On the generalized sums of Mersenne, Fermat, Cullen and Woodall Numbers

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Amelia Carolina Sparavigna

Politecnico di Torino

Abstract Recently we have discussed Mersenne and Fermat numbers using generalized sums. Here we discuss Cullen and Woodall numbers, which are similar to Mersenne and Fermat numbers. The generalized sums are given for them. Recursive relations are given accordingly.

Keywords Generalized sums, Mersenne numbers.

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In two recent papers we have discussed some properties of the Mersenne numbers [1,2] and of the Fermat numbers [3], using an approach based on the generalized sums [4-8].
In [2], in particular, the generalized sum of the Mersenne numbers and the group based on this sum is proposed. Mersenne numbers are $M_n = 2^n - 1$. These numbers form a group with the following generalized sum:

$$(1) \quad M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n$$

Using (1), for the Mersenne numbers we can imagine the following recursive relation:

$$M_{n+1} = M_n \oplus M_1 = M_n + M_1 + M_n M_1$$

Being $M_1 = 1$:

$$M_{n+1} = 2 M_n + 1$$

With a Fortran program, we have 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, 4194303, 8388607, 16777215, 33554431, 67108863, 134217727, 268435455, 536870911, 1073741823, 2147483647, 4294967295, 8589934591, 17179869183, 34359738367, 68719476735, 137438953471, 274877906943, 549755813887, 1099511627775, 2199023255551, 4398046511103, 8796093022207, 17592186044415, 35184372088831, 70368744177663, 140737488355327, 281474976710655, 562949953421311, 1125899906842623, in agreement to http://oeis.org/A000225 for the first 32 numbers.

The sum (1) is associative. The neutral element is $M_0=2^0-1=0$ and the opposites of the numbers are given by $0=M_n \oplus Opposite(M_n)$.

$$Opposite(M_n)=-\frac{M_n}{M_n+1}=M_{-n}$$

Explicitly:

$$Opposite(2^n-1)=2^{-n}-1$$

These numbers are the Mersenne numbers with a negative exponent. If we use them, we can have a group associated to the Mersenne numbers.

Fermat numbers are $F_n=2^n+1$ [9]. These numbers have the following generalized sum [3]:

$$(2) \quad F_m \oplus F_n=(1-F_m)+(1-F_n)+F_m F_n$$

Using (2), for the Fermat numbers we can imagine the following recursive relation:

$$F_{n+1}=F_n \oplus F_1=(1-F_n)+(1-F_1)+F_n F_1$$

Since $F_1=2^1+1=3$:

$$F_{n+1}=F_n \oplus F_1=(1-F_n)-2+3F_n=2F_n-1$$

Using a Fortran program we have: 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, 4194305, 8388609, 16777217, 33554433, 67108865, 134217729, 268435457, 536870913, 1073741825, 2147483649, 4294967297, 8589934593, 17179869185, 34359738369, 68719476737, 137438953473, 274877906945, 549755813889, 1099511627777, 2199023255553, 4398046511105, 8796093022209, 17592186044417, 35184372088833, 70368744177665, 140737488355329, 281474976710657, 562949953421313, 1125899906842625, in agreement to http://oeis.org/A000051 for the first 32 numbers.

The sum (2) is associative. The neutral element is $F_0=2^0+1=2$ and the opposites of the numbers are $Opposite(F_n)=F_{-n}$ [3].

Similar to the Fermat numbers, we have the Cullen numbers. The Woodall numbers are similar to the Mersenne numbers [8,9]. Let us find the generalized sums of them.
The Cullen numbers are:

$$C_n = 2^n n + 1$$

Let us find the generalized sum, as we did in [2,3]:

$$C_{m+n} = 2^{m+n}(m+n)+1$$

$$C_{m+n} = 2^m 2^n n + 2^m 2^n m + 2^m - 2^m + 2^n - 2^n + 1 = 2^m(2^n n + 1) + 2^n(2^m m + 1) - 2^n - 2^m + 1$$

$$C_{m+n} = 2^m C_n + 2^n C_m - 2^n - 2^m + 1 = 2^m(C_n - 1) + 2^n(C_m - 1) + 1$$

Let us write the generalized sum as:

$$\text{(3)} \quad C_m \oplus C_n = 2^m(C_n - 1) + 2^n(C_m - 1) + 1$$

The neutral element of this sum is $C_0 = 2^0 0 + 1 = 1$. Using (3):

$$C_m \oplus C_0 = 2^m(C_0 - 1) + 2^0(C_m - 1) + 1 = C_m$$

We have $C_1 = 2^1 1 + 1 = 3$. Recursive relation is:

$$C_{n+1} = C_n \oplus C_1 = 2^n(C_1 - 1) + 2^1(C_n - 1) + 1 = 2^{n+1} + 2(C_n - 1) + 1$$

