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Observers for linear systems by the time-integrals and moving average of the output

Laura Menini, Corrado Possieri and Antonio Tornambè

***Abstract*—In this paper, it is shown that, under some mild assumptions, it is possible to design observers for linear, time-invariant, continuous-time and discrete-time systems by feeding classical linear observers (as, *e.g.*, the Kalman filters and the Luenberger observer) with the successive integrals and the moving average of the measured output, respectively. The main interest in these observers relies on the fact that both the integral and the moving average exhibit low-pass behaviors, thus allowing the design of observers that are less sensitive to high-frequency noise. Examples are reported all throughout the paper to corroborate the theoretical results and to highlight the improved filtering properties of the proposed observers.**

***Index Terms*—Observer design, linear systems.**

I. Introduction

The problem of estimating the current state of a dynamic system from the inputs and outputs measurements is crucial in several applications, as, *e.g.*, the design of output feedback controllers. Its roots go back to the introduction of state-space approaches for the design of controllers and for the analysis of linear, time-invariant plants [1]. The birth of the theory of deterministic state observers can be traced back to [2], [3], which outlined how the inputs and outputs of a continuous-time plant may be used to construct an estimate of the system state. Since those seminal papers on continuous-time systems, a lot of works have extended the deterministic observation theory to discrete-time [4], [5], time-varying [6], [7], and, more recently, hybrid linear systems [8] (see [9] for a survey).

The approach classically used to design a state observer for linear systems is to build a replica of the plant and to feedback the difference between the outputs of the plant and of the replica through a linear gain [10]. This has been proved successful in several applications [11], [12], [13]; it allows the independent design of the observer and of the state feedback, leading to stabilizing output feedback controllers [14].

When dealing with linear plants, whose dynamics and outputs are affected by noise, one of the most used approaches to determine an estimate of the current state of the system is the Kalman filter [15], [16], [17], [18], which was first presented and developed in [19], [20], [21]. For time-invariant, linear systems, the Kalman filter essentially operates by feeding back the output error to a replica of the plant through a linear gain, to be found by solving a Riccati equation depending on the covariance matrices of the noises [22], [23]. For linear systems, affected by Gaussian, independent, zero-mean noises, with known distributions, the Kalman filter minimizes the mean square of the estimation error [24].

Since the early works of Kalman [20] and Luenberger [2], several attempts have been made to improve the convergence properties of the observer and to reject noises. For instance, in [25], Utkin proposed an observer for continuous-time linear system that achieves finite time convergence to zero of the output error by feeding back to the system replica the estimation error via a discontinuous switched signal. Other examples of observers for linear systems, largely used in practical applications [26], are the unknown input observers, first introduced in [27]; a technique to design a linear unknown input observer for continuous-time linear systems has been proposed in [28], whereas, in [29], a $\mathcal{H}_\infty$ approach is proposed for robust estimation of unknown inputs and state variables. On the other hand, in [30], [31], parameter-dependent approaches are given for robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering of uncertain systems.

Due to the large impact of observation theory on the fields of estimation and control, state observers for linear systems are still being studied largely [32], [33].

One of the main objectives of this paper is to characterize the integrals (for continuous-time systems) and the moving average (for discrete-time systems) of the output of a linear plant. The interest on these filters relies on the fact that they are perhaps the most simple low-pass numerical filter used in practical application to reduce the effects of high-frequency noise on the output. It turns out that, under some mild assumptions, the output of such filters preserves the observability properties of the system and that it can be expressed as a linear function of the state response of the system and of the successive time-integrals of the input, thus allowing, in principle, the coupling of the tools given in this paper with any other technique able to design observers for linear systems (the Kalman filters and the Luenberger observers are mainly preferred here due to their simplicity). The advantage of the use of such tools is that they improve the performance of other estimation techniques by reducing the effects of high-frequency additive measurement noise without affecting the observability properties of the system, as confirmed by several simulations reported all throughout the paper.

The paper is organized as follows. In Section II, the notation

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used in this paper is introduced and a hint on the main idea pursued to design observers is given. In Section III, some properties of the time-integrals of the output of a linear continuous-time system are derived. Such results are used in Section IV to design linear observers for continuous-time systems based on the time-integrals of the output. Following the same scheme, some properties of the moving average of the output of a discrete-time system are derived in Section V and used in Section VI to design observers for discrete-time systems. The case of detectable but not observable systems is dealt with in Section VII. Finally, conclusions are drawn in Section VIII.

II. Notation and problem statement

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{R}^{>0}$, $\mathbb{Z}^{>0}$, $\mathbb{R}^{\geq 0}$, and $\mathbb{Z}^{\geq 0}$ denote the sets of real, integer, positive real, positive integer, non-negative real, and non-negative integer numbers, respectively. The symbol $\mathrm{block\_diag}(A_1,\dots,A_n)$ denotes the block diagonal matrix, whose blocks are $A_1,\dots,A_n$. Given $A\in\mathbb{R}^{n\times n}$, $\sigma(A)$ denotes the *spectrum* of the matrix A, *i.e.*, the set of its eigenvalues. Given $M\in\mathbb{R}^{m\times n}$, with $m\geq n$ and such that $\mathrm{rank}(M)=n$, let $M^\dagger=(M^\top M)^{-1}M^\top$ denote the *Moore-Penrose pseudo-inverse* of M. Let $\mathbb{E}\{\cdot\}$ be the *expected value* of the random variable at argument. Given a *multi-index* $\ell=[\,\ell_1\ \cdots\ \ell_n\,]^\top$, $\ell_i\in\mathbb{Z}^{\geq 0}$, $i=1,\dots,n$, let $|\ell|:=\sum_{i=1}^n\ell_i$ and $\ell!:=\ell_1!\cdots\ell_n!$ be the *length* and the *factorial* of ℓ, respectively. Given $x=[\,x_1\ \cdots\ x_n\,]^\top$, define accordingly $x^\ell:=x_1^{\ell_1}\cdots x_n^{\ell_n}$ and $\frac{\partial^\ell}{\partial x^\ell}:=\frac{\partial^{|\ell|}}{\partial x_1^{\ell_1}\cdots x_n^{\ell_n}}$. Let $\mathbb{R}[x]$ be the set of all the polynomials in x with coefficients in $\mathbb{R}$, and let $\mathbb{R}^\infty[[x]]$ be the set of all the scalar functions $h:\mathbb{R}^n\to\mathbb{R}$ such that there exists an open (possibly, small) subset $\mathcal{U}$ of $\mathbb{R}^n$ containing $x=0$ such that h is $\mathcal{C}^\infty$ at each $x\in\mathcal{U}$; each function h in $\mathbb{R}^\infty[[x]]$ is referred to as *smooth*. The *formal power Taylor series* of $h\in\mathbb{R}^\infty[[x]]$ is

$$H(x)=\sum_{|\ell|=0}^{+\infty}\frac{1}{\ell!}\left.\frac{\partial^\ell h(x)}{\partial x^\ell}\right|_{x=0}x^\ell. \qquad (1)$$

Let $\mathbb{R}^\omega[[x]]$ be the set of all the $h\in\mathbb{R}^\infty[[x]]$ that are *analytic* at $x=0$, *i.e.*, such that, for each x in a neighborhood $\mathcal{B}\subset\mathcal{U}$ of $x=0$, the series given in (1) converges and $h(x)=H(x)$, for each $x\in\mathcal{B}$. Let $\mathbb{R}_0^\omega[[x]]$ be the set of all the functions h in $\mathbb{R}^\omega[[x]]$ that vanish at $x=0$ and $\mathbb{R}_0^\omega[[x]]^p$ be the set of all the p-dimensional vector functions whose entries are in $\mathbb{R}_0^\omega[[x]]$. For $h\in\mathbb{R}_0^\omega[[x]]^p$, let $\frac{\partial h(x)}{\partial x}$ denote the Jacobian matrix of h.

Consider the following linear and time-invariant system:

$$\Delta x(t)=A\,x(t)+B\,u(t), \qquad (2a)$$
$$y(t)=C\,x(t)+D\,u(t), \qquad (2b)$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and $y\in\mathbb{R}^p$ are the state, input and output vectors, respectively, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$, and $t\in\mathbb{T}$, with $\mathbb{T}$ being either $\mathbb{R}^{\geq 0}$ or $\mathbb{Z}^{\geq 0}$. If $\mathbb{T}=\mathbb{R}^{\geq 0}$, then the input $u(t)$ is assumed to be a function of t being continuously differentiable a sufficiently large number of times. If $\mathbb{T}=\mathbb{R}^{\geq 0}$, then the symbol $\Delta x(t)$ denotes the time-derivative of the state $x(t)$, *i.e.*, $\Delta x(t)=\dot{x}(t)$, whereas, if $\mathbb{T}=\mathbb{Z}^{\geq 0}$, then the symbol $\Delta x(t)$ denotes the one-step forward-shifted value of $x(t)$, *i.e.*, $\Delta x(t)=x(t+1)$.

Given a pair (C,A) and an integer $N\geq 1$,

$$O_N(C,A):=[\ C^\top\ \ \dots\ \ (A^\top)^{N-1}C^\top\ ]^\top$$

denote the *observability matrix* of order N of pair (C,A). Note that, by [34], system (2) is observable if and only if $\mathrm{rank}(O_n(C,A))=n$ for both $\mathbb{T}=\mathbb{R}^{\geq 0}$ and $\mathbb{T}=\mathbb{Z}^{\geq 0}$.

A. Rationale of the proposed observer design technique

The main objective of this paper is to design state observers for system (2), based on either some successive time-integrals of the output-response $y(t)$, if $\mathbb{T}=\mathbb{R}^{\geq 0}$, or some successive moving averages of the output-response $y(t)$, if $\mathbb{T}=\mathbb{Z}^{\geq 0}$.

The interest in this class of observers for $\mathbb{T}=\mathbb{R}^{\geq 0}$ relies on the following observation: if the output measurements are affected by a high-frequency noise, *i.e.*, the measured output is $y(t)+A\cos(\omega t)$, $\omega\gg 1$, then the integral of such a signal is $\int_0^t y(\gamma)\,\mathrm{d}\gamma-\frac{A}{\omega}\sin(\omega t)$, whence it is affected by a disturbance with smaller amplitude. Therefore, it may be convenient, in practical applications, to design an observer that considers the integral of the output-response rather than the output-response itself. As shown in the subsequent Section IV, such an observer requires, apart from the integrals of the input and output of the system, additional constants c that depend on the initial condition (a method to estimate such constants is given in the subsequent Sub-section IV-A). Hence, the proposed observer for $\mathbb{T}=\mathbb{R}^{\geq 0}$ has the structure depicted in Fig. 1.

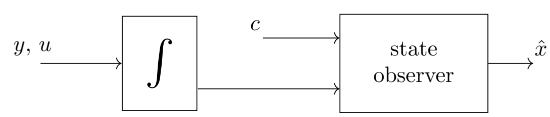


Fig. 1. Conceptual model of the proposed observer for $\mathbb{T}=\mathbb{R}^{\geq 0}$.

A similar reasoning holds for $\mathbb{T}=\mathbb{Z}^{\geq 0}$ since the moving average allows to efficiently filter additive disturbances on the output (see the subsequent Section VI).

III. Derivatives and integrals of the output of a continuous-time linear system

Throughout this section, it is assumed that $\mathbb{T}=\mathbb{R}^{\geq 0}$, so that the dynamics given in (2) read as follows:

$$\dot{x}(t)=A\,x(t)+B\,u(t), \qquad (3a)$$
$$y(t)=C\,x(t)+D\,u(t). \qquad (3b)$$

The objective of this section is to characterize the successive time-derivatives and time-integrals of the output-response $y(t)$ as functions of the state-response $x(t)$ of system (3) and of the time-derivatives and time-integrals of the input u.

A. Time-derivatives of the output-response

Let $y_i(t):=\frac{d^i}{dt^i}y(t)$ denote the i-th time-derivative of the output-response $y(t)$ of system (3), $i\in\mathbb{Z}^{\geq 0}$. The following proposition can be easily proved by the analysis carried out in Section 5.1.2 of [34], and gives a closed-form expression for the time-derivatives of the output-response $y(t)$ of system (3).

Proposition 1. *Let $x(t)$ and $y_k(t)$ be the state-response and the k-th time-derivative of the output-response $y(t)$ of system* (3)*, respectively; for $k \in \mathbb{Z}^{\geq 0}$ and $t \in \mathbb{R}^{\geq 0}$, one has*

$$y_k(t) = CA^k\, x(t) + D\, u_k(t) + \textstyle\sum_{i=0}^{k-1} CA^{k-i-1} B\, u_i(t), \quad (4)$$

where $u_i(t) := \frac{d^i}{dt^i} u(t)$, $i = 0, \ldots, k$.

B. Time-integrals of the output-response

Let system (3) be given, let $y_0(t) := y(t)$ (as in the previous sub-section) and let

$$y_{-(i+1)}(t) := \textstyle\int_0^t y_{-i}(\gamma)\, \mathrm{d}\gamma \quad (5)$$

be the $(i+1)$-th time-integral of the output-response $y(t)$ of system (3), $i \in \mathbb{Z}^{\geq 0}$. Note that $y_{-i}(0) = 0$, for $i \in \mathbb{Z}$, $i \geq 1$.

Proposition 2. *Assume* $\det(A) \neq 0$*; let $x(t)$ and $y_{-k}(t)$ be the state-response and the k-th time-integral of the output-response $y(t)$ of system* (3)*, respectively. For each $k \in \mathbb{Z}$, $k \geq 1$, and $t \in \mathbb{R}^{\geq 0}$, one has*

$$\begin{aligned} y_{-k}(t) &= CA^{-k}\, x(t) + D\, u_{-k}(t) \\ &\quad - \textstyle\sum_{i=1}^{k} CA^{i-k-1} B\, u_{-i}(t) + \sum_{j=1}^{k} \frac{c_{-j}}{(k-j)!}\, t^{k-j}, \end{aligned} \quad (6)$$

where $u_{-(i+1)}(t) := \int_0^t u_{-i}(\gamma)\, \mathrm{d}\gamma$, $i = 0, \ldots, k-1$, and $c_{-1}, \ldots, c_{-k}$ are constant vectors depending on $x(0) = x^o$.

