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Original
On the uniform algebraic observability of multi-switching linear systems / Menini, L.; Possieri, C.; Tornambe, A.. - In: INTERNATIONAL JOURNAL OF CONTROL. - ISSN 0020-7179. - (2019), pp. 1-11. [10.1080/00207179.2019.1695949]

## Availability:

This version is available at: 11583/2777913 since: 2020-01-08T13:31:59Z
Publisher:
Taylor and Francis Ltd.
Published
DOI:10.1080/00207179.2019.1695949

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# On the uniform algebraic observability of multi-switching linear systems 

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## ARTICLE HISTORY

Compiled January 8, 2020


#### Abstract

The main objective of this paper is the characterization of the notion of uniform algebraic observability for multi-switching linear systems. In particular, it is shown that a multi-switching linear system is 'generically' uniformly algebraically observable if and only if the number of available measurements is greater than the number of switching signals acting on the systems. This result is established without requiring any knowledge about the (possibly, stochastic) process generating the switching pattern. Moreover, since such systems can be used to model sensor networks, the results given in this paper can 'generically' be used to design simple observers for sensor networks able to cope with sensor failures and data losses.


## KEYWORDS

Uniform algebraic observability, multi-switching systems; observer design.

## 1. Introduction

The fast advancement of reliable wireless communications and of low-cost electronic devices paved the way to the development of multi-sensor networks, i.e., huge groups of dedicated sensors that sense, process, record and communicate the physical conditions of a plant (Akyildiz, Su, Sankarasubramaniam, \& Cayirci, 2002; Biswas, Jain, Ghosh, \& Agrawal, 2006; Sinopoli, Sharp, Schenato, Schaffert, \& Sastry, 2003). Applications of such networks span from health care monitoring (Peiris, 2013) and air pollution analysis (Tsujita, Yoshino, Ishida, \& Moriizumi, 2005) to environmental monitoring (Longhi \& Marrocco, 2017; N. Wang, Zhang, \& Wang, 2006) and safeness of constructions (Hubbard et al., 2015). The main advantages of such networks with respect to classical sensing protocols are their ability to cope with sensor failures (Schenato, Sinopoli, Franceschetti, Poolla, \& Sastry, 2007) and their scalability (Yick, Mukherjee, \& Ghosal, 2008).

Due to the large employment of such sensor networks in practical applications, an extensive research effort has been spent to characterize their properties. In order to capture the behavior of these network in presence of data loss and sensor failures, these systems are usually modeled as plants with non-switching, continuous-time dynamics and with switching outputs (Li, Ugrinovskii, \& Orsi, 2007; Ugrinovskii, 2013). In

[^0]particular, several approaches have been proposed to design the switching signal that governs the output response of the system. For instance, in Hespanha, Naghshtabrizi, and Xu (2007); Matei, Martins, and Baras (2009), the switching signal is governed by a Markov chain whose states correspond to the possible communication topologies, whereas in Langbort, Gupta, and Murray (2006); Subbotin and Smith (2009), the switching signal is governed by independent, two-state Markov processes describing the status of the individual links. An alternative method to model sensor networks is given in Manjunatha, Verma, and Srividya (2008), by using a fuzzy logic approach.

Several approaches have been given in the literature to merge data coming from different sensors. For instance, in Subbotin and Smith (2009), the data fusion is carried out by using Kalman filters, in Ugrinovskii (2013), the estimation is carried out by using $H_{\infty}$ techniques, in Jiao and Wu (2018), the measurements are processed by using a fuzzy model, in X. Wang, Hu, Jia, and Tang (2018), support vector machines are used, whereas, in Garin and Schenato (2010), a linear consensus algorithms is proposed to compute averages of local estimates.

The main objective of this paper is to provide necessary and sufficient conditions ensuring that the uniform algebraic observability is a 'generic' property for multiswitching systems.

In particular, it is shown that if the number of outputs is greater than the number of switching signals, then multi-switching linear systems are 'generically' uniformly observable with respect to the switching signals. On the other hand, it is proved that if the number of outputs is lower than or equal to the number of switching signals, then multi-switching systems are 'generically' not uniformly observable. Furthermore, in the case of uniformly observable systems, algebraic geometry techniques are proposed to compute the matrix relating the time derivatives of the output with the state of the system and it is shown how to use such a matrix to design a state observer. The main interest in this class of systems relies on the fact that they are able to replicate the output response of those switching plants that are usually employed to model multisensor networks (see the subsequent Lemmas 1 and 2). Thus, characterizing their observability properties is the first step toward the construction of reliable and simple observers that are able to deal with sensor and link failures.

## 2. The considered class of multi-switching systems

Notation: The sets of real and integer numbers are denoted $\mathbb{R}$ and $\mathbb{Z}$, respectively; given $a \in \mathbb{R}$ and $b \in \mathbb{Z}$, define $\mathbb{R}_{\geqslant a}:=\{s \in \mathbb{R}: s \geqslant a\}$ and $\mathbb{Z}_{\geqslant b}:=\{s \in \mathbb{Z}: s \geqslant b\} ; \mathbb{R}[x]$ denotes the ring of all the polynomials in $x:=\left[\begin{array}{lll}x_{1} & \ldots & x_{n}\end{array}\right]^{\top}$ with coefficients in $\mathbb{R}$, and $\mathbb{R}(x)$ denotes the field of all the rational functions in $x$ with coefficients in $\mathbb{R} ; I_{n}$ is the $n \times n$ identity matrix and $0_{m, n}$ is the $m \times n$ zero-matrix. Given a function $\alpha(\varepsilon)$ of a single variable $\varepsilon, \lim _{\varepsilon \backslash \varepsilon^{\star}} \alpha(\varepsilon)$ denotes the one-sided limit of $\alpha(\varepsilon)$ as $\varepsilon$ approaches $\varepsilon^{\star}$ from above; $\lim _{\varepsilon \backslash 0} \alpha\left(\varepsilon^{\star}-\varepsilon\right)$ and $\lim _{\varepsilon \backslash 0} \alpha\left(\varepsilon^{\star}+\varepsilon\right)$ are the left and the right one-sided limits of $\alpha(\varepsilon)$ at $\varepsilon^{\star}$, respectively. Given two matrices $C \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times n}$, the observability matrix associated with pair $(C, A)$ is: $\mathbb{O}(C, A):=\left[\begin{array}{lll}C^{\top} & \vdots & \left(C A^{n-1}\right)^{\top}\end{array}\right]^{\top}$. Given a vector field $f(x) \in \mathbb{R}^{n}$ and a function $h(\sigma, x) \in \mathbb{R}$, the successive directional derivatives of $h$ along $f$ are defined as: $L_{f}^{0} h=h$ and $L_{f}^{j+1} h=\left(\frac{\partial}{\partial x} L_{f}^{j} h\right) f, j \in \mathbb{Z}_{\geqslant 0}$. Consider a plant whose dynamics are described by

$$
\begin{equation*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t}=A x(t) \tag{1a}
\end{equation*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector at time $t \in \mathbb{R} ; t=0$ is the initial time. The measured output vector $y(t)=\left[\begin{array}{lll}y_{1}(t) & \ldots & y_{m}(t)\end{array}\right]^{\top} \in \mathbb{R}^{m}$ is taken by an observation network at time $t \in \mathbb{R}$ and is given by

$$
\begin{equation*}
y(t)=\tilde{C}_{\sigma(t)} x(t) \tag{1b}
\end{equation*}
$$

where $\tilde{C}_{j} \in \mathbb{R}^{m \times n}, j=0, \ldots, S$, and $\sigma: \mathbb{R} \rightarrow\{0, \ldots, S\}, S \in \mathbb{Z}_{\geqslant 1}$, is a signal (i.e, a function of time) to be specified.

Definition 1. Consider a signal $s: \mathbb{R} \rightarrow \mathcal{S}$, where $\mathcal{S}$ is a finite subset of $\mathbb{Z}$. Since the co-domain $\mathcal{S}$ of the signal $s(t)$ is a finite set, if $s(t)$ is a continuous function on $\mathbb{R}$, then $s(t)$ is necessarily constant, i.e., $s(t)=k$, for some $k \in \mathcal{S}$. If $s(t)$ is not continuous, then let $\hat{t} \in \mathbb{R}$ be the time of one of its discontinuities; since $\mathcal{S}$ is a finite set, if one of the two limits $\lim _{\varepsilon \searrow 0} s(\hat{t}+\varepsilon)$ and $\lim _{\varepsilon \searrow 0} s(\hat{t}+\varepsilon)$ exists, then it is necessarily finite; in addition, always because $\mathcal{S}$ is a finite set, a discontinuity can only be of three types:
(1.1) removable discontinuity at $t_{a} \in \mathbb{R}$, i.e., the two limits $\lim _{\varepsilon} \searrow_{0} s\left(t_{a}-\varepsilon\right)$ and $\lim _{\varepsilon \searrow 0} s\left(t_{a}+\varepsilon\right)$ exist and coincide, but $s\left(t_{a}\right) \neq \lim _{\varepsilon \searrow 0} s\left(t_{a}+\varepsilon\right)$; such a discontinuity can be removed by redefining the signal $s(t)$ so that $s\left(t_{a}\right)=\lim _{\varepsilon} \searrow 0 s\left(t_{a}+\varepsilon\right)$;
(1.2) jump discontinuity at $t_{b} \in \mathbb{R}$, i.e., the two $\operatorname{limits}_{\lim }^{\varepsilon} \searrow 0 s\left(t_{b}-\varepsilon\right)$ and $\lim _{\varepsilon \searrow 0} s\left(t_{b}+\varepsilon\right)$ exist, but they do not coincide $\left(i . e, \lim _{\varepsilon \searrow 0} s\left(t_{b}-\varepsilon\right) \neq \lim _{\varepsilon \searrow 0} s\left(t_{b}+\varepsilon\right)\right)$; in such a case, the time $t_{b}$ is called a switching time; the signal $s(t)$ can always be redefined so that $s\left(t_{b}\right)=\lim _{\varepsilon \searrow 0} s\left(t_{b}+\varepsilon\right)$, whence so to guarantee its right-continuity at the switching times;
(1.3) essential discontinuity at $t_{c} \in \mathbb{R}$, i.e., at least one of the two limits $\lim _{\varepsilon} \searrow_{0} s\left(t_{c}-\right.$ $\varepsilon)$ and $\lim _{\varepsilon \searrow 0} s\left(t_{c}+\varepsilon\right)$ does not exist; in such a case, the time $t_{c}$ is called a Zeno switching time; if $t_{c} \in \mathbb{R}$ is a Zeno switching time and $\lim _{\varepsilon \searrow 0} s\left(t_{c}+\varepsilon\right)$ exists, then the signal $s(t)$ can always be redefined so that $s\left(t_{c}\right)=\lim _{\varepsilon \searrow 0} s\left(t_{c}+\varepsilon\right)$, whence so to guarantee its right-continuity at such Zeno switching times.
If $s(t)$ has no removable discontinuities, has countable discontinuities (either Zeno or not), and is right-continuous at each discontinuity time, then it is a switching signal.

