On the role of dependence in residual lifetimes

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Abstract

Consider a vector (X, Y) that describes the failure times of two non-independent components of a system. Assuming that the first component has survived up to a given time $t > 0$, i.e., assuming $X > t$, one can define the corresponding residual lifetime under different assumptions on the failure of the second component, having lifetime Y. In particular, one can observe that the second component has not failed before a time $s \geq 0$ (maybe different from t), thus the conditioned residual lifetime $\widetilde{X}_t = [X - t | X > t, Y > s]$ can be considered, or one can not observe Y, and in this case the conditioned residual lifetime $X_t = [X - t | X > t]$ has to be studied. This note deals with conditions on the survival copula of (X, Y) such that $\widetilde{X}_t$ and X_t are comparable according to the main reliability stochastic orders. Similar conditions, based on the connecting copula of (X, Y), are described also for the conditioned inactivity times $\widetilde{X}^t = [t - X | X \leq t, Y \leq s]$ and $X^t = [t - X | X \leq t]$.

Keywords: stochastic orders, copulas, dependence notions.

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1 Introduction

Stochastic comparisons between univariate random lifetimes or residual lifetimes have been defined and applied in a variety of contexts in the last decades (see, e.g., the monographs Müller and Stoyan, 2002, Shaked and Shanthikumar, 2007, and Belzunce et al., 2015, for detailed properties and applications of the main univariate stochastic orders). It is a remarkable fact that most of the univariate stochastic orders considered in reliability literature are based on comparisons between the marginal distributions of the involved lifetimes, without taking care of possible effects due to dependence. However, when one considers a pair of components having non-independent lifetimes, and observes that one of them survived up to a time $t > 0$, the knowledge about the failure of the other component can strongly modify the distribution of the residual lifetime of the first. This fact has been clearly pointed out, for example, in Belzunce et al. (2016), where different counterexamples have been provided to show that the usual stochastic order between $[X - t|X > t, Y > t]$ and $[Y - t|X > t, Y > t]$, for any $t \geq 0$, does not imply, neither is implied by, the stochastic order between $[X - t|X > t]$ and $[Y - t|Y > t]$.

Thus, given a vector (X, Y) that describes the failure times of two non-independent components of a system, clearly the distribution of the residual life $X_t = [X - t|X > t]$ differs from that of $\widetilde{X}_t = [X - t|X > t, Y > s]$ when dependence exists among X and Y. In particular, it is rather intuitive to assume that $\widetilde{X}_t$ stochastically increases as the dependence among X and Y increases according to some positive dependence notion (and, viceversa, decreases under negative dependence). Also, heuristic arguments allow one to assume that $\widetilde{X}_t$ is greater than X_t, according to some stochastic order, under positive dependence among X and Y.

The aim of this note is to survey and formalize these intuitions, providing conditions on the copula or on the survival copula of the vector (X, Y), when it exists, such that $\widetilde{X}_t$ and X_t defined as above are stochastically comparable according to the main stochastic order considered in reliability literature, i.e., the usual stochastic order, the likelihood ratio order, the hazard rate order, the reversed hazard order and the mean residual life order. Similar results are provided also for the inactivity times $X^t = [t - X|X \leq t]$ and $\widetilde{X}^t = [t - X|X \leq t, Y \leq s]$. It should be pointed out that, while conditions for some of these comparisons are known or easy to prove, the same can not be asserted for other orders, and interesting equivalences come out, as described along the paper.

The organization of this note is the following. In Section 2 we briefly recall some useful notions. In Section 3 we provide conditions for stochastic orders between the residual lifetimes $\widetilde{X}_t$ and X_t, as well as conditions for stochastic orders between the inactivity times $\widetilde{X}^t$ and X^t. A brief discussion about the assumptions that should be satisfied by the copulas for the stochastic comparisons is provided in Section 4. Finally, possible applications of the main results in risk theory are briefly mentioned in Section 5.

Along this note the following notation is adopted. The terms "increasing" and "decreasing" are used in place of "nondecreasing" and "nonincreasing", respectively. For any random variable or random vector X and an event A, the notation $[X|A]$ describes the random variable whose distribution is the conditional distribution of X given A. Given a univariate or bivariate distribution function F, the corresponding univariate or bivariate survival function is denoted $\overline{F}$. To simplify notations, we will also assume that the involved lifetimes have support on the whole set $\mathbb{R}^+$, even if the stated results hold for lifetimes having finite supports.