Proof. By definition, (6) holds for $k = 0$. Now, assume that (6) holds for some $k \in \mathbb{Z}^{\geq 0}$. Thus, since A is invertible, one has that $x(t) = A^{-1}\, \dot{x}(t) - A^{-1} B\, u(t)$, whence

$$\begin{aligned} y_{-(k+1)}(t) &= \textstyle\int_0^t CA^{-k} x(\gamma)\, \mathrm{d}\gamma + D\, u_{-k-1}(t) \\ &\quad - \textstyle\sum_{i=2}^{k+1} CA^{i-k-2} B\, u_{-i}(t) + \sum_{j=1}^{k} \frac{c_{-j}}{(k-j+1)!}\, t^{k-j+1} \\ &= CA^{-k-1}\, x(t) - CA^{-k-1}\, x(0) + D\, u_{-k-1}(t) \\ &\quad - \textstyle\sum_{i=1}^{k+1} CA^{i-k-2} B\, u_{-i}(t) + \sum_{j=1}^{k} \frac{c_{-j}}{(k-j+1)!}\, t^{k-j+1}. \end{aligned}$$

Letting $c_{-k-1} = -CA^{-k-1}\, x(0)$, (6) holds by induction. □

Given $h \in \mathbb{R}_0^\omega[[x]]^p$, let $L_{Ax} h(x) := \frac{\partial h(x)}{\partial x} Ax$ be the *directional derivative* of h along the vector field Ax, $L_{Ax} h(x) \in \mathbb{R}_0^\omega[[x]]^p$; such a notation can be iterated as $L_{Ax}^{i+1} h(x) := L_{Ax}(L_{Ax}^i h(x))$, $i \in \mathbb{Z}^{\geq 0}$. A *directional integral* of $h \in \mathbb{R}_0^\omega[[x]]^p$ along the vector field Ax is, if any, a vector function $k \in \mathbb{R}^\omega[[x]]^p$ such that $L_{Ax} k(x) = h(x)$. Since $L_{Ax} k(x) = h(x)$ implies $L_{Ax}(k(x) + b) = h(x)$, for any constant vector b, function $k(x)$ can be chosen so that $k(0) = 0$, *i.e.*, $k \in \mathbb{R}_0^\omega[[x]]^p$. It is worth pointing out that the directional derivative of h along the vector field Ax always exists and is uniquely defined, whereas, although the time-integral of $y(t)$ given in (5) always exists and is uniquely defined, a directional integral of h along the vector field Ax need not either exist or be uniquely defined. As shown later in this section, the existence of a directional integral of h along the vector field Ax and its uniqueness in the class of linear functions is guaranteed by the condition $\det(A) \neq 0$. In case of existence of a function $k(x)$ such that $L_{Ax} k(x) = h(x)$, since $y(t) = h(x(t))$, where $x(t)$ and $y(t)$ are the state-response and the output-response of the considered system, respectively, one has that $k(x(t)) - \int_0^t y(\gamma)\, \mathrm{d}\gamma$ is a constant, which need not be zero.

The following lemma gives some properties of the directional integrals of a scalar function along the vector field Ax.

Lemma 1. *Let $A \in \mathbb{R}^{n \times n}$ and $h, \omega, k_1, k_2 \in \mathbb{R}_0^\omega[[x]]$.*

(1.1) If $L_{Ax}\omega(x) = 0$, then $L_{Ax} b(\omega(x)) = 0$, $\forall b \in \mathbb{R}_0^\omega[[\omega]]$.

(1.2) If $L_{Ax}\omega(x) = 0$ and $L_{Ax} k_1(x) = h(x)$, then $L_{Ax}(k_1(x) + \omega(x)) = h(x)$.

(1.3) If $L_{Ax} k_1(x) = h(x)$ and $L_{Ax} k_2(x) = h(x)$, then $L_{Ax}(k_1(x) - k_2(x)) = 0$.

Proof. Statement (1.1) follows from $L_{Ax} b(\omega(x)) = \frac{\partial b(\omega)}{\partial \omega}\big|_{\omega=\omega(x)} L_{Ax}\omega(x)$, which implies that if $L_{Ax}\omega(x) = 0$, then $L_{Ax} b(\omega(x)) = 0$. Statements (1.2) and (1.3) follow from the linearity of the directional derivative along Ax. □

If $\omega(x) \in \mathbb{R}^\omega[[x]]$ satisfies $L_{Ax}\omega(x) = 0$, then it is an *analytic first integral* associated with Ax; any scalar constant is a (trivial) analytic first integral associated with Ax, for any A.

Corollary 1. *Let $A \in \mathbb{R}^{n \times n}$ and $h \in \mathbb{R}_0^\omega[[x]]$; k is the* unique *directional integral in $\mathbb{R}_0^\omega[[x]]$ of $h(x)$ along the vector field Ax (*i.e.*, k is the* unique *solution in $\mathbb{R}^\omega[[x]]$ of $L_{Ax} k(x) = h(x)$ satisfying $k(0) = 0$) only if there is no non-constant analytic first integral associated with the vector field Ax.*

Some polynomials $\omega_1, \ldots, \omega_\upsilon \in \mathbb{R}[x]$ are *algebraically independent* if there is no non-zero polynomial $p \in \mathbb{R}[z_1, \ldots, z_\upsilon]$, where $z_1, \ldots, z_\upsilon$ are scalar variables, such that $p(\omega_1(x), \ldots, \omega_\upsilon(x))$ is the zero polynomial in $\mathbb{R}[x]$. Given a *monomial order* in $\mathbb{R}[x]$, a polynomial $p \in \mathbb{R}[x]$ is *monic* if the coefficient of the leading monomial of p is equal to 1.

Theorem 1 (see Lemma 3.3 of [35])**.** *Let a monomial order be fixed. For any $A \in \mathbb{R}^{n \times n}$, there is a finite number $\upsilon \in \mathbb{Z}$, $\upsilon \geq 1$, of algebraically independent monic polynomials $\omega_1, \ldots, \omega_\upsilon \in \mathbb{R}[x]$ such that each analytic first integral $\omega(x) \in \mathbb{R}^\omega[[x]]$ associated with the vector field Ax can be expressed as $\omega(x) = b(\omega_1(x), \ldots, \omega_\upsilon(x))$, where $b \in \mathbb{R}^\omega[[z_1, \ldots, z_\upsilon]]$.*

If the only analytic first integrals associated with the vector field Ax are the constants, then $\upsilon = 1$ and $\omega_1(x) = 1$.

In the remainder of this section, without loss of generality (since the observability of linear systems is not affected by the input [36]), only the unforced case is considered for simplicity.

A polynomial $p_\ell \in \mathbb{K}[x]$ is homogeneous of degree ℓ with respect to the *standard dilation* [37] if all the monomials of p_ℓ have the same degree ℓ; given $p \in \mathbb{K}[x]$, let p_ℓ be the sum of all terms of p of degree ℓ; each p_ℓ is homogeneous, $p = \sum_\ell p_\ell$ is a finite sum, and the p_ℓ's are called the *homogeneous components* of p. Similarly, for any analytic function $k \in \mathbb{R}_0^\omega[[x]]^p$, just considering its Taylor series expansion about the origin, one has that k can be expressed through the following (possibly, infinite) sum $k = \sum_{\ell \geq 1} p_\ell$, where p_ℓ is homogeneous of degree ℓ.

Lemma 2. *If $k \in \mathbb{R}_0^\omega[[x]]^p$ satisfies $L_{Ax} k(x) = Cx$, then there is $C_{-1} \in \mathbb{R}^{p \times n}$ such that $C_{-1} A = C$.*

Proof. If $k = \sum_{\ell \geq 1} p_\ell$ satisfies $L_{Ax} k(x) = Cx$, one has $\sum_{\ell \geq 1} L_{Ax} p_\ell(x) = Cx$. If p_ℓ is homogeneous of degree

ℓ with respect to the standard dilation, then $L_{Ax}p_\ell(x)$ is still homogeneous of degree ℓ with respect to the standard dilation, because Cx is homogeneous of degree 1 and the entries of the Jacobian matrix of p_ℓ are homogeneous of degree $\ell-1$; therefore, the equation $\sum_{\ell\geq 1} L_{Ax}p_\ell(x) = Cx$ becomes $L_{Ax}p_1(x) = Cx$ and $L_{Ax}p_\ell(x) = 0$, $\ell \in \mathbb{Z}, \ell \geq 2$. Letting $p_1(x) = C_{-1}x$, one has $C_{-1}Ax = Cx$, which yields $C_{-1}A = C$, by the arbitrariness of x. □

If $k = \sum_{\ell\geq 1} p_\ell$ satisfies $L_{Ax}k(x) = Cx$, then $p_1(x) = C_{-1}x$ is a linear directional integral of Cx along the vector field Ax, and p_ℓ is a polynomial first integral associated with the vector field Ax, for any $\ell \geq 2$; this implies that

$$k(x) = C_{-1}x + \omega(x),$$

where $\omega(x) = \sum_{\ell\geq 1} p_\ell(x) = k(x) - C_{-1}x$ is an analytic first integral associated with the vector field Ax.

Let $\{\omega_1, \dots, \omega_v\} \subset \mathbb{R}[x]$ be a maximal set of algebraically independent monic polynomial first integrals associated with the vector field Ax, so that each analytic first integral $\omega(x) \in \mathbb{R}_0^\omega[[x]]$ associated with Ax can be expressed as $\omega(x) = b(\omega_1(x), \dots, \omega_v(x))$, where $b \in \mathbb{R}^\omega[[z_1, \dots, z_v]]$.

Lemma 3. *If there is $C_{-1} \in \mathbb{R}^{p\times n}$ such that $C_{-1}A = C$, all the solutions $k \in \mathbb{R}^\omega[[x]]$ (possibly, $k \in \mathbb{R}_0^\omega[[x]]$) of $L_{Ax}k(x) = Cx$ are given by:*

$$k(x) = C_{-1}x + b(\omega_1(x), \dots, \omega_v(x)),$$

where $b \in \mathbb{R}^\omega[[z_1, \dots, z_v]]^p$ is arbitrary.

Note that, by construction, for any $x^o \in \mathbb{R}^n$, one has that

$$k(\mathrm{e}^{At}x^o) = \int_0^t C\mathrm{e}^{A\gamma}x^o \,\mathrm{d}\gamma + b(\omega_1(\mathrm{e}^{At}x^o), \dots, \omega_v(\mathrm{e}^{At}x^o)),$$

where $b(\omega_1(\mathrm{e}^{At}x^o), \dots, \omega_v(\mathrm{e}^{At}x^o))$ is constant, $b(\omega_1(\mathrm{e}^{At}x^o), \dots, \omega_v(\mathrm{e}^{At}x^o)) = c$; hence, if $y(t) = C\mathrm{e}^{At}x^o$ and $x(t) = \mathrm{e}^{At}x^o$ are the output-response and the state-response, respectively, of the (unforced) linear system under consideration from the initial state $x(0) = x^o$, then

$$k(x(t)) = \int_0^t y(\gamma)\,\mathrm{d}\gamma + c.$$

Now, if $k_1, k_2 \in \mathbb{R}^\omega[[x]]$ are two directional integrals of Cx along the vector field Ax, since

$$k_i(x(t)) \;\;=\;\; \int_0^t y(\gamma)\,\mathrm{d}\gamma + c_i, \quad i = 1, 2,$$

$k_1(x(t)) - k_2(x(t))$ is constant as a function of t, although $k_1(x) - k_2(x)$ need not be constant as a function of x.

Since $C_{-1}A = C$ if and only if each row of C is a linear combination of the rows of A, one has the following lemma.

Lemma 4. *Given pair (C, A), one has:*

(4.1) there is C_{-1} such that $C_{-1}A = C$ if and only if

$$\mathrm{rank}(\begin{bmatrix} A \\ C \end{bmatrix}) = \mathrm{rank}(A), \tag{7}$$

(4.2) there is a unique C_{-1} such that $C_{-1}A = C$ if and only if $\det(A) \neq 0$*, and, in such a case, $C_{-1} = CA^{-1}$.*

By Statement (4.2) of Lemma 4, condition $\det(A) \neq 0$ is necessary to have uniqueness of the solution of $C_{-1}A = C$. If $\det(A) \neq 0$ and system (3) is observable, then

$$\mathrm{rank}(\begin{bmatrix} A \\ C \end{bmatrix}) = \mathrm{rank}(A) = n,$$

whence (7) holds. On the contrary, if $\det(A) = 0$, then 0 is one of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and $\mathrm{rank}(A) < n$. As well known [34], pair (C, A) is observable if and only if

$$\mathrm{rank}(\begin{bmatrix} A - \lambda_i E \\ C \end{bmatrix}) = n, \tag{8}$$

where E is the $n\times n$ identity matrix. Letting $\lambda_i = 0$ in (8), one has $\mathrm{rank}(\begin{bmatrix} A \\ C \end{bmatrix}) = n$, whence (7) does not hold. This proves that condition $\det(A) \neq 0$ is also necessary if pair (C, A) (*i.e.*, system (3)) is observable, as stated in the following lemma.

Lemma 5. *Under the assumption that pair (C, A) is observable, condition (7) holds if and only if* $\det(A) \neq 0$*.*

Lemma 5 motivates the condition $\det(A) \neq 0$ assumed in Proposition 2. The following theorem shows that, under the assumption $\det(A) \neq 0$, system (3) is observable through the measured output Cx if and only if it is observable through the fictitious outputs CA^ix, with $i \in \mathbb{Z}$.

Theorem 2. *Consider system* (3) *and assume that* $\det(A) \neq 0$*. Let $C_i := CA^i$, $i \in \mathbb{Z}$. Pair (C, A) is observable if and only if pair (C_i, A) is observable.*

Proof. Consider the observability matrices of order n, $O_n(C, A)$ and $O_n(C_{-i}, A)$, and note that $O_n(C, A) = O_n(C_{-i}, A)A^i$. The lemma follows from the non-singularity of the dynamic matrix A. □

The following corollary shows that, under the assumption that (C, A) is observable and $\det(A) \neq 0$, $k_i(x)$ is linear (*i.e.*, $k_i(x) = C_{-i}x$ for some matrix C_{-i}) and satisfies $L_{Ax}^i k_i(x) = Cx$ if and only if $C_{-i} = CA^{-i}$.

Corollary 2. *Let pair (C, A) be observable and assume that* $\det(A) \neq 0$*. For each $i \in \mathbb{Z}$, $i \geq 1$, the function $k_i(x) = CA^{-i}x$ is the unique i-th directional integral of Cx along the vector field Ax in $\mathbb{R}_0^\omega[[x]]$ being linear.*

Proof. By Lemmas 3, 4, and 5, $k_1(x) = CA^{-1}x$ is the unique directional integral of Cx along the vector field Ax in $\mathbb{R}_0^\omega[[x]]$ being linear. Hence, assume that $k_i(x) = CA^{-i}x$ is such that $L_{Ax}^i k_i(x) = Cx$ and let $C_i = CA^{-i}$. By Theorem 2, if (C, A) is observable, then (C_i, A) is observable. Thus, by Lemmas 3, 4, and 5, $k_{i+1}(x) = CA^{-(i+1)}x$ is the unique directional integral of $CA^{-i}x$ along the vector field Ax in $\mathbb{R}_0^\omega[[x]]$ being linear, and the statement follows. □

Under the assumptions of Corollary 2, a directional integral of Cx need not be unique in $\mathbb{R}_0^\omega[[x]]$. By Lemma 3, if $\{1\}$ is not a maximal set $\{\omega_1, \dots, \omega_v\} \subset \mathbb{R}[x]$ of algebraically independent monic (with respect to some monomial order) polynomial first integrals associated with Ax, then there are infinitely many analytic (but just one linear) directional integrals of Cx along the vector field Ax in $\mathbb{R}_0^\omega[[x]]$.

