqslant 0}$, $i = 1,\ldots,n$, let $|\alpha| := \sum_{i=1}^{n} \alpha_i$ and let $\alpha! := \alpha_1! \cdots \alpha_n!$. Letting $x = [\ x_1 \ \ \cdots \ \ x_n\ ]^\top$ be a vector of variables and $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ be a multi-index, define the *multivariate partial derivative*

$$\frac{\partial^\alpha}{\partial x^\alpha} := \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}.$$

Let $\mathbb{R}[x]$ be the *ring* of the polynomials in x. Given $p_1,\ldots,p_\ell \in \mathbb{R}[x]$, the set

$$\langle p_1,\ldots,p_\ell\rangle := \{p = \sum_{i=1}^{\ell} a_i p_i,\, a_i \in \mathbb{R}[x],\, i = 1,\ldots,\ell\}$$

is the *ideal in* $\mathbb{R}[x]$ *generated by* $p_1,\ldots,p_\ell$, whereas

$$\mathbb{V}(p_1,\ldots,p_\ell) := \{x \in \mathbb{R}^n : p_i(x) = 0,\, i = 1,\ldots,\ell\} \quad (1)$$

is the *variety generated by* $p_1,\ldots,p_\ell$. Similarly, given an ideal $\mathcal{I}$ in $\mathbb{R}[x]$, the *variety of the ideal* $\mathcal{I}$ is the set

$$\mathbb{V}(\mathcal{I}) := \{x \in \mathbb{R}^n : p(x) = 0,\, \forall p \in \mathcal{I}\}.$$

Given ideals $\mathcal{I}_a$, $\mathcal{J}_b$ in $\mathbb{R}[x]$, let

$$\sqrt{\mathcal{I}_a} := \{p \in \mathbb{R}[x] : \exists N \in \mathbb{Z}_{\geqslant 0},\, N \geqslant 1, \text{such that } p^N \in \mathcal{I}_a\},$$

be the *radical of* $\mathcal{I}_a$, whereas the *sum of* $\mathcal{I}_a$ *and* $\mathcal{J}_b$ is

$$\mathcal{I}_a + \mathcal{J}_b := \{p \in \mathbb{R}[x] : \exists p_a \in \mathcal{I}_a,\, p_b \in \mathcal{J}_b \text{ s.t. } p = p_a + p_b\}.$$

A *monomial order* $\succ$ is a total well ordering relation among the monomials $x^\alpha \in \mathbb{R}[x]$. Once a monomial order $\succ$ is fixed (such as the LEX order defined in [33]), any polynomial $p \in \mathbb{R}[x]$ can be rewritten as $p = c_1 x^{\beta_1} + c_2 x^{\beta_2} + \cdots + c_m x^{\beta_m}$, where $c_i \in \mathbb{R} \setminus \{0\}$, $i = 1,\ldots,m$, and $x^{\beta_1} \succ x^{\beta_2} \succ \cdots \succ x^{\beta_m}$. Thus, the *leading term* and the *leading coefficient of* p are $\mathrm{LT}(p) = c_1 x^{\beta_1}$ and $\mathrm{LC}(p) = c_1$, respectively. Once a monomial order $\succ$ is fixed, the set $\mathcal{G}_\mathcal{I} = \{g_1,\ldots,g_s\}$, with $\langle g_1,\ldots,g_s\rangle = \mathcal{I}$, is a *Gröbner basis* of an ideal $\mathcal{I}$ in $\mathbb{R}[x]$ if

$$\langle \mathrm{LT}(g_1),\ldots,\mathrm{LT}(g_s)\rangle = \langle \mathrm{LT}(\mathcal{I})\rangle,$$

where $\mathrm{LT}(\mathcal{I}) := \{cx^\alpha : \exists p \in \mathcal{I} \text{ such that } \mathrm{LT}(p) = cx^\alpha\}$.

A function $q : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* in its domain of definition $\mathcal{U} \subset \mathbb{R}^n$ (assumed to be open) if q is $\mathcal{C}^\infty$ at each $x \in \mathcal{U}$. If $x_0 \in \mathcal{U}$, a power series can be associated with the smooth function q locally about x_0 as follows:

$$Q(x) = \sum_{|\alpha|=0}^{+\infty} \tfrac{1}{\alpha!}\big(\tfrac{\partial^\alpha}{x^\alpha} q(x)\big)\big|_{x=x_0} (x - x_0)^\alpha. \quad (2)$$

A smooth function q is *locally analytic* in x_0 if there exists an open neighborhood $\mathcal{B} \subset \mathcal{U}$ of x_0 such that the series in (2) converges and $q(x) = Q(x)$, for each $x \in \mathcal{B}$; q is *analytic in* $\mathcal{U}$ if it is locally analytic in each $x_0 \in \mathcal{U}$.

Given $i \in \mathbb{Z}_{\geqslant 0}$ and smooth functions $h : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, define the *i-th directional derivative of* h *along* f as $L_f^0 h(x) = h(x)$ and $L_f^i h(x) = (\frac{\partial}{\partial x} L_f^{i-1} h(x)) f(x)$, and the *i-th directional successor of* h *along* f as $\Delta_f^0 h(x) = h(x)$ and $\Delta_f^i h(x) = \Delta_f^{i-1} h \circ f(x)$, $i \in \mathbb{Z}_{\geqslant 0}$.

Given analytic functions $q_j : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$, $j = 1,2,\ldots$, define the *analytic set*

$$\mathbb{V}(q_1, q_2 \ldots) = \{x \in \mathcal{U} : q_j(x) = 0,\, i = 1,2,\ldots\}.$$

III. Invariant sets for continuous-time systems

Consider the continuous-time, time-invariant system

$$\dot{x}(t) = f(x(t)),\, t \in \mathbb{R}, \quad (3)$$

where $x = [\ x_1 \ \ \ldots \ \ x_n\ ]^\top$, and the *vector field* $f : \mathcal{U} \to \mathbb{R}^n$ is analytic in $\mathcal{U}$, with $\mathcal{U}$ being an open subset of $\mathbb{R}^n$. Assume, without loss of generality, that the initial time is 0. The *response* at time t of system (3) from the initial state $x(0) = x_0 \in \mathcal{U}$ is denoted $x(t) = \phi_f(t, x_0)$; $\phi_f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}^n$ is the *CT-flow* associated with f and satisfies (see [34])

$$\begin{aligned} &\phi_f(0, x_0) = x_0, \\ &\tfrac{\partial}{\partial t}\phi_f(t, x_0) = f(\phi_f(t, x_0)),\, \forall t \in \mathbb{T}_{x_0}, \end{aligned}$$

where $\mathbb{T}_{x_0}$ is the *maximal interval of existence* of $\phi_f(t, x_0)$.

Intuitively, a subset $\mathcal{S}$ of $\mathcal{U}$ is f*-invariant* if the response of system (3) from any $x_0 \in \mathcal{S}$ remains in $\mathcal{S}$ for all $t \in \mathbb{T}_{x_0}$, as formalized in the following definition.

Definition 1. *A set* $\mathcal{S} \subset \mathcal{U}$ *is* positively *(respectively,* negatively*)* f-invariant *if, for each* $x_0 \in \mathcal{S}$, $\phi_f(t, x_0) \in \mathcal{S}$, $\forall t \in \mathbb{T}_{x_0} \cap \mathbb{R}_{\geqslant 0}$ *(respectively,* $t \in \mathbb{T}_{x_0} \cap \mathbb{R}_{\leqslant 0}$*). A set* $\mathcal{S}$ *is* f-invariant *if it is both positively and negatively* f*-invariant.*

Hence, consider the following problem.

Problem 1. *Let functions* $h_j : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$, $j = 1,\ldots,m$, *analytic in* $\mathcal{U}$*, be given. Find the largest* f*-invariant set contained in* $\mathbb{V}(h_1,\ldots,h_m)$.

The availability of tools to determine a solution to Problem 1 is crucial in several control applications and, in particular, for the application of the next two theorems.

Theorem 1 (Lyapunov stability, [4])**.** *Let* 0 *be an equilibrium of system* (3) *and assume that* $\mathcal{U} = \mathbb{R}^n$. *Let* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *be a differentiable, radially unbounded, positive definite function such that* $L_f V(x) \leqslant 0$, $\forall x \in \mathbb{R}^n$. *If the largest* f*-invariant set in* $\mathbb{V}(L_f V(x))$ *is* $\{0\}$*, then* 0 *is globally asymptotically stable.*

Theorem 2 (Invariance principle, [35])**.** *Let* $\Omega \subset \mathcal{U}$ *be a compact, positively* f*-invariant set. Let* $V : \Omega \to \mathbb{R}$ *be analytic in* Ω *and such that* $L_f V(x) \leqslant 0$, $\forall x \in \Omega$. *Let* $\mathcal{S} \subset \Omega$ *be the largest* f*-invariant set in* $\mathbb{V}(L_f V(x))$. *Every response* $x(t)$ *of system* (3) *starting in* Ω *approaches* $\mathcal{S}$*,* i.e.*,*

$$\lim_{t\to\infty}\left(\inf_{z\in\mathcal{S}} \|x(t) - z\|\right) = 0,$$

where $\|\cdot\|$ *is any norm in* $\mathbb{R}^n$.

In order to apply Theorem 1, one has to guarantee that $\mathcal{S} = \{0\}$ is the solution to Problem 1 with $h = L_f V(x)$. On the other hand, in order to apply Theorem 2 to prove attractiveness of the set $\mathcal{S}$, one has to guarantee that $\mathcal{S}$ is the largest invariant set that is contained in $\mathbb{V}(L_f V(x))$, *i.e.*, that $\mathcal{S}$ is the solution to Problem 1 with $h = L_f V(x)$, and that the set Ω is positively invariant. It is worth noticing that if, in addition to the hypotheses of Theorem 2, the function V is positive definite and there is $c \in \mathbb{R}$, $c > 0$, such that $\{x \in \mathbb{R}^n : V(x) \leqslant c\}$ is a subset of $\mathcal{U}$ and is compact, then, by classical Lyapunov results [4], the set $\Omega := \{x \in \mathbb{R}^n : V(x) \leqslant c\}$ is positively invariant (and contractive).

Remark 1. *The LaSalle invariance principle recalled in Theorem 2 has already been used in the literature [36], [37] to determine estimates of the region of attraction of an equilibrium point via polynomial optimization techniques, such as sum of squares tools. The main difference between such techniques and the ones given in this paper is that the former allow one to determine positively f-invariant sets that constitute inner estimates of the region of attraction of an asymptotically stable equilibrium point, whereas the latter allow to determine algebraically the largest f-invariant set $\mathcal{S}$ contained in a variety (such as,* e.g.*, $\mathbb{V}(L_f V(x))$),* i.e.*, under the hypotheses of Theorem 2, they allow one to characterize the set to which the trajectories of system* (3) *converge rather than its region of attraction.*

In the following Subsection III-A, a method to determine the solution to Problem 1 in the analytic case is proposed. An algorithm, which uses the algebraic geometry tools recalled in Section II, is given in Subsection III-B to apply such a technique in the polynomial case.