2 Notations and useful notions

Let (X,Y) be a pair of absolutely continuous random lifetimes having support on $S_2 = \mathbb{R}^+ \times \mathbb{R}^+$, having joint distribution function $F(x,y) = P[X \le x, Y \le y]$ and corresponding marginal distribution functions F_X and F_Y. Assuming that $X > t$ and $Y > s$ for given $t, s \ge 0$, we denote

$$X_t = [X - t | X > t] \qquad \text{and} \qquad \widetilde{X}_t = [X - t | X > t, Y > s].$$

On the contrary, assuming that $X \le t$ and $Y \le s$ for given $t, s \ge 0$, we denote

$$X^t = [t - X | X \le t] \qquad \text{and} \qquad \widetilde{X}^t = [t - X | X \le t, Y \le s].$$

It is easy to verify that, for $x \ge 0$,

$$\overline{F}_{X_t}(x) = \frac{\overline{F}(x+t,0)}{\overline{F}(t,0)} \qquad \text{and} \qquad \overline{F}_{\widetilde{X}_t}(x) = \frac{\overline{F}(x+t,s)}{\overline{F}(t,s)}, \tag{2.1}$$

while, for $x \le t$,

$$\overline{F}_{X^t}(x) = \frac{F(t-x,+\infty)}{F(t,+\infty)} \qquad \text{and} \qquad \overline{F}_{\widetilde{X}^t}(x) = \frac{F(t-x,s)}{F(t,s)}. \tag{2.2}$$

As described in the next sections, conditions for stochastic comparisons among the above defined residual lifetimes and inactivity times can be described in terms of the connecting copula of (X,Y), or in terms of the corresponding survival copula, whose definitions are recalled here. Let (X,Y) be a pair of random lifetimes as above.

- The function $C : [0,1]^2 \to [0,1]$ such that, for all $(x,y) \in S_2$,

  $$F(x,y) = C(F_X(x), F_Y(x))$$

  is said to be the *connecting copula* of (X,Y). In this case, it also holds

  $$C(u,v) = F(F_X^{-1}(u), F_Y^{-1}(v)) \ \text{ for all } u,v \in [0,1].$$

- The function $\overline{C} : [0,1]^2 \to [0,1]$ such that, for all $(x,y) \in S_2$,

  $$\overline{F}(x,y) = \overline{C}(\overline{F}_X(x), \overline{F}_Y(y))$$

  is said to be the *survival copula* of (X,Y). In this case, it also holds

  $$\overline{C}(u,v) = \overline{F}(\overline{F}_X^{-1}(u), \overline{F}_Y^{-1}(v)) \ \text{ for all } u,v \in [0,1].$$

We observe that from a mathematical viewpoint, survival copulas and connecting copulas turn out in any case to be copulas, and that they both describe the dependence structure of (X, Y). Also, it is well known that if the marginal distributions are continuous then the connecting copula C and the survival copula $\overline{C}$ are unique. For this reason we will assume here, and everywhere throughout the paper, continuity of the marginal distributions for the vector (X, Y). We address the readers to the monographs Nelsen (2006) and Durante and Sempi (2015) for further details.

Some of the most relevant stochastic orders of interest in reliability field, that will be used throughout this paper to compare random lifetimes or inactivity times, are recalled now. For it, let X_1 and X_2 be two absolutely continuous random variables having common support $\mathbb{R}^+$, distribution functions F_1 and F_2 and density functions f_1 and f_2, respectively. We say that X_1 is smaller than X_2

- in the *usual stochastic order* (denoted by $X_1 \leq_{ST} X_2$) if $\overline{F}_1(t) \leq \overline{F}_2(t)$ for all $t \in \mathbb{R}^+$;
- in the *hazard rate order* (denoted by $X_1 \leq_{HR} X_2$) if the ratio $\overline{F}_2(t)/\overline{F}_1(t)$ is increasing in $t \in \mathbb{R}^+$;
- in the *reversed hazard rate order* (denoted by $X_1 \leq_{RHR} X_2$) if the ratio $F_2(t)/F_1(t)$ is increasing in $t \in \mathbb{R}^+$;
- in the *likelihood ratio order* (denoted by $X_1 \leq_{LR} X_2$) if the ratio $f_2(t)/f_1(t)$ is increasing in $t \in \mathbb{R}^+$;
- in the *mean residual life order* (denoted by $X_1 \leq_{MRL} X_2$) if $E[X_1 - t | X_1 > t] \leq E[X_2 - t | X_2 > t]$ for all $t \in \mathbb{R}^+$, i.e., if the ratio $\int_t^{+\infty} \overline{F}_2(s) ds \,/\, \int_t^{+\infty} \overline{F}_1(s) ds$ is increasing in $t \in \mathbb{R}^+$.

We address the reader to Shaked and Shanthikumar (2007) for a detailed description of these stochastic orders and to Barlow and Proschan (1975) for a list of examples of applications in reliability theory. Here, in particular, we just point out that:

- $X_1 \leq_{HR} X_2$ if, and only if, $[X_1 - t | X_1 > t] \leq_{ST} [X_2 - t | X_2 > t]$ for all $t \in \mathbb{R}^+$,
- $X_1 \leq_{RHR} X_2$ if, and only if, $[t - X_1 | X_1 \leq t] \geq_{ST} [t - X_2 | X_2 \leq t]$ for all $t \in \mathbb{R}^+$
- $X_1 \leq_{LR} X_2$ if, and only if, $[X | a \leq X \leq b] \leq_{ST} [X_2 | a \leq X_2 \leq b]$ for all $a \leq b, a, b \in \mathbb{R}^+$.