IV. State estimation by output integration for continuous-time systems

The following Assumption 1 ensures existence and uniqueness of the *linear* directional integral of the output of system (3) along Ax.

Assumption 1. As for system (3), pair (C, A) is observable and $\det(A) \neq 0$.

Remark 1. Assumption 1 is met by any observable linear system that admits the origins as unique equilibrium point if the input vanishes identically for all times $t \in \mathbb{R}^{\geq 0}$.

The following example shows what may happen when Assumption 1 does not hold.

Example 1. Assume that $A = \text{block\_diag}(A_a, A_b)$, $C = [\ C_a \quad C_b\ ]$, and $x = [\ x_a^\top \quad x_b^\top\ ]^\top$, with consistent dimensions; let $B = 0$ and $D = 0$. If $A_b = 0$, then any linear function $k_b x_b$ of x_b is a linear first integral associated with Ax, whence if $C_{-1} A = C$, then $C_{-1}x + k_b x_b$, $k_b \neq 0$, is linear directional integral of Cx along Ax, being different from $C_{-1}x$. On the other hand, if $A_b = 0$ and $C_b \neq 0$, then let $C_{-1} = [\ C_{-1,a} \quad C_{-1,b}\ ]$. Clearly, $L_{Ax}C_{-1}x = L_{A_a x_a}C_{-1,a}x_a + L_{A_b x_b}C_{-1,b}x_b = C_{-1,a}A_a x_a$ cannot never be equal to $Cx = C_a x_a + C_b x_b$ since $C_b \neq 0$, whence there is no linear directional integral of Cx along the vector field Ax. Clearly, Assumption 1 does not hold in both cases.

Following Propositions 1 and 2, define, for $i \in \mathbb{Z}$, $j \in \mathbb{Z}^{\geq 0}$,

$$R_j := \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ CB & D & 0 & 0 & 0 \\ CAB & CB & D & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{j-2}B & CA^{j-1}B & \cdots & CB & D \end{bmatrix},$$

$$S_j := \begin{bmatrix} CA^{-1}B-D & CA^{-2}B & \cdots & CA^{1-j}B \\ 0 & CA^{-1}B-D & \cdots & CA^{2-j}B \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & CA^{-1}B-D \end{bmatrix},$$

$$T_j(t) := \begin{bmatrix} E & \cdots & \frac{t^{j-3}}{(j-3)!}E & \frac{t^{j-2}}{(j-2)!}E \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & E & tE \\ 0 & \cdots & 0 & E \end{bmatrix},$$

$$y_{i,i+j}(t) := \begin{bmatrix} y_i(t) \\ y_{i+1}(t) \\ \vdots \\ y_{i+j}(t) \end{bmatrix}, \quad O_{i,i+j} := \begin{bmatrix} CA^i \\ CA^{i+1} \\ \vdots \\ CA^{i+j} \end{bmatrix},$$

$$u_{i,i+j}(t) := \begin{bmatrix} u_i(t) \\ u_{i+1}(t) \\ \vdots \\ u_{i+j}(t) \end{bmatrix}, \quad c_{i,i+j} := \begin{bmatrix} c_i \\ c_{i+1} \\ \vdots \\ c_{i+j} \end{bmatrix},$$

$$S_{j,0} := \begin{bmatrix} S_j & 0 \\ 0 & -D \end{bmatrix}, \quad T_{j,0}(t) := \begin{bmatrix} T_j(t) \\ 0 \end{bmatrix},$$

where 0 is the zero matrix and E is the identity matrix, both of consistent dimensions, and $c_i = 0$, for all $i \in \mathbb{Z}^{\geq 0}$.

In the following theorem, it is shown how a sort of *static* state observer for system (3) can be designed, provided that the successive time-derivatives and time-integrals of the output-response $y(t)$ and of the input $u(t)$ are available (such a strong assumption will be removed later in the paper).

Theorem 3. *Let system* (3) *be given, let Assumption 1 hold and assume that the signals* $y_{0,n-1}(t)$, $y_{1-n,0}(t)$, $u_{0,n-1}(t)$, $u_{1-n,0}(t)$ *are available for all* $t \in \mathbb{R}^{\geq 0}$. *Let*

$$\bar{x} = O_{0,n-1}^{\dagger}(y_{0,n-1} - R_n u_{0,n-1}), \tag{9a}$$
$$\hat{x} = O_{1-n,0}^{\dagger}(y_{1-n,0} + S_{n,0}u_{1-n,0} - T_{n,0}\bar{c}), \tag{9b}$$
$$\bar{c} = T_n^{-1}(y_{1-n,-1} - O_{1-n,-1}\bar{x} + S_n u_{1-n,-1}), \tag{9c}$$

where the dependence on t *has been omitted. Thus, one has*
(3.1) $\bar{c}(t)$ *is constant and* $\bar{c}(t) = c_{1-n,-1}$ *for all* $t \in \mathbb{R}^{\geq 0}$,
(3.2) $\bar{x}(t)$ *and* $\hat{x}(t)$ *are equal to* $x(t)$ *for all times* $t \in \mathbb{R}^{\geq 0}$.

Proof. By Propositions 1 and 2, since $\det(A) \neq 0$, one has

$$y_{0,n-1} = O_{0,n-1}x + R_n u_{0,n-1}, \tag{10a}$$
$$y_{1-n,0} = O_{1-n,0}x - S_{n,0}u_{1-n,0} + T_{n,0}c_{1-n,-1}, \tag{10b}$$
$$y_{1-n,-1} = O_{1-n,-1}x - S_n u_{1-n,-1} + T_n c_{1-n,-1}. \tag{10c}$$

Since (C, A) is observable, by Theorem 2, the matrices $O_{0,n-1}$ and $O_{1-n,0}$ have full rank, whence, by (10a) and (10c), one has that $\bar{x}(t)$ and $\bar{c}(t)$ are equal to $x(t)$ and $c_{1-n,-1}$, respectively, for all $t \in \mathbb{R}^{\geq 0}$ [38]. Thus, by (10b), one has that $\hat{x}(t) = x(t)$, for all times $t \in \mathbb{R}^{\geq 0}$. □

In Theorem 3, it is shown how to exactly reconstruct the state of system (3) from the successive time-derivatives and time-integrals of the output-response y. In fact, the state x is reconstructed with the two formulae (9a) and (9b):

(i) the reconstruction $\bar{x}$ of x in (9a) only uses the time-derivatives $y_{0,n-1}$ and $u_{0,n-1}$ of y and u, respectively;

(ii) the reconstruction $\hat{x}$ of x in (9b) only uses the time-integrals $y_{1-n,0}$ and $u_{1-n,0}$ of $y(t)$ and of $u(t)$, respectively, and the knowledge of $\bar{c}$, which by Statement (3.1) of Theorem 3 is constant and equal to $c_{1-n,-1}$;

(iii) the constant $\bar{c}$ can be obtained by (9c) as a function of $\bar{x}$ and of the time-integrals of y and of u.

The static state observer for system (3) given in Theorem 3 assumes the knowledge of the successive time-derivatives and time-integrals of $y(t)$ and of $u(t)$. The goal of the next subsection is to incorporate in the scheme (9) suitable dynamic filters to compute the integrals and to avoid the use of the derivatives, so to obtain an observer just using $u(\cdot)$ and $y(\cdot)$.

A. An observer using integrals and moving average

With the aim of avoiding the use of too many symbols, with a little abuse of notation, in the following, the symbols $\bar{x}$, $\bar{c}$ and $\hat{x}$ are used to denote the estimates of the corresponding exact values given by (9). In particular, $\bar{x}(t)$ is a first "rough" estimate of $x(t)$, $\bar{c}(t)$ is a non-constant estimate of the constant vector $c_{1-n,-1}$ and $\hat{x}(t)$ is the proposed estimate of the state.

In absence of noise, a feasible solution to implement the observer (9) using only the measured $u(\cdot)$ and $y(\cdot)$ would be that of computing the estimate $\bar{x}(t)$ of $x(t)$ by using high-gain observers [39], [40] based on estimates of the vectors $y_{0,n-1}$ and $u_{0,n-1}$. The computation of the integrals in the vectors $y_{1-n,-1}$ and $u_{1-n,-1}$ can be simply realized by using a chain of integrators (see the subsequent (14d) and (14e)); it is important to stress that the integrals thus obtained are highly *insensitive* to high-frequency noise. On the contrary, since high-gain observers are highly *sensitive* to high-frequency noise, it is more convenient, in practical applications, to obtain the estimate $\bar{x}(t)$ through a Luenberger observer based on the measures of $u(t)$ and $y(t)$, which does not require the knowledge of the time-derivatives of the output-response $y(t)$ and of the input $u(t)$ (in particular, the subsequent Luenberger observer (14c)). In view of (9c), define the function $\phi(t, U, Y, x)$ as follows:

$$\phi := \begin{cases} T_n^{-1}(t)(Y - O_{1-n,-1}x + S_n U), & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

where $Y \in \mathbb{R}^{p(n-1)}$ and $U \in \mathbb{R}^{m(n-1)}$ are auxiliary variables used just to define ϕ. In particular, by Statement (3.1) of Theorem 3, if U and Y are replaced by $y_{1-n,-1}(t)$ and $u_{1-n,-1}(t)$, respectively, one obtains $c_{1-n,-1} = \phi(t, u_{1-n,-1}(t), y_{1-n,-1}(t), x(t))$. Since, in practical applications, the estimate $\bar{x}(t)$ need not be equal to $x(t)$ for all $t \in \mathbb{R}^{\geq 0}$, one has that

$$\bar{c}(t) := \phi(t, u_{1-n,-1}(t), y_{1-n,-1}(t), \bar{x}(t)) \tag{12}$$

need not be constant. Although, by Statement (3.1) of Theorem 3, the signal $\phi(t, u_{1-n,-1}(t), y_{1-n,-1}(t), x(t))$ is constant and equals $c_{1-n,-1}$, its estimate $\bar{c}(t)$ is severely non-constant even after the "settling time" for the estimate $\bar{x}(t)$ in presence of high-frequency noise affecting the measured $u(\cdot)$ and $y(\cdot)$. Hence, an improved estimate $\bar{c}_f$ of $c_{1-n,-1}$ can be obtained by filtering $\bar{c}(t)$ through a *low-pass filter* such as the *moving average* with averaging period equal to τ, for some $\tau \in \mathbb{R}^{>0}$:

$$\bar{c}_f(t) = \tfrac{1}{\tau} \int_{t-\tau}^{t} \phi(\gamma, u_{1-n,-1}(\gamma), y_{1-n,-1}(\gamma), \bar{x}(\gamma))\, \mathrm{d}\gamma. \tag{13}$$

In particular, by initializing $\bar{c}_f(0) = 0$, the moving average (13) can be computed as in the subsequent (14b) (note that $\phi(t, U, Y, x) = 0$ if $t < 0$). Finally, by (10b), one has that $y_{1-n,0}(t) = O_{1-n,0}x(t) - S_{n,0}u_{1-n,0}(t) + T_{n,0}(t)c_{1-n,-1}$. Therefore, it is possible to obtain the estimate $\hat{x}(t)$ of the state $x(t)$ of system (3) by using another Luenberger observer based on the output $y_{1-n,0}(t)$ (the subsequent Luenberger observer (14a)). The use of the vector $y_{1-n,0}(t)$ rather than of its last (block) entry $y(t)$ improves significantly the performance of the observer in presence of high-frequency noise (see the examples at the end of this section).

Summarizing, the proposed state observer is represented in Fig. 2; it consists of the following Luenberger observer (the dependence on t is omitted):

$$\begin{aligned} \dot{\hat{x}} &= A\,\hat{x} + B\,u \\ &\quad + K\,(y_{1-n,0} - O_{1-n,0}\,\hat{x} + S_{n,0}\,u_{1-n,0} - T_{n,0}\,\bar{c}_f), \end{aligned} \tag{14a}$$

where $y_{1-n,0} = [\, y_{1-n,-1}^\top \;\; y^\top \,]^\top$, $u_{1-n,0} = [\, u_{1-n,-1}^\top \;\; u^\top \,]^\top$, K is such that the eigenvalues of the matrix $A - KO_{1-n,0}$ have negative real part, and the estimate $\bar{c}_f(t)$ of $c_{1-n,-1}$ is given by

$$\begin{aligned} \dot{\bar{c}}_f(t) &= \tfrac{1}{\tau}\big(\phi(t, u_{1-n,-1}(t), y_{1-n,-1}(t), \bar{x}(t)) \\ &\quad - \phi(t-\tau, u_{1-n,-1}(t-\tau), y_{1-n,-1}(t-\tau), \bar{x}(t-\tau))\big), \end{aligned} \tag{14b}$$

where ϕ is the function given in (11), whereas $\bar{x}(t)$, $u_{1-n,-1}(t)$ and $y_{1-n,-1}(t)$ are the block-entries of the state-response of the following system:

$$\dot{\bar{x}} = A\,\bar{x} + B\,u + L\,(y - C\,\bar{x} - D\,u), \tag{14c}$$

$$\dot{u}_{1-n,-1} = \begin{bmatrix} 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \\ 0 & 0 & \cdots & 0 \end{bmatrix} u_{1-n,-1} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ E \end{bmatrix} u, \tag{14d}$$

$$\dot{y}_{1-n,-1} = \begin{bmatrix} 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \\ 0 & 0 & \cdots & 0 \end{bmatrix} y_{1-n,-1} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ E \end{bmatrix} y, \tag{14e}$$

where 0 is the zero matrix, E is the identity matrix, both of consistent dimensions, L is such that the eigenvalues of $A - LC$ have negative real part, and $u_{1-n,-1}(0) = 0$, $y_{1-n,-1}(0) = 0$, $\bar{c}_f(0) = 0$, $\tau \in \mathbb{R}^{>0}$. System (14) is referred to as the *state observer without reset*.

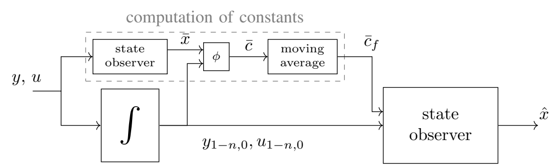


Fig. 2. Structure of the dynamic observer (14).

The following theorem states that system (14) is an asymptotic state observer for system (3).

