Hadamard star configurations

HADAMARD STAR CONFIGURATIONS

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Abstract. Bocci, Carlini, and Kileel have shown that the square-free Hadamard product of a finite set of points Z that all lie on a line ℓ in $\mathbb{P}^n$ produces a star configuration of codimension n. In this paper we introduce a construction using the Hadamard product to construct star configurations of codimension c. In the case that $c = n = 2$, our construction produces the star configurations of Bocci, Carlini, and Kileel. We will call any star configuration that can be constructed using our approach a Hadamard star configuration. Our main result is a classification of Hadamard star configurations.

1. Introduction

During the past decade, star configurations in $\mathbb{P}^n$ have been identified as an interesting family of projective varieties. Roughly speaking (formal definitions are postponed until the next section), a star configuration is a union of linear spaces that have been intersected in a prescribed manner. Under the appropriate conditions, a star configuration can be a set of points in $\mathbb{P}^n$. These varieties and their corresponding ideals have been found to have interesting extremal properties (e.g., in relation to the containment problem for symbolic powers of ideals [4]). At the same time, there are nice descriptions of the minimal generators and minimal graded free resolutions of these ideals (see [8]). Other papers that have studied star configurations include [5, 9, 12].

Quite recently (and perhaps, surprisingly) it has been shown that star configurations of points also arise when one studies the Hadamard product of projective varieties. Given two varieties X and Y in $\mathbb{P}^n$, the Hadamard product of X and Y, denoted $X \star Y$, is the closure of the rational map from $X \times Y$ to $\mathbb{P}^n$ given by

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \to [a_0b_0 : \cdots : a_nb_n].$$

Note that the name is inspired by the Hadamard product of two matrices in linear algebra where one multiplies two matrices of the same size entry-wise. The notion of a Hadamard product of varieties first appeared in [6, 7] in the study of the geometry of Boltzmann machines. The Hadamard product is also a useful tool in tropical geometry (see [11, Theorem 5.5.11]). More recently Bocci, Kileel, and the first author [3] developed some of the properties of Hadamard products of linear spaces. Additional results on Hadamard products can be found in [1, 2, 10]. It was in the paper of Bocci, Carlini, and Kileel (see [3, Theorem 4.7]) that a connection to star configurations was made. In particular, if ℓ is a line in $\mathbb{P}^n$ and Z is a set of points on ℓ, then under some suitable hypotheses, the

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Hadamard product of Z with itself at most $\min\{n, |Z|\}$ times is a star configuration of points.

It is natural to ask if the Hadamard product can be used to construct star configurations of various codimensions, not just codimension n. We introduce such a construction in this paper. A star configuration that can be constructed via our approach will be called a *Hadamard star configuration*. When $n = 2$, we show that our Hadamard star configurations are the star configurations produced by Bocci, et al.'s procedure.

It can be shown that there are star configurations that cannot be Hadamard star configurations (see Example 2.12). It thus behooves us to ask if one can determine if a given star configuration is a Hadamard star configuration. One of the main results of this paper is a classification of Hadamard star configurations (see Theorem 3.1 and Remark 3.3).

Our paper is structured as follows. In the next section we present the definitions and results mentioned in the introduction. In particular, we introduce Hadamard star configurations. In section three we present our classification result. In the final section, we examine the problem of constructing Hadamard star configurations.

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2. Background results

In this section, we introduce the relevant background. Throughout this paper $S = \mathbb{C}[x_0, \ldots, x_n]$. We will denote by S_i the homogeneous degree i piece of S. Given $I \subset S$, a homogeneous ideal, we let $V(I) \subset \mathbb{P}^n$ denote the variety defined by the vanishing locus of all elements of I.

We begin by defining a star configuration.

Definition 2.1. A set $\mathcal{L} = \{L_1, \ldots, L_r\} \subset S_1$ is *generally linear* if $r \geq n+1$ and if any choice of $n+1$ distinct elements in $\mathcal{L}$ are linearly independent.