According to Definition 1, the discontinuity times (i.e., either the switching times or the Zeno switching times) of a switching signal are assumed to be countable, whence they can be bi-univocally associated with an integer at subscript and denoted $t_{i}, i \in \mathbb{Z}$; hence, there is no discontinuity in the open interval

$$
T_{j}:=\left(t_{j}, \inf \left\{t_{i}: t_{j}<t_{i}\right\}\right), \quad j \in \mathbb{Z}
$$

No minimum (or average) dwell time is assumed.
The switching signal $\sigma(t)$ is characterized by the next assumption, which is extended throughout the paper to any other switching signal, without an explicit reference.

Assumption 1. (1.1) The signal $\sigma: \mathbb{R} \rightarrow\{0, \ldots, S\}, S \in \mathbb{Z}_{\geqslant 1}$, is a switching signal defined according to Definition 1;
(1.2) the signal $\sigma(t)$ is not measured and, in particular, the discontinuity times $t_{i}$ are not known before they happen;
(1.3) the number $S+1$ of the possible values taken by the switching signal $\sigma(t)$ is not known.

The following lemma shows that the output (1b) of system (1) can be expressed as a linear function of $x$, with coefficients being polynomials in $\sigma(t)$ of degree $S$.

Lemma 1 (Menini, Possieri, and Tornambe (2014)). There exist polynomials $p_{0}, \ldots, p_{S} \in \mathbb{R}[\sigma]$ having degree $S$ such that he output of system (1) is given by

$$
\begin{equation*}
y(t)=\left(p_{0}(\sigma(t)) \tilde{C}_{0}+\ldots+p_{S}(\sigma(t)) \tilde{C}_{S}\right) x(t) \tag{2}
\end{equation*}
$$

The following lemma shows that the representation (2) can be simplified by introducing additional switching signals (satisfying Definition 1), thus obtaining an expression that is linear in $x$, with coefficients being polynomials of degree 1 (linear) in such switching signals.

Lemma 2. There exist $N \in \mathbb{Z}_{\geqslant 1}$ switching signals $\sigma_{j}: \mathbb{R} \rightarrow\left\{0, \ldots, S_{j}\right\}, j=1, \ldots, N$, and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$ such that the output of system (1) can be expressed as

$$
\begin{equation*}
y(t)=\left(C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}\right) x(t) \tag{3}
\end{equation*}
$$

Proof. Let $N=S$. Hence, let $S_{j}=1$ and define

$$
\sigma_{j}(t):= \begin{cases}1, & \text { if } \sigma(t)=j  \tag{4}\\ 0, & \text { if } \sigma(t) \neq j\end{cases}
$$

$j=1, \ldots, N$. Thus, for each $t$ such that $\sigma(t)=0$, one has

$$
C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}=C_{0}
$$

whereas, for each $t$ such that $\sigma(t)=j, j \in\{1, \ldots, S\}$,

$$
C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}=C_{0}+C_{j}
$$

Therefore, one has that (3) holds with $C_{0}=\tilde{C}_{0}$ and $C_{j}=\tilde{C}_{j}-\tilde{C}_{0}, j=1, \ldots, N$.
By Lemmas 1 and 2, the following system is considered in the remainder of the paper (with $t \in \mathbb{R}$ ) instead of (1):

$$
\begin{align*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t} & =A x(t)  \tag{5a}\\
y(t) & =\left(C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}\right) x(t) \tag{5b}
\end{align*}
$$

where $\sigma_{j}: \mathbb{R} \rightarrow\left\{0, \ldots, S_{j}\right\}, S_{j} \in \mathbb{Z}_{\geqslant 1}, j=1, \ldots, N$, are switching signals defined according to Definition 1, and satisfying Assumption 1. At each discontinuity time $t_{i}$, at least one of the switching signals $\sigma_{j}\left(t_{i}\right)$ has a discontinuity. A system in the form (5) is referred to as multi-switching.

The main objective of this paper is to give necessary and sufficient conditions ensuring that the uniform algebraic observability is a 'generic' property for the multiswitching system (5) (see the subsequent Definition 2), despite the fact that the switching signals $\sigma_{1}, \ldots, \sigma_{N}$ are not measured (whence, their discontinuity times are not known) and their co-domains are not known.

By Assumption 1, the following statements hold:
(i) if $t^{\star} \in T_{i}$, then $\frac{\mathrm{d}^{k} y(t)}{\mathrm{d} t^{k}}$ is continuous at $t=t^{\star}$ and

$$
\left.\frac{\mathrm{d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}}=\left(C_{0}+\sum_{j=1}^{N} \sigma_{j}\left(t_{i}\right) C_{j}\right) A^{k} x\left(t^{\star}\right)
$$

(ii) if $t^{\star}=t_{i}$ and $t_{i}$ is Zeno, then $\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}-\varepsilon}$ does not exists, but $\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}-\varepsilon}$ exists,

$$
\begin{equation*}
\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}+\varepsilon}=\left(C_{0}+\sum_{j=1}^{N} \sigma_{j}\left(t_{i}\right) C_{j}\right) A^{k} x\left(t^{\star}\right) . \tag{6}
\end{equation*}
$$

Similarly, if $t^{\star}=t_{i}$ and $t_{i}$ is not Zeno, then $\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}+\varepsilon}$ exists and is given by (6), whereas

$$
\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(t)}{\mathrm{d} t^{k}}\right|_{t=t^{\star}-\varepsilon}=\left(C_{0}+\sum_{j=1}^{N} \sigma_{j}\left(t_{i-1}\right) C_{j}\right) A^{k} x\left(t^{\star}\right)
$$

Therefore, the signal $\frac{\mathrm{d}^{k} y(t)}{\mathrm{d} t^{k}}$ need not be defined nor be continuous at the discontinuity times $t=t_{i}$. Hence, for the above reasons, let $y^{(k)}(t)$ denote the following rightcontinuous function of time:

$$
y^{(k)}(t):=\left.\lim _{\varepsilon \searrow 0} \frac{\mathrm{~d}^{k} y(\tau)}{\mathrm{d} \tau^{k}}\right|_{\tau=t+\varepsilon},
$$

which is defined at all $t \in \mathbb{R}$, is continuous at each time $t \neq t_{i}$, and only right-continuous at the discontinuity times $t=t_{i}$.

Let $p_{A}(\lambda)=\lambda^{n}+a_{1} \lambda^{n-1}+\cdots+a_{n}$ be the characteristic polynomial of $A$. Hence, by the Cayley-Hamilton Theorem (Kailath, 1980), $p_{A}(A):=A^{n}+a_{1} A^{n-1}+\cdots+a_{n} I_{n}=0$.

Lemma 3. For each $t \in \mathbb{R}$ (including the switching times $t_{i}$ )

$$
y^{(n)}(t)+a_{1} y^{(n-1)}(t)+\cdots+a_{n} y^{(0)}(t)=0
$$

Proof. By construction, one has $y^{(k)}(t)=\left(C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}\right) A^{k} x(t)$, for all $t \in \mathbb{R}$. Therefore, it holds that

$$
\begin{aligned}
& y^{(n)}(t)+a_{1} y^{(n-1)}(t)+\cdots+a_{n} y^{(0)}(t)= \\
& \quad\left(C_{0}+\sigma_{1}(t) C_{1}+\ldots+\sigma_{N}(t) C_{N}\right) p_{A}(A) x(t)=0
\end{aligned}
$$

According to the above lemma, the signal $y_{j}^{(n)}(t)+a_{1} y_{j}^{(n-1)}(t)+\cdots+a_{n} y_{j}^{(0)}(t)$ is a continuous function of $t \in \mathbb{R}$ (actually, it is constant and equal to 0 ), although its addends $y^{(n)}(t), a_{1} y^{(n-1)}(t), \ldots, a_{n} y^{(0)}(t)$ are only right-continuous.

It is now possible to formalize the notion of uniform algebraic observability.
Definition 2. System (5) is uniformly (with respect $\sigma_{1}, \ldots, \sigma_{N}$ ) algebraically observable if there exist $K \in \mathbb{Z}_{\geqslant 0}$ and $F \in \mathbb{R}^{n \times m(K+1)}$ such that, letting

$$
\check{x}(t)=F\left[\begin{array}{c}
y^{(0)}(t)  \tag{7}\\
\vdots \\
y^{(K)}(t)
\end{array}\right]
$$

one has $x(t)=\check{x}(t)$ for all times $t \in \mathbb{R}$.
The right-hand side of equation (7) is continuous at each $t \in \mathbb{R}, t \neq t_{i}$, and rightcontinuous at each $t=t_{i}$, since the functions $y^{(j)}(t)$ are right-continuous. In addition,
since $x(t)$ is a function being continuous also at the discontinuity times $t_{i}$ (at which $y^{(j)}(t)$ is only right-continuous), the condition $x(t)=\check{x}(t)$ for all $t \neq t_{i}$ implies that the right-hand side of (7) cannot have discontinuities, whence it is a continuous function of time on the whole $\mathbb{R}$, when system (5) is uniformly algebraically observable; clearly, if $y^{(0)}(t), \ldots, y^{(K)}(t)$ are replaced in (7) by some estimates $\hat{y}^{(0)}(t), \ldots, \hat{y}^{(K)}(t)$ for the computation of an estimate $\hat{x}(t)$ of $\check{x}(t)$, even infinitesimal errors in the recognition of some discontinuity time will imply peaking in the resulting estimation errors.

In Definition 2, observable means that $x(t)$ can be expressed as a function of $y^{(0)}(t), \ldots, y^{(K)}(t)$, for some $K$, algebraic means that such a function is linear, whereas uniform means that such a function is independent of the switching signals.

The objective of the following sections is the study of the uniform algebraic observability of system (5). The two fundamental parameters in such a study are the number $m$ of the entries of the output vector $y(t)$ and the number $N$ of the switching signals $\sigma_{j}$; it will be shown that if $m \geqslant N+1$, then system (5) is uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$, whereas if $m \leqslant N$, then system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$; in particular, it will be shown that both statements hold independently of the dimension $n$ of the state vector and of the sets of the values taken by the switching signals.

## 3. Review of algebraic geometry tools

Given $p_{1}, \ldots, p_{\ell} \in \mathbb{R}[x]$, the set

$$
\mathbf{V}\left(p_{1}, \ldots, p_{\ell}\right):=\left\{x \in \mathbb{R}^{n}: p_{j}(x)=0, j=1, \ldots, \ell\right\}
$$

is the affine variety of $\mathbb{R}^{n}$ generated by $p_{1}, \ldots, p_{\ell}$, whereas

$$
\left\langle p_{1}, \ldots, p_{\ell}\right\rangle:=\left\{\sum_{j=1}^{\ell} q_{j} p_{j}: q_{j} \in \mathbb{R}[x], j=1, \ldots, \ell\right\}
$$

is the ideal generated by $p_{1}, \ldots, p_{\ell}$ (the set $\left\{p_{1}, \ldots, p_{\ell}\right\}$ is a basis of such an ideal). An ideal that can be generated by a single polynomial is said to be principal.

Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x]$. The set $\mathbf{V}(\mathcal{I}):=\left\{x \in \mathbb{R}^{n}: p(x)=0, \forall p \in \mathcal{I}\right\}$ is a variety of $\mathbb{R}^{n}$ and $\mathbf{V}(\mathcal{I})=\mathbf{V}\left(p_{1}, \ldots, p_{\ell}\right)$, for any basis $\left\{p_{1}, \ldots, p_{\ell}\right\}$ of $\mathcal{I}$. Given any subset $\mathcal{S}$ of $\mathbb{R}^{n}, \mathbf{I}(\mathcal{S}):=\{p \in \mathbb{R}[x]: p(x)=0, \forall x \in \mathcal{S}\}$ is an ideal of $\mathbb{R}[x]$ even if $\mathcal{S}$ is not a variety and $\mathcal{S} \subseteq \mathbf{V}(\mathbf{I}(\mathcal{S}))$. In particular, $\overline{\mathcal{S}}:=\mathbf{V}(\mathbf{I}(\mathcal{S}))$ is the smallest variety of $\mathbb{R}^{n}$ containing $\mathcal{S}$, which is called the Zariski closure of $\mathcal{S}$.

Definition 3. Let $\mathcal{W} \subseteq \mathbb{R}^{\ell}$ be the set of all $\beta \in \mathbb{R}^{\ell}$ for which some property does not hold; it is said that such a property holds for 'almost all' $\beta \in \mathbb{R}^{\ell}$ (briefly, it is 'generic' in $\mathbb{R}^{\ell}$ ) if the Zariski closure $\overline{\mathcal{W}}$ of $\mathcal{W}$ does not coincide with $\mathbb{R}^{\ell}$.

Assume that a property is 'generic' in $\mathbb{R}^{\ell}$; if such a property holds at $\beta^{\circ} \in \mathbb{R}^{\ell}$, then there exists a neighborhood $\mathcal{B}$ of $\beta^{\circ}$ such that the property holds for all $\beta \in \mathcal{B}$; on the other hand, if the property does not hold at $\beta^{\circ} \in \mathbb{R}^{\ell}$, then, for any arbitrarily small neighborhood $\mathcal{B}$ of $\beta^{\circ}$, there exists another $\hat{\beta}^{\circ} \in \mathcal{B}$ such that the property holds at $\hat{\beta}^{\circ}$.

If $\mathfrak{a}=\left[\mathfrak{a}_{1} \ldots \mathfrak{a}_{n}\right]^{\top}, \mathfrak{a}_{j} \in \mathbb{Z}_{\geqslant 0}, j=1, \ldots, n$, is a multi-index, then $x^{\mathfrak{a}}:=x_{1}^{\mathfrak{a}_{1}} \cdots x_{n}^{\mathfrak{a}_{n}}$ is a monomial and the total degree (briefly, degree) of $x^{\mathfrak{a}}$, denoted $\operatorname{deg}\left(x^{\mathfrak{a}}\right)$, is defined as $\operatorname{deg}\left(x^{\mathfrak{a}}\right):=\sum_{i=1}^{n} \mathfrak{a}_{i}=:|\mathfrak{a}|$. A monomial order in $\mathbb{R}[x]$, denoted $\succ$, is a total, well
order on the set of monomials $x^{\mathfrak{a}} \in \mathbb{R}[x]$. The monomial order used in this paper is the Lexicographic monomial order (briefly, Lex order) (Cox, Little, \& O'Shea, 2015). Given any $p \in \mathbb{R}[x]$ and any monomial order $\succ$, the leading term $\operatorname{LT}(p)=c x^{\mathfrak{a}}$ is the greatest term in $p, \operatorname{LC}(p)=c$ is the leading coefficient and $\operatorname{LM}(p)=x^{\mathfrak{a}}$ is the leading monomial of $p$; hence, $\operatorname{deg}(p):=\operatorname{deg}(\operatorname{LT}(p))=|\mathfrak{a}|$. A polynomial $r \in \mathbb{R}[x]$ is reduced with respect to $\left\{p_{1}, \ldots, p_{\ell}\right\}$ if either $r=0$ in $\mathbb{R}[x]$ or no monomial of $r$ is divisible by any $\operatorname{LT}\left(p_{j}\right), j=1, \ldots, \ell$. Given an ideal $\mathcal{I}$ of $\mathbb{R}[x]$ and a monomial order $\succ$, a finite set $\mathcal{G}=\left\{g_{1}, \ldots, g_{\ell}\right\} \subseteq \mathcal{I}$ is a Gröbner basis of $\mathcal{I}$ if $\left\langle\operatorname{LT}\left(g_{1}\right), \ldots, \operatorname{LT}\left(g_{\ell}\right)\right\rangle=\langle\operatorname{LT}(\mathcal{I})\rangle$, where $\operatorname{LT}(\mathcal{I})=\left\{c x^{\mathfrak{a}}: \exists f \in \mathcal{I}: \operatorname{LT}(f)=c x^{\mathfrak{a}}\right\}$. By the Hilbert Basis Theorem, each ideal of $\mathbb{R}[x]$ admits a finite Gröbner basis. A Gröbner basis is reduced if $\mathrm{LC}\left(g_{j}\right)=1$, $j=1, \ldots, \ell$, and $g_{j}$ is reduced with respect to $\left\{g_{1}, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{\ell}\right\}, j=1, \ldots, \ell$. Each ideal $\mathcal{I} \neq\langle\emptyset\rangle$ has a unique reduced Gröbner basis, with respect to the chosen monomial order $\succ$ (Cox et al., 2015). Let $\mathcal{I}$ be an ideal of $\mathbb{R}\left[x_{b}, x_{c}\right]$. The elimination ideal of $\mathcal{I}$ that eliminates $x_{b}$ is $\mathcal{I} \cap \mathbb{R}\left[x_{c}\right]$.

Given two polynomial rings $\mathcal{R}_{a}$ and $\mathcal{R}_{b}$, if an element (e.g., a polynomial or an ideal) has been defined taking $\mathcal{R}_{a}$ as the ambient ring, the coercion is that operation for which such an element can be thought of as if it had been defined with $\mathcal{R}_{b}$ being the ambient ring, when possible and convenient.

## 4. Some algebraic tools for the observation problem for polynomial systems

In this section, it is shown how the tools reviewed in Section 3 can be used to characterize the observability of non-linear polynomial systems (see also Menini et al., 2014; Menini, Possieri, \& Tornambe, 2016; Menini, Possieri, \& Tornambe, 2019).

Consider the following polynomial system:

$$
\begin{equation*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t}=f(x(t)), \quad y(t)=h(x(t)) \tag{8a}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}, f \in \mathbb{R}^{n}[x]$ and $h \in \mathbb{R}^{m}[x]$. Let $y^{(j)}(t):=\frac{\mathrm{d}^{j} y(t)}{\mathrm{d} t^{j}}, j \in \mathbb{Z}_{\geqslant 0}$ (note that the definition of $y^{(j)}(t)$ given in the following Section 2 coincides with the definition above in absence of discontinuities, as in this section). Roughly speaking, since $h$ and $f$ are polynomial, pair $(h, f)$ is observable if the state vector $x(t)$ at time $t \in \mathbb{R}$ can be expressed as a function of $y^{(0)}(t), \ldots, y^{(L)}(t)$, for some sufficiently high $L \in \mathbb{Z}, L \geqslant n$ (see Theorem 1 of Inouye, 1977); this is correlated with the injectivity of the following observation map:

$$
y_{e, L}:=\left[\begin{array}{c}
y^{(0)} \\
\vdots \\
y^{(L)}
\end{array}\right]=\left[\begin{array}{c}
h(x) \\
\vdots \\
L_{f}^{L} h(x)
\end{array}\right]=: H_{L}(x) .
$$

Definition 4. If there exists a neighborhood $\mathcal{B}_{x \circ}$ of $x^{\circ}$ and a function $x=F_{L}\left(y_{e, L}\right)$ from $\mathcal{B}_{x^{\circ}}$ to $\mathbb{R}^{n}$ such that $y_{e, L}=H_{L}\left(F_{L}\left(y_{e, L}\right)\right)$, for all $y_{e, L} \in \mathcal{A}_{x^{\circ}}$, where $\mathcal{A}_{x^{\circ}}$ is the image of $\mathcal{B}_{x^{\circ}}$ through $H_{L}(\cdot)$, then pair $(h, f)$ is locally observable about $\left(x^{\circ}, H_{L}\left(x^{\circ}\right)\right)$. If $\mathcal{B}_{x^{\circ}}$ coincides with the whole $\mathbb{R}^{n}$, then pair $(h, f)$ is globally observable. If $\mathcal{B}_{x^{\circ}}$ coincides with the whole $\mathbb{R}^{n}$ and $F_{L}\left(y_{e, L}\right)$ is a linear function of $y_{e, L}$, then pair $(h, f)$ is algebraically observable.

Assumption 2. There exists $L \in \mathbb{Z}_{\geqslant n}$ and $x^{\circ} \in \mathbb{R}^{n}$ such that $\operatorname{rank}\left(J_{L}\left(x^{\circ}\right)\right)=n$, where $J_{L}(x):=\frac{\partial H_{L}(x)}{\partial x}$.

Assumption 2 guarantees the local observability of pair $(h, f)$, as stated in the following lemma, whose proof is straight-forward by the Implicit Function Theorem (Burckel, 1980), and which holds if one is working over both $\mathbb{R}$ and $\mathbb{C}$.

Lemma 4. If Assumption 2 holds for some $L \in \mathbb{Z}_{\geqslant 0}$, then it holds with $L$ substituted by $L+j, j \in \mathbb{Z}_{\geqslant 0}$. Under Assumption 2, pair $(h, f)$ is locally observable about $\left(x^{\circ}, H_{L}\left(x^{\circ}\right)\right)$.

Consider the following ideal of $\mathcal{R}_{a}=\mathbb{R}\left[x, y_{e, L}\right]$ :

$$
\mathcal{I}_{L}=\left\langle y_{e, L}-H_{L}(x)\right\rangle .
$$

The set of all polynomial constraints existing among the entries of $y_{e, L}$ (called output embeddings) is given by $\mathcal{J}_{L}=\mathcal{I}_{L} \cap \mathbb{R}\left[y_{e, L}\right]$. In particular, in order to find if there exists a (possibly, local) left inverse of $x=H_{L}\left(y_{e, L}\right)$, first coerce $\mathcal{J}_{L}$ into $\mathbb{R}\left[y_{e, L}\right]$, define the quotient ring $\mathbb{R}\left[y_{e, L}\right] / \mathcal{J}_{L}$, and let $\mathbb{K}\left\{y_{e, L}\right\}$ be the field of the rational functions having numerator and denominator in $\mathbb{R}\left[y_{e, L}\right] / \mathcal{J}_{L}$. Hence, fix $\mathcal{R}_{b}=\mathbb{K}\left\{y_{e, L}\right\}[x]$ as ambient ring and coerce $\mathcal{I}_{L}$ into $\mathcal{R}_{b}$; finally, consider the ideals $\mathcal{K}_{L, j}=\mathcal{I}_{L} \cap \mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right], j=1, \ldots, n$.

Lemma 5. Under Assumption 2, the ideal $\mathcal{I}_{L}$ thought of as an ideal of $\mathcal{R}_{b}$ is zerodimensional; in particular, there exists $q_{j} \in \mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right]$, of degree (as a polynomial in $\left.x_{j}\right)$ greater than or equal to 1 , such that $\mathcal{K}_{L, j}=\left\langle q_{j}\right\rangle, j=1, \ldots n$.

