On Repunits

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Amelia Carolina Sparavigna

Politecnico di Torino

Abstract Here we discuss the repunits. An operation of addition of these numbers is proposed. A recursive formula is given accordingly. Symmetric repunits are also defined.

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As explained in [1], the term "repunit" was coined by Beiler in a book of 1966 [2], for the numbers defined as:

$$R_n = \frac{10^n - 1}{10 - 1}$$

The sequence of repunits starts with 1, 11, 111, 1111, 11111, 111111, ... (sequence A002275 in the OEIS, https://oeis.org/A002275). As we can easily see, these numbers are linked to q-integers and Mersenne numbers [3-7]. A q-integer is defined as [3]:

$$[n] = \frac{q^n - 1}{q - 1}$$

so we have the Mersenne numbers for q=2. The repunits are the q-integers for q=10 :

$$[n]_{q=10} = \frac{10^n - 1}{10 - 1}$$

We can use the same approach for the repunits of that proposed in [4-6]. Let us consider the following operation (generalized sum):

$$R_{m+n} = R_m \oplus R_n$$

defined in the following manner:

$$(1) \quad R_m \oplus R_n = R_m + R_n + (10-1) R_m R_n$$

This is the addition of the q-units as given in [4,5]. The neutral element for (1) is $R_0 = 0$, so that: $R_m \oplus R_0 = R_m + R_0 + (10-1) R_m R_0 = R_m$.

The recursive relation for the repunits, given according to (1) and starting from $R_1 = 1$, is:

$$R_m \oplus R_1 = R_m + R_1 + (10-1) R_m R_1$$

That is: `11, 111, 1111, 11111, 111111, 1111111, 11111111, and so on.`

In [8], we have discussed the symmetric q-integers, which are defined as [3]:

$$[n]_s = \frac{q^n - q^{-n}}{q - q^{-1}}$$

We can define the “symmetric” repunits as:

$$R_{n,s} = \frac{10^n - 10^{-n}}{10 - 10^{-1}} = 2 \frac{\sinh(n \ln 10)}{10 - 10^{-1}}$$

The sequence is: 1, 10.1, 101.01, 1010.101, 10101.0101, 101010.10101, etc.

In this case, the addition is defined [8]:

$$R_{m,s} \oplus R_{n,s} = R_{m,s} \cosh(n \ln 10) + R_{n,s} \cosh(m \ln 10)$$

or

$$R_{m,s} \oplus R_{n,s} = R_{m,s} \sqrt{1 + k^2 (R_{n,s})^2} + R_{n,s} \sqrt{1 + k^2 (R_{m,s})^2}$$

where $k = \frac{1}{2} \left(10 - \frac{1}{10}\right)$. Let us note that $R_{1,s} = \frac{10 - 10^{-1}}{10 - 10^{-1}} = 1$.

The recursive formula for the symmetric repunits is:

$$R_{n+1,s} = R_{n,s} \oplus R_{1,s} = R_{n,s} \sqrt{1 + k^2} + \sqrt{1 + k^2 (R_{n,s})^2}$$

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