A. Solution to Problem 1 in the analytic case

The main objective of this subsection is to provide a solution to Problem 1 when f and $h_1, \ldots, h_m$ are analytic functions, but not necessarily polynomial. To this end, consider the following remark.

Remark 2. *Let $\alpha : \mathbb{T} \to \mathbb{R}^m$ be analytic in some interval $\mathbb{T}$, possibly coincident with $\mathbb{R}$. Let $\hat{\mathbb{T}}$ be any open subinterval of $\mathbb{T}$. One has that $\alpha(t)$ is equal to c for all $t \in \mathbb{T}$, for some constant $c \in \mathbb{R}^m$, if and only if $\alpha(t)$ is equal to c for all $t \in \hat{\mathbb{T}}$. Therefore, for any t_0 in the interior of $\mathbb{T}$, $\alpha(t)$ is equal to c for all $t \in \mathbb{T}$ if and only if $\alpha(t_0) = c$ and $\left.\frac{\mathrm{d}^k \alpha(t)}{\mathrm{d}t^k}\right|_{t=t_0} = 0$, $\forall k \in \mathbb{Z}$, $k > 0$.*

Following Remark 2, the next lemma provides necessary and sufficient conditions ensuring that a function h vanishes identically along the CT-flow $\phi_f(t, x_0)$.

Lemma 1. *Let $f : \mathcal{U} \to \mathbb{R}^n$ and $h : \mathcal{U} \to \mathbb{R}$ be analytic in $\mathcal{U}$. For each $x_0 \in \mathcal{U}$, one has $h(\phi_f(t, x_0)) = 0$, $\forall t \in \mathbb{T}_{x_0}$ if and only if $L_f^i h(x_0) = 0$, $\forall i \in \mathbb{Z}_{\geqslant 0}$.*

Proof. Since f and h are analytic in $\mathcal{U}$, for any $x_0 \in \mathcal{U}$, one has that $h(\phi_f(t, x_0))$ is an analytic function of t in $\mathbb{T}_{x_0}$ Hence, since $0 \in \mathbb{T}_{x_0}$, $h(\phi_f(t, x_0)) = 0$ for all $t \in \mathbb{T}_{x_0}$ if and only if $h(\phi_f(t, x_0)) = 0$ for all $t \in \mathcal{T}_{x_0}$, where $\mathcal{T}_{x_0} \subset \mathbb{T}_{x_0}$ is a sufficiently small neighborhood of 0. Since

$$h(\phi_f(t, x)) = \sum_{i=0}^{+\infty} \tfrac{t^i}{i!} L_f^i h(x), \quad \forall t \in \mathcal{T}_{x_0},$$

one has that $h(\phi_f(t, x_0))$ vanishes identically if and only if $L_f^i h(x_0) = 0$, $\forall i \in \mathbb{Z}_{\geqslant 0}$. □

The results given in Lemma 1 are used in the following lemma to prove that the largest positively (respectively, negatively) f-invariant subset of $\mathbb{V}(q_1, q_2, \ldots)$ is the largest f-invariant subset of $\mathbb{V}(q_1, q_2, \ldots)$.

Lemma 2. *If $\mathcal{S} \subset \mathcal{U}$ is the largest positively (respectively, negatively) f-invariant set contained in $\mathbb{V}(q_1, q_2, \ldots)$, then it is the largest f-invariant set contained in $\mathbb{V}(q_1, q_2, \ldots)$.*

Proof. Let $x_0 \in \mathcal{S}$ be fixed. Thus, let $\mathbb{T}_{x_0} \subset \mathbb{R}$ be the maximal interval of existence of $\phi_f(t, x_0)$. Since $\mathcal{S}$ is the largest positively f-invariant set contained in $\mathbb{V}(q_1, q_2, \ldots)$, there exists $T \in \mathbb{R}_{\geqslant 0}$ such that $q_j(\phi_f(t, x_0)) = 0$ for all $t \in [0, T]$, $j = 1, 2, \ldots$. By Lemma 1, if there exists a response $x(t)$ of system (3) such that $q_j(x(t)) = 0$, $\forall t \in [0, T]$, for some $j \in \mathbb{Z}_{\geqslant 0}$, then $q_j(x(t)) = 0$, $\forall t \in \mathbb{T}_{x_0}$. Therefore, since x_0 can be chosen arbitrarily in $\mathcal{S}$ and there does not exist $\bar{x} \in \mathbb{V}(q_1, q_2, \ldots) \setminus \mathcal{S}$ such that $L_f^i q_j(\bar{x}) = 0$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and $j \in \mathbb{Z}_{\geqslant 0}$ (otherwise $\bar{x}$ would be in $\mathcal{S}$), $\mathcal{S}$ is the largest f-invariant set contained in $\mathbb{V}(q_1, q_2, \ldots)$. □

By using Lemma 2, the following theorem shows how to determine the solution to Problem 1 in the analytic case.

Theorem 3. *Let $f : \mathcal{U} \to \mathbb{R}^n$ and $h_j : \mathcal{U} \to \mathbb{R}$, $j = 1, \ldots, m$, be analytic functions in $\mathcal{U}$. The largest f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is*

$$\mathcal{S} = \mathbb{V}(h_1, \ldots, h_m, L_f h_1, \ldots, L_f h_m, L_f^2 h_1, \ldots).$$

Proof. By Lemma 2, the largest f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is

$$\mathcal{S} = \{x \in \mathcal{U} : h_j(\phi_f(t, x)) = 0, \forall t \in \mathbb{T}_x, j = 1, \ldots, m\}.$$

By Lemma 1, $h_j(\phi_f(t, x)) = 0$ for all $t \in \mathbb{T}_x$ if and only if $L_f^i h_j(x) = 0, \forall i \in \mathbb{Z}_{\geqslant 0}$. □

The following example shows how Theorem 3 can be used to determine the solution to Problem 1.

Example 1. *Consider the following system*

$$\dot{x}_1 = \sin(x_2), \qquad \dot{x}_2 = -\sin(x_1), \tag{4}$$

and let $h(x) = \sin(x_1)$. It can be easily derived that

$$L_f^i h(x) = \begin{cases} \sin(x_1) q_i(x), & \text{if } i \text{ is even}, \\ \sin(x_2) q_i(x), & \text{if } i \text{ is odd}, \end{cases}$$

where $q_i(x)$ is a globally analytic function for each $i \in \mathbb{Z}_{\geqslant 0}$ (namely, $q_0(x) = 1$, $q_1(x) = \cos(x_1)$, $q_2 = \sin^2(x_2) + \cos(x_1)\cos(x_2)$, and so on). Therefore, since $\mathbb{V}(\cos(x_1)) \cap \mathbb{V}(\sin(x_1)) = \emptyset$, the largest invariant w.r.t. system (4) *contained in $\mathbb{V}(h)$ is*

$$\mathcal{S} = \{x \in \mathbb{R}^2 : x_1 = k_1 \pi, x_2 = k_2 \pi, k_1, k_2 \in \mathbb{Z}\},$$

which is the set of all the equilibria of system (4)*. Fig. 1 depicts the phase plot of system* (4) *and the largest invariant set w.r.t. system* (4) *contained in $\mathbb{V}(h)$.*

In order to apply Theorem 3, one has to compute the intersection of infinitely many sets, *i.e.*

$$\mathcal{S} = \bigcap_{i \in \mathbb{Z}_{\geqslant 0}} \bigcap_{j=1}^{m} \mathbb{V}(L_f^i h_j(x)). \tag{5}$$

Hence, consider the following proposition, which show how these computations can be simplified.

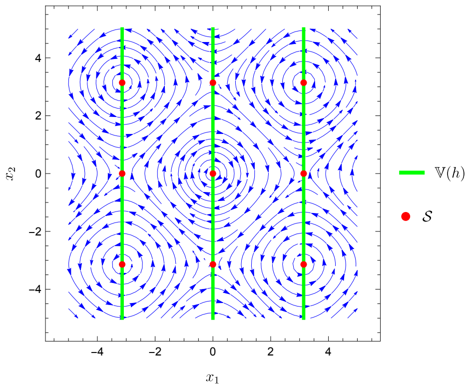

Fig. 1. Phase plot of system (4) (blue), algebraic set $\mathbb{V}(h)$ (green) and invariant set $\mathcal{S}$ (red).

Proposition 1. *Assume that there exist a set $\{q_1,\dots,q_\ell\}$ of analytic functions in $\mathcal{U}$ such that*

$$L_f q_k = \sum_{\chi=1}^{\ell} \alpha_{k,\chi} q_\chi,$$

for some $\alpha_{k,\chi}$, $k=1,\dots,\ell$, being analytic in $\mathcal{U}$. Hence, there exist functions $\beta_{i,j,k}$ being analytic in $\mathcal{U}$ such that $L_f^i q_j = \sum_{k=1}^{\ell}\beta_{i,j,k} q_k$, $j=1,\dots,\ell$, $i\in\mathbb{Z}_{\geqslant 0}$. Furthermore, if $q_i(x_0)=0$, $i=1,\dots,\ell$, then $q_i(\phi_f(t,x_0))=0$, $i=1,\dots,\ell$, for all $t\in\mathbb{T}_{x_0}$.

Proof. The proof is carried out by induction. By assumption, there exist analytic functions $\alpha_{j,k}$ such that

$$L_f q_j = \sum_{\chi=1}^{\ell} \alpha_{j,\chi} q_\chi.$$

Thus, the statement holds for $i=1$. Assume now that, for $i\in\mathbb{Z}_{\geqslant 0}$, there exist analytic functions $\beta_{i,j,k}$ such that $L_f^i q_j = \sum_{k=1}^{\ell}\beta_{i,j,k} q_k$, $j=1,\dots,\ell$. Therefore,

$$L_f^{i+1} q_j = \sum_{k=1}^{\ell} (L_f \beta_{i,j,k})\, q_k + \sum_{k=1}^{\ell}\beta_{i,j,k}\sum_{\chi=1}^{\ell}\alpha_{k,\chi} q_\chi,$$

which is a linear combination of $q_1,\dots,q_\ell$, with analytic coefficients in $\mathcal{U}$. Thus, if $q_j(x_0)=0$ for all $j=1,\dots,\ell$, then $L_f^{i+1} q_j(x_0)=0$ for all $i\in\mathbb{Z}_{\geqslant 0}$, $j\in\{1,\dots,m\}$. Hence, the proof follows by Lemma 1. □

By Proposition 1, if there exists $N\in\mathbb{Z}_{\geqslant 0}$ such that each $L_f^{N+1} h_j$, $j\in\{1,\dots,m\}$, can be expressed as a linear combination of $L_f^i h_j$, $j=1,\dots,m$, $i=0,\dots,N$, with analytic coefficients, then

$$\mathcal{S} = \bigcap_{i=0}^{N}\bigcap_{j=1}^{m} \mathbb{V}(L_f^i h_j(x)). \tag{6}$$

Therefore, if such an assumption holds, then the computations to be carried out so to find a solution to Problem 1 involve just a finite number of intersections, as shown in the next example.