Proof. The ideal $\mathcal{I}_{L}$ is zero dimensional in $\mathcal{R}_{b}$. In fact, if $\mathcal{I}_{L}$ was not zero-dimensional, then for each $x^{\circ} \in \mathbb{R}^{n}$, the map $x=F_{L}\left(y_{e, L}\right)$ would have an infinite number of complex left inverses, but this would be an absurd by Lemma 4.

Since any ideal of $\mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right]$ is principal (since $x_{j}$ is a single variable), there are three possible cases: (i) $\mathcal{K}_{L, j}=\langle\emptyset\rangle$, (ii) $\mathcal{K}_{L, j}=\langle 1\rangle$, and (iii) $\mathcal{K}_{L, j}=\left\langle q_{j}\right\rangle$, for some non-constant polynomial $q_{j} \in \mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right]$. Case (i) cannot happen, because otherwise $\mathcal{I}_{L}$ is not zero-dimensional. Case (ii) cannot happen, because in that case for almost all $y_{e, L} \in \mathbf{V}\left(\mathcal{J}_{L}\right)$, the equation $y_{e, L}=H_{L}(x)$ has no (either real or complex) solutions, but this is false because such an equation has at least one real solution for almost all $y_{e, L} \in \mathbf{V}\left(\mathcal{J}_{L}\right)$. Hence, there is a non-constant $q_{j} \in \mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right]$ such that $\mathcal{K}_{L, j}=$ $\left\langle q_{j}\right\rangle$.

By Lemma 5 , for each $j \in\{1, \ldots, n\}$, there exists a non-constant polynomial $q_{j} \in$ $\mathbb{K}\left\{y_{e, L}\right\}\left[x_{j}\right]$ such that $\mathcal{K}_{L, j}=\left\langle q_{j}\right\rangle$, which shows that $x_{j}$ is an algebraic function over $\mathbb{K}\left\{y_{e, L}\right\}$. If degree $\left(q_{j}\right)=1$, then $x_{j}$ is a linear function over $\mathbb{K}\left\{y_{e, L}\right\}$, whence it is defined for almost all values of $y_{e, L} \in \mathbf{V}\left(\mathcal{J}_{L}\right)$.

Example 1. Let $n=1, m=1, f=x^{2}-\frac{1}{2} x$ and $h=x^{2}$. One has $L_{f} h=2 x^{3}-x^{2}$. Consider the ambient ring $\mathcal{R}_{a}=\mathbb{R}\left[x, y^{(0)}, y^{(1)}\right]$ and the ideal

$$
\mathcal{I}_{1}=\left\langle y^{(0)}-h, y^{(1)}-L_{f} h\right\rangle=\left\langle-x^{2}+y^{(0)},-2 x^{3}+x^{2}+y^{(1)}\right\rangle
$$

One has $\mathcal{J}_{1}=\mathcal{I}_{1} \cap \mathbb{R}\left[y^{(0)}, y^{(1)}\right]=\left\langle g_{1}\right\rangle$, where:

$$
g_{1}\left(y^{(0)}, y^{(1)}\right)=4\left(y^{(0)}\right)^{3}-\left(y^{(0)}\right)^{2}-2 y^{(0)} y^{(1)}-\left(y^{(1)}\right)^{2}
$$

which shows that $y^{(0)}$ and $y^{(1)}$ are algebraically dependent functions of time $t$, i.e., $g_{1}\left(y^{(0)}(t), y^{(1)}(t)\right)=0, \forall t \in \mathbb{R}$. In order to take into account such a constraint, instead of working over $\mathcal{R}_{a}$, one has to work over the ambient ring $\mathcal{R}_{b}=\mathbb{K}\left\{y^{(0)}, y^{(1)}\right\}[x]$, where $\mathbb{K}\left\{y^{(0)}, y^{(1)}\right\}$ is the field of rational functions having numerator and denominator in the quotient ring $\mathbb{R}\left[y^{(0)}, y^{(1)}\right] /\left\langle g_{1}\left(y^{(0)}, y^{(1)}\right)\right\rangle$. The ideal $\mathcal{I}_{1}$ can be coerced into $\mathcal{R}_{b}$; choosing the Lex order in $\mathbb{R}\left[y^{(0)}, y^{(1)}\right]$, with $y^{(0)} \succ y^{(1)}$, the reduced Gröbner basis of $\mathcal{I}_{1}$ is $\mathcal{G}_{1}=\left\langle x-\frac{y^{(0)}+y^{(1)}}{2 y^{(0)}}\right\rangle$. If $y^{(0)}=0$, by the constraint $g_{1}\left(y^{(0)}, y^{(1)}\right)=0$, one has $y^{(1)}=0$; hence, $x$ can be globally represented by:

$$
x= \begin{cases}0, & \text { if } y^{(0)}=0 \\ \frac{y^{(0)}+y^{(1)}}{2 y^{(0)}}, & \text { if } y^{(0)} \neq 0\end{cases}
$$

Another global expression of the left inverse is $x=\left(\frac{y^{(0)}+y^{(1)}}{2}\right)^{1 / 3}$, and it is worth pointing out that $\left(\frac{y^{(0)}+y^{(1)}}{2 y^{(0)}}\right)^{3}=\frac{y^{(0)}+y^{(1)}}{2}$ along the variety $\mathbf{V}\left(g_{1}\right)$, since

$$
\left(\frac{y^{(0)}+y^{(1)}}{2 y^{(0)}}\right)^{3}-\frac{y^{(0)}+y^{(1)}}{2}=-\frac{1}{8} \frac{y^{(0)}+y^{(1)}}{\left(y^{(0)}\right)^{3}} g_{1}\left(y^{(0)}, y^{(1)}\right)=0 .
$$

Pair $(h, f)$ is therefore globally observable, although the map $y=h(x)$ alone is not sufficient for the reconstruction of $x$.

In the following section, it is shown how the algebraic geometry tools reviewed in Section 3 can be employed to study the uniform observability of a switching linear system under different assumptions on the switching signal.

## 5. A motivating example

Consider system (1) characterized by $n=2, m=1, N=1$ and by the following randomly generated matrices:

$$
A=\left[\begin{array}{cc}
\frac{7}{9} & 1 \\
\frac{3}{5} & 9
\end{array}\right], \quad C_{0}=\left[\begin{array}{cc}
\frac{2}{5} & \frac{1}{9}
\end{array}\right], \quad C_{1}=\left[\begin{array}{cc}
\frac{1}{9} & \frac{4}{9}
\end{array}\right]
$$

the characteristic polynomial of matrix $A$ is $p_{A}(\lambda)=\lambda^{2}-\frac{88}{9} \lambda+\frac{32}{5}$. Let $f(x)=A x$, $h\left(\sigma_{1}, x\right)=\left(C_{0}+C_{1} \sigma_{1}\right) x$, and $y(t)=h\left(\sigma_{1}(t), x(t)\right)$, and compute

$$
\begin{aligned}
h\left(\sigma_{1}, x\right) & =\frac{5 \sigma_{1}+18}{45} x_{1}+\frac{4 \sigma_{1}+1}{9} x_{2}, \\
L_{f} h\left(\sigma_{1}, x\right) & =\frac{143 \sigma_{1}+153}{405} x_{1}+\frac{185 \sigma_{1}+63}{45} x_{2}, \\
L_{f}^{2} h\left(\sigma_{1}, x\right) & =\frac{49960 \sigma_{1}+20664}{18225} x_{1}+\frac{15128 \sigma_{1}+5256}{405} x_{2}, \\
L_{f}^{3} h\left(\sigma_{1}, x\right) & =\frac{4025824 \sigma_{1}+1421856}{164025} x_{1}+\frac{6176800 \sigma_{1}+2149344}{18225} x_{2},
\end{aligned}
$$

and so forth; hence, one has $y^{(j)}(t)=L_{f}^{j} h\left(\sigma_{1}(t), x(t)\right), j \in \mathbb{Z}_{\geqslant 0}$, where the switching signal $\sigma_{1}(t)$ takes values in $\{0,1\}$.
(i) If the switching signal is measured and the set of the values taken by the switching signal $\sigma_{1}(t)$ is not known before they are actually taken, then it is convenient to take $\mathcal{R}_{a}:=\mathbb{R}\left(\sigma_{1}\right)\left[x_{1}, x_{2}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right]$ as ambient ring, (where $L \in \mathbb{Z}_{\geqslant 0}$ denotes
the maximum number of time-derivatives that are used for the solution of the observability problem), which means that $\sigma_{1}$ is taken as a known parameter and that all the considered polynomials have coefficients that are rational functions of $\sigma_{1}$. Define the following ideals of $\mathcal{R}_{a}$ :

$$
\mathcal{I}_{j}^{a}=\left\langle y^{(0)}-h\left(\sigma_{1}, x\right), \ldots, y^{(j)}-L_{f}^{j} h\left(\sigma_{1}, x\right)\right\rangle,
$$

for $j \in \mathbb{Z}_{\geqslant 0}$. The monomials of $\mathcal{R}_{a}$ are ordered according to the Lex order, with $x_{1} \succ x_{2} \succ y^{(L)} \succ \ldots \succ y^{(0)}$. The reduced Gröbner basis $\mathcal{G}_{1}^{a}$ of $\mathcal{I}_{1}^{a}$ is $\mathcal{G}_{1}^{a}=\left\{g_{0}^{a}, g_{1}^{a}\right\}$, where:

$$
\begin{aligned}
& g_{0}^{a}=x_{1}+\frac{\left(8100 \sigma_{1}+2025\right) y^{(1)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}+\frac{\left(-74925 \sigma_{1}-25515\right) y^{(0)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441} \\
& g_{1}^{a}=x_{2}+\frac{\left(-2025 \sigma_{1}-7290\right) y^{(1)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}+\frac{\left(6435 \sigma_{1}+6885\right) y^{(0)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}
\end{aligned}
$$

the reduced Gröbner basis $\mathcal{G}_{2}^{a}$ of $\mathcal{I}_{2}^{a}$ is $\mathcal{G}_{2}^{a}=\mathcal{G}_{1}^{a} \cup\left\{g_{2}^{a}\right\}$, where:

$$
g_{2}^{a}=y^{(2)}-\frac{88}{9} y^{(1)}+\frac{32}{5} y^{(0)} ;
$$

the reduced Gröbner basis $\mathcal{G}_{3}^{a}$ of $\mathcal{I}_{3}^{a}$ is $\mathcal{G}_{3}^{a}=\mathcal{G}_{2}^{a} \cup\left\{g_{3}^{a}\right\}$, where:

$$
g_{3}^{a}=y^{(3)}-\frac{36128}{405} y^{(1)}+\frac{2816}{45} y^{(0)} .
$$

In general, one has that the reduced Gröbner basis $\mathcal{G}_{j+1}^{a}$ of $\mathcal{I}_{j+1}^{a}$ is $\mathcal{G}_{j+1}^{a}=\mathcal{G}_{j}^{a} \cup\left\{g_{j+1}^{a}\right\}$, where $g_{j+1}^{a}=y^{(j+1)}+a_{j} y^{(1)}+b_{j} y^{(0)}$, for some constants $a_{j}, b_{j} \in \mathbb{R}$, for $j \in \mathbb{Z} \geqslant 1$. From the polynomials $g_{0}^{a}, g_{1}^{a}$, one deduces that the state variables $x_{1}$ and $x_{2}$ can be expressed as linear functions of $y^{(0)}$ and $y^{(1)}$ (i.e., of the measured output and of its first time-derivative), with coefficients that are rational functions of $\sigma_{1}$ :

$$
\begin{aligned}
& x_{1}=-\frac{\left(8100 \sigma_{1}+2025\right) y^{(1)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}-\frac{\left(-74925 \sigma_{1}-25515\right) y^{(0)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441} \\
& x_{2}=-\frac{\left(-2025 \sigma_{1}-7290\right) y^{(1)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}-\frac{\left(6435 \sigma_{1}+6885\right) y^{(0)}}{5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441}
\end{aligned}
$$

since the other polynomials $g_{j}^{a}, j \geqslant 2$, are independent of $x_{1}, x_{2}$ and of the switching signal $\sigma_{1}$, one can conclude that no improvements in the above formulas could be obtained by taking into account the time-derivatives of $y(t)$ of order greater than 1 . Since the denominator $5465 \sigma_{1}^{2}+29030 \sigma_{1}+9441$ does not vanish for $\sigma_{1} \in\{0,1\}$, the above expressions can used to compute $x_{1}, x_{2}$ from the knowledge of $y^{(0)}, y^{(1)}$, but this can be done only if the switching signal $\sigma_{1}(t)$ is measured.

It is worth pointing out that, by the formal substitutions $y^{(k)} \rightarrow \lambda^{k}, k \in \mathbb{Z}_{\geqslant 0}$, in the polynomial embeddings $g_{j}^{a}=0, j \in \mathbb{Z}_{\geqslant 2}$, one obtains the characteristic polynomials associated with them: $p_{g_{2}^{a}}(\lambda)=\lambda^{2}-\frac{88}{9} \lambda+\frac{32}{5}$ (i.e., the characteristic polynomial $p_{A}(\lambda)$ of $\left.A\right), p_{g_{2}^{a}}(\lambda)=\lambda^{3}-\frac{36128 \lambda}{405}+\frac{2816}{45}=\left(\lambda+\frac{88}{9}\right)\left(\lambda^{2}-\frac{88 \lambda}{9}+\frac{32}{5}\right)$ (i.e., a multiple of $\left.p_{A}(\lambda)\right)$ and so forth $\left(p_{g_{j}^{a}}(\lambda)=\lambda^{j+1}+a_{j} \lambda+b_{j}\right.$ being a multiple of $\left.p_{A}(\lambda)\right)$. This, in particular, implies that the signals $\alpha_{2}(t)=y^{(2)}(t)-\frac{88}{9} y^{(1)}(t)+\frac{32}{5} y^{(0)}(t)$ and $\alpha_{3}(t)=$ $y^{(3)}(t)-\frac{36128}{405} y^{(1)}(t)+\frac{2816}{45} y^{(0)}(t)$ and so forth (obtained from the embeddings with the substitutions $\left.y^{(k)} \rightarrow y^{(k)}(t), k \in \mathbb{Z}_{\geqslant 0}\right)$ are continuous and equal to zero for all $t \in \mathbb{R}$, including the switching times $t_{i}$.
(ii) If the switching signal is measured and its co-domain is known before they are actually taken, one has (letting $\sigma_{1}=0$ in the expressions computed above)

$$
x_{1}=\frac{225}{1049} y^{(1)}+\frac{2835}{1049} y^{(0)}, \quad x_{2}=\frac{810}{1049} y^{(1)}-\frac{765}{1049} y^{(0)}
$$

and (letting $\sigma_{1}=1$ in the expressions computed above)

$$
x_{1}=-\frac{10125}{43936} y^{(1)}+\frac{12555}{5492} y^{(0)}, \quad x_{2}=\frac{9315}{43936} y^{(1)}-\frac{1665}{5492} y^{(0)} ;
$$

hence, by multiplying the first equations by $1-\sigma_{1}$, the second equations by $\sigma_{1}$, and adding the results, one obtains the following polynomial expressions:

$$
\begin{aligned}
& x_{1}=-\left(\frac{225}{1049}+\frac{735525 \sigma_{1}}{46088864}\right) y^{(1)}-\left(\frac{2399625 \sigma_{1}}{5761108}-\frac{2835}{1049}\right) y^{(0)}, \\
& x_{2}=\left(\frac{810}{1049}-\frac{25867825 \sigma_{1}}{46088864}\right) y^{(1)}+\left(\frac{245499 \sigma_{1}}{5661108}-\frac{765}{1049}\right) y^{(0)} .
\end{aligned}
$$

From the analysis carried out in the cases (i) and (ii), one concludes that in order to express $x_{1}$ and $x_{2}$ as linear functions of $y^{(0)}, \ldots, y^{(L)}$, for some $L \in \mathbb{Z}_{>0}$, there are actually two possibilities: the first one, which is rational in $\sigma_{1}$ and holds independently of the values taken by $\sigma_{1}$, and the second one, which is polynomial in $\sigma_{1}$, but obtained (valid) only because one has exploited in such a computation the knowledge of the possible values taken by $\sigma_{1}$. Therefore, there is no formula expressing $x_{1}$ and $x_{2}$ as linear functions of $y^{(0)}, \ldots, y^{(L)}$, being independent of $\sigma_{1}$.
(iii) One can try to answer the following question: 'is it possible to express pair $x_{1}, x_{2}$ as a function of $y^{(0)}, \ldots, y^{(L)}$, possibly not linear, but independent of $\sigma_{1}$, without knowing the set of the values taken by $\sigma_{1}$ ?'. To answer such a question, it is convenient to take $\mathcal{R}_{b}:=\mathbb{R}\left[\sigma_{1}, x_{1}, x_{2}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right]$ as ambient ring, which means that $\sigma_{1}$ is treated, instead of as a known parameter, as a single variable, like all the other variables. The ideals $\mathcal{I}_{j}^{a}, j \in \mathbb{Z} \geqslant 1$, which have been defined in case (i) as subsets of $\mathcal{R}_{a}$, can be coerced into $\mathcal{R}_{b}$. The monomials of $\mathcal{R}_{b}$ are ordered according to the Lex order, with $\sigma_{1} \succ x_{1} \succ x_{2} \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)}$. Define the elimination ideals $\mathcal{J}_{j}^{b, 1}=\mathcal{I}_{j}^{a} \cap \mathbb{R}\left[x_{1}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right]$. The reduced Gröbner basis $\mathcal{G}_{1}^{b, 1}$ of $\mathcal{J}_{1}^{b, 1}$ is $\mathcal{G}_{1}^{b, 1}=\emptyset ;$ the reduced Gröbner basis $\mathcal{G}_{j+1}^{b, 1}$ of $\mathcal{J}_{j+1}^{b, 1}$ is $\mathcal{G}_{j+1}^{b, 1}=\mathcal{G}_{j}^{b, 1} \cup\left\{g_{j+1}^{b, 1}\right\}$, where $g_{j+1}^{b, 1}=g_{j+1}^{a}$, with the polynomials $g_{j+1}^{a}$ being those computed in the case (i). This allows one to conclude that, if the set of the values taken by $\sigma_{1}$ is not known, it is not possible to express $x_{1}, x_{2}$ as functions of $y^{(0)}, \ldots, y^{(L)}$, being independent of $\sigma_{1}$, also allowing non-linearities in such expressions.
(iv) One can try to answer the following question: 'is it possible to express pair $x_{1}, x_{2}$ as a function of $y^{(0)}, \ldots, y^{(L)}$, possibly not linear, being independent of $\sigma_{1}$, but knowing the set of the values taken by $\sigma_{1}$ ?'. To answer such a question, it is convenient to take $\mathcal{R}_{c}:=\mathbb{R}\left[\sigma_{1}, x_{1}, x_{2}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right] /\left\langle\sigma_{1}\left(\sigma_{1}-1\right)\right\rangle$ as ambient ring, which means that $\sigma_{1}$ is treated, instead of as a known parameter, as a single variable whose domain is not the whole $\mathbb{R}$ but $\{0,1\}$ (the roots of $\sigma_{1}\left(\sigma_{1}-1\right)$ ). The ideals $\mathcal{I}_{j}^{a}$, $j \in \mathbb{Z}_{\geqslant 1}$, which have been defined as subsets of $\mathcal{R}_{a}$, can be coerced into $\mathcal{R}_{c}$. Fix the Lex order with $\sigma_{1} \succ x_{1} \succ x_{2} \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)}$ and define the elimination ideals $\mathcal{J}_{j}^{c, 1}=\mathcal{I}_{j}^{a} \cap \mathbb{R}\left[x_{1}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right]\left(\right.$ respectively, $\left.\mathcal{J}_{j}^{c, 2}=\mathcal{I}_{j}^{a} \cap \mathbb{R}\left[x_{2}, y^{(0)}, y^{(1)}, \ldots, y^{(L)}\right]\right)$, which are obtained by eliminating the variables $\sigma_{1}$ and $x_{2}$ (respectively, $\sigma_{1}$ and $x_{1}$ ) from $\mathcal{I}_{j}^{a}, j \in \mathbb{Z}_{\geqslant 1}$; these ideals are principal. The reduced Gröbner basis $\mathcal{G}_{1}^{c, 1}$ of $\mathcal{J}_{1}^{c, 1}$
is $\mathcal{G}_{1}^{c, 1}=\left\{g_{1}^{c, 1}\right\}$ (respectively, $\mathcal{G}_{1}^{c, 2}$ of $\mathcal{J}_{1}^{c, 2}$ is $\mathcal{G}_{1}^{c, 2}=\left\{g_{1}^{c, 2}\right\}$ ), where:

$$
\begin{aligned}
g_{1}^{b, 1} & =46088864 x_{1}^{2}+\left(20506725 y^{(1)}-229920120 y^{(0)}\right) x_{1} \\
& +18225\left(5 y^{(1)}-63 y^{(0)}\right)\left(25 y^{(1)}-248 y^{(0)}\right) \\
g_{1}^{b, 2} & =46088864 x_{2}^{2}+\left(47583720 y^{(0)}-45359595 y^{(1)}\right) x_{2} \\
& +2025\left(18 y^{(1)}-17 y^{(0)}\right)\left(207 y^{(1)}-296 y^{(0)}\right)
\end{aligned}
$$