Hence, the hazard rate order and the reversed hazard rate order are equivalent to compare residual lifetimes and inactivity times of items, respectively, at any age $t \geq 0$. Analogously, the likelihood ratio order can be used to compare both residual lifetimes and inactivity times, while this is not the case for the weaker usual stochastic order. Moreover, the following relationships are well known:

$$\begin{array}{ccccc}
X_1 \leq_{LR} X_2 & \Rightarrow & X_1 \leq_{HR} X_2 & \Rightarrow & X_1 \leq_{MRL} X_2 \\
\Downarrow & & \Downarrow & & \Downarrow \\
X_1 \leq_{RHR} X_2 & \Rightarrow & X_1 \leq_{ST} X_2 & \Rightarrow & E(X_1) \leq E(X_2).
\end{array}$$

Finally, we recall the definition of TP_2 functions: a non-negative function $g : \mathbb{R}^2 \to \mathbb{R}^+$ is said to be Totally Positive of order 2 (TP_2) if

$$g(x_1, y_1)g(x_2, y_2) - g(x_1, y_2)g(x_2, y_1) \geq 0 \text{ for all } x_1 \leq x_2 \textit{ and } y_1 \leq y_2.$$

For an exhaustive monograph on properties and applications of TP_2 functions in probability and statistics we refer the reader to Karlin (1968).

3 Stochastic comparisons of residual lifetimes

Let (X, Y) be a vector of absolutely continuous random lifetimes having survival function

$$\overline{F}(x, y) = \exp(-x - y) + \frac{\sin(2\pi \exp(-x)) \sin(2\pi \exp(-y))}{8\pi^2}, \quad x, y \geq 0. \tag{3.1}$$

Note that, marginally, X and Y are exponentially distributed, and are connected by the survival copula described in subsequent Example 4.3. It is not difficult to verify that, fixed $s \geq 0$, the corresponding residual lifetimes X_t e $\widetilde{X}_t$, whose survival functions are defined in (2.1), are not always comparable according to the usual stochastic order. See, for example, Figure 1, which gives the difference $\overline{F}_{X_t}(x) - \overline{F}_{\widetilde{X}_t}(x)$ for $t = 0.03$ and $s = 0.06$.

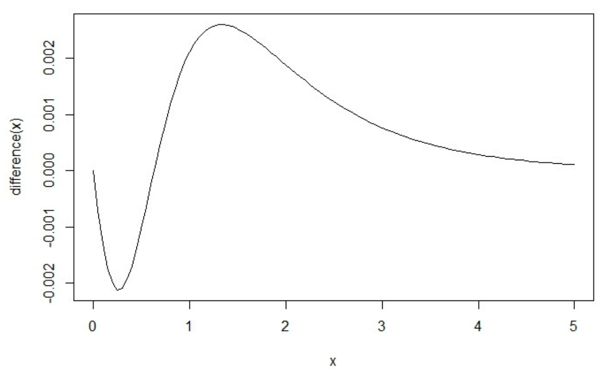

Figure 1: Plot of the difference $\overline{F}_{X_t}(x) - \overline{F}_{\widetilde{X}_t}(x) = \overline{F}(x + t, 0)/\overline{F}(t, 0) - \overline{F}(x + t, s)/\overline{F}(t, s)$ for $\overline{F}$ defined as in (3.1), $t = 0.03$ and $s = 0.06$.

Thus, stochastic orders between X_t e $\widetilde{X}_t$ are not always satisfied. Conditions on the survival copula $\overline{C}$ of the vector (X, Y) such that X_t and $\widetilde{X}_t$ are stochastically ordered for all $t, s \geq 0$ are described in this section. The first proposition, which is easy to prove, deals with the usual stochastic order and the hazard rate order.

Proposition 3.1. *Let (X, Y) be a pair of absolutely continuous random lifetimes having survival copula $\overline{C}$. The following statements are equivalent.*

(a) $\frac{\overline{C}(u,v)}{u}$ *is decreasing in u for all $v \in [0, 1]$;*

(b) $X_t \leq_{HR} \widetilde{X}_t$ *for all* $t, s \in \mathbb{R}^+$*;*

(c) $X_t \leq_{ST} \widetilde{X}_t$ *for all* $t, s \in \mathbb{R}^+$*.*

Proof. [(a) $\Rightarrow$ (b)]. Observe that $X_t \leq_{HR} \widetilde{X}_t$ for all $t, s \in \mathbb{R}^+$ if, and only if, the ratio $\overline{F}_{X_t}(x)/\overline{F}_{\widetilde{X}_t}(x)$ is increasing in x, i.e., by (2.1), if and only if $\overline{F}(x+t,s)/\overline{F}(x+t,0)$ is increasing in x, that can be restated as

$$\overline{F}(t_1,0)\overline{F}(t_2,s) \geq \overline{F}(t_1,s)\overline{F}(t_2,0) \ \forall\ 0 \leq t_1 \leq t_2 \text{ and } s \geq 0. \tag{3.2}$$

Inequality (3.2) is the same as

$$\overline{C}(\overline{F}_X(t_1),\overline{F}_Y(0))\ \overline{C}(\overline{F}_X(t_2),\overline{F}_Y(s)) \geq \overline{C}(\overline{F}_X(t_1),\overline{F}_Y(s))\ \overline{C}(\overline{F}_X(t_2),\overline{F}_Y(0)) \tag{3.3}$$

for all $0 \leq t_1 \leq t_2$ and $s \geq 0$. Letting $\overline{F}_X(t_1) = z, \overline{F}_X(t_2) = u$ and $\overline{F}_Y(s) = v$, and observed that $\overline{F}_X(t_1) \geq \overline{F}_X(t_2)$, the inequality (3.3) can be rewritten as

$$\overline{C}(z,1)\overline{C}(u,v) \geq \overline{C}(z,v)\overline{C}(u,1) \text{ for all } u,v,z \in [0,1] \text{ and } u \leq z,$$

which is the statement (a).