Example 2. *Consider the following system (p. 214 of [38]):*

$$\dot{x}_1 = -\sin(x_1)\left(\tfrac{1}{10}\cos(x_1)+\cos(x_2)\right), \tag{7a}$$
$$\dot{x}_2 = \sin(x_2)\left(\cos(x_1)-\tfrac{1}{10}\cos(x_2)\right), \tag{7b}$$

with $\mathcal{U}=\mathbb{R}^2$. The objective of this example is to determine the largest f-invariant set contained in $\mathbb{V}(h)$, where $h(x)=\sin(x_1)\sin(x_2)$. One has

$$L_f h(x) = -\tfrac{1}{10}\sin(x_1)\sin(x_2)(\cos^2(x_1)+\cos^2(x_2)).$$

Therefore, since there exists an analytic function α such that $L_f h(x)=\alpha(x)\,h(x)$, by Proposition 1, the largest f-invariant set contained in $\mathbb{V}(h)$ is $\mathcal{S}=\mathbb{V}(h)$. Fig. 2 depicts the phase plot of system (7) and $\mathbb{V}(h)=\mathcal{S}$.

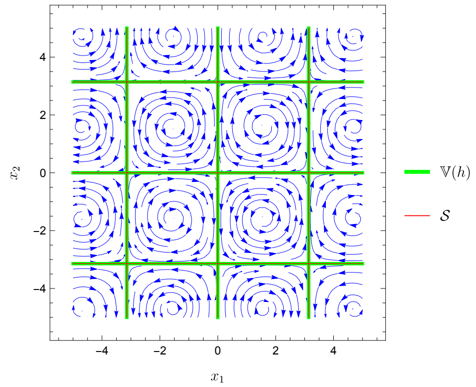

Fig. 2. Phase plot of system (7) (blue curves), algebraic set $\mathbb{V}(h)$ (green) and f-invariant set $\mathcal{S}$ (red).

In order to determine if $L_f^{N+1}h$ can be expressed as a linear combination of $h_1,\dots,h_m,\dots,L_f^N h_1,\dots,L_f^N h_m$, one has to solve a sort of ideal membership problem. If both f and h are *elementary* functions, such a problem can be addressed by recasting the functions $h_1,\dots,h_m,\dots,L_f^N h_1,\dots,L_f^N h_m, L_f^{N+1}h_1,\dots,L_f^{N+1}h_m$ in polynomial form by using the methods given in [39], [21] and by solving an ideal membership problem [33], [40], as shown in the following example.

Example 3. *Consider the system given in Ex. 8, Ch. 9 of [38]*

$$\dot{x}_1 = \sin(x_1)\sin(x_2), \qquad \dot{x}_2 = -\cos(x_1)\cos(x_2), \tag{8}$$

with $\mathcal{U}=\mathbb{R}^2$, and let $h(x)=x_2$. The objective is to determine the set $\mathcal{S}$ that is the solution to Problem 1. By computing

$$L_f h(x) = -\cos(x_1)\cos(x_2),$$

it is immediate to see that it cannot be expressed as $\alpha(x)\,h(x)$, for some $\alpha(x)$ analytic on the whole $\mathbb{R}^2$. Similarly, it is not trivial to determine whether

$$L_f^2 h(x) = \cos(x_2)\sin^2(x_1)\sin(x_2) - \cos^2(x_1)\cos(x_2)\sin(x_2)$$

can be expressed as a linear combination of h and $L_f h$. However, algebraic geometry can be used to solve this problem, as suggested in Chapter 6 of [33]. Introduce the auxiliary single variables c_1,c_2,s_1,s_2, and fix in $\mathbb{R}[c_1,c_2,s_1,s_2,x_2]$ the LEX order $\succ$, with $c_1\succ c_2\succ s_1\succ s_2\succ x_2$. Consider the polynomial $-c_1c_2$ obtained by replacing $\cos(x_1)$ and $\cos(x_2)$ in $L_f h(x)$ with c_1 and c_2, respectively; consider the ideal $\mathcal{J}_1$ generated by x_2, c_1c_2 and by the $s_i^2+c_i^2-1$, $i=1,2$, where the polynomial $s_i^2+c_i^2-1$ arises from the equality $\sin^2(x_i)+\cos^2(x_i)=1$, with similar substitutions. In the same *way, consider the polynomial* $p_2 = c_2 s_1^2 s_2 - c_1^2 c_2 s_2$*, which arises from* $L_f^2 h(x)$*. By computing* $p_2 \% \mathcal{J}_1 = c_2 s_2 \neq 0$*, one obtains that* p_2 *cannot be expressed as a polynomial sum of the generators of* $\mathcal{J}_1$*, whence* $L_f^2 h$ *cannot be expressed as a linear combination of* h *and* $L_f h$*. Thus, compute* $L_f^3 h(x)$ *and its associated polynomial* $p_3 = -c_2 c_1^3 s_2^2 - c_2^3 c_1 s_1^2 + 5 c_2 c_1 s_1^2 s_2^2 + c_2^3 c_1^3$*. Hence, by letting*

$$\mathcal{J}_2 = \langle x_2, c_1 c_2, c_2 s_1^2 s_2 - c_1^2 c_2 s_2, s_1^2 + c_1^2 - 1, s_2^2 + c_2^2 - 1 \rangle,$$

one obtains that $p_3 \% \mathcal{J}_2 = 0$*, whence* $L_f^3 h$ *can be expressed as a linear combination of* h*,* $L_f h$*,* $L_f^2 h$*. Thus, by Proposition 1, the solution to Problem 1 is given by*

$$\mathcal{S} = \mathbb{V}(h, L_f h, L_f^2 h) = \{x \in \mathbb{R}^2 : \cos(x_1) = 0, x_2 = 0\}.$$

Fig. 3 shows the phase plot of system (8)*,* $\mathbb{V}(h)$ *and* $\mathcal{S}$*.*

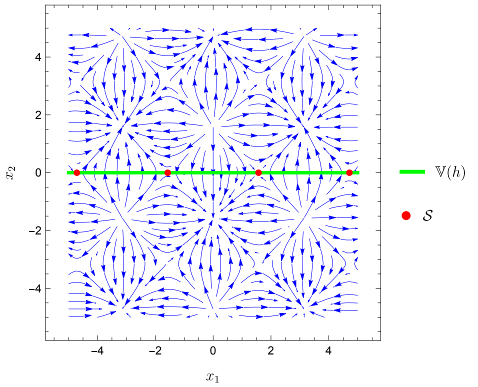

Fig. 3. Phase plot of system (8) (blue), algebraic set $\mathbb{V}(h)$ (green) and invariant set $\mathcal{S}$ (red).

Note that, although for many analytic systems there is $N \in \mathbb{Z}_{\geqslant 0}$ such that $L_f^{N+1} h_j$ can be expressed as a linear combination of $L_f^i h_j$, $j = 1, \dots, m$, $i = 0, \dots, N$, this need not always hold, as shown in the following counterexample, which shows that the solution to Problem 1 need not be computable as a finite intersection, as in (6).

Example 4. *Consider the system*

$$\dot{x} = -x,$$

with $\mathcal{U} = \mathbb{R}$*, and let*

$$h(x) = x \prod_{k=1}^{\infty} (1 - \exp(x^2 - k^2))^k.$$

By [41], the function $h : \mathbb{R} \to \mathbb{R}$ *is analytic in* $\mathbb{R}$*,* $L_f^i h(\pm i) \neq 0$ *and* $L_f^i h(\pm(i+j)) = 0$*, for* $i = 1, 2, \dots$*, and* $j = 1, 2, 3, \dots$ *. Since it results that* $L_f^{N+1} h(N+1) \neq 0$ *and* $h(N+1) = 0, \dots, L_f^N h(N+1) = 0$*,* $N \in \mathbb{Z}_{\geqslant 0}$*, there is no* $N \in \mathbb{Z}_{\geqslant 0}$ *such that* $L_f^{N+1} h$ *can be expressed as a linear combination of* $h, \dots, L_f^N h$*. Namely, for each* $N \in \mathbb{Z}_{\geqslant 0}$*,*

$$\bigcap\nolimits_{i=0}^{N} \mathbb{V}(L_f^i h(x)) = \{0, \pm(N+1), \pm(N+2), \dots\},$$

whereas the largest f*-invariant set contained in* $\mathbb{V}(h)$ *is*

$$\mathcal{S} = \bigcap\nolimits_{i=0}^{\infty} \mathbb{V}(L_f^i h(x)) = \{0\},$$

i.e.*, the set* $\mathcal{S}$ *cannot be determined with a finite number of intersections.*

In the following subsection, it is shown that, if both f and h are polynomial functions, then there always exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $L_f^{N+1} h_j$ can be expressed as a weighted sum of $h_1, \dots, h_m, \dots, L_f^N h_1, \dots, L_f^N h_m$, with polynomial weights, thus allowing the design of an algorithm to solve Problem 1.

B. Solution to Problem 1 in the polynomial case

The main objective of this subsection is to show that the computations required to determine the solution to Problem 1 are much simpler when the entries of f and $h_1, \dots, h_m$ are in $\mathbb{R}[x]$, which implies that $\mathcal{U} = \mathbb{R}^n$.

Consider the following lemma.

Lemma 3. *Let a countable (possibly, infinite) sequence of polynomials* $p_1, p_2, \dots$ *in* $\mathbb{R}[x]$ *be given. There exists a finite number of polynomials* $\hat{p}_1, \dots, \hat{p}_\ell$ *in* $\mathbb{R}[x]$ *such that:*

$$\bigcap_{i=1}^{\infty} \mathbb{V}(p_i) = \mathbb{V}(\hat{p}_1, \dots, \hat{p}_\ell). \tag{9}$$

Proof. Let $\mathcal{I}_1 = \langle p_1 \rangle$ and $\mathcal{I}_{i+1} = \mathcal{I}_i + \langle p_{i+1} \rangle$, $i \in \mathbb{Z}$, $i \geq 1$. Since $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \subset \dots$ is an ascending chain of ideals in $\mathbb{R}[x]$, by Theorem 7 at p. 80 of [33], there is $N \in \mathbb{Z}_{\geqslant 0}$, $N \geqslant 1$ such that $\mathcal{I}_N = \mathcal{I}_{N+j}$, $\forall j \in \mathbb{Z}_{\geqslant 0}$. By Theorem 4 at p. 190 of [33], $\forall i \in \mathbb{Z}_{\geqslant 0}$, $i \geqslant 1$,

$$\mathbb{V}(p_1) \cap \dots \cap \mathbb{V}(p_{i+1}) = \mathbb{V}(\mathcal{I}_i + \langle p_{i+1} \rangle).$$

Therefore, since $\mathcal{I}_i + \langle p_{i+1} \rangle = \mathcal{I}_N$, for all $i \in \mathbb{Z}$, $i \geqslant N$, and, by the Hilbert Basis Theorem [33], $\mathcal{I}_N$ is finitely generated, one has that (9) holds: one can take the polynomials $\hat{p}_1, \dots, \hat{p}_\ell$ as those constituting the reduced Gröbner basis of $\mathcal{I}_N$, with respect to any monomial order. □

Lemma 3 suggests that, in order to determine a solution to Problem 1 by using the expression given in (5), if both the entries of f and $h_1, \dots, h_m$ are polynomials, then $\mathcal{S}$ can be obtained by computing the intersection of a finite number of varieties. In the remainder of this subsection, it is shown how such computations can be carried out by using the algebraic geometry tools recalled in Section II.