So we have: 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, 1048577, 2228225, 4718593, 9961473, 20971521, 44040193, 92274689, 192937985, 402653185, 838860801, 1744830465, 3623878657, 7516192769, 15569256449, 32212254721, 66571993089, 137438953473, 283467841537, 584115552257, 1202590842881, 2473901162497, 5085241278465, 10445360463873, 21440476741633, 43980465111041, in agreement to http://oeis.org/A002064.
Of the Cullen numbers, we can also give another form of the generalized sum (3):

$$C_m \oplus C_n = 2^m \frac{m}{m}(C_n - 1) + 2^n \frac{n}{n}(C_m - 1) + 1 + \frac{(C_n - 1)}{m} - \frac{(C_n - 1)}{m} + \frac{(C_m - 1)}{n} - \frac{(C_m - 1)}{n}$$

$$C_m \oplus C_n = \frac{C_m}{m}(C_n - 1) + \frac{C_n}{n}(C_m - 1) + 1 - \frac{(C_n - 1)}{m} - \frac{(C_m - 1)}{n}$$

(4) $C_m \oplus C_n = \frac{1}{m}(C_m - 1)(C_n - 1) + \frac{1}{n}(C_n - 1)(C_m - 1) + 1$

The recursive relation assumes the form:

$$C_{n+1} = C_n \oplus C_1 = \frac{1}{1}(C_1 - 1)(C_n - 1) + \frac{1}{n}(C_n - 1)(C_1 - 1) + 1$$

$$C_{n+1} = C_n \oplus C_1 = 2(C_n - 1) + \frac{2}{n}(C_n - 1) + 1$$

In the case that we use the generalized sum (4), we have to remember that when m or n are equal to zero, we need to assume $(C_m - 1)/m = 1$, $(C_n - 1)/n = 1$.

In this manner:

$$C_m \oplus C_0 = \frac{1}{m}(C_m - 1)(C_0 - 1) + (C_m - 1) + 1 = C_m$$

since $C_0 = 1$.

The Woodall numbers are:

$$W_n = 2^n n - 1 \quad .$$

Let us find the generalized sum:

$$W_{m+n} = 2^{m+n}(m+n) - 1$$

$$W_{m+n} = 2^m 2^n n + 2^m 2^n m + 2^m - 2^m + 2^n - 2^n - 1 = 2^m(2^n n - 1) + 2^n(2^m m - 1) + 2^n + 2^m - 1$$

$$W_{m+n} = 2^m W_n + 2^n W_m + 2^n + 2^m - 1 = 2^m(W_n + 1) + 2^n(W_m + 1) - 1$$

Let us write the generalized sum as:

(5) $W_m \oplus W_n = 2^m(W_n + 1) + 2^n(W_m + 1) - 1$

The neutral element of this sum is $W_0 = 2^0 0 - 1 = -1$. Using (5):

$$W_m \oplus W_0 = 2^m(W_0 + 1) + 2^0(W_m + 1) - 1 = W_m$$

We have $W_1 = 2^1 1 - 1 = 1$. The recursive relation is:

$$W_{n+1} = W_n \oplus W_1 = 2^n(W_1 + 1) + 2^1(W_n + 1) - 1 = 2^{n+1} + 2(W_n + 1) - 1$$

So we have: 1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, 2228223, 4718591, 9961471, 20971519, 44040191, 92274687, 192937983, 402653183, 838860799, 1744830463, 3623878655, 7516192767, 15569256447, 32212254719, 66571993087, 137438953471, 283467841535, 584115552255, 1202590842879, 2473901162495, 5085241278463, 10445360463871, 21440476741631 43980465111039, in agreement to http://oeis.org/A003261.

Again, we can give another form of the generalized sum (5):

$$W_m \oplus W_n = 2^m \frac{m}{m}(W_n+1) + 2^n \frac{n}{n}(W_m+1) - 1 + \frac{(W_n+1)}{m} - \frac{(W_n+1)}{m} + \frac{(W_m+1)}{n} - \frac{(W_m+1)}{n}$$

$$W_m \oplus W_n = \frac{W_m}{m}(W_n+1) + \frac{W_n}{n}(W_m+1) - 1 + \frac{(W_n+1)}{m} + \frac{(W_m+1)}{n}$$

(6) $$W_m \oplus W_n = \frac{1}{m}(W_m+1)(W_n+1) + \frac{1}{n}(W_n+1)(W_m+1) - 1$$

The recursive relation assumes the form:

$$W_{n+1} = W_n \oplus W_1 = \frac{1}{n}(W_n+1)(W_1+1) + (W_n+1)(W_1+1) - 1$$

$$W_{n+1} = W_n \oplus W_1 = \frac{2}{n}(W_n+1) + 2(W_n+1) - 1$$

In the case that we use the generalized sum (6), we have to remember that when m or n are equal to zero, we need to assume $(W_m+1)/m=1$, $(W_n+1)/n=1$.

In this manner:

$$W_m \oplus W_0 = \frac{1}{m}(W_m+1)(W_0+1) + (W_m+1) - 1 = W_m$$

since $W_0 = -1$.

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