[(b) $\Rightarrow$ (c)]. It follows from relationships among stochastic orders.

[(c) $\Rightarrow$ (a)]. Observe that, by (2.1), the inequality $X_t \leq_{ST} \widetilde{X}_t$ holds for all $t, s \in \mathbb{R}^+$ if, and only if,

$$\frac{\overline{F}(x+t,0)}{\overline{F}(t,0)} \leq \frac{\overline{F}(x+t,s)}{\overline{F}(t,s)} \ \forall x,t,s \geq 0.$$

The latter is clearly equivalent to (3.2), i.e., to statement (a), as shown above. □

Remark 3.1. *Clearly, one has* $\widetilde{X}_t \leq_{HR} X_t$ *and* $\widetilde{X}_t \leq_{ST} X_t$ *for all* $t \in \mathbb{R}^+$ *if the monotonicity in statement (a) of Proposition 3.1 is reversed.*

Remark 3.2. *It is interesting to note that, even if in general the usual stochastic order is strictly weaker than the hazard rate order, they become equivalent in comparing* $\widetilde{X}_t$ *and* X_t*.*

It should be observed that the set of inequalities (3.2) holds if, and only if, $[Y|X > t]$ is stochastically increasing in t, i.e., if Y is *right tail increasing* in X (shortly, RTI($Y|X$)). The RTI($Y|X$) notion is a well-know property describing positive dependence among random variables; see, for example, Nelsen (2006), Chapter 5, or Colangelo et al. (2005) on its formal definition and applications in modeling dependence. The equivalence between RTI($Y|X$) notion and a condition on the copula of (X,Y) similar to statement (a) in Proposition 3.1 is also pointed out in Nelsen, Theorem 5.2.5.

The next property provides conditions on the survival copula $\overline{C}$ for the likelihood ratio and the reversed hazard orders between X_t and $\widetilde{X}_t$ for all $t, s \in \mathbb{R}^+$.

Proposition 3.2. *Let* (X,Y) *be a pair of absolutely continuous random lifetimes having survival copula* $\overline{C}$*. The following statements are equivalent.*

(a) $\frac{\partial \overline{C}(u,v)}{\partial u}$ *is decreasing in* u *for all* $v \in [0,1]$*;*

(b) $X_t \leq_{LR} \widetilde{X}_t$ *for all* $t, s \in \mathbb{R}^+$*;*

(c) $X_t \leq_{RHR} \widetilde{X}_t$ *for all* $t, s \in \mathbb{R}^+$*.*

Proof. [(a) $\Leftrightarrow$ (c)]. Assume that $X_t \leq_{RHR} \widetilde{X}_t$ holds for all $t, s \in \mathbb{R}^+$. Letting $v = \overline{F}_Y(s),\ u = \overline{F}_X(t)$ and $z = \overline{F}_X(x+t)$, and observing that $z \leq u$, the following list of equivalences holds.

$$
\begin{aligned}
X_t \leq_{RHR} \widetilde{X}_t \ &\Leftrightarrow\ F_{\widetilde{X}_t}(x)/F_{X_t}(x) \ \text{ is increasing in } x,\\
&\Leftrightarrow\ \frac{1-\overline{F}(x+t,s)/\overline{F}(t,s)}{1-\overline{F}(x+t,0)/\overline{F}(t,0)} \ \text{ is increasing in } x,\\
&\Leftrightarrow\ \frac{\overline{F}(t,s)-\overline{F}(x+t,s)}{\overline{F}(t,0)-\overline{F}(x+t,0)} \ \text{ is increasing in } x,\\
&\Leftrightarrow\ \frac{\overline{C}(\overline{F}_X(t),\overline{F}_Y(s))-\overline{C}(\overline{F}_X(x+t),\overline{F}_Y(s))}{\overline{F}_X(t)-\overline{F}_X(x+t)} \ \text{ is increasing in } x,\\
&\Leftrightarrow\ \frac{\overline{C}(u,v)-\overline{C}(z,v)}{u-z} \ \text{ is decreasing in } z,\ \text{ for } z \leq u\\
&\Leftrightarrow\ \overline{C}(z,v) \text{ is concave in } z \text{ for any fixed } v \in [0,1]
\end{aligned}
$$

The assertion now follows observing that concavity of $\overline{C}(z,v)$ in z is equivalent to the monotonicity of $\partial\overline{C}(u,v)/\partial u$.