Definition 2.2. Let $\mathcal{L} = \{L_1, \ldots, L_r\} \subset S_1$ and let c be an integer $1 \leq c \leq n$. We introduce the variety

$$\mathbb{X}_c(\mathcal{L}) = \bigcup_{1 \leq i_1 < \cdots < i_c \leq r} V(L_{i_1}, \ldots, L_{i_c}) \subset \mathbb{P}^n.$$

If $\mathcal{L}$ is generally linear we say that $\mathbb{X}_c(\mathcal{L})$ is a *codimension c star configuration*, or simply, a *star configuration*.

Remark 2.3. Note that the star configuration $\mathbb{X}_c(\mathcal{L})$ is completely determined by c and by the points $[L_1], \ldots, [L_r] \in \mathbb{P}S_1$. Alternatively, given $\mathcal{L} = \{L_1, \ldots, L_r\}$, we can consider the hyperplanes $H_i = V(L_i)$. These hyperplanes uniquely determine $\mathbb{X}_c(\mathcal{L})$, thus we will sometimes write $\mathbb{X}_c(H_1, \ldots, H_r)$.

Example 2.4. Let $\mathcal{L} = \{L_1, \ldots, L_5\} \subset S_1$ where $S = \mathbb{C}[x_0, x_1, x_2]$ be a generally linear set. If $c = 2$, then the star configuration $\mathbb{X}_2(\mathcal{L}) \subset \mathbb{P}^2$ is the 10 points of intersection of the five lines, as shown in Figure 1. If $c = 1$, then the star configuration $\mathbb{X}_1(\mathcal{L})$ is the union of the five lines in Figure 1. Notice that the lines resemble a star in this case.

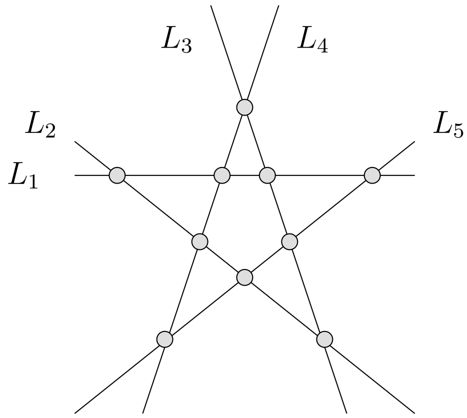

FIGURE 1. A star configuration of 10 points in $\mathbb{P}^2$

We now turn our attention to the required background on Hadamard products.

Definition 2.5. Let $P = [a_0 : \cdots : a_n]$ and $Q = [b_0 : \cdots : b_n]$ be two points in $\mathbb{P}^n$. If there exists some i such that $a_ib_i \neq 0$, then the *Hadamard product* of P and Q, denoted $P \star Q$, is given by

$$P \star Q = [a_0b_0 : \cdots : a_nb_n].$$

Otherwise, we say $P \star Q$ is not defined.

Definition 2.6. The *Hadamard product of two varieties* $X, Y \subset \mathbb{P}^n$ is given by

$$X \star Y = \overline{\{P \star Q \mid P \in X, Q \in Y, \text{ and } P \star Q \text{ is defined}\}},$$

where we mean the closure in $\mathbb{P}^n$ with respect to the Zariski topology. The r-th *Hadamard power* of X is $X^{\star r} = X \star X^{\star(r-1)}$ where $X^0 = [1 : \cdots : 1]$.

Remark 2.7. The Hadamard product can be defined in terms of the Segre product and a specific projection map, as in [3, Definition 2.2]. One can show that our definition is equivalent to the definition given above (see [3, Remarks 2.4 and 2.5]). The defining ideal of $I(X \star Y)$ can be computed using elimination; see [3, Remark 2.6] for an algorithm.