Theorem 4. *Let Assumption 1 hold; let $x(t)$ be the state-response of system* (3) *from $x(0) \in \mathbb{R}^n$ to $u(t)$ and let $\hat{x}(t)$ be the state-response in the $\hat{x}$-variables of system* (14) *from* $(\hat{x}(0), \bar{c}_f(0), \bar{x}(0), u_{1-n,-1}(0), y_{1-n,-1}(0))$ *in* $\mathbb{R}^n \times \{0\} \times \mathbb{R}^n \times \{0\} \times \{0\}$, *to $u(t)$ and $y(t) = Cx(t) + Du(t)$.*

(4.1) For all $x(0), \hat{x}(0), \bar{x}(0) \in \mathbb{R}^n$, one has that

$$\lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0. \tag{15}$$

(4.2) For each $\varepsilon \in \mathbb{R}^{>0}$, there are $\delta_1, \delta_2 \in \mathbb{R}^{>0}$ such that if $\|x(0) - \hat{x}(0)\| < \delta_1$ and $\|x(0) - \bar{x}(0)\| < \delta_2$, then $\|x(t) - \hat{x}(t)\| < \varepsilon$ for all times $t \in \mathbb{R}^{\geq 0}$.

Proof. Define $\chi := x - \bar{x}$, whose dynamics are given by $\dot{\chi} = (A - LC)\chi$. Therefore, letting $\underline{\lambda} = \max_{\lambda \in \sigma(A-LC)} \mathrm{Re}(\lambda)$, $\underline{\lambda} < 0$, there are $\alpha \in \mathbb{R}^{>0}$ and $\lambda \in (\underline{\lambda}, 0)$ such that $\|\chi(t)\| \leq \alpha \exp(\lambda t)\|\chi(0)\|$ for all $t \in \mathbb{R}^{\geq 0}$. Thus, one has

$$\phi(t, u_{1-n,-1}, y_{1-n,-1}, \bar{x}) = c_{1-n,-1} + T_n^{-1} O_{1-n,-1}\chi, \tag{16}$$

for all $t \in \mathbb{R}^{>0}$. Then, define $\zeta = x - \hat{x}$, whose dynamics, by Theorem 3 and (16), are given, for all $t \in \mathbb{R}^{\geq 0}$, by

$$\begin{aligned} \dot{\zeta}(t) &= (A - KO_{1-n,0})\zeta(t) \\ &\quad + \tfrac{1}{\tau} K T_{n,0}(t) \int_{\max(t-\tau,0)}^{t} T_n^{-1}(\gamma) O_{1-n,-1}\chi(\gamma)\, \mathrm{d}\gamma, \end{aligned} \tag{17}$$

Since $\|\chi(\gamma)\| \leq \alpha \exp(\lambda\gamma)\|\chi(0)\|$ for all $\gamma \in \mathbb{R}^{\geq 0}$, there exists $\beta \in \mathbb{R}^{\geq 0}$ such that $\|T_n^{-1}(\gamma) O_{1-n,-1}\chi(\gamma)\| \leq \beta \exp(\lambda\gamma)\|\chi(0)\|$. By the Dominated Convergence Theorem [41], there exists $\mu \in \mathbb{R}^{\geq 0}$ such that $\|T_{n,0}(t) \int_{t-\tau}^{t} T_n^{-1}(\gamma) O_{1-n,-1}\chi(\gamma)\, \mathrm{d}\gamma\| \leq \mu|\exp(\lambda t) - \exp(\lambda(t-\tau))|\|\chi(0)\|$, for all $t \geq \tau$. Hence, since the eigenvalues of $A - KO_{1-n,0}$ have negative real part, by [42], one has that (15) holds. In addition, there exists a symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ such that

$$(A - KO_{1-n,0})^\top P + P(A - KO_{1-n,0}) = -E.$$

Hence, define the positive definite function $V = \zeta^\top P\zeta$. By (17), there exists $\bar{\mu} \in \mathbb{R}^{>0}$ such that

$$\dot{V}(t) \le -\|\zeta(t)\|^2 + \tfrac{\bar{\mu}}{\tau}\|\zeta(t)\|\|P\|\|\chi(0)\|.$$

Since P is positive definite, there exist $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}^{>0}$ such that $\underline{\lambda}^2\|\zeta\|^2 \le V \le \overline{\lambda}^2\|\zeta\|^2$, whence by ISS arguments [42]:

$$\|\zeta(t)\| \le \max\left\{\tfrac{\overline{\lambda}}{\underline{\lambda}}\|\zeta(0)\|, \tfrac{2\bar{\mu}\|P\|\overline{\lambda}}{\tau\underline{\lambda}}\|\chi(0)\|\right\}, \quad t \in \mathbb{R}^{\ge 0}. \quad \square$$

Statement (4.1) of Theorem 4 guarantees that the state $\hat{x}$ of system (14) converges to the state x of system (3) independently of the initial conditions $x(0), \hat{x}(0), \bar{x}(0) \in \mathbb{R}^n$. On the other hand, Statement (4.2) of Theorem 4 guarantees that the estimation error $x(t) - \hat{x}(t)$ can be made arbitrarily small for all $t \in \mathbb{R}^{\ge 0}$ by letting $x(0) - \hat{x}(0)$ and $x(0) - \bar{x}(0)$ be sufficiently small. Thus, the combination of these two properties ensures asymptotic stability of the estimation error.

The following remark allows one to evaluate the filtering properties of the proposed observer.

Remark 2. Let the assumptions of Theorem 4 hold. Let $\zeta = x - \hat{x}$, and let the measured output of system (3) be affected by an additive noise $d(t)$, *i.e.*, $y = C\,x + D\,u + d$. The transfer function from d to ζ is given by the following expression:

$$(sE - A + KO_{1-n,0})^{-1}K\Big(-\begin{bmatrix}\frac{1}{s^{n-1}}E\\ \vdots\\ E\end{bmatrix} + \tfrac{1}{\tau}Q(\tau,s)$$
$$\cdot\big(O_{1-n,-1}(sE - A + LC)^{-1}L + \begin{bmatrix}\frac{1}{s^{n-1}}E\\ \vdots\\ \frac{1}{s}E\end{bmatrix}\big)\Big), \quad (18)$$

where

$$Q(\tau,s) = \begin{bmatrix} E - e^{-\tau s}T_n(\tau) \\ 0 \end{bmatrix}\Big(sE - \begin{bmatrix} 0 & E & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & E\\ 0 & 0 & \cdots & 0\end{bmatrix}\Big)^{-1}.$$

Note that, the state observers given in (14a) and in (14c) can be Kalman filters (*i.e.*, Luenberger observers with K and L computed as optimal gains). In the following, it is shown, by several numerical examples, that the overall state observer (14) has an improved performance with respect to the Kalman filter (14c) if the signal $y(t)$ is affected by high-frequency bounded noise. Furthermore, it is worth noticing that the Luenberger observers in (14a) and in (14d) can be substituted with other observers able to estimate the state of a linear plant. For instance, a state observer for system (25) based on the time-integrals of the output-response $y(t)$ can be built by using the sliding-mode techniques given in [25], [43], as detailed in the following. Let Assumption 1 hold and assume that $\operatorname{rank}(C) = p < n$. Letting N_a be a basis of $\ker(C)$ and letting N_b be a basis of $\ker(CA^{-1})$, define the matrices $Z_a = \left[\begin{smallmatrix} N_a^\top \\ C\end{smallmatrix}\right]$, $Z_b = \left[\begin{smallmatrix} N_b^\top \\ CA^{-1}\end{smallmatrix}\right]$, $\bar{A} = Z_aAZ_a^{-1}$, $\bar{B} = Z_aB$, $\bar{C} = CZ_a^{-1}$, $\check{A} = Z_bAZ_b^{-1}$, $\check{B} = Z_bB$, $\check{C} = CA^{-1}Z_b^{-1}$. Let $\bar{A}$ and $\check{A}$ be partitioned as

$$\bar{A} = \begin{bmatrix}\bar{A}_{1,1} & \bar{A}_{1,2}\\ \bar{A}_{2,1} & \bar{A}_{2,2}\end{bmatrix}, \quad \check{A} = \begin{bmatrix}\check{A}_{1,1} & \check{A}_{1,2}\\ \check{A}_{2,1} & \check{A}_{2,2}\end{bmatrix}, \quad (19)$$

respectively, and let L and K be such that the eigenvalues of $(\bar{A}_{1,1} + L\,\bar{A}_{2,1})$ and $(\check{A}_{1,1} + K\,\check{A}_{2,1})$ have negative real part (the existence of such values is guaranteed by Assumption 1 and Corollary 2), and let $\kappa_a = \left[\begin{smallmatrix} L \\ -E\end{smallmatrix}\right]$, $\kappa_b = \left[\begin{smallmatrix} K \\ -E\end{smallmatrix}\right]$. Thus, letting $\rho \in \mathbb{R}^{\ge 0}$ be a sufficiently large integer, consider the sliding-mode observer, based on the first directional integral of the output along Ax, given by

$$\hat{x}(t) = Z_b^{-1}\check{x}(t), \quad (20a)$$

where $\check{x}(t)$ is the state-response of the following system

$$\dot{\check{x}}(t) = \check{A}\,\check{x}(t) + \check{B}\,u(t) + \kappa_b\rho\,\mathrm{sign}(\check{C}\check{x} - \hat{y}(t)),$$

where $\hat{y}$ is an estimate of $\check{C}x$, obtained as

$$\hat{y}(t) = y_{-1}(t) - \bar{c}_f(t) - (D - CA^{-1}B)u_{-1}(t), \quad (20b)$$

where $u_{-1}(t)$, $y_{-1}(t)$, and $\bar{c}_f(t)$ are the entries of the state-response of the system

$$\dot{u}_{-1} = u, \quad (20c)$$
$$\dot{y}_{-1} = y, \quad (20d)$$
$$\dot{\bar{c}}_f(t) = \tau^{-1}(\phi(t, u_{-1}(t), y_{-1}(t), Z_a^{-1}\bar{x}(t))$$
$$- \phi(t-\tau, u_{-1}(t-\tau), y_{-1}(t-\tau), Z_a^{-1}\bar{x}(t-\tau))), \quad (20e)$$

where ϕ is the function given in (11) and $\bar{x}(t)$ is the state-response of the following system:

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u + \kappa_a\rho\,\mathrm{sign}(\bar{C}\,\bar{x} + D\,u - y). \quad (20f)$$

Note that system (20f) is the sliding-mode observer given in [25] for the dynamics given in (25), systems (20c) and (20d) compute the time-integral of the measured output $y(t)$ and of the input $u(t)$, respectively, whereas system (20b) is a sliding-mode observer for the integral of the output. The following theorem establishes that system (20) is a state observer for (3).

Theorem 5. *Let Assumption 1 hold; let $x(t)$ be the state-response of system* (3) *from $x(0) \in \mathbb{R}^n$ to $u(t)$ and let $\check{x}(t)$ be the state-response in the $\check{x}$-variables of system* (20) *from* $(\check{x}(0), \bar{c}_f(0), \bar{x}(0), u_{-1}(0), y_{-1}(0))$ *in* $\mathbb{R}^n \times \{0\} \times \mathbb{R}^n \times \{0\} \times \{0\}$, *to $u(t)$ and $y(t) = Cx(t) + Du(t)$.*

(5.1) For all $\varsigma \in \mathbb{R}^{>0}$, there exists $\rho \in \mathbb{R}^{>0}$ such that, for all $x(0), \hat{x}(0), \bar{x}(0) \in \mathbb{R}^n$ such that $\|x(0) - \check{x}(0)\| \le \varsigma$ and $\|x(0) - \check{x}(0)\| \le \varsigma$, one has

$$\lim_{t\to\infty}(x(t) - \hat{x}(t)) = 0, \quad (21)$$

where $\hat{x}$ is obtained from $\check{x}$ by using (20a).

(5.2) If, additionally, $\ker(C) \subseteq \ker(CA^{-1})$*, then there is a finite time Ξ such that $CA^{-1}(x(t) - \hat{x}(t)) = 0$ for all $t \geqslant \Xi$.*

Proof. Assume, without loss of generality, that system (3) is already in the coordinate associated with (20a), so that $Z_b = E$ and $\check{x} = \hat{x}$. Letting $\chi = x - \bar{x}$, by [43], for all $\varsigma \in \mathbb{R}^{>0}$, there exists $\rho \in \mathbb{R}^{>0}$ such that, if $|x(0) - \bar{x}(0)| \le \varsigma$, then there are

$\alpha \in \mathbb{R}^{>0}$ and $\lambda < 0$ such that $\|\chi(t)\| \le \alpha \exp(\lambda t)\|\chi(0)\|$ for all $t \in \mathbb{R}^{\ge 0}$. Moreover, by (16) and since $T_1 = E$, one has

$$\phi(t, u_{1-n,-1}, y_{1-n,-1}, \bar{x}) = c_{-1} + CA^{-1}\chi, \tag{22}$$

for all $t \in \mathbb{R}^{>0}$. Therefore, it results that, for all $t \ge \tau$,

$$\begin{aligned} \bar{c}_f &= \tfrac{1}{\tau}\textstyle\int_{t-\tau}^{t} \phi(\gamma, u_{1-n,-1}, y_{1-n,-1}, \bar{x})\mathrm{d}\gamma \\ &= c_{-1} + \tfrac{1}{\tau}\textstyle\int_{t-\tau}^{t} CA^{-1}\chi(\gamma)\mathrm{d}\gamma. \end{aligned}$$

Thus, define $\zeta = \check{x} - x$, partitioned as in (19), $\zeta = [\ \zeta_1^\top \quad \zeta_2^\top\ ]^\top$, whose dynamics are given by

$$\begin{aligned} \dot{\zeta}_1 &= \check{A}_{1,1}\zeta_1 + \check{A}_{1,2}\zeta_2 + K\rho\,\mathrm{sign}(\zeta_2 + \bar{c}_f - c_{-1}), \\ \dot{\zeta}_2 &= \check{A}_{2,1}\zeta_1 + \check{A}_{2,2}\zeta_2 - \rho\,\mathrm{sign}(\zeta_2 + \bar{c}_f - c_{-1}). \end{aligned}$$

Thus, letting $\vartheta = \zeta_2 + \bar{c}_f - c_{-1}$, for all $t \ge \tau$, one has

$$\begin{aligned} \vartheta^\top \dot{\vartheta} &= \vartheta^\top(\check{A}_{2,1}\zeta_1 + \check{A}_{2,2}\zeta_2) - \rho\|\vartheta\| \\ &\quad + \tfrac{1}{\tau}\vartheta^\top CA^{-1}(\chi(t) - \chi(t-\tau)) \\ &\le -\|\vartheta\|(\rho - \|\check{A}_{2,1}\zeta_1 - \check{A}_{2,2}\zeta_2\| + \tfrac{2\,\alpha}{\tau}\|CA^{-1}\|). \end{aligned}$$

Therefore, by the same reasoning used in Section 3.2 of [43], if ρ is sufficiently large, then ϑ will converge to zero in finite time, thus inducing a sliding motion. During sliding, both $\dot{\vartheta}$ and ϑ vanish identically, thus implying that

$$\begin{aligned} \dot{\zeta}_1 &= \check{A}_{1,1}\zeta_1 - \tfrac{1}{\tau}\check{A}_{1,2}\textstyle\int_{t-\tau}^{t} CA^{-1}\chi(\gamma)\mathrm{d}\gamma + K\varphi, \\ \varphi &= \check{A}_{2,1}\zeta_1 + \tfrac{1}{\tau}CA^{-1}(\chi(t) - \chi(t-\tau)) \\ &\quad - \tfrac{1}{\tau}\check{A}_{2,2}\textstyle\int_{t-\tau}^{t} CA^{-1}\chi(\gamma)\mathrm{d}\gamma, \end{aligned}$$

where φ is the so-called equivalent output error injection that is required to maintain the sliding motion. Hence, since the eigenvalues of $(\check{A}_{1,1} + K\check{A}_{2,1})$ have negative real part and $\lim_{t\to\infty}\chi(t) = 0$, the statement follows by classical bounded-input bounded-state stability arguments [42].