[(a) $\Leftrightarrow$ (b)]. Let us denote as $w_v(u) = \partial\overline{C}(u,v)/\partial u$ and let us observe that

$$
\frac{\partial\overline{F}(x,s)}{\partial x} = \frac{\partial\overline{C}(\overline{F}_X(x),\overline{F}_Y(s))}{\partial x} = w_{\overline{F}_Y(s)}(\overline{F}_X(x))(-f_X(x)).
$$

Therefore we can write

$$
f_{X_t}(x) = \frac{h_0(x+t)}{\overline{F}(t,0)} = \frac{w_1(\overline{F}_X(x+t))(-f_X(x+t))}{\overline{F}(t,0)},
$$

and

$$
f_{\widetilde{X}_t}(x) = \frac{h_0(x+t)}{\overline{F}(t,s)} = \frac{w_{\overline{F}_Y(s)}(\overline{F}_X(x+t))(-f_X(x+t))}{\overline{F}(t,s)}.
$$

Thus, for fixed t,

$$
\begin{aligned}
X_t \leq_{LR} \widetilde{X}_t \ &\Leftrightarrow\ f_{\widetilde{X}_t}(x)/f_{X_t}(x) \ \text{ is increasing in } x \text{ for all } t,\\
&\Leftrightarrow\ \frac{w_{\overline{F}_Y(s)}(\overline{F}_X(x+t))(-f_X(x+t))}{w_1(\overline{F}_X(x+t))(-f_X(x+t))}\frac{\overline{F}(t,s)}{\overline{F}(t,0)} \ \text{ is increasing in } x \text{ for all } t,s \geq 0\\
&\Leftrightarrow\ \frac{w_{\overline{F}_Y(s)}(z)}{w_1(z)} \ \text{ is decreasing in } z \text{ for all } t,s \geq 0,\\
&\Leftrightarrow\ \frac{w_v(z)}{w_1(z)} \ \text{ is decreasing in } z \text{ for any } v \in [0,1].
\end{aligned}
$$

Having observed that $w_1(z) = \partial\overline{C}(z,1)/\partial z = 1$, the last condition is equivalent to statement (a).

[(b) $\Leftrightarrow$ (c)]. It follows from previous equivalences.

□

Remark 3.3. *It is interesting to note that, even if in general the reversed hazard rate order is strictly weaker than the likelihood ratio order, they become equivalent in comparing $\widetilde{X}_t$ and X_t.*

Remark 3.4. *The monotonicity property stated in condition (a) of Proposition 3.2 is equivalent to the notion of stochastic monotonicity: Y is said to be* stochastically increasing *in X (shortly SI($Y|X$)) if $[Y|X=t]$ stochastically increases in t. As proved in Nelsen, 2007, SI($Y|X$) holds if, and only if, $\partial C(u,v)/\partial u$ is decreasing in u for all $v \in [0,1]$, where C is the connecting copula of (X,Y). In turns, the latter is equivalent to affirm that $\partial \overline{C}(u,v)/\partial u$ decreases in u for all $v \in [0,1]$. Using the same arguments as in the proof of Theorem 5.2.12 in Nelsen (2006), one can verify that property (a) in Proposition 3.2 strictly implies property (a) in Proposition 3.1. This implication also means that, in comparing $\widetilde{X}_t$ and X_t, the reversed hazard order implies the hazard order, while in general such two orders are not related each other.*

The last proposition deals with comparison of $\widetilde{X}_t$ and X_t in the mean residual life order.

Proposition 3.3. *Let (X,Y) be a pair of absolutely continuous random lifetimes having survival copula $\overline{C}$. Then $X_t \leq_{MRL} \widetilde{X}_t$ for all $t,s \in \mathbb{R}^+$ if the marginal distribution F_X is concave and*

$$\frac{\int_0^z \overline{C}(u,v)du}{z^2} \text{ is decreasing in } z \text{ for all } v \in [0,1] \tag{3.4}$$

Proof. Observe that (3.4) is equivalent to

$$\frac{\int_0^z \overline{C}(u,v)du}{\int_0^z \overline{C}(u,1)du} \text{ is decreasing in } z \text{ for all } v \in [0,1],$$

which in turn is equivalent to

$$\frac{\int_z^1 \overline{C}(1-u,1)du}{\int_z^1 \overline{C}(1-u,1-v)du} \text{ is decreasing in } z \text{ for all } v \in [0,1]. \tag{3.5}$$

Let now (U,V) be a random vector having marginal uniform distributions on $[0,1]$ and whose survival function is $\overline{F}_{(U,V)}(u,v) = \overline{C}(1-u,1-v)$, for $(u,v) \in [0,1]^2$. Then (3.5) is the same as

$$U \leq_{MRL} [U|V>v] \quad \forall\, v \in [0,1].$$

Observe now that, by assumptions, F_X^{-1} is increasing and convex. Thus, by Theorem 2.A.19 in Shaked and Shanthikumar (2007), which states the closure of the mean residual life order with respect to increasing and convex transformations, one has

$$F_X^{-1}\big(U\big) \leq_{MRL} F_X^{-1}\big([U|V>v]\big) \quad \forall\, v \in [0,1],$$

which is equivalent to

$$\frac{\int_x^{+\infty} \overline{C}(1-F_X(r),1)dr}{\int_x^{+\infty} \overline{C}(1-F_X(r),1-v)dr} \quad \text{decreases in } x \geq 0 \text{ for all } \; v \in [0,1]. \tag{3.6}$$