When using the Hadamard product, we sometimes need to ensure we have enough non-zero coordinates in our points. The following notation shall prove useful.

Definition 2.8. Let Δ_i be the set of points of $\mathbb{P}^n$ which have at most $i+1$ non-zero coordinates, that is, Δ_i is the set of points which have at least $n-i$ zero coordinates. In particular, $\Delta_n = \mathbb{P}^n$.

We now introduce a special set of linear forms.

Definition 2.9. A set of linear forms $\mathcal{L} = \{L_1, \ldots, L_r\} \subset S_1$ is a *Hadamard set* if there exist an $L = a_0x_0 + \cdots + a_nx_n \in S_1$ and points $P_1, \ldots, P_r \in \mathbb{P}^n$ with $P_i = [p_0(i) : \cdots : p_n(i)] \in \mathbb{P}^n \setminus \Delta_{n-1}$, that is, $p_j(i) \neq 0$ for any i, j, such that

$$[L_i] = \left[\frac{a_0}{p_0(i)}x_0 + \cdots + \frac{a_n}{p_n(i)}x_n\right] \in \mathbb{P}S_1.$$

We say that $\mathcal{L}$ is a *strong Hadamard set* if, in addition,

$$\sum_{j=0}^{n} a_j p_j(i) = 0 \text{ for } i = 1, \dots, r.$$

Remark 2.10. Recall that the support of a linear form L is the set of variables appearing in L with a non-zero coefficient. The set $\mathcal{L} = \{L_1, \dots, L_r\}$ is a Hadamard set if and only if all the linear forms L_i have the same support. To see why, just define L with the same support as the elements of $\mathcal{L}$ and all coefficients equal one; the P_i are then defined by taking the inverse of the coefficients of the linear forms L_i. However a necessary and sufficient condition to be a strong Hadamard set is more complicated, as shown in Theorem 3.1.

We are now able to define our main object of interest.

Definition 2.11. A star configuration $\mathbb{X}_c(\mathcal{L})$ is a *Hadamard star configuration* if $\mathcal{L}$ is a strong Hadamard set.

Example 2.12. Not every star configuration is a Hadamard star configuration. For example, let $S = \mathbb{C}[x_0, x_1, x_2]$ and consider $\mathcal{L} = \{x_0, x_1, x_2\}$. It is easy to see that the star configuration

$$\mathbb{X}_2(\mathcal{L}) = \{[1:0:0], [0:1:0], [0:0:1]\},$$

which consists of the three coordinate points of $\mathbb{P}^2$, is not a Hadamard star configuration. Indeed, by Remark 2.10, $\mathcal{L}$ is not even a Hadamard set.

We use the name Hadamard because of the connection with the Hadamard product, as shown in the next two lemmas.

Lemma 2.13. *Let $L = a_0 x_1 + \cdots + a_n x_n \in S_1$ and $P = [p_0 : \cdots : p_n] \in \mathbb{P}^n \setminus \Delta_{n-1}$. Then*

$$P \star V(L) = V(L') \text{ where } L' = \tfrac{a_0}{p_0} x_0 + \cdots + \tfrac{a_n}{p_n} x_n.$$

Proof. For any $Q = [q_0 : \cdots : q_n] \in V(L)$, the linear form L' vanishes on the point $P \star Q = [p_0 q_0 : \cdots : p_n q_n]$; in fact

$$\frac{a_0}{p_0}(p_0 q_0) + \cdots + \frac{a_n}{p_n}(p_n q_n) = a_0 q_0 + \cdots + a_n q_n = 0.$$

Conversely, for any $R = [r_0 : \cdots : r_n] \in V(L')$, the equation

$$\frac{a_0}{p_0}(r_0) + \cdots + \frac{a_n}{p_n}(r_n) = 0$$

implies $R' = [\frac{r_0}{p_0} : \cdots : \frac{r_n}{p_n}] \in V(L)$. Since $R = P \star R'$, we get $R \in P \star V(L)$. □