Furthermore, under the above positions [43], there exists $\bar{\Xi} \in \mathbb{R}^{>0}$ such that the output estimation error $Cx(t) - C\bar{x}(t)$ vanishes identically for all $t \geqslant \bar{\Xi}$, thus implying that $\chi(t) \in \ker(C)$ for all $t \geqslant \bar{\Xi}$. Hence, if additionally $\ker(C) \subset \ker(CA^{-1})$, then $\bar{c}_f(t) = c_{-1}$ for all $t \geqslant \bar{\Xi} + \tau$. Thus, there exists a finite time Ξ such that $\xi_2(t) = 0$ for all $t \geqslant \Xi$. □

B. The proposed observer

The state observer given in (14) uses information obtained from the time-integrals of the output-response $y(t)$ and of the input $u(t)$ of system (3) to determine estimates of some constants related to the state $x(t)$. If some noise is added to the output-response $y(t)$, these estimates may be subject to a drift (see the subsequent Fig. 4(b)). Therefore, it is suggested, when implementing such a state observer, to periodically reset the states $u_{1-n,0}$, $y_{1-n,0}$ and $\bar{c}_f$, as detailed below. In practice, consider the state observer given by:

$$\dot{\hat{x}} = A\,\hat{x} + B\,u + K(\hat{y}_{1-n,0} - O_{1-n,0}\,\hat{x}), \tag{23a}$$

where K makes the eigenvalues of $A - KO_{1-n,0}$ have negative real part and $\hat{y}_{1-n,0}$ is an estimate of $O_{1-n,0}x$ given by

$$\hat{y}_{1-n,0} = y_{1-n,0,i} + S_{n,0}u_{1-n,0,i} - T_{n,0}(\eta_i)\bar{c}_{f,i}, \tag{23b}$$

where $i = 1$, if $\mathrm{mod}(t, 2\theta) \le \theta$, or $i = 2$, if $\mathrm{mod}(t, 2\theta) > \theta$, where $u_{1-n,0,i}(t) = [\ u_{1-n,-1,i}(t)^\top \quad u(t)^\top\ ]^\top$, $y_{1-n,0,i}(t) = [\ y_{1-n,-1,i}(t) \quad y(t)^\top\ ]^\top$, whereas $\eta_i(t)$, $u_{1-n,-1,i}(t)$, $y_{1-n,-1,i}(t)$, and $\bar{c}_{f,i}(t)$ are the block-entries of the state-response of the following system:

$$\dot{\eta}_i = 1, \tag{23c}$$

$$\dot{u}_{1-n,-1,i} = \begin{bmatrix} 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \\ 0 & 0 & \cdots & 0 \end{bmatrix} u_{1-n,-1,i} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ E \end{bmatrix} u, \tag{23d}$$

$$\dot{y}_{1-n,-1,i} = \begin{bmatrix} 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \\ 0 & 0 & \cdots & 0 \end{bmatrix} y_{1-n,-1,i} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ E \end{bmatrix} y, \tag{23e}$$

$$\begin{aligned} \dot{\bar{c}}_{f,i}(t) &= \tau^{-1}(\phi(\eta_i(t), u_{1-n,-1,i}(t), y_{1-n,-1,i}(t), \bar{x}(t)) \\ &\quad - \phi(\eta_i(t) - \tau, u_{1-n,-1,i}(t-\tau), y_{1-n,-1,i}(t-\tau), \bar{x}(t-\tau))), \end{aligned} \tag{23f}$$

where ϕ is the function given in (11), $i = 1, 2$, and $\eta_i, u_{1-n,-1,i}, y_{1-n,-1,i}, \bar{c}_{f,i}$ are reset to their initial condition (assumed to be zero) according to the following periodic logic with period $2\theta \in \mathbb{R}^{>0}$:

$$\eta_i(t) = 0,\ u_{1-n,-1,i}(t) = 0,\ y_{1-n,-1,i}(t) = 0,\ \bar{c}_{f,i}(t) = 0, \tag{23g}$$

where $i = 1$ if $\mathrm{mod}(t, 2\theta) = \theta$, or $i = 2$, if $\mathrm{mod}(t, 2\theta) = 0$. The proposed observer still uses the dynamics (14c) to compute the estimate $\bar{x}(t)$:

$$\dot{\bar{x}} = A\,\bar{x} + B\,u + L\,(y - C\,\bar{x} - D\,u). \tag{23h}$$

where L is such that the eigenvalues of $A - L\,C$ have negative real part. System (23) is referred to as *state observer with reset*.

Note that the dynamics of system (23) essentially match with the ones of the observer given in (14) for $t \in [k\theta, (k+1)\theta]$, $k \in \mathbb{Z}^{\ge 0}$. The main difference between systems (23) and (14) is that a portion of the state of the former is reset to its initial condition every θ times. In particular, the subsystem of system (23) that is labeled with 1 (respectively, 2) is reset to zero if $t = (2k+1)\theta$ (respectively, $t = 2k$), $k \in \mathbb{Z}^{\ge 0}$. The advantage of these resets is that they allow one to reduce the errors due to the numerical integration of the signals $y(t)$ and $u(t)$ (which naturally increase with the integration times). It is worth noticing that the signals $\bar{c}_{f,1}$ and $\bar{c}_{f,2}$ are no more estimates of the same constant $c_{1-n,-1}$. As a matter of fact, by Theorem 4, the state-response in the $\bar{c}_f$-variables of system (14) is an estimate of $c_{1-n,-1} = -O_{1-n,-1}x(0)$. On the other hand, due to the presence of the resets, the signal $\bar{c}_{f,1}(t)$ is an estimate of $-O_{1-n,-1}x((2k+1)\theta)$ for all $t \in [(2k+1)\theta, (2k+3)\theta]$, whereas $\bar{c}_{f,2}(t)$ is an estimate of $-O_{1-n,-1}x(2k\theta)$ for all $t \in [2k\theta, (2k+2)\theta]$, $k \in \mathbb{Z}^{\ge 0}$. Therefore, $\bar{c}_{f,1}(t)$ and $\bar{c}_{f,2}(t)$ are estimates of piecewise constant signals, whose behaviors depend on the state of system (3) at the reset times. In particular, since $\bar{c}_{f,1}(t)$ is reset to zero at $t = (2k+1)\theta$, there is an initial transient behavior in the obtained estimates of $-O_{1-n,-1}x((2k+1)\theta)$ at the beginning of each interval $[(2k+1)\theta, (2k+3)\theta]$ (a similar behavior can be noticed for the signal $\bar{c}_{f,2}(t)$ at the beginning of each interval $[2k\theta, (2k+2)\theta]$), $k \in \mathbb{Z}^{\ge 0}$. However, since the estimate $\hat{y}_{1-n,0}$ of $O_{1-n,0}x$ neglects the signal $\bar{c}_{1,f}(t)$

(respectively, $\bar{c}_{2,f}(t)$) for all $t \in [(2k+1)\theta, (2k+2)\theta]$ (respectively, $t \in [2k\theta, (2k+1)\theta]$), $k \in \mathbb{Z}^{\geq 0}$, such a transient behavior is neglected by the observer given in (23). Therefore, system (23) provides an estimate of the state of system (3) avoiding saturations due to floating point computations of the polynomial entries of $T_{n,0}$ and T_n^{-1}, and neglecting the transient behavior of the moving average (23f).

The resetting logic given in (23g) can be used to reset the states u_{-1}, y_{-1}, and $\bar{c}_f$ of the sliding mode observer given in (20). In the following, the observer obtained by coupling system (20) with the resetting logic given in (23g) is referred to as *sliding observer with resets*.

The following example illustrates the application of the observer with resets and compares its performance with those of a Kalman filter and of the observer without resets.

Example 2. Consider the mechanical system depicted in Fig. 3 and assume that the only measurable output is the position of the body having mass m_2. The dynamics of the system are

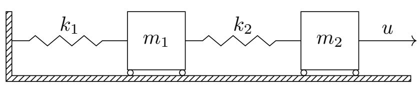


Fig. 3. Mechanical system considered in Example 2.

given by

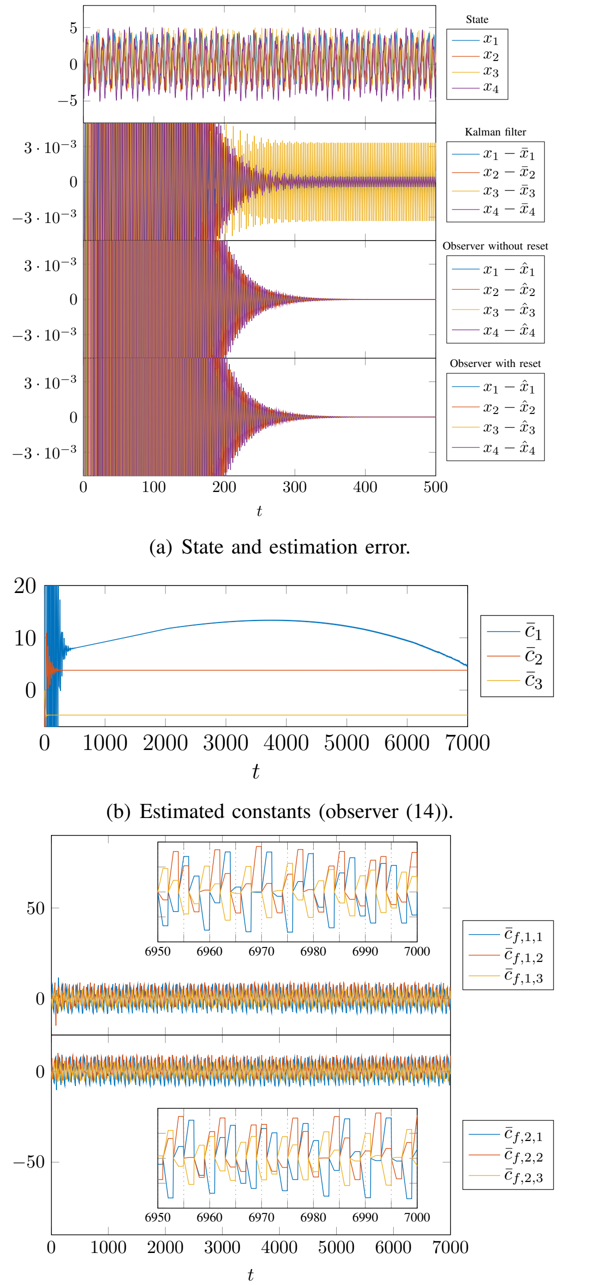
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -k_2/m_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} u, \tag{24a}$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x, \tag{24b}$$

where $x \in \mathbb{R}^4$, $y \in \mathbb{R}$, and $u \in \mathbb{R}$ denotes the force applied to the body having mass m_2. Numerical simulations have been carried out to compare the performances of the state observers (14) and (23) with the performance of a Kalman filter, assuming the following (normalized) values of the parameters $k_1 = 1.6$, $k_2 = 1.1$, $m_1 = 1$, $m_2 = 0.6$, $x(0) = x^0 = [\,1 \quad -1 \quad 3 \quad -4\,]^\top$, $u(t) = \min(t, 1)$, and that $y(t)$ is affected by the additive noise $d(t) = 4\sin(100\pi t)$. The gains L and K of the Kalman filters (see below) have been designed with the objective of minimizing the steady state estimation errors. In particular, the gain L has been tuned in order to minimize $\max_{t\in[400,500]} \|x(t) - \bar{x}(t)\|$. Hence, assuming such a value for the parameter L, the gain K has been tuned in order to minimize $\max_{t\in[400,500]} \|x(t) - \hat{x}(t)\|$. The estimate $\bar{x}(t)$ obtained by the tuned Kalman filter is used by both (14) and (23) to reconstruct the state of (24). The parameters of (14) and (23) are $\tau = 1$, $\theta = 2$,

$$L = \begin{bmatrix} 0.071 & 0.0057 & 0.26 & 0.034 \end{bmatrix}, \quad K = \begin{bmatrix} 0.021 & -0.11 & 0.028 & 0 \\ 0.31 & -0.14 & 0.23 & 0 \\ 0.36 & -0.59 & 0.19 & 0 \\ 0.79 & -0.079 & 0.0062 & 0 \end{bmatrix}.$$

Fig. 4 depicts the time behavior of the state-response of system (24) from the initial state x^o and with input u, the estimated constants $\bar{c}_f$, and the estimation errors obtained by using the Kalman filter (14c) $\equiv$ (23h) and the state observers (14) and (23). As shown in Fig. 4(a), the proposed state observers (14) and (23) are both less sensitive to high-frequency noise than the Kalman filter. However, as shown in Fig. 4(b), the state-response in the $\bar{c}_f$ variables of system (14)



(a) State and estimation error.

(b) Estimated constants (observer (14)).

(c) Estimates $\bar{c}_{f,i}$ (observer with resets (23)).