Let a monomial order $\succ$ be fixed. Given an ideal $\mathcal{I}$ in $\mathbb{R}[x]$, let $\mathcal{G}_{\mathcal{I}} = \{p_1, \dots, p_\ell\}$ be the reduced Gröbner basis of $\mathcal{I}$ with respect to $\succ$. The *directional derivative* $L_f \mathcal{I}$ *of* $\mathcal{I}$ *along* f is the ideal defined as follows:

$$L_f \mathcal{I} := \langle L_f p_1, \dots, L_f p_\ell \rangle.$$

Given a polynomial p in some ideal $\mathcal{I}$, the next lemma states that the directional derivative along f of p belongs to the sum of $\mathcal{I}$ and of its directional derivative $L_f \mathcal{I}$.

Lemma 4. *If* $f \in \mathbb{R}^n[x]$ *and* $\mathcal{I}$ *is an ideal in* $\mathbb{R}[x]$*, then* $p \in \mathcal{I}$ *implies that*

$$L_f p \in (\mathcal{I} + L_f \mathcal{I}).$$

Proof. If $p \in \mathcal{I}$, then, letting $\langle p_1, \dots, p_\ell \rangle =: \mathcal{I}$, there exist $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}[x]$ such that $p = \sum_{j=1}^{\ell} \alpha_j p_j$. Thus, one has

$$L_f p = \sum_{j=1}^{\ell} (L_f \alpha_j) p_j + \sum_{j=1}^{\ell} \alpha_j (L_f p_j).$$

Since $\mathcal{I}$ is an ideal in $\mathbb{R}[x]$, one has that $\sum_{j=1}^{\ell} (L_f \alpha_j) p_j \in \mathcal{I}$ and $\sum_{j=1}^{\ell} \alpha_j (L_f p_j) \in L_f \mathcal{I}$. The statement follows by the definition of $\mathcal{I} + L_f \mathcal{I}$. □

The following statement extends the results of Lemma 4 to the i-th directional derivative of a polynomial.

Lemma 5. *If $f \in \mathbb{R}^n[x]$ and $\mathcal{I}$ is an ideal in $\mathbb{R}[x]$, define the sequence of ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \dots$ iteratively as*

$$\mathcal{I}_0 = \mathcal{I}, \tag{10a}$$

$$\mathcal{I}_{i+1} = \mathcal{I}_i + L_f \mathcal{I}_i, \; i \in \mathbb{Z}_{\geqslant 0}. \tag{10b}$$

Hence, if $p \in \mathcal{I}$, then $L_f^i p \in \mathcal{I}_i$, $\forall i \in \mathbb{Z}_{\geqslant 0}$.

Proof. The lemma is proved by induction. The base of the induction is proved by Lemma 4 by letting $i = 1$. As for the inductive step, assume that for some $i \in \mathbb{Z}_{\geqslant 0}$, if $p \in \mathcal{I}_0$, then $L_f^i p \in \mathcal{I}_i$. Since $L_f^{i+1} p = L_f(L_f^i p)$, $L_f^i p \in \mathcal{I}_i$ and $\mathcal{I}_{i+1} = \mathcal{I}_i + L_f \mathcal{I}_i$, by Lemma 4, one has that $L_f^{i+1} p = L_f(L_f^i p) \in \mathcal{I}_i + L_f \mathcal{I}_i = \mathcal{I}_{i+1}$. □

Consider the ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \dots$ defined as in (10). The following lemma states that there is an element $\mathcal{I}_N$ of such a sequence such that $\mathcal{I}_{N+j} = \mathcal{I}_N$ for all $j \in \mathbb{Z}_{\geqslant 0}$.

Lemma 6. *If $f \in \mathbb{R}^n[x]$ and $\mathcal{I}$ is an ideal in $\mathbb{R}[x]$, define $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \dots$ as in* (10). *Hence, there is $N \in \mathbb{Z}_{\geqslant 0}$, such that $L_f^i h_j \in \mathcal{I}_N$ for all $i \in \mathbb{Z}_{\geqslant 0}$ and all $j \in \{1, \dots, m\}$.*

Proof. Since $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \dots$ is an ascending chain of ideals in $\mathbb{R}[x]$, by Theorem 7 at p. 80 of [33], there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{I}_{N+j} = \mathcal{I}_N$ for all $j \in \mathbb{Z}_{\geqslant 0}$. By Lemma 5, $L_f^i h_j \in \mathcal{I}_i \subset \mathcal{I}_N$, for all $i \in \mathbb{Z}_{\geqslant 0}$. □

By combining the results given in Theorem 3 and Lemmas 3 and 6, the following lemma shows that if $\mathcal{I}_0 = \langle h_1, \dots, h_m \rangle$, then $\mathbb{V}(\mathcal{I}_N)$ is the solution to Problem 1.

Lemma 7. *Let $\mathcal{I} = \langle h_1, \dots, h_m \rangle$ and define $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \dots$ as in Lemma 6. Hence, the solution to Problem 1 is $\mathbb{V}(\mathcal{I}_N)$.*

The following example shows how to use the results given in Lemma 7 to determine the solution to Problem 1.

Example 5. *The objective of this example is to study the stability of the origin of*

$$\dot{x} = f(x) := \begin{bmatrix} -x_1^3 - x_2 \\ x_1^5 \end{bmatrix}.$$

Consider the candidate Lyapunov function

$$V(x) = x_1^6 + 3x_2^2,$$

which is such that $L_f V(x) = -6x_1^8 \leqslant 0$. Therefore, the origin of the considered system is stable. By computing the ideals $\mathcal{I}_i$ as in Lemma 7, one obtains that $\mathcal{I}_{8+j} = \mathcal{I}_8$ for all $j \in \mathbb{Z}_{\geqslant 0}$. Let $h(x) = -6x_1^8$. Fix the LEX order, with $x_1 \succ x_2$. Letting $\mathcal{I}_0 = \langle h \rangle$, the reduced Gröbner basis of $\mathcal{I}_0$ is $\mathcal{G}_{\mathcal{I}_0} = \{g_{0,1}\}$, with $g_{0,1}(x) = x_1^8$. Compute $L_f g_{0,1}(x) = -8x_1^{10} - 8x_1^7 x_2$; letting $\mathcal{I}_1 = \mathcal{I}_0 + \langle L_f g_{0,1} \rangle$, the reduced Gröbner basis of $\mathcal{I}_1$ is $\mathcal{G}_{\mathcal{I}_1} = \{g_{1,1}, g_{1,2}\}$, with $g_{1,1}(x) = g_{0,1}(x) = x_1^8$ and $g_{1,2}(x) = x_1^7 x_2$. By repeating such computations as in (10), *one obtains that $\mathcal{I}_{8+j} = \mathcal{I}_8$ for all $j \in \mathbb{Z}_{\geqslant 0}$. The reduced Gröbner basis of $\mathcal{I}_8$ is $\mathcal{G}_{\mathcal{I}_8} = \{g_{8,1}, \dots, g_{8,9}\}$, where*

$$\begin{aligned} g_{8,1}(x) &= x_1^8, & g_{8,2}(x) &= x_1^7 x_2, & g_{8,3}(x) &= x_1^6 x_2^2, \\ g_{8,4}(x) &= x_1^5 x_2^3, & g_{8,5}(x) &= x_1^4 x_2^4, & g_{8,6}(x) &= x_1^3 x_2^5, \\ g_{8,7}(x) &= x_1^2 x_2^6, & g_{8,8}(x) &= x_1 x_2^7, & g_{8,9}(x) &= x_2^8. \end{aligned}$$

Thus, the largest f-invariant set contained in $\mathbb{V}(L_f V(x))$ is $\mathcal{S} = \mathbb{V}(\mathcal{I}_8) = \{0\}$ and, by Theorem 1, the origin is globally asymptotically stable.

As shown by Example 5, in order to apply the method proposed in Lemma 7 to solve Problem 1, one has to compute a large number of reduced Gröbner bases. In the following, it will be shown that such computations can be reduced by exploiting the concept of the radical of an ideal. In fact, letting $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \dots$ be defined as in (10), let $\mathcal{K}_i = \sqrt{\mathcal{I}_i}$, $i \in \mathbb{Z}_{\geqslant 0}$. By Lemma 6, there is $\hat{N} \in \{0, \dots, N\}$ such that $\mathcal{K}_{\hat{N}+j} = \mathcal{K}_{\hat{N}}$, $j \in \mathbb{Z}_{\geqslant 0}$. Thus, consider the next lemma, whose proof follows by Lem. 5 at p. 182 of [33].

Lemma 8. *Assume that $f \in \mathbb{R}^n[x]$ and $h_1, \dots, h_m \in \mathbb{R}[x]$. Let $\mathcal{I} = \langle h_1, \dots, h_m \rangle$, define $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \dots$ as in* (10), *and let $\mathcal{K}_i = \sqrt{\mathcal{I}_i}$, $i \in \mathbb{Z}_{\geqslant 0}$. There is $\hat{N} \in \mathbb{Z}_{\geqslant 0}$ such that $L_f^i h_j \in \mathcal{J}_{\hat{N}}$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{1, \dots, m\}$.*

The following lemma shows how to determine an ideal $\mathcal{K}_{\tilde{N}}$ containing $L_f^i h_j$ for $i \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{1, \dots, m\}$.

Lemma 9. *Assume that $f \in \mathbb{R}^n[x]$ and $h \in \mathbb{R}^m[x]$. Thus, define the sequence of ideals $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots$ as*

$$\mathcal{K}_0 = \sqrt{\langle h_1, \dots, h_m \rangle}, \tag{11a}$$

$$\mathcal{K}_{i+1} = \sqrt{\mathcal{K}_i + L_f \mathcal{K}_i}, \; i \in \mathbb{Z}_{\geqslant 0}. \tag{11b}$$

Hence, there is $\tilde{N} \in \mathbb{Z}_{\geqslant 0}$ such that $L_f^i h_j \in \mathcal{K}_{\tilde{N}}$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and all $j \in \{1, \dots, m\}$.