Lemma 2.14. *Let $\mathcal{L} = \{L_1, \dots, L_r\} \subset S_1$.*

(i) *$\mathcal{L}$ is a Hadamard set for L and $P_1, \dots, P_r$ if and only if $V(L_i) = P_i \star V(L)$ for all $1 \le i \le r$.*

(ii) *$\mathcal{L}$ is a strong Hadamard set for L and $P_1, \dots, P_r$ if and only if $V(L_i) = P_i \star V(L)$ and $P_i \in V(L)$ for all $1 \le i \le r$.*

Proof. We prove only (ii) since (i) can also be obtained from this proof. Let $\mathcal{L} = \{L_1, \ldots, L_r\}$ be a strong Hadamard set for $L \in S_1$ and $P_1, \ldots, P_r \in \mathbb{P}^n \setminus \Delta_{n-1}$. If $L = a_0x_0 + \cdots + a_nx_n$ and $P_i = [p_0(i) : \cdots : p_n(i)]$, by definition of a Hadamard set, $L_i = \frac{a_0}{p_0(i)}x_0 + \cdots + \frac{a_n}{p_n(i)}x_n$ for each i. Lemma 2.13 then implies $V(L_i) = P_i \star V(L)$. The condition $\sum_{j=0}^n a_jp_j(i) = 0$ is simply the condition that each $P_i \in V(L)$. Reversing this argument gives the reverse implication. □

We round out this section by explaining the connection to [3]. We first need the following variant of the Hadamard product.

Definition 2.15. If $\mathbb{X}$ is a finite set of points in $\mathbb{P}^n$, then the r-th *square-free Hadamard product* of $\mathbb{X}$ is

$$\mathbb{X}^{\underline{\star} r} = \{P_1 \star \cdots \star P_r \mid P_i \in \mathbb{X} \text{ and } P_i \neq P_j\}.$$

Bocci, Carlini, and Kileel then proved the following result (we have specialized their result).

Theorem 2.16 ([3, Theorem 4.7])**.** *Let ℓ be a line in $\mathbb{P}^n$ such that $\ell \cap \Delta_{n-2} = \varnothing$, and let $\mathbb{X} \subset \ell$ be a set of $m > n$ points with $\mathbb{X} \cap \Delta_{n-1} = \varnothing$. Then $\mathbb{X}^{\underline{\star} n}$ is a star configuration of $\binom{m}{n}$ points of $\mathbb{P}^n$.*

When $n = 2$, we now show that the star configurations produced by the above result are always Hadamard star configurations. Moreover, even if $n \neq 2$, if we assume one extra condition on the line ℓ, we can create more Hadamard star configurations. We have the following:

Theorem 2.17. *Let ℓ be a line in $\mathbb{P}^n$ such that $\ell \cap \Delta_{n-2} = \varnothing$, and let $\mathbb{X} \subset \ell$ be a set of $m > n$ points with $\mathbb{X} \cap \Delta_{n-1} = \varnothing$.*

(i) *If $[1 : 1 : \cdots : 1] \in \ell$, then $\mathbb{X}^{\underline{\star} n}$ is a Hadamard star configuration.*
(ii) *If $n = 2$, then $\mathbb{X}^{\underline{\star} 2}$ is a Hadamard star configuration.*

Proof. Let $\mathbb{X} = \{P_1, \ldots, P_m\} \subset \ell$ with $m > n$.

(i). If $[1 : \cdots : 1] \in \ell$, then $\ell \subset \ell^{\star 2} \subset \ell^{\star 3} \subset \cdots \subset \ell^{\star(n-1)}$ (see, for example, [10] after Definition 4.6). By [3, Corollary 3.7], the variety $\ell^{\star(n-1)}$ is a hyperplane defined by one linear form, say

$$\ell^{\star(n-1)} = V(L) \text{ where } L = a_0x_0 + \cdots + a_nx_n.$$

Note that since $\mathbb{X} \subset \ell \subset \ell^{\star(n-1)}$, every point $P \in \mathbb{X}$ satisfies $L(P) = 0$.