the reduced Gröbner basis $\mathcal{G}_{j+1}^{c, k}$ of $\mathcal{J}_{j+1}^{c, k}$ is $\mathcal{G}_{j+1}^{c, k}=\mathcal{G}_{j}^{c, k} \cup\left\{g_{j+1}^{c, k}\right\}$, where $g_{j+1}^{c, k}=g_{j+1}^{a}$, for each $k \in\{1,2\}$; also in this case, since the polynomials $g_{j}^{c, k}, j \in \mathbb{Z}_{\geqslant 2}$, are independent of $x_{1}, x_{2}$ and of the switching signal $\sigma_{1}$, no improvements in the above formula could be obtained by taking into account the time-derivatives of $y(t)$ of order greater than 1. The discriminants of $g_{1}^{b, 1}$ and $g_{1}^{b, 2}$, thought of as polynomials of $x_{1}$ and $x_{2}$, respectively, are $\Delta_{1}=\left(3269 y^{(1)}+85320 y^{(0)}\right)^{2}$ and $\Delta_{2}=\left(573705 y^{(1)}-436408 y^{(0)}\right)^{2}$, respectively, which imply that pair $x_{1}, x_{2}$ can be locally expressed as a function of $y^{(0)}, y^{(1)}$, which is independent of the switching signal $\sigma_{1}$; in particular, one obtains:

$$
\begin{aligned}
& x_{1}=\frac{100440 y^{(0)}-10125 y^{(1)}}{43936} \\
& x_{2}=-\frac{765 y^{(0)}-810 y^{(1)}}{1049}
\end{aligned}
$$

$$
x_{1}=\frac{2835 y^{(0)}-225 y^{(1)}}{1049}
$$

$$
x_{2}=-\frac{13320 y^{(0)}-9315 y^{(1)}}{43936}
$$

Such solutions coincide with the ones that can be computed directly from pair ( $C_{0}, A$ ) (corresponding to $\sigma_{1}=0$ ),

$$
x_{1}=\frac{2835 y^{(0)}-225 y^{(1)}}{1049}, \quad x_{2}=-\frac{765 y^{(0)}-810 y^{(1)}}{1049}
$$

and from pair $\left(C_{0}+C_{1}, A\right)$ (corresponding to $\left.\sigma_{1}=1\right)$

$$
x_{1}=\frac{100440 y^{(0)}-10125 y^{(1)}}{43936}, \quad \quad x_{2}=-\frac{13320 y^{(0)}-9315 y^{(1)}}{43936}
$$

Such solutions are clearly useless if $\sigma_{1}$ is not measured and the set of the values taken by $\sigma_{1}$ is not known a priori, although they are apparently independent of $\sigma_{1}$.
$(v)$ To consider the observation problem under the assumption that $\sigma_{1}$ is not measured and its co-domain is not known, consider the additional output $z=\left(D_{0}+\sigma_{1} D_{1}\right) x$, with the following expressions of $D_{0}$ and $D_{1}$ being randomly generated

$$
D_{0}=\left[\begin{array}{cc}
\frac{1}{2} & \frac{3}{5}
\end{array}\right], \quad \quad D_{1}=\left[\begin{array}{cc}
\frac{9}{2} & \frac{7}{10}
\end{array}\right]
$$

Hence, let $k\left(\sigma_{1}, x\right)=\left(D_{0}+D_{1} \sigma_{1}\right) x$, and compute

$$
\begin{aligned}
k\left(\sigma_{1}, x\right) & =\frac{9 \sigma_{1}+1}{2} x_{1}+\frac{7 \sigma_{1}+6}{10} x_{2} \\
L_{f} k\left(\sigma_{1}, x\right) & =\frac{1764 \sigma_{1}+337}{450} x_{1}+\frac{108 \sigma_{1}+59}{10} x_{2}
\end{aligned}
$$

and so forth; one has $z^{(j)}(t)=L_{f}^{j} k\left(\sigma_{1}(t), x(t)\right), j \in \mathbb{Z}_{\geqslant 0}$. It is convenient to take $\mathcal{R}_{d}:=\mathbb{R}\left[\sigma_{1}, x_{1}, x_{2}, y^{(0)}, \ldots, y^{(L)}, z^{(0)}, \ldots, z^{(L)}\right]$ as ambient ring. Fix the Lex order with $\sigma_{1} \succ x_{1} \succ x_{2} \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)} \succ z^{(L)} \succ \ldots \succ z^{(1)} \succ z^{(0)}$ and define

$$
\mathcal{I}_{1}^{d}=\left\langle y^{(0)}-h, y^{(1)}-L_{f} h, z^{(0)}-k, z^{(1)}-L_{f} k\right\rangle \subset \mathcal{R}_{d}
$$

The reduced Gröbner basis $\mathcal{G}_{1}^{d}$ of $\mathcal{I}_{1}^{d} \cap \mathbb{R}\left[x_{1}, x_{2}, y^{(0)}, \ldots, y^{(L)}, z^{(0)}, \ldots, z^{(L)}\right]$ is $\mathcal{G}_{1}^{d}=$ $\left\{g_{1}^{c}, g_{2}^{c}, g_{3}^{c}\right\}$,

$$
\begin{aligned}
g_{1}^{c} & =1005365203 x_{1}+485568675 y_{1}-2961637020 y^{(0)} \\
& -69131250 z_{1}+95248350 z_{0} \\
g_{2}^{c} & =1005365203 x_{2}+1408481055 y_{1}-804144060 y^{(0)} \\
& -526932900 z_{1}+368357950 z_{0} \\
g_{3}^{c} & =856575\left(y^{(1)}\right)^{2}-8375400 y^{(1)} y^{(0)}-23490 y^{(1)} z^{(1)} \\
& -1298250 y^{(1)} z^{(0)}+5482080\left(y^{(0)}\right)^{2} \\
& +1527930 y^{(0)} z^{(1)}-150336 y^{(0)} z^{(0)}-60300\left(z^{(1)}\right)^{2} \\
& +589600 z^{(1)} z^{(0)}-385920\left(z^{(0)}\right)^{2} .
\end{aligned}
$$

Clearly, $g_{1}^{c}$ and $g_{2}^{c}$ allow one to express $x_{1}$ and $x_{2}$, respectively, as linear functions of $y^{(0)}, y^{(1)}, z^{(0)}, z^{(1)}$, being independent of $\sigma_{1}$, which therefore can be employed when $\sigma_{1}$ is not measured and the set of the values taken by $\sigma_{1}$ is not known.

Note that $\mathcal{I}_{1}^{d} \cap \mathbb{R}\left[y^{(0)}, \ldots, y^{(L)}, z^{(0)}, \ldots, z^{(L)}\right] \quad$ is principal and $\mathcal{I}_{1}^{d} \cap$ $\mathbb{R}\left[y^{(0)}, \ldots, y^{(L)}, z^{(0)}, \ldots, z^{(L)}\right]=\left\langle g_{3}^{c}\right\rangle$. Since $g_{3}^{c}$ is not linear, no linear embedding (independent of $\sigma_{1}$ ) of the considered system is in $\mathcal{I}_{1}^{d}$. To have linear embeddings independent of $\sigma_{1}$, define the following ideal of $\mathcal{R}_{d}$ :

$$
\begin{aligned}
& \mathcal{I}_{2}^{d}=\left\langle y^{(0)}-h\left(\sigma_{1}, x\right), y^{(1)}-L_{f} h\left(\sigma_{1}, x\right), y^{(2)}-L_{f} h^{2}\left(\sigma_{1}, x\right)\right. \\
&\left.z^{(0)}-k\left(\sigma_{1}, x\right), z^{(1)}-L_{f} k\left(\sigma_{1}, x\right), z^{(2)}-L_{f}^{2} k\left(\sigma_{1}, x\right)\right\rangle .
\end{aligned}
$$

The two eliminations ideals $\mathcal{I}_{2}^{d, y} \cap \mathbb{R}\left[y^{(0)}, \ldots, y^{(L)}\right]$ and $\mathcal{I}_{2}^{d, z} \cap \mathbb{R}\left[z^{(0)}, \ldots, z^{(L)}\right]$ are principal and generated by

$$
p_{2}^{y}=y^{(2)}-\frac{88}{9} y^{(1)}+\frac{32}{5} y^{(0)}, \quad \quad p_{2}^{z}=z^{(2)}-\frac{88}{9} z^{(1)}+\frac{32}{5} z^{(0)}
$$

Note that the coefficients of these polynomials are those of the characteristic polynomial of the matrix $A$ (see Lemma 3).

## 6. 'Generic' properties of matrices with entries being rational functions of parameters

Let $\beta=\left[\begin{array}{lll}\beta_{1} & \ldots & \beta_{\ell}\end{array}\right]^{\top}$ be a vector of real parameters and let $M(\beta)$ be a matrix of dimensions $n \times n$, whose entries are rational functions of $\beta$ (i.e., $\left.M_{i, j}(\beta) \in \mathbb{R}(\beta)\right)$. The following lemma gives necessary and sufficient conditions for the 'generic' invertibility of the matrix $M(\beta)$.

Lemma 6. The matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^{\ell}$ if and only if the rational function $\operatorname{det}(M(\beta))$ is not identically zero.

Proof. Since $M_{j, k} \in \mathbb{R}(\beta)$, there exist co-prime $N_{j, k}, D_{j, k} \in \mathbb{R}[\beta]$ such that $M_{j, k}(\beta)=$ $\frac{N_{j, k}(\beta)}{D_{j, k}(\beta)}$. Define the algebraic variety $\mathcal{V}:=\mathbb{V}\left(D_{1,1}, \ldots, D_{1, n}, \ldots, D_{n, n}\right)$ of $\mathbb{R}^{\ell}$, which satisfies $\mathcal{V} \neq \mathbb{R}^{\ell}$ since $\left\langle D_{1,1}, \ldots, D_{n, n}\right\rangle \neq\langle\emptyset\rangle$. Clearly, $M(\hat{\beta})$ is defined for all spe-
cializations $\hat{\beta} \in \mathbb{R}^{\ell} \backslash \mathcal{V}$, whence it is defined for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^{\ell}$, because the Zariski closure of $\mathcal{V}$ coincides with $\mathcal{V}$, whence it is not equal to $\mathbb{R}^{\ell}$. Since the entries of $M(\beta)$ are in $\mathbb{R}(\beta)$, one has that $\operatorname{det}(M(\beta)) \in \mathbb{R}(\beta)$, whence there exist $N, D \in \mathbb{R}[\beta]$ such that $\operatorname{det}(M(\beta))=\frac{N(\beta)}{D(\beta)}$. Clearly, $\operatorname{det}(M(\hat{\beta}))$ is defined (i.e., $D(\hat{\beta}) \neq 0)$ for all specializations $\hat{\beta} \in \mathbb{R}^{\ell} \backslash \mathcal{V}$. Define the algebraic variety $\mathcal{W}:=\mathcal{V} \cup \mathbb{V}(N)$, where $\mathcal{V} \cup \mathbb{V}(N) \neq \mathbb{R}^{\ell}$ since $\mathcal{V} \neq \mathbb{R}^{\ell}$ and $\mathbb{V}(N) \neq \mathbb{R}^{\ell}$ due to the fact that $\operatorname{det}(M(\beta))$ is not identically zero. Now, $\operatorname{det}(M(\hat{\beta})) \neq 0$ (i.e., $M(\hat{\beta})$ is non-singular) for all specializations $\hat{\beta} \in \mathbb{R}^{\ell} \backslash \mathcal{W}$, whence the matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^{\ell}$, because the Zariski closure of $\mathcal{W}$ coincides with $\mathcal{W}$, whence it is not equal to $\mathbb{R}^{\ell}$.