Proof. By induction, it is proved that $L_f^i h_j \in \mathcal{K}_i$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and all $j \in \{1, \dots, m\}$. Let $i = 0$. Clearly, $h_j \in \langle h_1, \dots, h_m \rangle$, whence by Lemma 5 at p. 182 of [33], $h_j \in \sqrt{\langle h_1, \dots, h_m \rangle} = \mathcal{K}_0$. Now, for any $i \in \mathbb{Z}_{\geqslant 0}$, assume that $L_f^i h_j \in \mathcal{K}_i$. By Lemma 4, since $L_f^i h_j \in \mathcal{K}_i$, one has that $L_f^{i+1} h_j = L_f L_f^i h_j \in (\mathcal{K}_i + L_f \mathcal{K}_i)$, whence

$$L_f^{i+1} h_j \in \sqrt{\mathcal{K}_i + L_f \mathcal{K}_i} = \mathcal{K}_{i+1},$$

by Lemma 5 at p. 182 of [33]. Since $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$ is an ascending chain of ideals in $\mathbb{R}[x]$, by Theorem 7 at p. 80 of [33], there exists $\tilde{N} \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{K}_{\tilde{N}+j} = \mathcal{K}_{\tilde{N}}$ for all $j \in \mathbb{Z}_{\geqslant 0}$. □

In the following theorem, whose proof follows directly from Theorem 3 and Lemma 9, a method is proposed to determine a solution to Problem 1 in the polynomial case.

Theorem 4. *Assume that $f \in \mathbb{R}^n[x]$ and $h_1, \dots, h_m \in \mathbb{R}[x]$, and define $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots$ as in* (11). *Letting $\tilde{N}$ be such that*

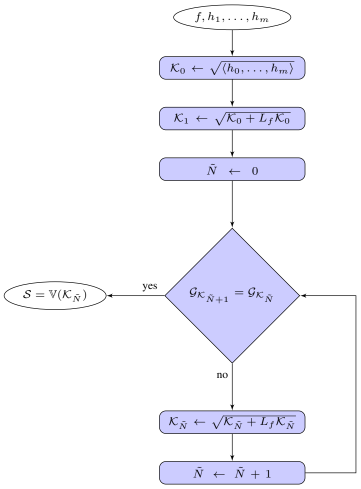

Fig. 4. Illustration of Algorithm 1.

$\mathcal{K}_{\tilde{N}+j} = \mathcal{K}_{\tilde{N}}$ *for all* $j \in \mathbb{Z}_{\geqslant 0}$*, the largest* f*-invariant set contained in* $\mathbb{V}(h_1, \dots, h_m)$ *is*

$$\mathcal{S} = \mathbb{V}(\mathcal{K}_{\tilde{N}}).$$

The following Algorithm 1 summarizes the computations that have to be carried out to solve Problem 1 in the polynomial case by using Theorem 4.

Algorithm 1 Solution to Problem 1

Input: $f \in \mathbb{R}^n[x]$ and $h_1, \dots, h_m \in \mathbb{R}[x]$

Output: the largest f-invariant set contained in $\mathbb{V}(h_1, \dots, h_m)$

1: $\mathcal{K}_0 \leftarrow \sqrt{\langle h_1, \dots, h_m \rangle}$, $\tilde{N} \leftarrow 0$, $\mathcal{K}_1 \leftarrow \sqrt{\mathcal{K}_0 + L_f \mathcal{K}_0}$
2: **while** $\mathcal{K}_{\tilde{N}+1} \neq \mathcal{K}_{\tilde{N}}$ **do**
3: $\quad \tilde{N} \leftarrow \tilde{N} + 1$, $\mathcal{K}_{\tilde{N}+1} \leftarrow \sqrt{\mathcal{K}_{\tilde{N}} + L_f \mathcal{K}_{\tilde{N}}}$
4: **return** $\mathbb{V}(\mathcal{K}_{\tilde{N}})$

Note that in order to apply Algorithm 1, just the knowledge of the polynomial f and $h_1, \dots, h_m$ is required. Once a monomial order is fixed, Step 2 of Algorithm 1 can be carried out by comparing the reduced Gröbner bases $\mathcal{G}_{\mathcal{K}_{\tilde{N}+1}}$ and $\mathcal{G}_{\mathcal{K}_{\tilde{N}}}$ of the respective ideals $\mathcal{K}_{\tilde{N}+1}$ and $\mathcal{K}_{\tilde{N}}$. As a matter of fact, by Theorem 5 at p. 120 of [33], for a given monomial order, each ideal has a unique reduced Gröbner basis, whence $\mathcal{K}_{\tilde{N}+1} = \mathcal{K}_{\tilde{N}}$ if and only if $\mathcal{G}_{\mathcal{K}_{\tilde{N}+1}} = \mathcal{G}_{\mathcal{K}_{\tilde{N}}}$. Note that, by Lemma 9, such an algorithm entails a finite number of iterations. The following Fig. 4 illustrates Algorithm 1.

The following example is an application of Algorithm 1.

Example 6. *Consider again the system and the Lyapunov function* V *given in Example 5. By using Algorithm 1 with input* f *and* $L_f V$*, one obtains that* $\mathcal{K}_0 = \langle x_1 \rangle$*,* $\mathcal{K}_1 = \langle x_1, x_2 \rangle$ *and* $\mathcal{K}_2 = \langle x_1, x_2 \rangle$*. Thus, since* $\mathcal{G}_{\mathcal{K}_2} = \mathcal{G}_{\mathcal{K}_1}$*, the algorithm terminates by returning* $\mathcal{S} = \mathbb{V}(\mathcal{K}_1) = \{0\}$*. Note that, in order to obtain the same result by using Lemma 6, one has to compute the reduced Gröbner bases of* 9 *ideals, whereas, by using Algorithm 1, just 3 reduced Gröbner bases have to be computed.*

The following remark details how to compute the restriction of the dynamics (3) to an f-invariant set $\mathcal{S}$.

Remark 3. *By [42], if the set* $\mathcal{S}$ *is* f*-invariant and* $\bar{x} \in \mathcal{S}$ *is not a critical point, then* f *is tangent to* $\mathcal{S}$ *in* $\bar{x}$*, thus allowing to easily determine the restriction of the dynamics of system* (3) *to the set* $\mathcal{S}$*. As a matter of fact, define*

$$d_{\bar{x}}(p) := \frac{\partial p}{\partial x_1}(\bar{x})(x_1 - \bar{x}_1) + \cdots + \frac{\partial p}{\partial x_n}(\bar{x})(x_n - \bar{x}_n),$$

and the tangent space of $\mathcal{S}$ in $\bar{x}$ *as*

$$T_{\bar{x}}(\mathcal{S}) := \mathbb{V}(\langle d_{\bar{x}} p : p \in \mathbf{I}(\mathcal{S}) \rangle).$$

By [33], if $\mathcal{S} = \langle p_1, \dots, p_\ell \rangle$*, then* $T_{\bar{x}}(\mathcal{S}) = \mathbb{V}(d_{\bar{x}} p_1, \dots, d_{\bar{x}} p_\ell)$*, which is a translation of a linear subspace of* $\mathbb{R}^n$*,* i.e.*, there exist* $b_1, \dots, b_s \in \mathbb{R}^n$ *(dependent on* $\bar{x}$*) such that* $T_{\bar{x}}(\mathcal{S}) = \bar{x} + \operatorname{Span}(b_1, \dots, b_s)$*. Therefore, for each non-critical* $\bar{x} \in \mathcal{S}$*, one has* $f(\bar{x}) \in T_{\bar{x}}(\mathcal{S})$*,* i.e.*, there exist* $c_1, \dots, c_s \in \mathbb{R}$ *(dependent on* $\bar{x}$*) such that*

$$f(\bar{x}) = \bar{x} + \sum_{j=1}^{s} c_j b_j.$$

This parametrization constitutes the restriction of the dynamics (3) *to* $\mathcal{S}$*.*

IV. Invariant sets for discrete-time systems

Consider the discrete-time, time-invariant system

$$x(t+1) = f(x(t)), \; t \in \mathbb{Z}, \tag{12}$$

where $x = [\, x_1 \;\; \dots \;\; x_n \,]^\top$, and $f : \mathcal{U} \to \mathcal{U}$ is analytic in $\mathcal{U}$, with $\mathcal{U}$ being an open subset of $\mathbb{R}^n$. Assume, without loss of generality, that the initial time is 0. The *response* of system (12) from the initial state $x(0) = x_0 \in \mathcal{U}$ is denoted $x(t) = \psi_f(t, x_0)$, where $\psi_f : \mathbb{Z} \times \mathcal{U} \to \mathbb{R}^n$ is the *DT-flow* associated with f, which satisfies [34]

$$\psi_f(0, x_0) = x_0,$$
$$\psi_f(t+1, x_0) = f(\psi_f(t, x_0)), \; \forall t \in \mathbb{T}_{x_0},$$

where $\mathbb{T}_{x_0} \subset \mathbb{Z}$ is the *maximal interval of existence* of $\psi_f(t, x_0)$. The notion of invariance for discrete-time systems differs from the one for continuous-time systems, as formalized in the following definition.

Definition 2. *The set* $\mathcal{S} \subset \mathbb{R}^n$ *is*

- positively f-invariant *if* $f(\mathcal{S}) \subset \mathcal{S}$*;*
- negatively f-invariant *if* $f(\mathcal{S}) \supset \mathcal{S}$*;*
- f-invariant *if* $f(\mathcal{S}) = \mathcal{S}$*.*

The main goal of this section, formalized in the following problem, is to determine the largest positively f-invariant set contained in an analytic set.

Problem 2. *Let functions* $h_j : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$*,* $j = 1, \dots, m$*, be analytic in* $\mathcal{U}$*. Find the largest positively* f*-invariant set contained in* $\mathbb{V}(h_1, \dots, h_m)$*.*

Note that, for discrete-time systems, the attention can be focused just on positively f-invariant sets, when considering the following two theorems.

Theorem 5 (Invariance principle, [43], [5]). *Let Ω be a subset of $\mathcal{U}$ and let $V : \Omega \to \mathbb{R}$ be a continuous function such that $\Delta_f V(x) - V(x) \leqslant 0$ for all $x \in \Omega$. Let $\mathcal{S}$ be the largest positively f-invariant set contained in $\mathbb{V}(\Delta_f V(x) - V(x)) \cap \bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω. If $\psi_f(t, x_0) \in \Omega$ for all $t \in \mathbb{Z}_{\geqslant 0}$ and $\psi_f(t, x_0)$ is bounded, then there exists $c \in \mathbb{R}$ such that $x(t) \to \mathcal{S} \cap V^{-1}(c)$ as $t \to \infty$.*

Theorem 6 (Lyapunov stability, [44]). *Let $\mathcal{U} = \mathbb{R}^n$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable, radially unbounded, and positive definite. If $\Delta_f V(x) - V(x) \leqslant 0$ for all $x \in \mathbb{R}^n$, then any response of system* (12) *tends to the largest positively f-invariant set contained in $\mathbb{V}(\Delta_f V(x) - V(x))$.*

Note that, in order to apply both Theorems 5 and 6, one has to determine the largest positively f-invariant set contained in $\mathbb{V}(\Delta_f V(x) - V(x))$.