For $i = 1, \ldots, m$, let L_i be the linear form such that $V(L_i) = P_i \star V(L)$. As shown in the proof of [3, Theorem 4.7], these linear forms are generally linear. Moreover, $\mathbb{X}^{\underline{\star} n}$ is the star configuration given by $\mathcal{L} = \{L_1, \ldots, L_m\}$. We have now shown that $\mathcal{L}$ is a strong Hadamard set by Lemma 2.14.

(ii). When $n = 2$, then $\ell = \ell^{\star(n-1)}$ is given by a linear form $L = a_0x_0 + \cdots + a_nx_n$. We can then repeat the above argument. □

Remark 2.18. If $[1 : \cdots : 1] \notin \ell \subset \mathbb{P}^n$, then $\ell^{\star(n-1)}$ is still given by a linear form L, and $\mathbb{X}^{\underline{\star} n}$ is still the star configuration given by $\mathcal{L} = \{L_1, \ldots, L_m\}$ where $V(L_i) = P_i \star V(L)$. So, in particular, $\mathcal{L}$ is a Hadamard set. However, it may not be a strong Hadamard set because there is no guarantee that L vanishes on every $P_i \in \mathbb{X}$. For instance, let $\ell \subset \mathbb{P}^3$ be the line through $[0:1:2:1]$, $[0:1:2:1]$ and $[2:-1:0:1]$. We have that $\ell^{\star 2}$ is the plane through $[0:0:2:1]$, $[0:-1:0:1]$ and $[2:0:0:1]$, and this plane does not contain the line ℓ.

3. Characterization of Hadamard star configurations

As shown in Example 2.12, not every star configuration can be a Hadamard star configuration. It is then natural to ask if we can classify what star configurations are Hadamard star configurations. The main result of this section is a solution to this question.

The classification of Hadamard star configurations reduces to classifying strong Hadamard sets.

Theorem 3.1. *Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be a generally linear set where $L_i = a_0(i)x_0 + \cdots + a_n(i)x_n$ for $i = 1, \dots, r$. Then $\mathcal{L}$ is a strong Hadamard set if and only if*

1. *$a_j(i) \neq 0$ for all i, j; and*
2. *there exists an $\mathbf{a} \in \mathbb{C}^{n+1}$ without zero coordinates such that $M\mathbf{a} = \mathbf{0}$, where*

$$M = \begin{pmatrix} \frac{1}{a_0(1)} & \cdots & \frac{1}{a_n(1)} \\ \vdots & \vdots & \vdots \\ \frac{1}{a_0(r)} & \cdots & \frac{1}{a_n(r)} \end{pmatrix}.$$

Moreover, if $n = 2$, condition (2) is equivalent to the condition that $\operatorname{rk}(M) \leq 2$.

Proof. Assume that $\mathcal{L}$ is a strong Hadamard set. Thus there exists a linear form $L = a_0x_0 + \cdots + a_nx_n$ and r points $P_i = [p_0(i) : \cdots : p_n(i)] \in \mathbb{P}^n \smallsetminus \Delta_{n-1}$ such that $P_i \star V(L) = V(L_i)$ for $i = 1, \dots, r$. By Lemma 2.13, this implies that

$$a_j(i) = \frac{a_j}{p_j(i)} \text{ for all } 1 \leq i \leq r \text{ and } 0 \leq j \leq n.$$

Since $\mathcal{L}$ is generally linear, then $a_j \neq 0$ for all $j = 0, \dots, n$. To see this, assume for a contradiction, that $a_n = 0$. Then the above expression for $a_n(i)$ implies $a_n(i) = 0$ for all $i = 1, \dots, r$. Thus $L_1, \dots, L_r$ are linear forms not involving x_n, and hence any $n+1$ elements of $\{L_1, \dots, L_r\}$ are linear dependent, contradicting our assumption that the elements of $\mathcal{L}$ are generally linear.