Letting $v = F_Y(s)$, from (3.6) follows that the ratio $\int_x^{+\infty} \overline{F}(r,0)dr / \int_x^{+\infty} \overline{F}(r,s)dr$ decreases in x for all $t,s \geq 0$, thus also

$$\frac{\int_{x+t}^{+\infty} \overline{F}(r,0)dr}{\int_{x+t}^{+\infty} \overline{F}(r,s)dr} \cdot \frac{\overline{F}(t,s)}{\overline{F}(t,0)} = \frac{\int_x^{+\infty} \overline{F}_{X_t}(r)dr}{\int_x^{+\infty} \overline{F}_{\widetilde{X}_t}(r)dr} \quad \text{decreases in } x \geq 0 \text{ for all } t,s \geq 0.$$

The latter means $X_t \leq_{MRL} \widetilde{X}_t$. ∎

It should be observed that the inequality $X_t \leq_{MRL} \widetilde{X}_t$ for all t immediately follows without additional assumptions on the marginal distribution F_X if the copula satisfies property (a) in Proposition 3.1, since the mean residual order follows from the hazard rate order.

Examples of copulas for which the conditions described in the previous statements are satisfied are described in the next section.

By using the same techniques as those in the previous proofs one can also prove the statement that follows, which describes conditions for stochastic comparisons between the inactivity times defined in (2.2). Being similar to the previous ones, the proof of Proposition 3.4 is omitted; we just point out that, instead of the mean residual life order, the *mean inactivity time* order ($\leq_{MIT}$) and its closure under increasing concave transformations proved in Li and Xu (2006) are used to prove point (d).

Proposition 3.4. *Let (X, Y) be a pair of absolutely continuous random lifetimes having connecting copula C. Then*

a) $X^t \leq_{HR} \widetilde{X}^t$ for all $t \in \mathbb{R}^+$ if, and only if, C is such that

$$\frac{C(u,v)}{u} \text{ is decreasing in } u \text{ for all } v \in [0,1].$$

b) $X^t \leq_{RHR} \widetilde{X}^t$ for all $t \in \mathbb{R}^+$ if, and only if, C is such that

$$\frac{\partial C(u,v)}{\partial u} \text{ is decreasing in } u \text{ for all } v \in [0,1],$$

c) $X^t \leq_{LR} \widetilde{X}^t$ for all $t \in \mathbb{R}^+$ if, and only if, C is such that

$$\frac{\partial C(u,v)}{\partial u} \text{ is decreasing in } u \text{ for all } v \in [0,1],$$

d) $X^t \leq_{MRL} \widetilde{X}^t$ for all $t \in \mathbb{R}^+$ if, and only if, the marginal distribution F_X is convex on its finite support and C is such that

$$\frac{\int_0^z C(u,v)du}{z^2} \text{ is decreasing in } z \text{ for all } v \in [0,1]$$

Observe that conditions for the stochastic inequality $X^t \leq_{ST} \widetilde{X}^t$ to hold for all $t, s \in \mathbb{R}^+$ follow from point (a) in Proposition 3.4. Also, it is interesting to observe that the conditions on the dependence for the various orders are the same as those previous propositions, but expressed in terms of the connecting copula instead of in terms of properties of the survival copula.

We conclude this section pointing out that all the statements above can be generalized to the case of more than two dependent random lifetimes, since the assumptions on the copula deal with its marginal behavior for any fixed value of the other components. For example, Proposition 3.1 can be easily generalized as follow.

Proposition 3.5. *Let $(X, Y_1, \ldots, Y_n)$ be a vector of absolutely continuous random lifetimes having survival copula $\overline{C}$. The following statements are equivalent.*

(a) $\frac{\overline{C}(u,v_1,\ldots,v_n)}{u}$ *is decreasing in* u *for all* $v_i \in [0,1], i = 1,...,n$;

(b) $[X-t|X>t] \leq_{HR} [X-t|X>t, Y_1 > s_1, \ldots, Y_n > s_n]$ *for all* $t, s_1, \ldots, s_n \in \mathbb{R}^+$;

(c) $[X-t|X>t] \leq_{ST} [X-t|X>t, Y_1 > s_1, \ldots, Y_n > s_n]$ *for all* $t, s_1, \ldots, s_n \in \mathbb{R}^+$;.

4 Examples and counterexamples

This section is devoted to a brief discussion on the assumptions that should be satisfied by the survival copula of (X,Y) (or by its connecting copula) for existence of a stochastic comparison among X_t and $\widetilde{X}_t$ (or among X^t and $\widetilde{X}^t$) for all $t, s \in \mathbb{R}^+$. Given a generic copula C, the following three properties have been considered in Section 3.

(a) $\partial C(u,v) \ / \ \partial u$ is decreasing in u for all $v \in [0,1]$;

(b) $C(u,v) \ / \ u$ is decreasing in u for all $v \in [0,1]$;

(c) $\int_0^z C(u,v)du \ / \ z^2$ is decreasing in z for all $v \in [0,1]$.

