Advances in the recovery of binary sparse signals

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***Abstract*—Recently, the recovery of binary sparse signals from compressed linear systems has received attention due to its several applications. In this contribution, we review the latest results in this framework, that are based on a suitable non-convex polynomial formulation of the problem. Moreover, we propose novel theoretical results. Then, we show numerical results that highlight the enhancement obtained through the non-convex approach with respect to the state-of-the-art methods.**

I. Problem statement and theoretical results

The recovery of finite-valued sparse signals from linear measurements has received an increasing attention in the last years. By finite-valued signal we mean a vector $x \in \mathcal{A}^n$, where $\mathcal{A}$ is a known alphabet, i.e., a finite set of symbols. This problem is relevant in a number of applications, e.g., digital image recovery [1], security [2], digital communications [3], [4], discrete control design [5], localization [6], [7], and spectrum sensing in cognitive radio networks [8], [9], [10], [11].

Classical recovery methods, which assume $x \in \mathbb{R}^n$, can be used to recover finite-valued signals as well. However, if $\mathcal{A}$ is known, this information can be exploited to enhance the recovery accuracy. At first glance, the knowledge of $\mathcal{A}$ seems to recast the problem into a combinatorial problem. However, it has been shown that by considering the convex hull of $\mathcal{A}$, the problem is still convex and provides a more precise estimation. The seminal work [12] analyses this approach in the compressed sensing (CS) framework, with a Basis Pursuit formulation of the problem. Previously, this approach was exploited in [13], [4].

In the literature on the recovery of finite-valued sparse signals, many contributions are devoted to the binary case $\mathcal{A} = \{0, 1\}$, which is intrinsically the simplest case, but has several applications, ranging from digital communications to image processing and localization. We refer to [13], [14], [15], [16], [17], [18], [19], [20] for analyses and applications of the binary alphabet.

The aim of this contribution is to illustrate recent and novel results on the recovery of binary sparse signals from compressed linear measurements. Specifically, we retrieve the key result in [20], and we extend it to further considerations and formulations.

Let $\widetilde{x} \in \{0, 1\}^n$ be the desired signal. Given measurements $y = A\widetilde{x}$, with $A \in \mathbb{R}^{m,n}$, $m < n$, in [20], the following non-convex, polynomial optimization problem is proposed to recover $\widetilde{x}$:

$$\min_{x\in[0,1]^n} \frac{1}{2}\|y - Ax\|_2^2 + \lambda \sum_{i=1}^{n}\left(x_i - \frac{1}{2}x_i^2\right), \quad \lambda > 0. \tag{1}$$

This problem is a kind of Lasso [21], [22] with concave, polynomial penalty $\sum_{i=1}^{n}\left(x_i - \frac{1}{2}x_i^2\right)$. When measurements are noise-free, $\widetilde{x}$ is the unique binary solution of (1) under mild assumptions, as proven in [20].

Assumption 1.

(a) $A_{[m]}^T A_{[m]} - \lambda I \succ 0$;
(b) $\widetilde{x}$ *is* k*-sparse and* $k \leq m$;
(c) *the columns of* A *are in general position.*

Conditions (a) and (c) are fulfilled by random matrices with entries generated according to a continuous distribution, as discussed in [23].

Theorem 1. *Let* $\widetilde{x} \in \{0, 1\}^n$ *be* k*-sparse, and let* $y = A\widetilde{x}$*. Under assumptions 1.(a)-(b)-(c), if* λ *is sufficiently small, then* $\widetilde{x}$ *is the unique global solution of* (1) *over* $\{0, 1\}^n$.

The proof of this theorem can be found in [20].

The formulation (1) has then no bias in case of noise-free measurements, and is also robust to noise [24]. If the measurements are noise-free, the problem can be also formulated as constrained optimization problem:

$$\min_{x\in[0,1]^n} \sum_{i=1}^{n}\left(x_i - \frac{1}{2}x_i^2\right) \quad \text{s. t. } Ax = y. \tag{2}$$

As a difference to (1), this formulation is uniquely addressed to the noise-free case, while has the advantage of not requiring the tuning of λ. In CS, it is known that the every k-sparse vector is the unique solution of Basis Pursuit if and only if the null space property holds, that is, if for any $v \in \ker A \setminus \{0\}$, $\|v_S\|_1 \leq \|v_{S^c}\|_1$, S being any support with cardinality smaller or equal to k, and S^c being the complementary set (see, e.g., [22]).

Interestingly, we can prove that (2) is well posed under a weaker null space condition.

Theorem 2. *Every* k*-sparse vector in* $[0, 1]^n$ *is the unique solution of* (2) *if and only if for any* $v \in [-1, 1]^n \cap \ker A \setminus \{0\}$, $\|v_S\|_2^2 \leq \|v_{S^c}\|_1$.