Because $a_j \neq 0$ for all $j = 0, \dots, n$, we thus have that $a_j(i) \neq 0$ for all $1 \leq i \leq r$ and $0 \leq j \leq n$, thus proving (1).

For each $i = 1, \dots, r$, we also have $P_i \in V(L)$ since $\mathcal{L}$ is a strong Hadamard set. Since $p_j(i) = \frac{a_j}{a_j(i)}$ for all $1 \leq i \leq r$ and $0 \leq j \leq n$, we have

$$a_0p_0(i) + \cdots + a_np_n(i) = \frac{a_0^2}{a_0(i)} + \cdots + \frac{a_n^2}{a_n(i)} = 0$$

for all $i = 1, \dots, r$. But this means that $\mathbf{a} = \begin{pmatrix} a_0^2 & a_1^2 & \cdots & a_n^2 \end{pmatrix}^T \in \mathbb{C}^{n+1}$ satisfies $M\mathbf{a} = \mathbf{0}$. Since each $a_j^2 \neq 0$ for $j = 0, \dots, n$, the vector $\mathbf{a}$ shows that (2) is satisfied.

Now, by assuming that (1) and (2) hold, we need to find a linear form $L = a_0x_0 + \cdots + a_nx_n$ and points $P_i = [p_0(i) : \cdots : p_n(i)] \in \mathbb{P}^n \smallsetminus \Delta_{n-1}$ for $i = 1, \dots, r$, such that $V(L_i) = P_i \star V(L)$ for $i = 1, \dots, r$ and $P_i \in V(L)$ for $i = 1, \dots, r$.

By (2), there exists a vector $\mathbf{a} \in \mathbb{C}^{n+1}$ with no zero coordinates such that $M\mathbf{a} = \mathbf{0}$. Say $\mathbf{a}^T = \begin{pmatrix} b_0 & \cdots & b_n \end{pmatrix}$, and let $a_i^2 = b_i$; note that because $\mathbb{C}$ is algebraically closed, each $a_i \in \mathbb{C}$.

We now claim that $L = a_0x_0 + \cdots + a_nx_n$ and

$$P_i = \left[\frac{a_0}{a_0(i)} : \frac{a_1}{a_1(i)} : \cdots : \frac{a_n}{a_n(i)}\right] \text{ for } i = 1, \ldots, r$$

are the desired linear form and points. We first note that (1) implies that the points P_i are defined. As well, each $a_j \neq 0$, so each $P_i \in \mathbb{P}^n \smallsetminus \Delta_{n-1}$. By Lemma 2.13, $P_i \star V(L) = V(L_i)$ for each i since

$$L_i = a_0(i)x_0 + \cdots + a_n(i)x_n = \frac{a_0}{\frac{a_0}{a_0(i)}}x_0 + \cdots + \frac{a_n}{\frac{a_n}{a_n(i)}}x_n.$$

Finally, each $P_i \in V(L)$ since

$$a_0\left(\frac{a_0}{a_0(i)}\right) + \cdots + a_n\left(\frac{a_n}{a_n(i)}\right) = \frac{a_0^2}{a_0(i)} + \cdots + \frac{a_n^2}{a_n(i)} = 0,$$

where the final equality follows from the fact that $\mathbf{a}^T = \begin{pmatrix} a_0^2 & \cdots & a_n^2 \end{pmatrix}$ is a solution to $M\mathbf{a} = \mathbf{0}$.