Fig. 4. Example 2: state-response of system (24); estimation errors, estimates $\bar{c}_f$ and $\bar{c}_{f,i}$ obtained by state observers (14) and (23).

is affected by a drift that is induced by the numerical errors arising from the integration of the output-response $y(t)$ and of the input $u(t)$ (which can be reduced by using higher-order integration methods or smaller numerical tolerances). Such a drift is made negligible by the use of the observer given in (23). As a matter of fact, the periodic resets of the states $u_{1-n,-1,i}$, $y_{1-n,-1,i}$ and $\bar{c}_{f,i}$, $i = 1, 2$, allow one to keep such an error bounded and small, because the signals y and u are integrated for a limited amount of time. It is to be noted that the signals $\bar{c}_{f,1}(t)$ and $\bar{c}_{f,2}(t)$ are estimates of piecewise constant signals, as shown in Fig. 4(c). The improved filtering properties of the proposed observer are confirmed by the Bode plots of the transfer functions $\bar{f}$ and $\hat{f}$ from the additive error d acting on y to the estimation errors $\chi = x - \bar{x}$ and $\zeta = x - \hat{x}$, respectively. Fig. 5 depicts the magnitude of these transfer functions computed according to (18). As shown by such a

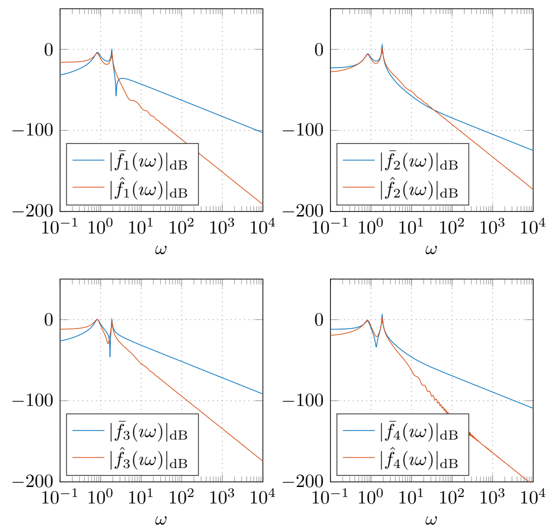


Fig. 5. Example 2: magnitude of the transfer functions from the additive noise $\boldsymbol{d}$ to the estimation errors $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$.

figure, the state observer given in (14) filters high-frequency additive noises better then the Kalman filter given in (14c).

In the following example, it is shown how to design a state observer based on the time-integrals of the output-response $y(t)$ by using the sliding-mode technique given in (20).

Example 3. Consider the two-tank system depicted in Fig. 6 and assume that the only measured output is the flow of the tank whose capacity is $\mathcal{C}_2$. The dynamics of the system are

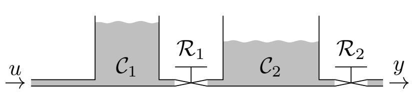


Fig. 6. Two-tank system considered in Example 3.

given by (3), where

$$A = \begin{bmatrix} -\frac{1}{\mathcal{R}_1\mathcal{C}_1} & \frac{1}{\mathcal{R}_1\mathcal{C}_1} \\ \frac{1}{\mathcal{R}_1\mathcal{C}_2} & -\frac{\mathcal{R}_1+\mathcal{R}_2}{\mathcal{C}_2\mathcal{R}_1\mathcal{R}_2} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{\mathcal{C}_1} \\ 0 \end{bmatrix} u, \tag{25a}$$

$$C = \begin{bmatrix} 0 & \frac{1}{\mathcal{R}_2} \end{bmatrix}, \qquad D = 0, \tag{25b}$$

x_i denotes the pressure in the i-th tank, $\mathcal{C}_i$ is the capacity of the i-th tank, $\mathcal{R}_i$ is the value of the i-th resistance, $i = 1, 2$.

The sliding observer with reset, obtained by coupling system (20) with the reset logic given in (23g), has been tested by a numerical simulation, assuming that $\mathcal{C}_1 = 1.019 \cdot 10^{-6} \frac{\mathrm{m}^3}{\mathrm{Pa}}$, $\mathcal{C}_2 = 11.3263 \cdot 10^{-6} \frac{\mathrm{m}^3}{\mathrm{Pa}}$, $\mathcal{R}_1 = 408.252 \cdot 10^3 \frac{\mathrm{Pa\,s}}{\mathrm{m}^3}$, $\mathcal{R}_2 = 3.306 \cdot 10^3 \frac{\mathrm{Pa\,s}}{\mathrm{m}^3}$, $x(0) = [\ 1 \quad 20\ ]$MPa, $L = -13.499 \cdot 10^{-6}$, $K = 496.856 \cdot 10^{-9}$, $\rho = 10^6$, $\tau = 0.3$s, $\theta = 0.6$s, and that $u(t)$ is a pulse signal with amplitude $1 \frac{\mathrm{m}^3}{\mathrm{s}}$, period 0.2s, and pulse width 30%. The output $y(t)$ is assumed to be affected by the additive noise $20 \sin(60\pi t)$. Fig. 7 depicts the results of such a numerical simulation. As shown by such a figure,

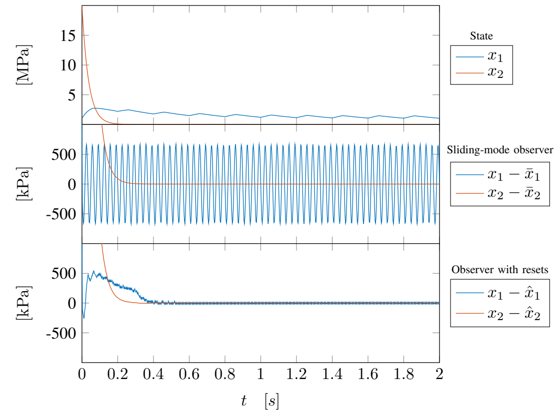


Fig. 7. Example 3: state-response of system (25); estimation errors obtained by using the sliding-mode observer (20).

the sliding observer with reset, which is based on the time-integrals of $u(t)$ and $y(t)$, performs better than the sliding-mode observer (20f) based on $u(t)$ and $y(t)$.

V. The Moving Average of the Output of a Discrete-Time System

Throughout this section, it is assumed that $\mathbb{T} = \mathbb{Z}^{\geq 0}$ so that the dynamics given in (2) read as follows:

$$x(t+1) = A\,x(t) + B\,u(t), \tag{26a}$$

$$y(t) = C\,x(t) + D\,u(t). \tag{26b}$$

The following assumption is made all throughout this section and is motivated by the subsequent Proposition 4.

Assumption 2. As for system (26), pair (C, A) is observable and $\det(A) \neq 0$.

The moving average is one of the most widely used methods in applications to filter high-frequency, zero-mean noises from discrete-time measurements [44]. In this section, the moving average of the output of system (26) is characterized.

Define, for all $j \in \mathbb{Z}^{\geq 0}$, $j \geq 1$, $t \geq N$, the *fictitious* output:

$$\tilde{y}_j(t) := y(t) - Du(t) + \textstyle\sum_{k=1}^{j} CA^{-k}B\,u(t+k-1), \tag{27}$$

and on its basis the following *auxiliary* output, which is the *moving average* of the fictitious output $\tilde{y}_j(t)$:

$$\xi_N(t) := \tfrac{1}{N+1}\textstyle\sum_{j=0}^{N} \tilde{y}_j(t-j), \quad t \geq N. \tag{28}$$

It is worth pointing out that, although the fictitious output $\tilde{y}_j(t)$ is a function of the future values $u(t+k-1)$, $k = 1, \ldots, j$, of the state (whence, it is not measurable at the current time t), the auxiliary output $\xi_N(t)$ does not depend on future values of the state (whence, it is measurable at the current time t, because $\tilde{y}_j(t-j)$ is measurable at time t). Note that $\tilde{y}_j(t)$ is a sort of non-causal, free output response of system (26) that is related to the future values of x.

The following technical lemma is used in the proof of the subsequent Proposition 3.

Lemma 6. *Let $x(t)$ and $y(t)$ be the state-response and output-response of system* (26)*, respectively, and assume* $\det(A) \neq 0$*; for each $j \in \mathbb{Z}^{\geq 0}$ and $t \in \mathbb{Z}^{\geq 0}$, one has:*

$$y(t) = CA^{-j}x(t+j) + Du(t) - \textstyle\sum_{k=1}^{j} CA^{-k}Bu(t+k-1). \tag{29}$$

Proof. By definition, $y(t) = C\,x(t) + D\,u(t)$ for all times $t \in \mathbb{Z}^{\geq 0}$, thus (29) holds for all $t \in \mathbb{Z}^{\geq 0}$ and $j = 0$. Assume now that (29) holds for all $t \in \mathbb{Z}^{\geq 0}$ and for some $j \in \mathbb{Z}^{\geq 0}$. Thus, since $\det(A) \neq 0$, one has $x(t+j) = A^{-1}x(t+j+1) - A^{-1}Bu(t+j)$, which yields

$$\begin{aligned} y(t) &= CA^{-j-1}x(t+j+1) - CA^{-j-1}Bu(t+j) \\ &\quad + Du(t) - \textstyle\sum_{k=1}^{j} CA^{-k}Bu(t+k-1) \\ &= CA^{-j-1}x(t+j+1) + Du(t) \\ &\quad - \textstyle\sum_{k=1}^{j+1} CA^{-k}Bu(t+k-1) \end{aligned}$$

Therefore, (29) holds by induction on j. □

The following proposition characterizes the relation between the output-response $\xi_N(t)$ and the state-response $x(t)$.

Proposition 3. *Let system* (26) *be given and assume* $\det(A) \neq 0$*. For all $N \in \mathbb{Z}^{\geq 0}$ and $t \in \mathbb{Z}^{\geq 0}$, $t \geq N$, one has:*

$$\xi_N(t) = \tfrac{1}{N+1} C(\textstyle\sum_{j=0}^{N} A^{-j})x(t). \tag{30}$$

Proof. By (27) and Lemma 6, one has $\tilde{y}_j(t-j) = CA^{-j}x(t)$. Thus, (30) follows by its definition. □

The only assumption needed to relate $\tilde{y}_j(t-j)$ with $x(t)$ by (30) is $\det(A) \neq 0$. In the following proposition, it is shown that such an assumption is indeed necessary to guarantee the existence of an analytic function relating $\tilde{y}_j(t-1)$ with $x(t)$, in the unforced case.

Proposition 4. *Consider the unforced system obtained from system* (26) *by letting $u = 0$. There exists $k \in \mathbb{R}_0^{\omega}[[x]]^p$ such that $k(x(t)) = \tilde{y}_j(t-1)$ only if* (7) *holds.*

Proof. Note that $x(t) = A\,x(t-1)$ and $\tilde{y}_j(t-1) = C\,x(t-1)$. Let $k(x) = \sum_{|\ell|=1}^{+\infty} H_\ell x^\ell$. Thus, $k(Ax) = \sum_{|\ell|=1}^{+\infty} H_\ell (Ax)^\ell$, whence if $k(x(t)) = k(Ax(t-1)) = Cx(t-1)$, then letting $C_{-1} = \begin{bmatrix} H_{e_1} & \cdots & H_{e_n} \end{bmatrix}$, where e_i is the multi-index coinciding with the i-th column of the identity matrix E, *i.e.*, such that $H_{e_i} = \frac{\partial^{e_i} k(x)}{\partial x^{e_i}}$, $i = 1, \dots, n$, one has $C_{-1}A = C$. By Lemma 4, there is such a C_{-1} if and only if (7) holds. □

If (C, A) is observable, then, by Lemma 5, condition (7) holds if and only if $\det(A) \neq 0$, thus showing that Assumption 2 is necessary to allow one to use the moving average of the output to estimate the state of system (26).

The following theorem characterizes the observability of system (26) from the auxiliary output $\xi_N(t)$.

Theorem 6. *Under Assumption 2, let $\xi_N(t)$ be defined as in* (28)*, for some $N \in \mathbb{Z}^{\geq 0}$. System* (26) *is observable from the auxiliary output $\xi_N(t)$ if and only if*

$$\operatorname{rank}(\textstyle\sum_{j=0}^{i} A^{-j}) = n. \tag{31}$$

Proof. Let $G_N = \frac{1}{N+1} C(\sum_{j=0}^{N} A^{-j})$ so that $\xi_N(t) = G_N x(t)$. Hence, the observability matrix of (G_N, A) is

$$\begin{aligned} \begin{bmatrix} G_N \\ \vdots \\ G_N A^{n-1} \end{bmatrix} &= \tfrac{1}{N+1} \begin{bmatrix} C(\sum_{j=0}^{N} A^{-j}) \\ \vdots \\ C(\sum_{j=0}^{N} A^{-j})A^{n-1} \end{bmatrix} \\ &= O_n(C, A)(\textstyle\sum_{j=0}^{N} A^{-j}). \end{aligned}$$

Since $O_n(C, A)$ is non-singular, system (26) is observable from $\xi_N(t)$ if and only if (31) holds. □

The following proposition provides necessary and sufficient conditions for condition (31) to hold.

Proposition 5. *Let $N \in \mathbb{Z}^{\geq 0}$ be fixed; under the assumption* $\det(A) \neq 0$*, condition* (31) *holds if and only if*

$$\sigma(A) \cap \left\{ s \in \mathbb{C} : \textstyle\sum_{j=0}^{N} \frac{1}{s^j} = 0 \right\} = \emptyset.$$

Proof. First, it is worth noticing that $\operatorname{rank}(\sum_{j=0}^{N} A^{-j}) = \operatorname{rank}(T(\sum_{j=0}^{N} A^{-j})T^{-1}) = \operatorname{rank}(\sum_{j=0}^{N} (TAT^{-1})^{-j})$ for any non-singular matrix $T \in \mathbb{C}^{n\times n}$, *i.e.*, (31) is independent of a linear change of basis. Thus, assume, without loss of generality, that A^{-1} is in the Jordan normal form, *i.e.*, $A^{-1} = \operatorname{block\_diag}(J_1, \dots, J_\nu)$, where, for $\ell = 1, \dots, \nu$,

$$J_\ell = \begin{bmatrix} \lambda_\ell & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_\ell \end{bmatrix}.$$

By [45], $A^{-j} = \operatorname{block\_diag}(J_1^j, \dots, J_\nu^j)$, where, for $j \in \mathbb{Z}^{\geq 0}$,

$$J_\ell^j = \begin{bmatrix} \lambda_\ell^j & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_\ell^j \end{bmatrix},$$

with $*$ denoting a generic element, possibly different from 0, $\ell = 1, \dots, \nu$. Therefore, $\sum_{j=0}^{N} A^{-j}$ has full rank if and only if $\sum_{j=0}^{N} \lambda_\ell^j \neq 0$ for each $\lambda_\ell \in \sigma(A^{-1})$. The statement follows since $\sigma(A) = \{\lambda \in \mathbb{C} : \frac{1}{\lambda} \in \sigma(A^{-1})\}$. □

Corollary 3. *Under Assumption 2, let $N \in \mathbb{Z}^{\geq 0}$ be fixed. Hence, system* (26) *is observable from the following output:*

$$\begin{bmatrix} \xi_N(t) \\ \xi_{N+1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{N+1} \sum_{j=0}^{N} \tilde{y}_j(t-j) \\ \frac{1}{N+2} \sum_{j=0}^{N+1} \tilde{y}_j(t-j) \end{bmatrix}. \tag{32}$$

Proof. By the same reasoning used in the proof of Theorem 6, system (26) is observable from the output (32) if and only if

$$\Omega_N := \begin{bmatrix} O_n(C, A) & 0 \\ 0 & O_n(C, A) \end{bmatrix} \begin{bmatrix} \sum_{j=0}^{N} A^{-j} \\ \sum_{j=0}^{N+1} A^{-j} \end{bmatrix} \tag{33}$$

has rank n. Since

$$\begin{bmatrix} -E & E \end{bmatrix} \begin{bmatrix} \sum_{j=0}^{N} A^{-j} \\ \sum_{j=0}^{N+1} A^{-j} \end{bmatrix} = A^{-(N+1)},$$

and $\det(A) \neq 0$, the rightmost matrix factor in the expression given in (33) has rank n. On the other hand, the rank of the first term in the expression given in (33) is $2\operatorname{rank}(O_n(C, A))$. By the Sylvester rank inequality [45], $\operatorname{rank}(\Omega_N) \geq 2\operatorname{rank}(O_n(C, A)) - n$. Since $\operatorname{rank}(O_n(C, A)) = n$, and hence $\operatorname{rank}(\Omega_N) = n$, system (26) is observable from the output (32). □

VI. STATE ESTIMATION FROM THE MOVING AVERAGE OF THE OUTPUT FOR DISCRETE-TIME SYSTEMS

In this section, a procedure is proposed to design a state observer for system (26) on the basis of the knowledge of the moving average of the output.

