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The groupoid of the Triangular Numbers and the generation of related integer sequences

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Here we discuss the binary operators of the set made by the triangular numbers, sequence A000217, in the On-Line Encyclopedia of Integer Sequences (OEIS). As we will see, by means of these binary operators we can obtain related integer sequences. Here we propose some of them. The sequences, except one, are given in OEIS.

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In [1], we find defined the triangular numbers as those which are counting dots arranged in equilateral triangles. Then, the n-th triangular number is the number of dots in the triangle with n dots on a side. It is equal to the sum of the natural numbers from 1 to n:

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The triangular numbers are forming the sequence A000217 in OEIS, the On-Line Encyclopedia of Integer Sequences [2,3].
Some properties of triangular numbers are given in [1] and [4]. One of the properties that we find in [1] is:

$$T_{n+m} = T_n + T_m + nm \quad (1)$$

Actually, we have another manner to write T_{n+m} , if we consider OEIS A000217 as a groupoid.
A groupoid is an algebraic structure made by a set with a binary operator [5]. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this operation is associative and we have a neutral element and opposite elements into the set, then the groupoid becomes a group. So, let us consider OEIS A000217 numbers and find binary operators between them.

As we did in some previous discussions (see for instance [6]), we can find a binary operator, which is satisfying the closure. We can follow the same approach as in [7-10].
We have:

$$2T_n = n^2 + n = (n+1)^2 - (n+1)$$

Let us use numbers A_n:, so that: $A_n = (n+1)$. Then:

$$A_{n+m} = (n+1) + (m+1) - 1$$

So we can define a binary operation such as: $A_{n+m} = A_n \oplus A_m = A_n + A_m - 1$.

Moreover, we have that: $2T_n = A_n^2 - A_n$; $A_n = \frac{1}{2} \pm \frac{1}{2}(1+8T_n)^{1/2}$ (2)

Let us consider in (2) the positive sign:

$$A_{n+m} = A_n + A_m - 1 = \frac{1}{2}(1+8T_n)^{1/2} + \frac{1}{2}(1+8T_m)^{1/2}$$

$$A_{n+m} = \frac{1}{2} + \frac{1}{2}(1+8T_{n+m})^{1/2}$$

So we have:

$$(1+8T_{n+m}) = [-1 + (1+8T_n)^{1/2} + (1+8T_m)^{1/2}]^2 =$$

$$(1+8T_n) + (1+8T_m) + 1 - 2(1+8T_n)^{1/2} - 2(1+8T_m)^{1/2} + 2(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

Then:

$$T_{n+m} = T_n + T_m + \frac{1}{4} - \frac{1}{4}(1+8S_n)^{1/2} - \frac{1}{4}(1+8T_m)^{1/2} + \frac{1}{4}(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

The binary operator, that is, the generalized sum for the triangular numbers is given as:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}[1 - (1+8T_n)^{1/2} - (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}] \quad (3)$$

Using (3) and (1), we have the following identity:

$$4nm = 1-(1+8T_n)^{1/2}-(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}$$

From the generalized sum (3), we have the recursive relation: $T_{n+1}=T_n \oplus T_1$.

Starting from number $T_1=1$, the generated sequence is 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on.
The recursive relation can be written, in this case with $T_1=1$, as:

$$T_{n+1}=T_n+1+\frac{1}{4}[-2-(1+8T_n)^{1/2}+3(1+8T_n)^{1/2}]$$

$$T_{n+1}=T_n+1+\frac{1}{2}[-1+(1+8T_n)^{1/2}]$$

Moreover, we have that $(1+8T_n)^{1/2}$ is the sequence of the odd numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, and so on.

Let us consider again (3), that is:

$$T_n \oplus T_m = T_n+T_m+\frac{1}{4}[1-(1+8T_n)^{1/2}-(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}]$$

in the form $T_{n+1}=T_n \oplus T_1$, but here we change the values of T_1 . Here in the following the sequences that we generate.

$T_1=0$, sequence 0, 0, 0, 0, 0, 0, … .

$T_1=1$, sequence 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, and so on. And this is OEIS A000217, the sequence of triangular numbers.

$T_1=3$, sequence 10, 21, 36, 55, 78, 105, 136, 171, 210, 253, 300, 351, 406, 465, 528, 595, 666, 741, 820, 903, … . Searching this sequence in OEIS, we can easily find that it is A014105, that is, the Second Hexagonal Numbers: $H_n=n(2n+1)$.

$T_1=4$, sequence 12, 24, 40, 60, 84, 112, 144, 180, 220, 264, 312, 364, 420, 480, 544, 612, 684, 760, 840, 924, … OEIS A046092 (four times triangular numbers).