Lemma 6 essentially states that $M(\beta)$ is 'generically' invertible if and only if its determinant is not identically zero. The following corollary shows that this condition is equivalent to require that there exists $\hat{\beta}^{\circ} \in \mathbb{R}^{\ell}$ such that $\operatorname{det}\left(M\left(\hat{\beta}^{\circ}\right)\right) \neq 0$.

Corollary 1. The matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^{\ell}$ if and only if there exists $\hat{\beta}^{\circ} \in \mathbb{R}^{\ell}$ such that $M\left(\hat{\beta}^{\circ}\right)$ is defined and non-singular.

Proof. By absurd assume that $\operatorname{det}(M(\beta))$ is identically zero. If there exists a specialization $\hat{\beta}^{\circ} \in \mathbb{R}^{h}$ such that $M\left(\hat{\beta}^{\circ}\right)$ is defined and non-singular, then $\operatorname{det}(M(\beta))$ is defined at $\beta=\hat{\beta}^{\circ}$ and $\operatorname{det}\left(M\left(\hat{\beta}^{\circ}\right)\right) \neq 0$, which is a contradiction.

## 7. 'Generic' uniform algebraic observability of multi-switching systems

By using the results established in Section 6, the conditions under which system (5) is 'generically' uniformly algebraically observable are studied in this section. Let $y_{j}$ denote the $j$-th entry of the output vector $y$, let $C_{\ell}^{[j]}$ denote the $j$-th row of the matrix $C_{\ell}$ so that, for $j=1, \ldots, m$,

$$
y_{j}(t)=\left(C_{0}^{[j]}+\sigma_{1}(t) C_{1}^{[j]}+\ldots+\sigma_{N}(t) C_{N}^{[j]}\right) x(t),
$$

Let $z_{j}=\left[\begin{array}{lll}y_{j}^{(0)} & \ldots & y_{j}^{(n-1)}\end{array}\right]^{\top}$, where $y_{j}^{(k)}$ is the $j$-th entry of $y^{(k)}, j=1, \ldots, m$, and let $z=\left[\begin{array}{lll}z_{1}^{\top} & \ldots & z_{m}^{\top}\end{array}\right]^{\top}$. Since

$$
\begin{equation*}
y_{j}^{(k)}(t)=\left(C_{0}^{[j]}+\sigma_{1}(t) C_{1}^{[j]}+\ldots+\sigma_{N}(t) C_{N}^{[j]}\right) A^{k} x(t), \tag{9}
\end{equation*}
$$

for all $t \in \mathbb{R}_{\geqslant 0}$ and $k \in \mathbb{Z}_{\geqslant 0}$, one has that

$$
z_{j}(t)=\left[\begin{array}{llll}
\mathbb{O}_{j, 0} & \mathbb{O}_{j, 1} & \ldots & \mathbb{O}_{j, N}
\end{array}\right]\left[\begin{array}{c}
x(t)  \tag{10}\\
\sigma_{1}(t) x(t) \\
\vdots \\
\sigma_{N}(t) x(t)
\end{array}\right],
$$

where $\mathbb{O}_{j, \ell}=\mathbb{O}\left(C_{\ell}^{[j]}, A\right)$ is the observability matrix of $\left(C_{\ell}^{[j]}, A\right), j=1, \ldots, m, \ell=$ $0, \ldots, N$. The following theorem provides conditions on the number of the entries of
the output vector of system (5) to guarantee its uniform algebraic observability for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$.

Theorem 1. The following two statements hold independently of the dimension $n$ of the state vector and of the sets of values taken by the switching signals:
(1.1) if $m \geqslant N+1$, system (5) is uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}, C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$;
(1.2) if $m \leqslant N$, system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}, C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$.

Proof. Under the assumption that $m \geqslant N+1$, let

$$
\Phi:=\left[\begin{array}{cccc}
\mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \ldots & \mathbb{O}_{1, N} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{O}_{N+1,0} & \mathbb{O}_{m, 1} & \ldots & \mathbb{O}_{N+1, N}
\end{array}\right]
$$

First, it is proved that $\Phi$ is invertible for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in$ $\mathbb{R}^{m \times n}$. Consider the specialization

$$
\begin{aligned}
\hat{A}^{\circ} & =\left[\begin{array}{cc}
0_{n-1,1} & I_{n-1} \\
0_{1,1} & 0_{1, n-1}
\end{array}\right], \\
\left(\hat{C}_{\ell}^{[j]}\right)^{\circ} & = \begin{cases}{\left[\begin{array}{ll}
1 & 0_{1, n-1}
\end{array}\right],} & \text { if } j=\ell+1, \\
0_{n, 1}, & \text { otherwise } .\end{cases}
\end{aligned}
$$

Hence, since $\mathbb{O}\left(\left(\hat{C}_{\ell}^{[j]}\right)^{\circ}, \hat{A}^{\circ}\right)=I_{n}$, if $j=\ell-1$, and $\mathbb{O}\left(\left(\hat{C}_{\ell}^{[j]}\right)^{\circ}, \hat{A}^{\circ}\right)=0_{n, n}$, otherwise, one obtains that the specialization $\Phi^{\circ}$ coincides with $I_{n(N+1)}$, whence it is non-singular. Therefore, by Corollary 1 , the matrix $\Phi$ is invertible for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$. Thus,

$$
\left[\begin{array}{c}
x(t)  \tag{11}\\
\sigma_{1}(t) x(t) \\
\vdots \\
\sigma_{N}(t) x(t)
\end{array}\right]=\Phi^{-1}\left[\begin{array}{c}
z_{1} \\
\vdots \\
z_{N+1}
\end{array}\right],
$$

whence, letting $\tilde{F}$ be the first row block of $\Phi^{-1}$, one has that $x=$ $\tilde{F}\left[\begin{array}{lll}z_{1}^{\top} & \ldots & z_{N+1}^{\top}\end{array}\right]^{\top}$. Thus, if $m \geqslant N+1$, then system (5) is uniformly algebraically observable, for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$.

On the other hand, assume, by absurd, that $m \leqslant N$ and that system (5) is uniformly algebraically observable, i.e., there exist $K \in \mathbb{Z}_{\geqslant 0}$ and $F \in \mathbb{R}^{n \times m(K+1)}$ such that (7) holds for all $t \in \mathbb{R}_{\geqslant 0}$, for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$. First, notice that, since (7) holds for all the switching patterns of $\sigma_{1}, \ldots, \sigma_{N}$, it must hold also for $\sigma_{1}=\ldots=\sigma_{N}=0$. Thus, since pair ( $C_{0}, A$ ) is 'generically' observable by Corollary 1 (see also Kailath, 1980), one has that $K \geqslant n-1$. Furthermore, by (9) and by the Cayley-Hamilton Theorem (Meyer, 2000), one has that $y^{(k)}(t)$ can be expressed as a linear combination of $y^{(0)}, \ldots, y^{(n-1)}$ for all $k \geqslant n$. Therefore, if there exist $K \in \mathbb{Z} \geqslant 0$ and a matrix $F \in \mathbb{R}^{n \times m(K+1)}$ such that (7) holds for all $t \in \mathbb{R}_{\geqslant 0}$, then there exists a matrix $\tilde{F} \in \mathbb{R}^{n \times m n}$ such that, for all $t \in \mathbb{R}_{\geqslant 0}$, one has $x(t)=\tilde{F} z(t)$. Hence, by
taking (10) into account, one has that

$$
x=\tilde{F}\left[\begin{array}{c}
\mathbb{O}_{1,0}+\sigma_{1} \mathbb{O}_{1,1}+\ldots+\sigma_{N} \mathbb{O}_{1, N} \\
\vdots \\
\mathbb{O}_{m, 0}+\sigma_{1} \mathbb{O}_{m, 1}+\ldots+\sigma_{N} \mathbb{O}_{m, N}
\end{array}\right] x,
$$

for all $x \in \mathbb{R}^{n},\left(\sigma_{1}, \ldots, \sigma_{N}\right) \in\left\{0, \ldots, S_{1}\right\} \times \cdots \times\left\{0, \ldots, S_{N}\right\}$. Hence, one has

$$
\begin{align*}
& I_{n}=\tilde{F}\left[\begin{array}{c}
\mathbb{O}_{1,0} \\
\vdots \\
\mathbb{O}_{m, 0}
\end{array}\right],  \tag{12a}\\
& I_{n}=\tilde{F}\left[\begin{array}{c}
\mathbb{O}_{1,0}+\mathbb{O}_{1, j} \\
\vdots \\
\mathbb{O}_{m, 0}+\mathbb{O}_{m, j}
\end{array}\right], \quad j=1, \ldots, N . \tag{12b}
\end{align*}
$$

Since $m \leqslant N$, the relations given in (12) imply that $\tilde{F}$ is such that (12a) holds and

$$
\tilde{F}\left[\begin{array}{ccc}
\mathbb{O}_{1,1} & \ldots & \mathbb{O}_{1, m}  \tag{13}\\
\vdots & \ddots & \vdots \\
\mathbb{O}_{m, 1} & \cdots & \mathbb{O}_{m, m}
\end{array}\right]=0_{n \times n}
$$

which is a contradiction. Therefore, if $m \leqslant N$, then system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$.

By Lemma 3, there exists a matrix $Q$, given by

$$
Q=\left[\begin{array}{llll}
\alpha_{n} I_{m} & \alpha_{n-1} I_{m} & \cdots & \alpha_{1} I_{m} \tag{14}
\end{array}\right],
$$

where $p_{A}(\lambda)=\lambda^{n}+a_{1} \lambda^{n-1}+\cdots+a_{n}$ is the characteristic polynomial of matrix $A$, such that

$$
\begin{equation*}
y^{(n)}=Q y_{e, n-1}, \tag{15}
\end{equation*}
$$

where $y_{e, n-1}=\left[\begin{array}{lll}\left(y^{(0)}\right)^{\top} & \ldots & \left(y^{(n-1)}\right)^{\top}\end{array}\right]^{\top}$.
In addition, by the proof of Theorem 1 , if $m \geqslant N+1$, then, for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$, there exists $F \in \mathbb{R}^{n \times m n}$ such that

$$
\begin{equation*}
x=F y_{e, n-1} . \tag{16}
\end{equation*}
$$

The actual computation of $F$ can be done by geometric arguments. Assume $m \geqslant N+$ 1 , and take $\mathcal{R}=\mathbb{R}\left[\sigma_{1}, \ldots, \sigma_{N}, x, y^{(0)}, \ldots, y^{(n-1)}\right]$ as ambient ring and fix the Lex order with $\sigma_{1} \succ \ldots \succ \sigma_{N} \succ x_{1} \succ \ldots \succ x_{n} \succ y_{1}^{(n-1)} \succ \ldots \succ y_{1}^{(0)} \succ \ldots \succ y_{m}^{(n-1)} \succ \ldots \succ y_{m}^{(0)}$.