In view of Proposition 3 and Corollary 3, note that, by (27), for $N \in \mathbb{Z}^{\geq 0}$ and $t \geq N$, one has

$$\begin{aligned}\xi_N(t) = \tfrac{1}{N+1}\textstyle\sum_{j=0}^{N}&(y(t-j) - Du(t-j)\\ &+ \textstyle\sum_{k=1}^{j} CA^{-k}Bu(t-j+k-1)), \qquad (34a)\\ \xi_{N+1}(t) = \tfrac{1}{N+2}\textstyle\sum_{j=0}^{N+1}&(y(t-j) - Du(t-j)\\ &+ \textstyle\sum_{k=1}^{j} CA^{-k}Bu(t-j+k-1)). \qquad (34b)\end{aligned}$$

Then, define

$$G_N := \tfrac{1}{N+1} C \textstyle\sum_{j=0}^{N} A^{-j}, \tag{34c}$$

and let Ω_N be defined as in (33). In the following theorem (similar to Theorem 3), a state observer for system (26), which uses the moving averages $\xi_N(t)$ and $\xi_{N+1}(t)$, is given.

Theorem 7. *Under Assumption 2, let $N \in \mathbb{Z}^{\geq 0}$ be fixed. Letting $\xi_N(t)$ and $\xi_{N+1}(t)$ be given by* (34) *for all $t \in \mathbb{Z}^{\geq 0}$, $t \geq N+1$, define, for all $j \in \mathbb{Z}^{\geq 0}$, $j \leq n-1$,*

$$\begin{aligned}\tilde{\xi}_N(t+j) &= \xi_N(t+j) - \textstyle\sum_{k=0}^{j-1} G_N A^{j-1-k} Bu(t+k),\\ \tilde{\xi}_{N+1}(t+j) &= \xi_{N+1}(t+j) - \textstyle\sum_{k=0}^{j-1} G_{N+1} A^{j-1-k} Bu(t+k).\end{aligned}$$

Hence, one has that, for all $t \in \mathbb{Z}^{\geq 0}$, $t \geq N$,

$$x(t) = \Omega_N^{\dagger}\left(\begin{bmatrix} \tilde{\xi}_N(t) \\ \vdots \\ \tilde{\xi}_N(t+n-1) \\ \tilde{\xi}_{N+1}(t) \\ \vdots \\ \tilde{\xi}_{N+1}(t+n-1) \end{bmatrix}\right). \tag{35}$$

Proof. By Proposition 3, it results that $\xi_N(t) = G_N x(t)$. Since $\tilde{\xi}_N(t+j) = G_N A^j x(t)$ and $\tilde{\xi}_{N+1}(t+j) = G_{N+1} A^j x(t)$, for all $j \in \mathbb{Z}^{\geq 0}$, $j \leq n-1$, one has that (35) follows by its definition, because system (26) is observable from $[\, G_N^\top \quad G_{N+1}^\top \,]^\top x(t)$, by Corollary 3. □

The state $x(t+n)$ of system (26) can be estimated by forward propagating the estimate $x(t)$ given by (35); to simulate the system and to compute $x(t)$, the knowledge of the output $y(\gamma)$ and of the input $u(\gamma)$ of system (26) for all times $\gamma \in [t-i-1, t+n-1]$, $\gamma \in \mathbb{Z}^{\geq 0}$, is required. If such measurements are affected by additive noises, it may be preferable, rather than directly using the inverse of the observability map, to use either a Luenberger observer or a Kalman filter to estimate the current state of system (26). Toward this end, define the symbols $\bar{\xi}_N := [\, \xi_N^\top(t) \quad \xi_{N+1}^\top(t) \,]^\top$ and $\bar{G}_N = [\, G_N^\top \quad G_{N+1}^\top \,]^\top$, and consider the following remarks.

Remark 3 (*Luenberger observer*). Let Assumption 2 hold, let $N \in \mathbb{Z}^{\geq 0}$ be given and let $\xi_N(t)$, $\xi_{N+1}(t)$, G_N be given by (34). A state observer for system (26) is given, for all times $t \in \mathbb{Z}^{\geq 0}$, $t \geq N+1$, by

$$\check{x}(t+1) = A\check{x}(t) + Bu(t) + S(\bar{\xi}_N(t) - \bar{G}_N \check{x}(t)), \tag{36}$$

where S is a matrix (which exists, by Corollary 3) such that the eigenvalues of $A - S\bar{G}_N$ have modulus less than 1. System (34), (36) is briefly referred to as *Luenberger observer with moving average* (36).

Remark 4 (*Kalman filter*). Let the assumptions of Remark 4 hold. Let $\hat{x}(t|\tau)$ denote the estimate of the state $x(t)$ of system (26) based on measurements of the outputs $\xi_N(\tau)$ and $\xi_{N+1}(\tau)$ up to time $\tau \in \mathbb{Z}^{\geq 0}$, $\tau \leq t$. The filter state is updated according to the following system:

$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + S(\bar{\xi}_N(t) - \bar{G}_N \hat{x}(t|t-1)), \tag{37a}$$

where the gain matrix S solves a Riccati equation related to the covariance errors of the (gaussian, white) noises that affect system (26) (see, *e.g.*, [23], [22]). The filter generates the estimate $\hat{x}(t|t)$ of the current state $x(t)$ of system (26) through the following output:

$$\hat{x}(t|t) = \hat{x}(t|t-1) + W(\bar{\xi}_N(t) - \bar{G}_N \hat{x}(t|t-1)), \tag{37b}$$

where W is a matrix related to the gain S. The existence of the matrices S and W is guaranteed by Corollary 3. System (34), (37) is briefly referred to as *Kalman filter with moving average* (37).

The following theorem guarantees that systems (36) and (37) are asymptotic state observers for (26).

Theorem 8. *Let Assumption 2 hold. Letting $x(t)$, $\check{x}(t)$ and $\hat{x}(t|t)$ be the state-responses of systems* (26), (36), *and* (37), *respectively, one has that*

$$\lim_{t\to\infty}(x(t) - \hat{x}(t)) = 0 \quad \textit{and} \quad \lim_{t\to\infty}(x(t) - \hat{x}(t|t)) = 0.$$

Proof. The proof follows directly by Theorem 7. □

In the following remark, the filtering properties of the state observers given in (36) and (37) are highlighted.

Remark 5. Let the assumptions of Theorem 8 hold. Let $\psi(t) = x(t) - \check{x}(t)$ and $\varpi(t) = x(t) - \hat{x}(t|t)$; assume that the output y is affected by an additive noise $d(t)$, *i.e.*, $y(t) = C\,x(t) + D\,u(t) + d(t)$. The transfer function from d to ψ is given by

$$(zE - A + S\bar{G}_N)^{-1} S \begin{bmatrix} \frac{1}{N+1}\frac{1-z^{-N-1}}{1-z^{-1}} E \\ \frac{1}{N+2}\frac{1-z^{-N-2}}{1-z^{-1}} E \end{bmatrix}, \tag{38}$$

whereas the transfer function from d to ϖ is

$$((E - W\bar{G}_N)(zE - A + S\bar{G}_N)^{-1} S + W) \begin{bmatrix} \frac{1-z^{-N-1}}{(N+1)(1-z^{-1})} E \\ \frac{1-z^{-N-2}}{(N+2)(1-z^{-1})} E \end{bmatrix}. \tag{39}$$

The following lemma characterizes the mean and covariance of the estimation errors in presence of random noise.

Lemma 7. *Let Assumption 2 hold and assume that $\{d(t)\}_{t=0}^{\infty}$ is a sequence of independent, identically distributed random variables with zero-mean (*i.e.*, $\mathbb{E}\{d(t)\} = 0$) and covariance matrix Σ (*i.e.*, $\mathbb{E}\{d(t)d^\top(t)\} = \Sigma$ and $\mathbb{E}\{d(t_1)d^\top(t_2)\} = 0$, for $t_1 \neq t_2$, $t_1, t_2 \in \mathbb{Z}^{\geq 0}$). Thus, letting*

$$\bar{\Sigma} = \begin{bmatrix} \frac{1}{N+1}\Sigma & \frac{1}{N+2}\Sigma \\ \frac{1}{N+2}\Sigma & \frac{1}{N+2}\Sigma \end{bmatrix},$$

one has that

$$\lim_{t\to\infty} \mathbb{E}\{\psi(t)\} = 0, \tag{40a}$$

$$\lim_{t\to\infty} \mathbb{E}\{\varpi(t)\} = 0, \tag{40b}$$

$$\lim_{t\to\infty} \mathbb{E}\{\psi(t)\psi^\top(t)\} = \lim_{k\to\infty} \sum_{j=0}^{k} R(k,j), \tag{40c}$$

$$\lim_{t\to\infty} \mathbb{E}\{\varpi(t)\varpi^\top(t)\} = \lim_{k\to\infty} \sum_{j=0}^{k} V(k,j) + W\bar{\Sigma}W^\top, \tag{40d}$$

where

$$R(k,j) = (A - S\bar{G}_N)^{k-j-1} S\bar{\Sigma}S^\top (A^\top - \bar{G}_N^\top S^\top)^{k-j-1},$$
$$V(k,j) = (E - W\bar{G}_N)R(k,j)(E - W\bar{G}_N)^\top.$$

Proof. First, note that $\mathbb{E}\{\frac{1}{N+1}\sum_{j=0}^{N} d(t-j)\} = \frac{1}{N+1}\sum_{j=0}^{N}\mathbb{E}\{d(t-j)\} = 0$, whereas $\mathbb{E}\{\frac{1}{(N+1)^2}\sum_{j_1=0}^{N} d(t-j_1)\sum_{j_2=0}^{N} d^\top(t-j_2)\} = \frac{1}{N+1}\Sigma$, and, similarly, $\mathbb{E}\{\frac{1}{(N+1)(N+2)}\sum_{j_1=0}^{N} d(t-j_1)\sum_{j_2=0}^{N+1} d^\top(t-j_2)\} = \frac{1}{N+2}\Sigma$. Thus, by considering that

$$\psi(t+1) = (A - S\bar{G}_N)\psi(t) + S\begin{bmatrix} \frac{1}{N+1}\sum_{j=0}^{N} d(t-j) \\ \frac{1}{N+2}\sum_{j=0}^{N+1} d(t-j) \end{bmatrix},$$

by [46], one has that

$$\mathbb{E}\{\psi(t+1)\} = (A - S\bar{G}_N)\mathbb{E}\{\psi(t)\},$$
$$\mathbb{E}\{\psi(t+1)\psi^\top(t+1)\} = S\bar{\Sigma}S^\top + (A - S\bar{G}_N)\mathbb{E}\{\psi(t)\psi^\top(t)\}(A - S\bar{G}_N)^\top.$$

Therefore, (40) follows by Remark 5 and by the fact that $\psi(t)$ and $d(t)$ are independent random variables, □

Lemma 7 highlights the filtering properties of the given observer, deriving from the fact that the entries of the matrix Σ appear in (40) are divided by either $N+1$ or $N+2$.

The following example illustrates the application of the state observers given in (36) and (37) and compares their estimates with the ones obtained by a Kalman filter.

Example 4. Consider the Markov chain depicted in Fig. 8 and assume that the only available measured output is the sum of the probabilities of staying in state 1 and 3. The dynamics of

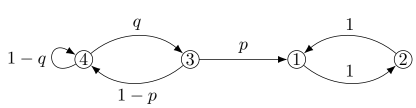


Fig. 8. The Markov chain considered in Example 4.

such a system are given by

$$x(t+1) = \begin{bmatrix} 0 & 1 & p & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 1-p & 1-q \end{bmatrix} x(t), \tag{41a}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} x(t), \tag{41b}$$

where $p, q \in [0,1]$, and the k-th entry of $x(t)$ denotes the probability of staying in state k at time t, $k = 1,\dots,4$. Numerical simulations have been carried out to compare the performances of the state observers given in (36) and (37) with the performance of a Kalman filter, assuming $p = \frac{1}{100}$, $q = \frac{1}{2}$, $x(0) = [\,\frac{1}{6}\ \frac{1}{2}\ 0\ \frac{1}{3}\,]^\top$. The constant $N = 14$ has been fixed. The gains of the Kalman filter have been designed by assuming that the dynamics in (41a) are affected by a noise $F\,d(t)$, where $F = [\ 1\ \ -1\ \ 0\ \ 0\ ]^\top$ and $d(t)$ is a Gaussian, white, zero-mean noise with covariance 10^{-3} and that the output of system (41) is affected by a a Gaussian, white, zero-mean noise with covariance 10^{-1}. The same assumptions have been made to design the gains S and W of (37):

$$S = \begin{bmatrix} -0.733 & 0.897 \\ 0.768 & -0.570 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} 0.768 & -0.570 \\ -0.733 & 0.897 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the eigenvalues of $A - S[\ C_{-i}^\top\ \ C_{-i-1}^\top\ ]^\top$ have modulus less than 1, the same gain has been used for system (36). A numerical simulation has been carried out letting the output $y(t)$ of system (41) be affected by a Gaussian, white, zero-mean noise with covariance 10^{-1} and letting the dynamics of the system be affected by the additive noise $F\,d(t)$, where F is the matrix given above and d is a Gaussian, white, zero-mean noise with covariance 10^{-3}, so that the Kalman filter is tuned to the noises affecting the system. Fig. 9 depicts the results of this simulation. As shown by Fig. 9, the performances of the