In the following Subsection IV-A, a method to determine the solution to Problem 2 in the analytic case is proposed. An algorithm, which uses the algebraic geometry tools recalled in Section II, is given in Subsection IV-B to apply such a technique in the polynomial case.

A. Solution to Problem 2 in the analytic case

The main objective of this section is to provide a solution to Problem 2 when both the entries of f and $h_1, \ldots, h_m$ are analytic in $\mathcal{U}$. Consider the following lemma.

Lemma 10 (Existence of responses, [45]). *The DT-flow $\psi_f(t, x_0)$ is well defined for each $(t, x_0) \in \mathbb{Z}_{\geqslant 0} \times \mathcal{U}$.*

Note that the statement of Lemma 10 holds just for $t \in \mathbb{Z}_{\geqslant 0}$. On the other hand, uniqueness of the responses of system (12) need not hold backward in time, since f need not be bijective. Consider the following theorem, which shows how to determine the solution to Problem 2 in the analytic case.

Theorem 7. *Let $f : \mathcal{U} \to \mathbb{R}^n$ and $h_j : \mathcal{U} \to \mathbb{R}$, $j = 1, \ldots, m$, be analytic functions in $\mathcal{U}$. The largest f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is*

$$\mathcal{S} = \mathbb{V}(h_1, \ldots, h_m, \Delta_f h_1, \ldots, \Delta_f h_m, \Delta_f^2 h_1, \ldots). \tag{13}$$

Proof. By the definition of the DT-flow $\psi_f(\cdot, \cdot)$ and by Lemma 10, for all $j \in \mathbb{Z}_{\geqslant 0}$ and $t \in \mathbb{Z}_{\geqslant 0}$, one has that

$$q_j(\psi_f(t, x_0)) = \Delta_f^t q_j(x)\big|_{x = x_0}.$$

Therefore, since the largest positively f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_s)$ is

$$\mathcal{S} = \{x \in \mathbb{R}^n : h_j(\psi_f(t, x)) = 0, \forall k \in \mathbb{Z}_{\geqslant 0}\},$$

this set equals the set given in (13). $\square$

In order to apply the technique proposed in Theorem 7 to solve Problem 2, one has to compute the intersection of infinitely many sets, *i.e.*,

$$\mathcal{S} = \bigcap\nolimits_{i \in \mathbb{Z}_{\geqslant 0}} \bigcap\nolimits_{j=1}^{m} \mathbb{V}(\Delta_f^i h_j(x)). \tag{14}$$

As for continuous-time systems, such computations can be simplified, as shown in the following proposition, whose proof is similar to the proof of Proposition 1.

Proposition 2. *Assume that there exist a set $\{q_1, \ldots, q_\ell\}$ of analytic functions in $\mathcal{U}$ such that*

$$\Delta_f q_k = \sum_{\chi=1}^{\ell} \alpha_{k,\chi} q_\chi,$$

for some $\alpha_{k,\chi}$, $k = 1, \ldots, \ell$, being analytic in $\mathcal{U}$. Hence, there exist $\beta_{i,j,k}$ being analytic in $\mathcal{U}$ such that, for $j = 1, \ldots, \ell$,

$$\Delta_f^i q_j = \sum_{k=1}^{\ell} \beta_{i,j,k} q_k.$$

Moreover, if $q_1(x_0) = \cdots = q_\ell(x_0) = 0$ then $q_1(x(k)) = \cdots = q_\ell(x(k))$, for all $k \in \mathbb{Z}_{\geqslant 0}$.

By Proposition 2, if there is $N \in \mathbb{Z}_{\geqslant 0}$ such that $\Delta_f^{N+1} h_j$ can be expressed as a linear combination of $\Delta_f^i h_j$, $j = 1, \ldots, m$, $i = 0, \ldots, N$, with analytic coefficients, then

$$\mathcal{S} = \bigcap\nolimits_{i=0}^{N} \bigcap\nolimits_{j=1}^{m} \mathbb{V}(\Delta_f^i h_j(x)), \tag{15}$$

i.e., as for continuous-time systems, just the intersection of a finite number of sets needs to be determined.

Example 7. *Consider the system [43]*

$$x_1(t+1) = \tfrac{x_2(t)}{1 + x_1^2(t)}, \qquad x_2(t+1) = \tfrac{x_1(t)}{1 + x_2^2(t)}, \tag{16}$$

with $\mathcal{U} = \mathbb{R}^2$. Consider the positive definite, radially unbounded, candidate Lyapunov function $V = \frac{1}{2}(x_1^2 + x_2^2)$. It can be easily seen that $\Delta_f V(x) - V(x) \leqslant 0$ for all $x \in \mathbb{R}^2$. Thus, letting

$$h(x) = \Delta_f V(x) - V(x),$$

by computing $\Delta_f h(x)$, $\Delta_f^2 h(x)$, and $\Delta_f^3 h(x)$ (whose explicit expressions are omitted for space reasons) it can be shown that $\Delta_f^3 h(x)$ can be expressed as a linear combination of h, $\Delta_f h(x)$, and $\Delta_f^2 h(x)$, with analytic coefficients. Therefore, by Proposition 2, the largest positively f-invariant set contained in $\mathbb{V}(h(x))$, is

$$\mathcal{S} = \mathbb{V}(h, \Delta_f h, \Delta_f^2 h) = \mathbb{V}(x_1 x_2).$$

By Theorem 6, trajectories of (16) *tend to $\mathcal{S}$. Fig. 5 depicts some trajectories of system* (16).

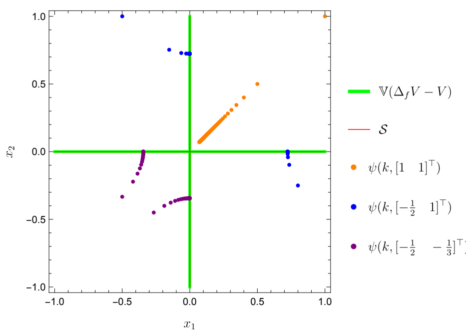

Fig. 5. Set $\mathbb{V}(h)$ (green), positively invariant set $\mathcal{S}$ (red), and trajectories of system (16).

In the following subsection, it is shown that, if both f and h are polynomial functions, then there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $\Delta_f^{N+1} h_j$ can be expressed as a linear combination of $h_1, \ldots, h_m, \ldots, \Delta_f^N h_1, \ldots, \Delta_f^N h_m$, thus allowing the design of an algorithm to solve Problem 2.

B. Solution to Problem 2 in the polynomial case

The main objective of this subsection is to show that the computations required to solve Problem 2 are simpler in case of polynomial systems, for which $\mathcal{U} = \mathbb{R}^n$. By Lemma 3, if $f \in \mathbb{R}^n[x]$ and $h_1, \ldots, h_m \in \mathbb{R}[x]$, then the set $\mathcal{S}$ given in (14) can be determined by computing a finite number of intersections. In the following, it is shown how such computations can be carried out by using the tools recalled in Section II.

Let a monomial order $\succ$ be fixed. Given an ideal $\mathcal{I}$ in $\mathbb{R}[x]$, let $\mathcal{G}_\mathcal{I} = \{p_1, \ldots, p_\ell\}$ be the reduced Gröbner basis of $\mathcal{I}$. The *directional successor of $\mathcal{I}$ along f* is

$$\Delta_f \mathcal{I} := \langle \Delta_f p_1, \ldots, \Delta_f p_\ell \rangle.$$

Given a polynomial p in some ideal $\mathcal{I}$, the next lemma states that the directional increment along f of p belongs to the sum of $\mathcal{I}$ and of its directional increment $\Delta_f \mathcal{I}$.

Lemma 11. *If $f \in \mathbb{R}^n[x]$ and $\mathcal{I}$ is an ideal in $\mathbb{R}[x]$, then $p \in \mathcal{I}$ implies that*

$$\Delta_f p \in (\mathcal{I} + \Delta_f \mathcal{I}).$$

Proof. If $p \in \mathcal{I}$, then, letting $\{p_1, \ldots, p_\ell\}$ be a basis of $\mathcal{I}$, there exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}[x]$ such that $p = \sum_{j=1}^{\ell} \alpha_j p_j$. Thus, $\Delta_f p(x) = \sum_{j=1}^{\ell} \Delta_f \alpha_j(x) \Delta_f p_j(x)$. $\square$

In view of Lemma 11, consider the following lemma.

Lemma 12. *If $f \in \mathbb{R}^n[x]$ and $h_1, \ldots, h_m \in \mathbb{R}[x]$, then define the sequence of ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ as*

$$\mathcal{I}_0 = \langle h_1, \ldots, h_m \rangle, \tag{17a}$$
$$\mathcal{I}_{i+1} = \mathcal{I}_i + \Delta_f \mathcal{I}_i, \; i \in \mathbb{Z}_{\geqslant 0}. \tag{17b}$$

There exists $N \in \mathbb{Z}_{\geqslant 0}$ such that

$$\Delta_f^i h \in \mathcal{I}_N, \; \forall i \in \mathbb{Z}_{\geqslant 0}, \; j \in \{1, \ldots, m\}. \tag{18}$$

Furthermore, the solution to Problem 2 is $\mathbb{V}(\mathcal{I}_N)$.

Proof. Note that the sequence $\mathcal{I}_0, \mathcal{I}_1, \ldots$ is such that

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \ldots.$$

Therefore, by Theorem 7 at p. 80 of [33], there is $N \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{I}_{N+j} = \mathcal{I}_N$ for all $j \in \mathbb{Z}_{\geqslant 0}$. Hence, by Lemma 11, one has that (18) holds, and hence, by Theorem 7, the largest f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is given by (13), which coincides with $\mathbb{V}(\mathcal{I}_N)$. $\square$

The following example shows how the tool given in Lemma 12 can be used to solve Problem 2.

Example 8. *Consider the following system*

$$x_1(t+1) = -2x_1^2(t)x_2(t) + 2x_2(t), \tag{19a}$$
$$x_2(t+1) = x_1(t)x_2^2(t) + \tfrac{1}{2}x_1(t). \tag{19b}$$

Consider the candidate Lyapunov function $V(x) = x_1^2 + 4x_2^2$. It can be easily seen that

$$\Delta_f V(x) - V(x) = 4x_2^2 x_1^2 (x_1^2 + x_2^2 - 1),$$

which is negative semi-definite about the origin. Now, one can compute the largest positively f-invariant set contained in $\mathbb{V}(\Delta_f V(x) - V(x))$ by using the sequence of ideals defined in Lemma 12. Thus, let $\mathcal{I}_0 = \langle 4x_2^2 x_1^2 (x_1^2 + x_2^2 - 1) \rangle$, $\mathcal{I}_1 = \mathcal{I}_0 + \Delta_f \mathcal{I}_0$ and $\mathcal{I}_2 = \mathcal{I}_1 + \Delta_f \mathcal{I}_1$. By computing the reduced Gröbner bases $\mathcal{G}_{\mathcal{I}_1}$ and $\mathcal{G}_{\mathcal{I}_2}$ of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, one has $\mathcal{G}_{\mathcal{I}_1} = \mathcal{G}_{\mathcal{I}_2}$, whence $\mathcal{I}_1 = \mathcal{I}_2$. In particular, the reduced Gröbner basis of $\mathcal{I}_1$ is $\mathcal{G}_{\mathcal{I}_1} = \{g_{1,1}, g_{1,2}\}$, where:

$$g_{1,1}(x) = x_1^3 x_2 + x_1 x_2^3 - x_1 x_2,$$
$$g_{1,2}(x) = 8x_1 x_2^9 + 4x_1 x_2^7 + 2x_1 x_2^5 - x_1 x_2^3 - x_1 x_2.$$

Thus, by Lemma 12, the largest positively f-invariant set contained in $\mathbb{V}(\Delta_f V(x) - V(x))$ is $\mathcal{S} = \mathbb{V}(\mathcal{I}_1)$.