It is easy to see that the relationships $(a) \Rightarrow (b) \Rightarrow (c)$ are satisfied. The first one is proved in Nelsen (2006), Section 5, while the second one easily follows from the Basic Composition Formula for integrals of TP_2 functions (see Karlin, 1968, Chapter 3). These relationships are strict (see Nelsen, 2006, for the first one, and Example 4.3 for the last one). Moreover, it should be pointed out that a sufficient condition for property (b) is that $C(u,v)$ is TP_2 in $(u,v) \in [0,1]^2$. Dealing with survival copulas, the TP_2 property of the survival copula of a vector (X,Y) is a well-known positive dependence property, sometime named *right corner set increasing* (shortly, RCSI(Y,X)), which is in fact stronger than the RTI$(Y|X)$ property (see, e.g., Nelsen, 2006, Corollary 5.2.16). Also, it is easy to verify that property (c) is satisfied when $\int_0^z C(u,v)du$ is TP_2 in $(z,v) \in [0,1]^2$.

We provide here examples of copulas satisfying the conditions described above. The first deals with the class of Archimedean copulas (or survival copulas). A bivariate copula is said to be *Archimedean* if it can be written as

$$C(u,v) = \phi\left(\phi^{-1}(u) + \phi^{-1}(v)\right) \quad \text{for all } u, v \in [0,1], \tag{4.1}$$

for a suitable decreasing and convex function $\phi : [0,\infty) \to [0,1]$ such that $\phi(0) = 1$ and with inverse function ϕ^{-1}. The inverse function ϕ^{-1} is usually called the *generator* of the Archimedean copula C. As pointed out in Nelsen (2006), many standard copulas (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. We also refer the reader to Müller and Scarsini (2005) and references therein for details, properties and examples of application of Archimedean copulas.

Example 4.1. *Let* $C(u,v)$ *be an Archimedean copula as defined in (4.1). Then property (a) is satisfied if, and only if, the derivative* $-\phi'(x) = -\ d\phi(x)/dx$ *exists and is log–convex. In fact, (a) holds if, and only if, for any fixed* $v \in [0,1]$ *one has that*

$$\frac{\partial C(u,v)}{\partial u} = \frac{\phi'(\phi^{-1}(u) + y)}{\phi'(\phi^{-1}(u))} \tag{4.2}$$

decreases in u*, where* $y = \phi^{-1}(u) \geq 0$*. Thus, (a) is the same as to have that the ratio* $\phi'(x+y)/\phi'(x)$ *is increasing in* x*. The latter is equivalent to the log–convexity of* $-\phi'$*.*

For what concerns property (b), it has been proved in Bassan and Spizzichino (2005), Proposition 6.1, that this property holds if, and only if, the function ϕ *is log–convex.*

For property (c), it holds if, and only if, for any $k \geq 0$ *the function* $h_k(u) = \phi(\phi^{-1}(u) + k)$*, defined on* $[0, 1]$*, satisfies*

$$x \; h_k(x) \leq 2 \int_0^x h_k(u)du \quad \text{for all} \quad x \in [0,1]. \tag{4.3}$$

Archimedean copulas that satisfy all these conditions are, e.g., the Clayton copula, for which $\phi(t) = (1+t)^{-\theta}, \theta \in [0, +\infty)$*, the Gumbel-Hougaard copula, for which* $\phi(t) = e^{-t^{1/\theta}}, \theta \in [1, +\infty)$*, and the Frank copula (for positive values of the parameter* θ*), for which* $\phi(t) = -\ln\big(1 + (e^{-\theta} - 1)e^{-t}\big)/\theta$*. An example of inverse generator* ϕ *which is not log-convex but satisfies condition (4.3) is* $\phi(t) = 1/(1+t^2)$ *(as one can verify with straightforward but long analytical calculations)*

Apart for Archimedean class, there exist many other families of copulas that satisfy the properties (a) to (c). The following example deals with the well-known *Farlie–Gumbel–Morgenstern* (FGM) family (see, e.g., Nelsen, 2006).

Example 4.2. *Let* C *be an FGM copula, i.e., let* $C(u, v) = uv[1 + \theta(1 - u)(1 - v)]$ *for* $\theta \in [-1, 1]$*. If the parameter* θ *is non-negative then* $C(u, v)$ *satisfies condition (a) above (thus also (b) and (c)). Validity of this assertion can be easily proved just observing that*

$$\frac{\partial C(u,v)}{\partial u} = v[1 + \theta(1-v)(1-2u)].$$

The following example shows that there exist copulas C for which property (b) does not hold while property (c) is satisfied, thus also that one can find vectors (X, Y) for which $X_t \leq_{MRL} \widetilde{X}_t$ holds for all $t, s \geq 0$ while the hazard rate and the usual stochastic orders do not hold. This is actually the case of the example at the beginning of Section 3, for which the assumptions of Proposition 3.3 are satisfied.