The condition $\|v_S\|_2^2 \leq \|v_{S^c}\|_1$ is weaker than $\|v_S\|_1 \leq \|v_{S^c}\|_1$ since $\|v_S\|_2^2 \leq \|v_S\|_1$ for any $v \in [-1, 1]^n$. Given the uniqueness, it results that the true $\widetilde{x}$ is the unique solution of (2). The proof is omitted for brevity.

II. Algorithms

Problems (1) and (2) are well posed to deal with the recovery of binary sparse signals. However, they are non-convex, therefore finding their solutions might be complex. Three efficient approaches can be considered: (a) ℓ_1 reweighting algorithms [20]; (b) ADMM-based algorithms [24]; (c) hierarchy of semi-definite relaxations, based on the polynomial optimization theory in [25], [26]. The methods (a) and (b) achieve only local minima. However, since they are very fast, they can be run with different initial conditions, which often provides the right binary solution. The approach (c), instead, is more complex, due to possible large dimensions of the involved semi-definite problems, while has rigorous recovery guarantees.

In Figure 1, we present some preliminary numerical results by applying the approach (a), denoted by RW.

III. Conclusion

In this work, we illustrate the efficiency of a non-convex polynomial approach to recover binary sparse signals from compressed linear measurements. Theoretical guarantees are provided, as well as numerical simulations. Extensions will envisage the presence of measurement noise and the case of larger (non-binary) alphabets.

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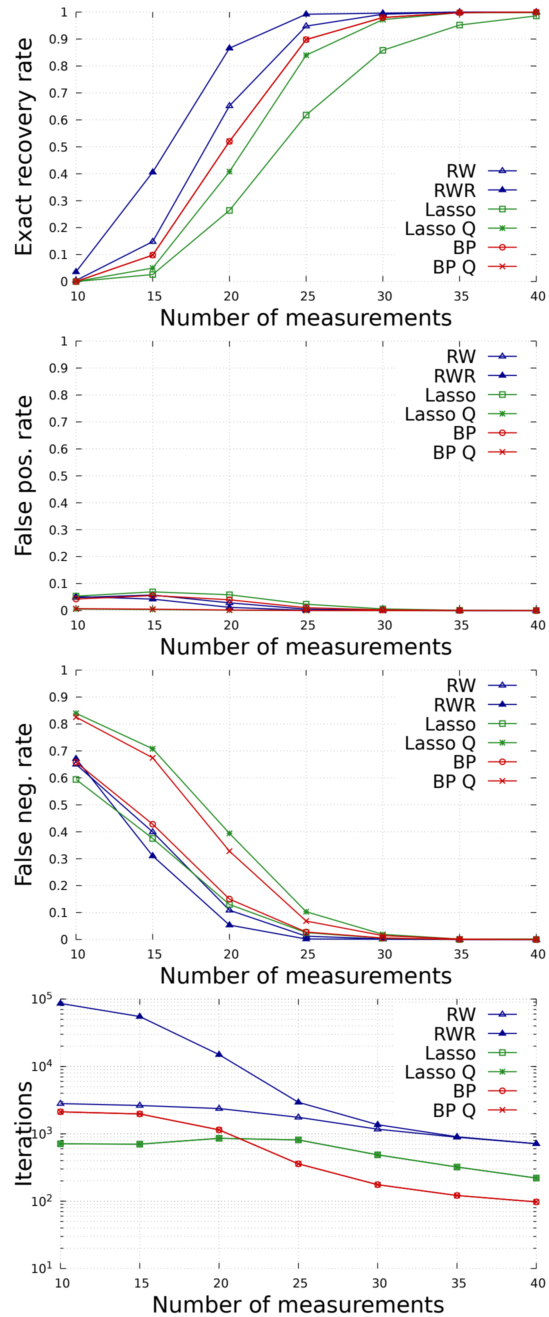

Figure 1: $n = 100$, $k = 5$, $m \in [10, 40]$. RW is the approach (a) [20], while RWR is its variant which considers 20 random initial conditions. The RW and RWR methods are shown to usually outperform the accuracy of classical methods Lasso and Basis Pursuit (in particular, in terms of exact recovery and false negative rate), at the price of a slower convergence rate. For Lasso, RW, RWR: $\lambda = 10^{-2}$. BP is for Basis Pursuit; Lasso Q and BP Q refer to the solutions obtained by quantizing the Lasso and BP solutions over $\{0, 1\}^n$ (this is particularly useful for Lasso, which produces biased solutions in the absence of noise). BP and Lasso are solved via ADMM.