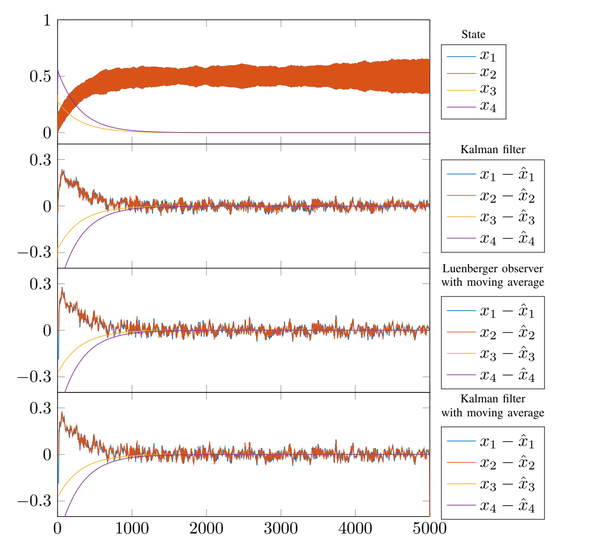


Fig. 9. Example 4: state-response of system (41), of the Kalman filter and of the state observers given in (36) and (37) when the output and the system dynamics are affected by a Gaussian additive noise.

proposed state observers are very similar to the performance of the Kalman filter, thus highlighting the effectiveness of the proposed state observers in the case of Gaussian noises with known covariance matrix. Another simulation has been carried out letting the output be affected by a noise uniformly distributed in $[-0.3, 0.3]$. Fig. 10 depicts the results of such a simulation. As shown by Fig. 10, the state observers given in (36) and (37) provide comparable estimates of the current state of system (41). Furthermore, both of them are less sensitive to uniformly distributed noises than the Kalman filter thanks to the filtering properties of the moving average. These two simulations highlight the effectiveness of the proposed observation scheme in presence of unknown measurement noise. In fact, if such a noise is Gaussian, then the proposed observer performs similarly to the Kalman filter, which is known to be

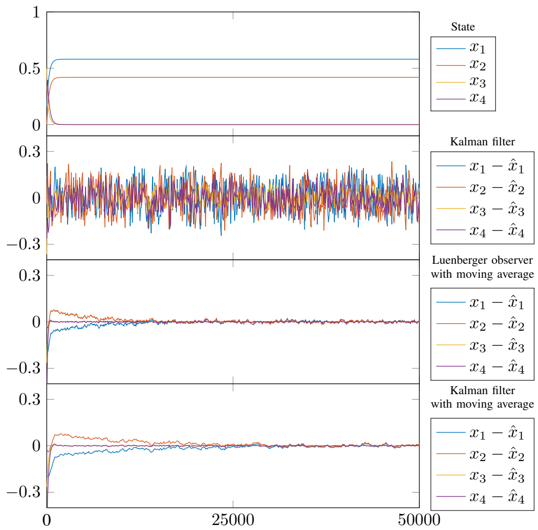


Fig. 10. Example 4: state-response of system (41), of the Kalman filter and of the state observers given in (36) and (37) when the output is affected by a uniformly distributed noise.

optimal. On the other hand, if the measurement noise is not Gaussian, then the proposed observer performs better than the Kalman filter showing improved filtering properties.

In real-world applications, continuous-time systems are controlled and monitored by digital devices. The main objective of the following example is to show how the tools given in Sections IV and VI can be adapted to deal with the sampled-data scenario.

Example 5. Consider the circuit depicted in Fig. 11 and assume that the only available measured output is the voltage across $\mathcal{C}$. The dynamics of such a system are given by

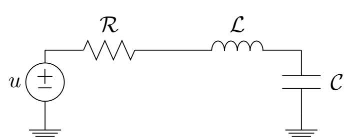


Fig. 11. The RLC electrical circuit considered in Example 5.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\mathcal{L}\mathcal{C}} & -\frac{\mathcal{R}}{\mathcal{L}} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{\mathcal{L}\mathcal{C}} \end{bmatrix} u, \tag{42a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \tag{42b}$$

where x_1 denotes the voltage across the capacitor and x_2 is its time-derivative. Assume that just samples of the input u and of the output y are measurable by ideal and synchronous samplers with sampling time δ_T. The techniques given in Sections IV and VI can be adapted to deal with sampled measurements. In particular, by integrating the dynamics given in (23) with a fixed-step numerical integration method (see, *e.g.*, [47], [48] for efficient implementations) with step equal to δ_T, it is possible to use the observer (23) to estimate the state of system (42) by taking into account just sampled measurements of the input and of the output (in the following, the resulting discrete-time observer is denoted *discretized observer*). On the other hand, assuming that the input u can be nicely approximated by its sampled counterpart fed to a zero-order holder, with sampling time δ_T, it is possible to firstly find a discrete-time system that approximates the trajectories of the continuous-time system (42) at the sampling times (see, *e.g.*, [49]) and to use system (37) to estimate the state of such a discrete-time system (in the following, this discrete-time observer is denoted *state observer for the discretized system*).

A numerical simulation has been carried out to compare the performances of these two state observers with the ones of a discretized Kalman filter, assuming the following (normalized) values for the circuit: $\mathcal{R} = \frac{1}{2}$, $\mathcal{C} = 1$, $\mathcal{L} = \frac{1}{2}$, and $\delta_T = 10^{-3}$. The initial condition of system (42) has been set to $x(0) = \begin{bmatrix} 5 & 0 \end{bmatrix}^\top$. The parameters of the discretized observer have been set to $\tau = 10^{-2}$, $\theta = 1$, $L = 10^{-3} \cdot \begin{bmatrix} 629.98 \\ 198.44 \end{bmatrix}$, $K = 10^{-3} \cdot \begin{bmatrix} -336.23 & 0 \\ -808.37 & 0 \end{bmatrix}$. On the other hand, the state observer for the discretized system has been designed by using (37), with $N = 10$, $S = 10^{-3} \cdot \begin{bmatrix} 0.49 & 0.49 \\ 0.23 & 0.23 \end{bmatrix}$, $W = 10^{-3} \cdot \begin{bmatrix} 0.49 & 0.49 \\ 0.23 & 0.23 \end{bmatrix}$. The input u has been generated by the following system:

$$\begin{aligned} \dot{\xi}_1 &= \sin(\xi_2), & \dot{\xi}_2 &= \sin(\xi_3), \\ \dot{\xi}_3 &= \sin(\xi_1), & u &= \xi_1, \end{aligned}$$

with $\xi(0) = \begin{bmatrix} 0.1 & -0.2 & 0.3 \end{bmatrix}^\top$, which exhibits a chaotic behavior [50]. A random noise, with zero-mean, and uniformly distributed in $[-1, 1]$, has been added to the samples of the output $y_D(k)$ that, afterwards, have been quantized as signed words, with 8-bit length and 4-bit fraction length. Fig. 12 depicts the results of such a simulation. As shown by such

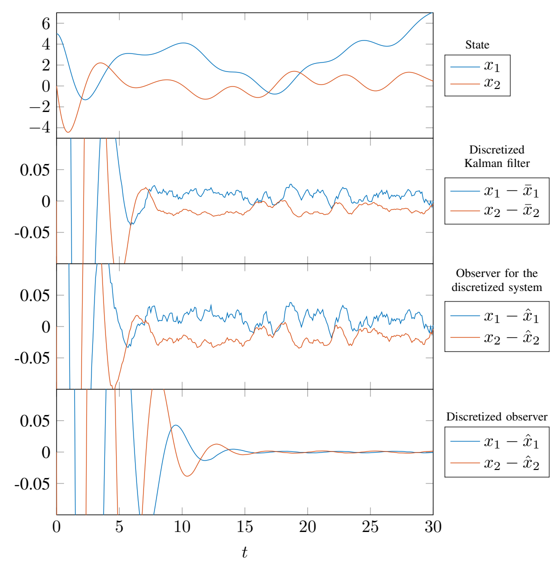


Fig. 12. Numerical simulation of the state-responses of system (42), of the discretized Kalman filter, of the state observer for the discretized system, and of the discretized observer.

a figure, both methods provide good estimates of the state of system (42) despite the presence of the additive noise, the quantization of the samples, and the approximation due to the sampling of the input u, In particular, while the state observer for the discretized system and the discretized Kalman filter have comparable performance, the discretized observer performs better than the other two systems. This is essentially due to the fact that the integral filters zero-mean noises better than the moving average and the classical Kalman filter.

VII. State observers for detectable systems

Assume that system (2) is detectable, but not observable. Thus, let W be a basis of $\ker(O_{n-1}(C,A))$ and let V be such that $\mathrm{rank}([\ W \quad V\ ]) = n$. Thus, define the matrix $T = [\ W \quad V\ ]$. By [34], using the linear change of coordinates $\breve{x} = Tx = [\ \breve{x}_i^\top \quad \breve{x}_o^\top\ ]^\top$, system (2) can be rewritten as

$$\Delta\breve{x}_i(t) = \breve{A}_{i,i}\,\breve{x}_i(t) + \breve{A}_{i,o}\,\breve{x}_o(t) + \breve{B}_i\,u(t), \tag{43a}$$

$$\Delta\breve{x}_o(t) = \breve{A}_{o,o}\,\breve{x}_o(t) + \breve{B}_o\,u(t), \tag{43b}$$

$$y(t) = \breve{C}_o\,\breve{x}_o(t) + \breve{D}\,u(t), \tag{43c}$$

where either $\sigma(\breve{A}_{i,i}) \subset \{s \in \mathbb{C} : \mathrm{Re}(s) < 1\}$, if $\mathbb{T} = \mathbb{R}^{\geq 0}$, or $\sigma(\breve{A}_{i,i}) \subset \{s \in \mathbb{C} : |s| < 1\}$, if $\mathbb{T} = \mathbb{Z}^{\geq 0}$, $\begin{bmatrix} \breve{A}_{i,i} & \breve{A}_{i,o} \\ 0 & \breve{A}_{o,o} \end{bmatrix} = TAT^{-1}$, $\begin{bmatrix} \breve{B}_i \\ \breve{B}_o \end{bmatrix} = TB$, $[\,0\ \breve{C}_o\,] = CT^{-1}$, $\breve{D} = D$, and pair $(\breve{C}_o, \breve{A}_{o,o})$ is observable (this is usually known as the *Kalman decomposition with respect to inobservability*). Hence, assuming that $\det(\breve{A}_{o,o}) \neq 0$, the techniques given in Sections IV and VI can be used to design a state observer for

$$\Delta\breve{x}_o(t) = \breve{A}_{o,o}\,\breve{x}_o(t) + \breve{B}_o\,u(t), \tag{44a}$$

$$y = \breve{C}_o\,\breve{x}_o(t) + \breve{D}\,u(t). \tag{44b}$$

Hence, let $\hat{\breve{x}}_o(t)$ be the estimate of the state of system (44) obtained by such a state observer and consider the system

$$\Delta\hat{\breve{x}}_i(t) = \breve{A}_{i,i}\,\hat{\breve{x}}_i(t) + \breve{A}_{i,o}\,\hat{\breve{x}}_o(t) + \breve{B}_i\,u(t). \tag{45}$$

By [34], $\hat{\breve{x}}_i(t)$ is an asymptotic estimate fo $\breve{x}_i(t)$, *i.e.*, $\lim_{t\to\infty} \breve{x}_i(t) - \hat{\breve{x}}_i(t) = 0$. Therefore, letting $\hat{\breve{x}}_o(t)$ be an estimate of the state of system (44) obtained by using the techniques given in Sections IV and VI for $\mathbb{T} = \mathbb{R}^{\geq 0}$ and for $\mathbb{T} = \mathbb{Z}^{\geq 0}$, respectively, and letting $\hat{\breve{x}}_i(t)$ be the state-response of system (45), an estimate of the state of system (2) can be obtained as $\hat{x} = T^{-1}[\ \hat{\breve{x}}_i^\top \quad \hat{\breve{x}}_o^\top\ ]^\top$.

VIII. Concluding remarks

It is well known that the time-integral and the moving average present a low-pass behavior; in this paper, it has been shown that their use allows the design of observers that are less sensitive to high-frequency noise than the Kalman filter and classical state observers, such as the Luenberger and the sliding-mode ones. The theoretical results have been corroborated by several numerical simulations that have highlighted the effectiveness of the proposed observers. In particular, it has been verified in simulation that the tools developed for purely continuous-time and discrete-time systems can be used to design efficient state observers based on sampled (possibly, noisy and quantized) measurements of the output, thus making the proposed observer suitable for practical applications.

In the continuous-time case, simulations of a mechanical system and a two tank system have shown increased filtering properties, for high-frequency noise, with respect to the Kalman filter and to a sliding-mode observer.

As for the discrete-time case, two simulations on an academic Markov chain have shown that, besides having increased filtering properties for uniformly distributed noise, the proposed observers perform quite similarly to the Kalman filter in the case of Gaussian noise, for which the latter is optimal.

Very relevant for practical applications is the performance shown in the case of a sampled-data system, namely a simple RLC circuit with quantized and noisy measurements, where it is evident that a great advantage is obtained with the proposed method when the approach is the widely used one of discretizing an observer designed in continuous-time.

Therefore, even if the implementation of the proposed observers requires more computational effort, they may be preferred to classic approaches [39], [51] due to their improved filtering properties with respect to high-frequency noises.

Note that, in order to implement both the continuous-time and the discrete-time observers proposed in this paper, one has to compute (off-line) a solution $C_{-k} = CA^{-k}$ to the linear equation $C = C_{-k}A^k$. If the matrix A has both large and small singular values, this problem may be numerically ill conditioned. However, in the literature, several techniques are given to specifically solve this problem. For instance, the matrix C_{-k} can be found: (i) by applying an iterated Tikhonov regularization [52]; (ii) by using a preconditioner for A [53]; (iii) by coupling the above techniques with a singular value decomposition [54]; (iv) by using a Newton-like method [55].

The focus of this paper is on linear systems, but a first idea on how to deal with nonlinear ones comes from the Flow Box Theorem [37]; for each $f \in \mathbb{R}^\infty[[x]]$ and each point x^o such that $f(x^o) \neq 0$ (in the following referred to as *regular point*), there exists a diffeomorphism $\varrho = \Phi(x)$ about x^o, $\Phi \in \mathbb{R}^\infty[[x]]$, such that $\phi(x^o) = 0$ and $(\frac{\partial\Phi}{\partial x} f) \circ \Phi^{-1}(\varrho) = e_1$, where e_1 is the first column of E, *i.e.*, each smooth vector field can be locally rectified about any regular point. Thus, locally about regular points, any smooth system is diffeomorphic to $\dot{\varrho} = e_1$, $y = h(\varrho)$, and hence a directional integral of the system is locally given by $k \circ \Phi(x)$, where $k(\varrho)$ satisfies $\frac{\partial k(\varrho)}{\partial \varrho_1} = h(\varrho)$, and, therefore, can be computed by direct integration or approximated by formal series. Based on the function $k \circ \Phi(x)$, observers based on the integral of the output can be designed for nonlinear plants about regular points. On the other hand, the characterization of directional integrals in the neighborhood of equilibria deserves more attention and requires further research as well as the characterization of the performance of the proposed observers in presence of process noise and non-modeled (possibly, nonlinear) dynamics.

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