To prove the final statement assume $n = 2$. Condition (1) implies that the matrix M given below exists:

$$M = \begin{pmatrix} \frac{1}{a_0(1)} & \frac{1}{a_1(1)} & \frac{1}{a_2(1)} \\ \vdots & \vdots & \vdots \\ \frac{1}{a_0(r)} & \frac{1}{a_1(r)} & \frac{1}{a_2(r)} \end{pmatrix}.$$

Obviously the condition (2) implies that $rk(M) \leq 2$. Now we prove by contradiction that $rk(M) \leq 2$ implies condition (2). Suppose that the equation $M\mathbf{x} = \mathbf{0}$ does not have a solution without zero coordinates. By (2), there is at least one solution to $M\mathbf{x} = \mathbf{0}$, say $\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$, with some $b_i = 0$. Without loss of generality, assume that $b_2 = 0$. So, for all $i = 1, \ldots, r$, we have

$$\frac{b_0}{a_0(i)} + \frac{b_1}{a_1(i)} = \frac{a_1(i)b_0 + a_0(i)b_1}{a_0(i)a_1(i)} = 0.$$

We thus have $a_1(i)b_0 + a_0(i)b_1 = 0$ for all $i = 1, \ldots, r$. But then every L_i vanishes at $[b_1 : b_0 : 0]$, contradicting the fact that $\mathbb{X}$ is a star configuration. So $M\mathbf{x} = \mathbf{0}$ must have a solution with no zero-coordinates. $\square$

Remark 3.2. Note that to check that condition (2) holds it is enough to check one of the following statements:

- the ideal of $\{\mathbf{x} \mid M\mathbf{x} = \mathbf{0}\} \subset \mathbb{C}^{n+1}$ does not contain the product $\Pi_0^n x_i$;
- there is no linear combination of the rows of M having exactly one non-zero entry.

Remark 3.3. From Theorem 3.1 it follows that if $\mathbb{X} = \mathbb{X}_c(L_1, \ldots, L_r) \subset \mathbb{P}^n$ is a star configuration with $L_i = a_0(i)x_0 + \cdots + a_n(i)x_n$ for all i, then $\mathbb{X}$ is a Hadamard star configuration if and only if

(1) $a_j(i) \neq 0$ for all i, j; and

(2) the matrix

$$M = \begin{pmatrix} \frac{1}{a_0(1)} & \cdots & \frac{1}{a_n(1)} \\ \vdots & \vdots & \vdots \\ \frac{1}{a_0(r)} & \cdots & \frac{1}{a_n(r)} \end{pmatrix}$$

is such that there exists an $\mathbf{x} \in \mathbb{C}^{n+1}$ without zero coordinates such that $M\mathbf{x} = 0$.

Moreover, if $n = 2$, the condition (2) is equivalent to $\mathrm{rk}(M) \leq 2$.

4. Toward explicit constructions of Hadamard star configurations in $\mathbb{P}^n$

In [3] the authors show how to construct zero dimensional star configurations by using a sufficiently general line $\ell \subset \mathbb{P}^n$. In particular, after one fixes points $P_1, \ldots, P_r \in \ell$ with no zero coordinates, one then computes all possible n-wise Hadamard products of n of the points without repetition. In this way one can explicitly produce the points of the star configuration without even knowing the set $\mathcal{L}$. But, if $n \geq 3$, there is no explicit description of the *hyperplanes* defining the star configuration. In what follows, we want to describe a star configuration in $\mathbb{P}^n$ using a hyperplane H and points lying on it.

Given a hyperplane $H \subset \mathbb{P}^n$ and points $P_i \in H$, it is not enough to assume that no zero coordinates appear in order to obtain a star configuration intersecting the hyperplanes $P_i \star H$. For example, it is easy to construct examples in which the hyperplanes $P_1 \star H, P_2 \star H$ and $P_3 \star H$ intersect in codimension smaller than three.