Define the following ideal $\mathcal{I}_{n-1}$ of $\mathcal{R}$ :

$$
\begin{aligned}
\mathcal{I}_{n-1}:=\left\langle y_{1}^{(0)}-\left(C_{0}^{[1]}+\sigma_{1} C_{1}^{[1]}+\right.\right. & \left.\ldots+\sigma_{N} C_{N}^{[1]}\right) x, \ldots, \\
& \left.y_{m}^{(n-1)}-\left(C_{0}^{[m]}+\sigma_{1} C_{1}^{[m]}+\ldots+\sigma_{N} C_{N}^{[m]}\right) A^{n-1} x\right\rangle .
\end{aligned}
$$

Compute the reduced Gröbner basis $\mathcal{G}_{n-1}$ of the elimination ideal $\mathcal{J}_{n-1}=\mathcal{I}_{n-1} \cap$ $\mathbb{R}\left[x, y^{(0)}, \ldots, y^{(n-1)}\right]$.

Proposition 1. Let $m \geqslant N+1$ and let the ideal $\mathcal{J}_{n-1}$ and its reduced Gröbner basis $\mathcal{G}_{n-1}$ be defined as above. For 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}$, the $j$-th entries of $x-F y_{e}$ belongs to $\mathcal{G}_{n-1}$ and, in particular, it is the only element of $\mathcal{G}_{n-1}$ depending on $x_{j}, j \in\{1, \ldots, n\}$.

Proof. The $j$-th entry of $x-F y_{e}$ is in $\mathcal{J}_{n-1}$ by construction; in particular, it is the only element of $\mathcal{G}_{n-1}$ depending on $x_{j}$ due to the elimination properties of the Lex order.

It is worth noticing that, although the results given in Proposition 1 have been stated in the 'generic' case, they can be actually used to check for uniform algebraic observability of system (5) when $A, C_{0}, \ldots, C_{n}$ are given matrices. In particular, letting the matrices $A, C_{0}, \ldots, C_{n}$ be given, system (5) is uniformly algebraically observable if and only if there exist polynomials $q_{1}, \ldots, q_{n}$ in $\mathcal{G}_{n-1}$ such that $\operatorname{LT}\left(q_{j}\right)=x_{j}, j=$ $1, \ldots, n$. Furthermore, if such polynomials exist, they can be directly used to compute a matrix $F$ such that (7) holds.

### 7.1. Observer design for multi-switching systems

In order to use (16) to estimate the current state of system (5), the signals $y^{(j)}(t)$ have to be estimated from $y(t)$. Several tools are available in the literature to compute these estimates, such as high-gain observers (Tornambe, 1992), sliding mode differentiators (Shtessel, Edwards, Fridman, \& Levant, 2014), and super-twisting algorithms (Moreno \& Osorio, 2012).

An alternative approach is to exploit the matrices $Q$ and $F$ to design a state observer for system (5). In particular, letting

$$
Q_{e}:=\left[\begin{array}{ll}
O_{\underline{m}(n-1), \underline{m}} & I_{\underline{m}(n-1)}  \tag{17}\\
\hdashline
\end{array}\right],
$$

the vector embedding $y^{(n)}=Q y_{e, n-1}$ admits the following state space representation (with $t \in \mathbb{R}$ ):

$$
\begin{equation*}
\frac{\mathrm{d} \xi(t)}{\mathrm{d} t}=Q_{e} \xi(t), \quad y(t)=C_{e} \xi(t), \quad x(t)=F \xi(t), \tag{18}
\end{equation*}
$$

where $C_{e}=\left[\begin{array}{llll}I_{m} & 0_{m \times m} & \ldots & 0_{m \times m}\end{array}\right]$. Therefore, letting $K$ be a matrix such that $Q_{e}-K C_{e}$ has all eigenvalues with negative real part (such a matrix exists since pair ( $C_{e}, Q_{e}$ ) is observable), an observer for system (5) that does not require the knowledge of $\sigma_{1}, \ldots, \sigma_{N}$ can be obtained by the design of a Luenberger observer for (18) and is given by (with $t \in \mathbb{R}$ ):

$$
\begin{equation*}
\frac{\mathrm{d} \hat{\xi}(t)}{\mathrm{d} t}=Q_{e} \hat{\xi}(t)+K\left(y(t)-C_{e} \hat{\xi}(t)\right), \quad \hat{x}(t)=F \hat{\xi}(t) \tag{19}
\end{equation*}
$$

where $\hat{\xi}(t)$ is the estimate of $\xi(t)$ at time $t$. As remarked above, at each switching time, the observer (19) exhibits a transient behavior, which, however, can be made arbitrarily short by letting the real part of the eigenvalues of $Q_{e}-K C_{e}$ (which can be assigned arbitrarily) be sufficiently negative.

## 8. A physically motivated example

Consider the mechanical system depicted in Figure 1, which consists of three bodies having (normalized) mass $\mathcal{M}_{1}=1 \mathrm{Kg}, \mathcal{M}_{2}=1.5 \mathrm{Kg}, \mathcal{M}_{3}=0.5 \mathrm{Kg}$ respectively, and three springs having (normalized) stiffness $\mathcal{K}_{1}=0.7 \mathrm{~N} / \mathrm{m}, \mathcal{K}_{2}=0.9 \mathrm{~N} / \mathrm{m}, \mathcal{K}_{3}=$ $0.8 \mathrm{~N} / \mathrm{m}$, respectively.


Figure 1. A simple mechanical system.
The dynamics of this plant are given by system (5a), with

$$
A=\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0  \tag{20}\\
-\frac{8}{5} & 0 & \frac{9}{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{3}{5} & 0 & -\frac{17}{15} & 0 & \frac{8}{15} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{8}{5} & 0 & -\frac{8}{5} & 0
\end{array}\right]
$$

Let the state of system (20) be measured by a sensor network with $N=2$ that provides the measurement vector

$$
y=\left[\begin{array}{c}
x_{1}-\sigma_{1} x_{3} \\
-x_{3}+\sigma_{1} x_{3} \\
-x_{5}+\sigma_{2} x_{5}
\end{array}\right],
$$

where the switching signals $\sigma_{1}, \sigma_{2}$ and their co-domain are not known. Note that, since $m=3$ and $N=2$, the assumptions of Theorem 1 are met. By using the methods given in Proposition 1, one obtains a matrix $F$ (whose explicit expression has been omitted for space constraints) such that (16) holds. Thus, an observer for system (20) based on the measurement vector $y$ and independent of $\sigma_{1}, \sigma_{2}$ is given by (19), where the matrix $Q$ is given by (14) and (17).

A numerical simulation has been carried out to test such an observer, assuming $\hat{y}_{e}(0)=0, x(0)=\left[\begin{array}{lllll}0.1 \mathrm{~m} & 0 \mathrm{~m} / \mathrm{s} & -0.3 \mathrm{~m} & 0 \mathrm{~m} / \mathrm{s} \quad 0.2 \mathrm{~m} & 0 \mathrm{~m} / \mathrm{s}\end{array}\right]^{\top}$, letting the switching signal be the one reported in the second plot of Figure 2, and designing $K$ so that the real part of all the eigenvalues of

$$
Q_{e}-K\left[\begin{array}{llll}
I_{3} & 0_{3} & \cdots & 0_{3}
\end{array}\right]
$$

is smaller than -30 . Figure 2 depicts the results of such a simulation.
As shown by such a figure, after an initial transient behavior, which occurs at each time instant for which one of the switching signals is not continuous, the observer (19) is able to reconstruct the current state of system (20), without requiring the knowledge of the switching signals $\sigma_{1}$ and $\sigma_{2}$.


Figure 2. Results of the numerical simulation.

## 9. Conclusions

In this paper, it has been shown that multi-switching systems are 'generically' uniformly algebraically observable if the number of available measurements is greater than the number of switching signals. Since these systems are able to exactly replicate the output response of those switching plant that are usually employed to model multi-sensor networks (see Lemmas 1 and 2), the results given in this paper allow the construction of reliable and simple observers that do not require any knowledge about the switching signals (i.e., about the failures occurring in the sensor-network).

The results given in this paper can be extended to the case of linear systems having time-varying and switching dynamical matrices. Namely, consider the system

$$
\begin{align*}
\frac{\mathrm{d} x(t)}{\mathrm{d} t} & =\left(A_{0}(t)+\sigma_{1}(t) A_{1}(t)+\ldots+\sigma_{N}(t) A_{N}(t)\right) x(t),  \tag{21a}\\
y(t) & =\left(C_{0}(t)+\sigma_{1}(t) C_{1}(t)+\ldots+\sigma_{N}(t) C_{N}(t)\right) x(t), \tag{21b}
\end{align*}
$$

where $A_{0}, \ldots, A_{N} \in \mathbb{R}^{n \times n}[t]$ and $C_{0}, \ldots, C_{N} \in \mathbb{R}^{m \times n}[t]$ are time-varying polynomial
matrices. Thus, define

$$
\begin{aligned}
& h_{0}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right)=\left(C_{0}(t)+\sigma_{1} C_{1}(t)+\ldots+\sigma_{N} C_{N}(t)\right) x, \\
& h_{i}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right)=\frac{\partial h_{i-1}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right)}{\partial t} \\
& \quad+\frac{\partial h_{i-1}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right)}{\partial x}\left(A_{0}(t)+\sigma_{1} A_{1}(t)+\ldots+\sigma_{N} A_{N}(t)\right) x,
\end{aligned}
$$

for $i=1, \ldots, L$. Hence, by using the tools detailed in Section 4, define the ideal

$$
\mathcal{I}_{L}:=\left\langle y_{0}-h_{0}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right), \ldots, y_{L}-h_{L}\left(\sigma_{1}, \ldots, \sigma_{N}, t, x\right)\right\rangle,
$$

and let $\mathcal{J}_{L}=\mathcal{I}_{L} \cap \mathbb{R}\left[y_{0}, \ldots, y_{L}\right]$. Hence, coercing $\mathcal{J}_{L}$ into $\mathbb{R}\left[y_{0}, \ldots, y_{L}\right]$, define the quotient ring $\mathbb{R}\left[y_{0}, \ldots, y_{L}\right] / \mathcal{J}_{L}$, and let $\mathbb{K}\left\{y_{e, N}\right\}$ be the field of the rational functions in $\mathbb{R}\left[y_{0}, \ldots, y_{L}\right] / \mathcal{J}_{L}$. Thus, fix $\mathcal{R}_{b}=\mathbb{K}\left\{y_{e, N}\right\}\left[x, t, \sigma_{1}, \ldots, \sigma_{N}\right]$ as ambient ring, coerce $\mathcal{I}_{L}$ into $\mathcal{R}_{b}$ and consider the ideals

$$
\mathcal{K}_{L, j}=\mathcal{I}_{L} \cap \mathbb{K}\left\{y_{e, N}\right\}\left[x_{j}\right], \quad j=1, \ldots, n .
$$

By the reasoning given in Section 4, these ideals are principal, i.e., there exist nonconstant polynomials $q_{j} \in \mathbb{K}\left\{y_{e, N}\right\}\left[x_{j}\right]$ such that $\mathcal{K}_{L, j}=\left\langle q_{j}\right\rangle, j=1, \ldots, n$. If $\operatorname{degree}\left(q_{j}\right)=1$, then such functions can be readily used to estimate the state of system (21); see Menini et al. (2019) for further details. However, in this case, it is not easy to find a lower bound on $L$ that guarantees 'generic' existence of linear $q_{j}$ 's.

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