As for continuous-time systems, the concept of radical of an ideal can be used to reduce the number of Gröbner bases to be computed, as shown in the following theorem.

Theorem 8. *Assume that $f \in \mathbb{R}^n[x]$ and $h \in \mathbb{R}^m[x]$. Define the sequence of ideals $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \ldots$ as*

$$\mathcal{K}_0 = \sqrt{\langle h_1, \ldots, h_m \rangle}, \tag{20a}$$
$$\mathcal{K}_{i+1} = \sqrt{\mathcal{K}_i + \Delta_f \mathcal{K}_i}, \; i \in \mathbb{Z}_{\geqslant 0}. \tag{20b}$$

There exists $\tilde{N} \in \mathbb{Z}_{\geqslant 0}$ such that $\Delta_f^i h_j \in \mathcal{K}_{\tilde{N}}$, $\forall i \in \mathbb{Z}_{\geqslant 0}$, $j \in \{1, \ldots, m\}$, and such that the largest positively f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is $\mathcal{S} = \mathbb{V}(\mathcal{K}_{\tilde{N}})$.

Proof. First, it is proved, by induction, that

$$\Delta_f^i h_j \in \mathcal{K}_i, \; \forall i \in \mathbb{Z}_{\geqslant 0}, \; \forall j \in \{1, \ldots, m\}. \tag{21}$$

Since, by Lemma 5 at p. 182 of [33], $\langle \mathcal{I} \rangle \subset \sqrt{\mathcal{I}}$, one has that (21) holds for $i = 0$. Assume now that (21) holds for some $i \in \mathbb{Z}_{\geqslant 0}$. Hence, by Lemma 11,

$$\Delta_f^{i+1} h_j \in (\mathcal{K}_i + \Delta_f \mathcal{K}_i) \subset \sqrt{\mathcal{K}_i + \Delta_f \mathcal{K}_i} = \mathcal{K}_{i+1},$$

thus concluding the induction.

Secondly, it is proved that there exists $\tilde{N} \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{K}_{\tilde{N}+j} = \mathcal{K}_{\tilde{N}}$, for all $j \in \mathbb{Z}_{\geqslant 0}$. By considering that

$$\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \ldots$$

is an ascending chain of ideals in $\mathbb{R}[x]$, by Theorem 7 at p. 80 of [33], there exists $\tilde{N} \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{K}_{\tilde{N}+j} = \mathcal{K}_{\tilde{N}}$ for all $j \in \mathbb{Z}_{\geqslant 0}$. The proof is concluded by the fact that, by Theorem 7, the largest positively f-invariant set contained in $\mathbb{V}(h_1, \ldots, h_m)$ is given by $\mathbb{V}(h_1, \ldots, h_m, \Delta_f h_1, \ldots)$. $\square$

The following Algorithm 2 resumes the computations to be carried out to solve Problem 2 in the polynomial case by using the method outlined in Theorem 8.

Note that, as for Algorithm 1, in order to apply Algorithm 2, just the knowledge of the polynomial f and $h_1, \ldots, h_m$ is required. The following Fig. 6 illustrates Algorithm 2.

The following remark details how to determine the restriction of the dynamics (12) to $\mathcal{S}$ in a particular case.

Algorithm 2 Solution to Problem 2.

Input: $f \in \mathbb{R}^n[x]$ and $h_1, \dots, h_m \in \mathbb{R}[x]$

Output: the largest positively f-invariant set contained in $\mathbb{V}(h_1, \dots, h_m)$

$\mathcal{K}_0 \leftarrow \sqrt{\langle h_1, \dots, h_m \rangle}$, $\tilde{N} \leftarrow 0$, $\mathcal{K}_1 \leftarrow \sqrt{\mathcal{K}_0 + \Delta_f \mathcal{K}_0}$
while $\mathcal{K}_{\tilde{N}+1} \neq \mathcal{K}_{\tilde{N}}$ **do**
  $\tilde{N} \leftarrow \tilde{N} + 1$, $\mathcal{K}_{\tilde{N}+1} \leftarrow \sqrt{\mathcal{K}_{\tilde{N}} + \Delta_f \mathcal{K}_{\tilde{N}}}$
return $\mathbb{V}(\mathcal{K}_{\tilde{N}})$

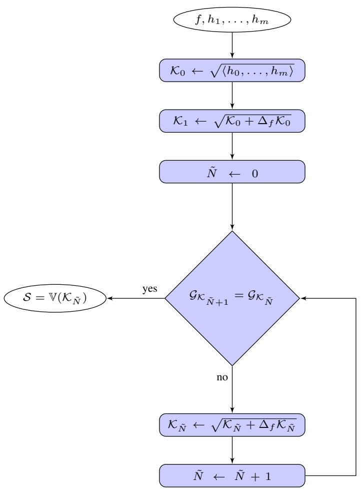

Fig. 6. Illustration of Algorithm 2.

Remark 4. *If the variety $\mathcal{S}$ is* rational *(*i.e.*, there exist $s \in \mathbb{Z}_{\geqslant 0}$ and rational mappings $\varpi : \mathcal{S} \dashrightarrow \mathbb{R}^s$, $\vartheta : \mathbb{R}^s \dashrightarrow \mathcal{S}$ such that $\vartheta \circ \varpi(x) = x$ for all $x \in \mathcal{S}$ and $\varpi \circ \vartheta(\sigma) = \sigma$ for all $\sigma \in \mathbb{R}^s$), then the restriction of the dynamics* (12) *to $\mathcal{S}$ can be easily determined. Namely, letting $s \in \mathbb{Z}_{\geqslant 0}$ be the dimension of $\mathcal{S}$, the dynamics of system* (12) *restricted to $\mathcal{S}$ are $\sigma(t+1) = \varpi \circ f \circ \vartheta \circ \sigma(t)$.*

V. Examples of application of the given methods

In this section, several examples are given in order to illustrate the application of the proposed methods.

The following example shows how Algorithm 1 can be used to determine the set of initial conditions that make the output of a linear system be identically zero.

Example 9. *Consider the linear system obtained either from* (3) *or from* (12) *by letting $f(x) = Ax$, with*

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Consider the linear function $h(x) = Cx$, where

$$C = [\ c_1 \quad c_2\ ].$$

Note that, if both f and h are linear, $f(x) = Ax$, $h(x) = Cx$, then $L_f h(x) = \Delta_f h(x) = CAx$, thus, in such a case, the outputs of Algorithms 1 and 2 are identical. By applying Algorithm 1 with inputs f and h, consider the ring $\mathbb{R}[a_1, a_2, c_1, c_2, x_1, x_2]$ and fix the lex order $>$, with $a_1 > a_2 > c_1 > c_2 > x_1 > x_2$. Hence, compute $\mathcal{K}_0 = \sqrt{\langle h \rangle}$, $\mathcal{K}_1 = \mathcal{K}_0 + L_{Ax}\mathcal{K}_0$ and $\mathcal{K}_2 = \mathcal{K}_1 + L_{Ax}\mathcal{K}_1$. The reduced Groebner basis of such ideals are:

$$\begin{aligned} \mathcal{G}_{\mathcal{K}_0} &= \{c_1 x_1 + c_2 x_2\}, \\ \mathcal{G}_{\mathcal{K}_1} &= \{(a_1 - a_2)c_2 x_2, c_1 x_1 + c_2 x_2\}, \\ \mathcal{G}_{\mathcal{K}_2} &= \{(a_1 - a_2)c_2 x_2, c_1 x_1 + c_2 x_2\}. \end{aligned}$$

Therefore, since $\mathcal{G}_{\mathcal{K}_2} = \mathcal{G}_{\mathcal{K}_1}$, the algorithm terminates with $\tilde{N} = 1$. One has the following cases:

(i) if $(a_1 - a_2)c_2 \neq 0$, $c_1 \neq 0$, then $\mathcal{S} = \{0\}$;
(ii) if $(a_1 - a_2)c_2 = 0$, $c_1 \neq 0$, then $\mathcal{S} = \mathbb{V}(c_1 x_1 + c_2 x_2)$;
(iii) if $(a_1 - a_2)c_2 \neq 0$, $c_1 = 0$, then $\mathcal{S} = \mathbb{V}(x_2)$;
(iv) if $(a_1 - a_2)c_2 = 0$, $c_1 = 0$, one has three cases:
  (iv.a) if $a_1 = a_2$, $c_2 \neq 0$, then $\mathcal{S} = \mathbb{V}(x_2)$;
  (iv.b) if $a_1 \neq a_2$, $c_2 = 0$, then $\mathcal{S} = \mathbb{R}^2$;
  (iv.c) if $a_1 = a_2$, $c_2 = 0$, then $\mathcal{S} = \mathbb{R}^2$.

The following example illustrates how the proposed methods can be used to study the stability of a mechanical system by using its energy as Lyapunov function.

Example 10. *Consider the mechanical system depicted in Fig. 7, whose dynamics are given by $\dot{x} = Ax$, where*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{m} & -\frac{d}{m} & \frac{k}{m} & \frac{d}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & \frac{d}{m} & -\frac{2k}{m} & -\frac{d}{m} \end{bmatrix},$$

$x(t) \in \mathbb{R}^4$ represents the positions and velocities of the two bodies having mass m, k is the stiffness of the springs and d is the damping coefficient of the damper.

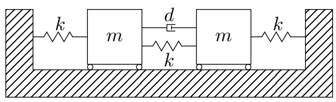

Fig. 7. A mechanical system.