Example 4.3. *Let* C *be a copula defined as in a Rodríguez-Lallena and Úbeda-Flores (2004), i.e., let*

$$C(u,v) = uv + \theta f(u)g(v)$$

where $\theta \in [0, 1]$ *and where* f *and* g *are two suitable absolutely continuous functions such that* $f(0) = f(1) = g(0) = g(1) = 0$*. In particular, let* $g(u) = f(u) = \frac{\sin(2\pi u)}{2\pi}$*. It can be verified that such two functions satisfy all the conditions for* C *to be a copula (see Rodríguez-Lallena and Úbeda-Flores, 2004, for details). In this particular case we have that* $C(u, v)/u$ *does not decrease in u. In fact,*

$$\frac{C(u,v)}{u} = v + \theta g(v)\frac{f(u)}{u}$$

decreases if $f(u)/u$ *is decreasing. But this is not the case for* $f(u) = (\sin(2\pi u))/(2\pi)$*, as one can easily verify.*

On the contrary, letting $F(z) = \int_0^z f(u)du$*, the ratio*

$$\frac{\int_0^z C(u,v)du}{z^2} = \frac{z^2 v/2 + \theta F(z) g(v)}{z^2} = \frac{v}{2} + \theta g(v)\frac{F(z)}{z^2} = \frac{v}{2} + \theta g(v)\frac{1-\cos(2\pi z)}{2\pi z^2}$$

is actually decreasing in $z \in [0,1]$

5 Applications in Risk Theory

Even if all the results discussed in previous sections are stated having in mind their possible applications in reliability theory, it is worth to mention that they can be applied also in a variety of other contexts. For example, in portfolio analysis and risk theory, where interaction among asset's values, or among risks, can not be ignored. For this reason, a number of co-risk measures (i.e., dependence-adjusted versions of measures usually employed to assess isolate risks) have been recently introduced and studied in the specialized literature; see, e.g., Mainik and Schaanning (2014) and Sordo et al. (2018) and references therein. The general idea under the definitions of these measures is to use the conditional distribution of a random value X, representing the risk of a particular component or the value of a financial asset, given that another component Y is under stress and assumes values greater than a fixed threshold.

Among other measures, an example is the *Conditional Value-at-Risk* ($\mathbf{CoVaR}_{\alpha,\beta}$), which has been defined to generalize the classical *Value-at-Risk* ($\mathbf{VaR}_\alpha$) measure of risk in the multivariate setting. Recall that, given a risk X, for a fixed $\alpha \in (0,1)$ the Value-at-Risk at α is defined as

$$\mathbf{VaR}_\alpha[X] = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}.$$

Given a vector of risks (X,Y), let $F_{[X|Y>s]}$ denote the conditional distribution of X given that $Y > s$. In order to take into account the effect on X of a possible stress on Y, the Conditional Value-at-Risk has been defined as

$$\mathbf{CoVaR}_{\alpha,\beta}[X|Y] = \mathbf{VaR}_\alpha[X|Y > \mathbf{VaR}_\beta[Y]] = F^{-1}_{[X|Y>\mathbf{VaR}_\beta[Y]]}(\alpha)$$

for every fixed $\alpha, \beta \in (0,1)$ (see Mainik and Schaanning, 2014).

Observe that, given two random values X_1 and X_2, it holds $X_1 \leq_{ST} X_2$ if, and only if, the corresponding quantile functions are ordered, i.e., if $F_{X_1}^{-1}(\alpha) \leq F_{X_2}^{-1}(\alpha)$ for all $\alpha \in (0,1)$. Thus, by Proposition 3.1 and letting $t = 0$ and $s = \mathbf{VaR}_\beta[Y]$, one immediately gets that if the survival copula $\overline{C}$ of the vector of risks (X,Y) satisfies condition (a) in that proposition, then

$$\mathbf{VaR}_\alpha[X] \leq \mathbf{CoVaR}_{\alpha,\beta}[X|Y]$$

for all $\alpha, \beta \in (0,1)$, i.e., one can assert that the Value-at-risk of X is always an underestimation of its Conditional Value-at-Risk (conditioned on Y).

Another interesting co-risk measure is the *Multivariate Tail Conditional Expectation* ($\mathbf{MTCE}_{\alpha,\beta}$) recently defined in Landsman et al. (2018), which is a generalization of the classical *Tail Conditional*

Expectation ($\mathbf{TCE}_\alpha$): given the vector of risks (X,Y), the tail conditional expectation at α of X is

$$\mathbf{TCE}_\alpha[X] = E[X|X > \mathbf{Var}_\alpha[X]],$$

while the multivariate tail conditional expectation at α and β of X given Y is defined as

$$\mathbf{MTCE}_{\alpha,\beta}[X|Y] = E[X|X > \mathbf{Var}_\alpha[X], Y > \mathbf{Var}_\beta[Y]].$$

Note that the inequality

$$E[X|X > \mathbf{Var}_\alpha[X]] \leq E[X|X > \mathbf{Var}_\alpha[X], Y > \mathbf{Var}_\beta[Y]] \tag{5.1}$$

holds for all $\alpha, \beta \in (0,1)$ if, and only if,

$$E[X|X > t] \leq E[X|X > t, Y > s]$$

for all $t, s \in \mathbb{R}$. Since the latter inequality follows by the mean residual life order between X_t and $\widetilde{X}_t$ as defined in previous sections, by Proposition 3.3 it follows that if the survival copula $\overline{C}$ of the vector of risks (X,Y) satisfies condition (a) in that proposition, then (5.1) holds, i.e., the tail conditional expectation at α of a risk X is always an underestimation of its multivariate tail conditional expectation at the same level α, for any $\beta \in (0,1)$.

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