Example 4.1. Consider the plane $H \subset \mathbb{P}^3$ defined by $4x_0 - 5x_1 + 2x_2 - x_3 = 0$ and the points

$$P_1 = [1:1:1:1], P_2 = [2:1:-2:-1], P_3 = [5:4:10:20] \in H.$$

By direct computations we obtain the planes

$$\begin{aligned} H_1 &= P_1 \star H : 4x_0 - 5x_1 + 2x_2 - x_3 = 0, \\ H_2 &= P_2 \star H : 2x_0 - 5x_1 - x_2 + x_3 = 0, \\ H_3 &= P_3 \star H : 16x_0 - 25x_1 + 4x_2 - x_3 = 0, \end{aligned}$$

and we see that $H_3 \supset H_1 \cap H_2$.

Definition 4.2. The *coordinate points* $E_0, \ldots, E_n \in \mathbb{P}^n$ are the points E_i having exactly one non-zero coordinate in position $i+1$.

The next theorem is the first step towards generalizing the work of [3]. In particular, the next result shows that if one mimics the construction of [3] for points on a hyperplane (instead of points on a line), and if one produces a star configuration, then there must be a special relationship between the points on the hyperplane and the coordinate points.

Theorem 4.3. *Let $H \subset \mathbb{P}^n$ be a hyperplane and $P_1, \ldots, P_r \in H$ be points such that $P_i \notin \Delta_{n-1}$ and H does not contain any of the coordinate points E_i. Set $H_i = P_i \star H$.*

If

$$\mathbb{X}_c(H_1, \ldots, H_r)$$

is a codimension $c \geq 2$ star configuration, then there is no rational normal curve containing the coordinates points $E_0, \ldots, E_n$ and the points P_i, P_j, and P_k for all possible choices of $1 \leq i < j < k \leq r$.

Proof. Assume $P_i = [p_0(i) : \cdots : p_n(i)] \in \mathbb{P}^n \setminus \Delta_{n-1}$, so
$$H_i : \frac{a_0}{p_0(i)} x_0 + \cdots + \frac{a_n}{p_n(i)} x_n = 0,$$
where $H = a_0x_0 + \cdots + a_nx_n$.

The hyperplane H_i corresponds to the point $H_i{}^\vee = \left[\frac{a_0}{p_0(i)} : \cdots : \frac{a_n}{p_n(i)}\right] \in (\mathbb{P}^n)^\vee$. If $\mathbb{X}_c(H_1, \ldots, H_r)$ is a star configuration, then H_i, H_j, H_k $(i < j < k)$ meet in codimension 3, that is, the points $H_i{}^\vee$, $H_j{}^\vee$, $H_k{}^\vee$ are not collinear.

Let $\sigma : (\mathbb{P}^n)^\vee \dashrightarrow (\mathbb{P}^n)^\vee$ be the standard Cremona trasformation $\sigma([y_0 : \cdots : y_n]) = \left[\frac{1}{y_0} : \cdots : \frac{1}{y_n}\right]$. By the well known properties of Cremona's maps, we have that $H_i{}^\vee$, $H_j{}^\vee$, $H_k{}^\vee$ are collinear if and only if the points P_i, P_j, P_k, $E_0, \ldots, E_n$ lie on a rational normal curve, and the result is proved. $\square$

Remark 4.4. Note that Theorem 4.3 only gives a necessary condition. In fact, the non-existence of the rational normal curves assures us that the three planes $P_i \star H, P_j \star H$, and $P_k \star H$ intersect in codimension three, that is, their equations are linearly independent. However, we do not know what happens for four or more planes H_i, that is, we do not know whether their equations are linearly independent. Hence we do not know whether the collection of *all* the linear forms is generally linear.

Remark 4.5. Note that in the case $n = 2$, the necessary condition of Theorem 3.3 is always satisfied since a line cannot intersect a rational normal curve of $\mathbb{P}^2$, that is, an irreducible conic, in three points. Note that the conic through P_i, P_j, E_0, E_1, E_2 is irreducible since the points P_i and P_j do not have zero coordinates.

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