$T_1=6$, sequence 21, 45, 78, 120, 171, 231, 300, 378, 465, 561, 666, 780, 903, 1035, 1176, 1326, 1485, 1653, 1830, 2016, … OEIS A081266 (Staggered diagonal of triangular spiral in

A051682).

$T_1=7$, sequence 23, 48, 82, 125, 177, 238, 308, 387, 475, 572, 678, 793, 917, 1050, 1192, 1343, 1503, 1672, 1850, 2037, … OEIS A062725.

$T_1=10$, sequence 36, 78, 136, 210, 300, 406, 528, 666, 820, 990, 1176, 1378, 1596, 1830, 2080, 2346, 2628, 2926, 3240, 3570, … OEIS A033585, that is, numbers: $2n(4n+1)$.

$T_1=11$, sequence 38, 81, 140, 215, 306, 413, 536, 675, 830, 1001, 1188, 1391, 1610, 1845, 2096, 2363, 2646, 2945, 3260, 3591, … OEIS A139276, that is, numbers $n(8n+3)$.

Of course, we can continue and obtain further sequences.

Let us remember that, in (2), we can consider the negative sign too. Then we have another binary operation:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}[1+(1+8T_n)^{1/2}+(1+8T_m)^{1/2}+(1+8T_n)^{1/2}(1+8T_m)^{1/2}]$$

Again, let us consider $T_{n+1}=T_n \oplus T_1$ as we did before.

$T_1=0$, sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, and so on. OEIS A000217, the sequence of triangular numbers.

$T_1=1$, sequence 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, … OEIS A000384, Hexagonal numbers $H_n=n(2n-1)$.

$T_1=3$, sequence 15, 36, 66, 105, 153, 210, 276, 351, 435, 528, 630, 741, 861, 990, 1128, 1275, 1431, 1596, 1770, 1953, … OEIS A062741, three times pentagonal numbers $3n(3n-1)/2$.

$T_1=4$, sequence 17, 39, 70, 110, 159, 217, 284, 360, 445, 539, 642, 754, 875, 1005, 1144, 1292, 1449, 1615, 1790, 1974, … OEIS A022266, numbers $n(9n-1)/2$.

$T_1=6$, sequence 28, 66, 120, 190, 276, 378, 496, 630, 780, 946, 1128, 1326, 1540, 1770, 2016, 2278, 2556, 2850, 3160, 3486, … OEIS A014635, numbers $2n(4n-1)$.

$T_1=7$, sequence 30, 69, 124, 195, 282, 385, 504, 639, 790, 957, 1140, 1339, 1554, 1785, 2032, 2295, 2574, 2869, 3180, 3507, … OEIS A139274, numbers $n(8n-1)$.

$T_1 = 10$, sequence 45, 105, 190, 300, 435, 595, 780, 990, 1225, 1485, 1770, 2080, 2415, 2775, 3160, 3570, 4005, 4465, 4950, 5460 …. This sequence is not present in OEIS.

$T_1 = 11$, sequence 47, 108, 194, 305, 441, 602, 788, 999, 1235, 1496, 1782, 2093, 2429, 2790, 3176, 3587, 4023, 4484, 4970, 5481, …. OEIS A178572, numbers with ordered partitions that have periods of length 5.

Of course, the approach here proposed can used for the generation of further integer sequences, using the binary operators given in the previous works [6-10]. It is possible that, among the generated sequences, news sequences are produced too.

References

[1] https://en.wikipedia.org/wiki/Triangular_number
[2] Sloane, N. J. A. Sequence A000217 in "The On-Line Encyclopedia of Integer Sequences."
[3] https://oeis.org/A000217
[4] Weisstein, Eric W. "Triangular Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html
[5] Stover, Christopher and Weisstein, Eric W. "Groupoid." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Groupoid.html
[6] Sparavigna, Amelia Carolina (2019). Composition Operations of Generalized Entropies Applied to the Study of Numbers, International Journal of Sciences 8(04): 87-92. DOI: 10.18483/ijSci.2044
[7] Sparavigna, Amelia Carolina. (2019, June 6). Binary Operators of the Groupoids of OEIS A093112 and A093069 Numbers (Carol and Kynea Numbers). Zenodo. http://doi.org/10.5281/zenodo.3240465
[8] Sparavigna, Amelia Carolina. (2019, June 16). Groupoids of OEIS A002378 and A016754 Numbers (oblong and odd square numbers). Zenodo. http://doi.org/10.5281/zenodo.3247003
[9] Sparavigna, Amelia Carolina. (2019, June 22). Groupoid of OEIS A001844 Numbers (centered square numbers). Zenodo. http://doi.org/10.5281/zenodo.3252339
[10] Sparavigna, Amelia Carolina. (2019, September 5). Groupoid of OEIS A003154 Numbers (star numbers or centered dodecagonal numbers). Zenodo. http://doi.org/10.5281/zenodo.3387054