Consider the candidate Lyapunov function

$$V = \tfrac{1}{2}(2k(x_1^2 - x_3 x_1 + x_3^2) + m(x_2^2 + x_4^2)),$$

which is the total energy of the system. It can be easily derived that $L_f V = -d(x_2 - x_4)^2 \leqslant 0$. Therefore, the origin of the considered system is stable (actually, the sub-level set $\{x \in \mathbb{R}^4 : V(x) \leqslant c\}$ is positively invariant for any $c \in \mathbb{R}_{\geqslant 0}$). By using Algorithm 1 with input $f(x) = Ax$ and $h(x) = L_f V(x)$, one obtains $\mathcal{K}_0 = \langle x_2 - x_4 \rangle$, $\mathcal{K}_1 = \langle x_2 - x_4, x_1 - x_3 \rangle$, and $\mathcal{K}_2 = \mathcal{K}_1$. Thus, the largest invariant set contained in $\mathbb{V}(L_f V)$ is $\mathcal{S} = \mathbb{V}(x_2 - x_4, x_1 - x_3)$. Thus, by Theorem 2, every solution of $\dot{x} = Ax$ approaches $\mathcal{S}$ as $t \to \infty$.

The methods given in this paper can be used even if in the largest f-invariant set contained in $\mathbb{V}(h_1, \dots, h_m)$ there is a limit cycle, as shown in the next example.

Example 11. *Consider system* (3) *with $f(x) = [\ -x_1(x_1^2 + x_2^2 - 1) \quad -x_2(x_1^2 + x_2^2 - 1)\ ]^\top$, which has been obtained by using [27, Alg. 1] and imposing $\mathbb{V}(x_1^2 + x_2^2 - 1)$ as limit cycle. Letting $V = (x_1^2 + x_2^2 - 1)^2$, note that $L_f V = -4(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \leqslant 0$, thus implying* *that sub-level sets of V are positively invariant. By using Algorithm 1 with input $f(x)$ and $h(x) = L_f V(x)$, one obtains $\mathcal{K}_0 = \langle (x_1^2+x_2^2-1)^2(x_1^2+x_2^2)\rangle$ and $\mathcal{K}_1 = \mathcal{K}_0$. Thus, the largest f-invariant set contained in $\mathbb{V}(L_f V)$ is*

$$\mathcal{S} := \mathbb{V}((x_1^2+x_2^2-1)^2(x_1^2+x_2^2)) = \mathbb{V}(x_1^2+x_2^2-1) \cup \{0\}.$$

The following example shows how the given tools can be used to study the behavior of an averaging algorithm.

Example 12. *Consider the equal-neighbor averaging model [46] $x(t+1) = Ax(t)$, with*

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Consider the candidate Lyapunov function

$$V = (x_1 - x_2)^2 + (x_1 - x_3)^2.$$

By using the "complete the squares" procedure [47], one obtains that $\Delta_f V - V$ can be written as

$$\Delta_f V - V = -\tfrac{31}{18}(x_1 - \tfrac{37}{62}x_2 - \tfrac{25}{62}x_3)^2 - \tfrac{89}{248}(x_2 - x_3)^2,$$

thus guaranteeing that the sub-level sets of V are positively invariant. By using Algorithm 2 with input $f(x) = Ax$ and $h(x) = \Delta_f V(x) - V(x)$, one obtains $\mathcal{K}_0 = \langle h \rangle$ and $\mathcal{K}_1 = \langle x_1 - x_3, x_2 - x_3\rangle = \mathcal{K}_2$. Thus, the largest positively invariant set with respect to $x(k+1) = Ax(k)$ contained in $\mathbb{V}(h)$ is $\mathcal{S} = \mathbb{V}(x_1 - x_2, x_2 - x_3)$. Hence, by Theorem 5, each bounded solution of the considered averaging algorithm tends to $\mathcal{S}$.

The following example shows how Algorithm 1 can be used, with minor modifications, to solve the output zeroing problem and to determine the zero dynamics of a nonlinear controlled system.

Example 13. *Consider the system given in Example 6.1.6 of [42], $\dot{x} = f(x) + g(x)u$, $y = \eta(x)$, where*

$$f(x) = \begin{bmatrix} x_2 \\ x_4 \\ x_1 x_4 \\ x_5 \\ x_3 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 1 & 0 \\ x_3 & x_2 \\ 0 & 1 \\ x_5 & x_2 \\ 1 & 1 \end{bmatrix}, \quad \eta(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$u(t) = [\ u_1(t) \quad u_2(t)\ ]^\top \in \mathbb{R}^2$ and $y(t) \in \mathbb{R}^2$ are the control and output vectors, respectively. The objective of this example is to determine the set of initial conditions and control inputs that make the output of the system be identically zero. By the same reasoning used to prove Lemma 1, one has that $y(t) = 0$ for all admissible $t \in \mathbb{R}$ if and only if all of its time derivatives vanish. Thus, by computing the time derivative of the output up to the third order, one obtains a set of polynomials in the state of the system and the time derivatives of the input, up to the second order. By computing the reduced Gröbner basis of the corresponding ideal, one obtains

$$\mathcal{G}_{\mathcal{I}} = \{u_{2,1}, u_{1,1}, u_{0,1}, x_5, x_4, u_{0,2} + x_3, x_2, x_1\},$$

where $u_{i,j} = \frac{d^i}{dt^i}u_j$, $i = 0, 1, 2$, $j = 1, 2$. Note that, by equating the elements of $\mathcal{G}$ to zero, one obtains set of Differential Algebraic Equations that can be solved by using any Computer Algebra System (see, e.g., Algorithm 6 of [48]). In particular, the solution to $\mathcal{G}_{\mathcal{I}} = 0$ is

$$u^\star(x) = \begin{bmatrix} 0 \\ -x_3 \end{bmatrix}, \quad x_1 = x_2 = x_4 = x_5 = 0. \qquad (22)$$

Thus, by letting $f^\star(x) = f(x) + g(x)u^\star(x)$ and letting $h(x) = [\ x_1 \quad x_2 \quad x_4 \quad x_5\ ]^\top$, and by using Algorithm 1 with input $f^\star$ and h, one obtains that the set $\mathcal{S} = \mathbb{V}(h)$ is the largest invariant set with respect to $\dot{x} = f^\star(x)$ contained in $\mathbb{V}(h)$. Therefore, the control input $u^\star$ and the set $\mathcal{S}$ solve the output zeroing problem. Thus, the method given in Remark 3 can be used to determine the zero dynamics of the system. In fact, $T_{\bar{x}}(\mathcal{S}) = \mathbb{V}(x_1, x_2, x_4, x_5)$ for each $\bar{x} \in \mathcal{S}$ and the restriction of $f^*(x)$ to $\mathcal{S}$ is

$$f^*(x) = [\ 0 \quad 0 \quad -x_3 \quad 0 \quad 0\ ]^\top,$$

whence the zero dynamics of the system are $\dot{x}_3 = -x_3$. Note that the same result has been obtained in [42], by using different arguments.

It is worth noticing that the procedure outlined in Example 13 does not require neither that the system is input affine nor that the number of inputs is equal to the number of outputs, as shown in the following example.

Example 14. *Consider the plant $\dot{x} = f(x, u)$, $y = \eta(x)$,*

$$f(x, u) = \begin{bmatrix} x_2 - u^3 \\ -u - x_1 \\ x_1 - x_3^3 \end{bmatrix}, \quad \eta(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

By computing the time derivatives of the output, the reduced Gröbner basis of the corresponding ideal is

$$\mathcal{G}_{\mathcal{I}} = \{u_0, x_2, x_1\}.$$

Thus, by letting $f^\star(x) = f(x, 0)$ and $h(x) = \eta(x)$, one obtains that the largest invariant set with respect to the system $\dot{x} = f^\star(x)$ contained in $\mathbb{V}(h)$ is $\mathcal{S} = \mathbb{V}(h)$. Therefore, $u^\star(x) = 0$ and $\mathcal{S}$ solve the output zeroing problem. The method given in Remark 3 can be used to determine the zero dynamics of the system. In fact, one has that $T_{\bar{x}}(\mathcal{S}) = \mathbb{V}(x_1, x_2)$ for each $\bar{x} \in \mathcal{S}$ and the restriction of the vector field $f^*(x)$ to $\mathcal{S}$ is $[\ 0 \quad 0 \quad -x_3^3\ ]^\top$. Thus, the zero dynamics of the system are $\dot{x}_3 = -x_3^3$.

The following example shows how Algorithm 2 can be used to characterize the observability of system (12) from the output. Consider the discrete-time system

$$x(t+1) = f(x(t)), \qquad y(t) = h(x(t)). \qquad (23)$$

System (23) is *zero-state observable* if $y(t) = 0$, $t \in \mathbb{Z}_{\geqslant 0}$, implies $x(t) = 0$, $k \in \mathbb{Z}_{\geqslant 0}$ [44].

Example 15. *Consider system (23), with*

$$f(x) = \begin{bmatrix} x_2 + x_4^2 \\ x_1 - 2x_2^2 \\ x_2 + x_3 \\ x_1 + x_3 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

By using Algorithm 2 with input f and h, one obtains that $\mathcal{K}_0 = \langle x_1, x_2\rangle$, $\mathcal{K}_1 = \langle x_1, x_2, x_4\rangle$, $\mathcal{K}_2 = \langle x_1, x_2, x_3, x_4\rangle$, *and* $\mathcal{K}_3 = \mathcal{K}_2$*. Thus, the algorithm returns as output* $\mathcal{S} = \mathbb{V}(\mathcal{K}_2) = \{0\}$, i.e.*, the largest positively* f*-invariant set such that* $y(t) = 0$ *for all* $t \in \mathbb{Z}_{\geqslant 0}$ *is* $\{0\}$*, whence the system is zero-state observable.*

VI. Conclusions

In this paper, some computational tools have been proposed to determine the largest invariant set contained in an analytic set, in the case of both continuous-time and discrete-time nonlinear systems. In particular, it has been shown that, if the vector field governing the dynamics of the system is analytic, such a set can be computed by determining the set of all the solutions of a system of (possibly, infinite) analytic equalities. On the other hand, if the vector field is polynomial and the algebraic set is a variety, then this set can be determined with a finite number of steps. Several applications to control problems have been reported. In fact, as shown in Section V, the techniques proposed in this paper can be used to determine the set of initial conditions that make the output of a linear system be identically zero, to study the stability of dynamical systems, to characterize the behavior of averaging algorithms, to solve the output zeroing problem, to determine the zero dynamics of a nonlinear controlled system, and to characterize the zero-state observability of autonomous nonlinear systems.

Future developments of the techniques given in this paper will deal with the extension of the proposed algorithms to the hybrid case, *i.e.*, to systems presenting both continuous-time and discrete-time dynamics.

Note that Algorithms 1 and 2 are computationally tractable in many cases of practical interest. In fact, although computing Gröbner bases is, in the worst case, an EXPSPACE–complete problem [49], in the generic case, Gröbner bases can be computed efficiently by the most recent algorithms and hence the algorithms given herein can be actually employed (see [50], [51] for further details).

Therefore, although the main limitation of the proposed approach is the computation of Gröbner bases of the ideals $\mathcal{K}_1, \ldots, \mathcal{K}_N$, it may be used to solve problems of practical interest. For instance, an implementation of Algorithm 1 in `Macaulay2` has been used to solve Problem 1 with f and h being polynomials in 10 variables of degree 6 within reasonable computing time.

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