POLITECNICO DI TORINO
Repository ISTITUZIONALE

On the k-regularity of the k-adic valuation of Lucas sequences

Original
On the k-regularity of the k-adic valuation of Lucas sequences / Murru, Nadir; Sanna, Carlo. - In: JOURNAL DE THÉORIE DES NOMBRES DE BORDEAUX. - ISSN 1246-7405. - 29:(2018), pp. 227-237.

Availability:
This version is available at: 11583/2719237 since: 2018-12-03T12:03:05Z

Publisher:
Institut de Mathématiques de Bordeaux Université de Bordeaux

Published
DOI:

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

12 October 2024

JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Nadir MURRU et Carlo SANNA

On the k-regularity of the k-adic valuation of Lucas sequences

Article à paraître, mis en ligne le 13 décembre 2017, 11 p.

© Société Arithmétique de Bordeaux, 2017, tous droits réservés.

L'accès aux articles de la revue « Journal de Théorie des Nombres de Bordeaux » (http://jtnb.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://jtnb.cedram.org/legal/). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Journal de Théorie des Nombres
de Bordeaux
Article mis en ligne le 13 décembre 2017

On the k-regularity of the k-adic valuation of Lucas sequences

par Nadir MURRU et Carlo SANNA

Résumé. Pour tous entiers $k \geq 2$ et $n \neq 0$, soit $\nu_k(n)$ le plus grand entier positif e tel que k^e divise n. De plus, soit $(u_n)_{n\geq 0}$ une suite de Lucas non dégénérée telle que $u_0 = 0$, $u_1 = 1$ et $u_{n+2} = au_{n+1} + bu_n$, pour certains entiers a et b. Shu et Yao ont montré que, pour tout nombre premier p, la suite $\nu_p(u_{n+1})_{n\geq 0}$ est p-régulière. Medina et Rowland ont déterminé le rang de $\nu_p(F_{n+1})_{n\geq 0}$, où F_n est le n-ième nombre de Fibonacci.

Nous montrons que si k et b sont premiers entre eux, alors $\nu_k(u_{n+1})_{n\geq 0}$ est une suite k-régulière. Si de plus k est un nombre premier, nous déterminons aussi le rang de cette suite. En outre, nous donnons des formules explicites pour $\nu_k(u_n)$, généralisant un théorème précédent de Sanna concernant les valuations p-adiques des suites de Lucas.

Abstract. For integers $k \geq 2$ and $n \neq 0$, let $\nu_k(n)$ denote the greatest nonnegative integer e such that k^e divides n. Moreover, let $(u_n)_{n\geq 0}$ be a nondegenerate Lucas sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$, for some integers a and b. Shu and Yao showed that for any prime number p the sequence $\nu_p(u_{n+1})_{n\geq 0}$ is p-regular, while Medina and Rowland found the rank of $\nu_p(F_{n+1})_{n\geq 0}$, where F_n is the n-th Fibonacci number.

We prove that if k and b are relatively prime then $\nu_k(u_{n+1})_{n\geq 0}$ is a k-regular sequence, and for k a prime number we also determine its rank. Furthermore, as an intermediate result, we give explicit formulas for $\nu_k(u_n)$, generalizing a previous theorem of Sanna concerning p-adic valuations of Lucas sequences.

1. Introduction

For integers $k \geq 2$ and $n \neq 0$, let $\nu_k(n)$ denote the greatest nonnegative integer e such that k^e divides n. In particular, if $k = p$ is a prime number then $\nu_p(\cdot)$ is the usual p-adic valuation. We shall refer to $\nu_k(\cdot)$ as the *k-adic valuation*, although, strictly speaking, for composite k this is not

Manuscrit reçu le 25 mai 2016, accepté le 13 septembre 2016.
2010 *Mathematics Subject Classification.* 11B37, 11B85, 11A99.
Mots-clefs. Lucas sequence, Fibonacci numbers, p-adic valuation, k-regular sequence, automatic sequence.

a "valuation" in the algebraic sense of the term, since it is not true that $\nu_k(mn) = \nu_k(m) + \nu_k(n)$ for all integers $m, n \neq 0$.

Valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [4, 6, 7, 8, 9, 10, 12, 14, 15, 18]). To this end, an important role is played by the family of k-regular sequences, which were first introduced and studied by Allouche and Shallit [1, 2, 3] with the aim of generalizing the concept of automatic sequences.

Given a sequence of integers $s(n)_{n\geq 0}$, its *k-kernel* is defined as the set of subsequences

$$\ker_k(s(n)_{n\geq 0}) := \{s(k^e n + i)_{n\geq 0} : e \geq 0,\ 0 \leq i < k^e\}.$$

Then $s(n)_{n\geq 0}$ is said to be *k-regular* if the $\mathbb{Z}$-module $\langle \ker_k(s(n)_{n\geq 0})\rangle$ generated by its k-kernel is finitely generated. In such a case, the *rank* of $s(n)_{n\geq 0}$ is the rank of this $\mathbb{Z}$-module.

Allouche and Shallit provided many examples of regular sequences. In particular, they showed that the sequence of p-adic valuations of factorials $\nu_p(n!)_{n\geq 0}$ is p-regular [1, Example 9], and that the sequence of 3-adic valuations of sums of central binomial coefficients

$$\nu_3\left(\sum_{i=0}^{n}\binom{2i}{i}\right)_{n\geq 0}$$

is 3-regular [1, Example 23]. Furthermore, for any polynomial $f(x) \in \mathbb{Q}[x]$ with no roots in the natural numbers, Bell [5] proved that the sequence $\nu_p(f(n))_{n\geq 0}$ is p-regular if and only if $f(x)$ factors as a product of linear polynomials in $\mathbb{Q}[x]$ times a polynomial with no root in the p-adic integers.

Fix two integers a and b, and let $(u_n)_{n\geq 0}$ be the Lucas sequence of characteristic polynomial $f(x) = x^2 - ax - b$, i.e., $(u_n)_{n\geq 0}$ is the integral sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$, for each integer $n \geq 0$. Assume also that $(u_n)_{n\geq 0}$ is nondegenerate, i.e., $b \neq 0$ and the ratio α/β of the two roots $\alpha, \beta \in \mathbb{C}$ of $f(x)$ is not a root of unity.

Using p-adic analysis, Shu and Yao [16, Corollary 1] proved the following result.

Theorem 1.1. *For each prime number p, the sequence $\nu_p(u_{n+1})_{n\geq 0}$ is p-regular.*

In the special case $a = b = 1$, i.e., when $(u_n)_{n\geq 0}$ is the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$, Medina and Rowland [11] gave an algebraic proof of Theorem 1.1 and also determined the rank of $\nu_p(F_{n+1})_{n\geq 0}$. Their result is the following.

Theorem 1.2. *For each prime number p the sequence $\nu_p(F_{n+1})_{n\geq 0}$ is p-regular. Precisely, for $p \neq 2, 5$ the rank of $\nu_p(F_{n+1})_{n\geq 0}$ is $\alpha(p) + 1$, where $\alpha(p)$ is the least positive integer such that $p \mid F_{\alpha(p)}$, while for $p = 2$ the rank is 5, and for $p = 5$ the rank is 2.*

In this paper, we extend Theorem 1.1 to k-adic valuations with k relatively prime to b; and we generalize Theorem 1.2 to nondegenerate Lucas sequences. Let $\Delta := a^2 + 4b$ be the discriminant of $f(x)$. Also, for each positive integer m relatively prime to b let $\tau(m)$ denote the *rank of apparition* of m in $(u_n)_{n\geq 0}$, i.e., the least positive integer n such that $m \mid u_n$ (which is well-defined, see, e.g., [13]).

Our first two results are the following.

Theorem 1.3. *If $k \geq 2$ is an integer relatively prime to b, then the sequence $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular.*

Theorem 1.4. *Let p be a prime number not dividing b, and let r be the rank of $\nu_p(u_{n+1})_{n\geq 0}$.*

- *If $p \mid \Delta$ then:*
  - *$r = 2$ if $p \in \{2, 3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$;*
  - *$r = 3$ if $p \in \{2, 3\}$ and $\nu_p(u_p) \neq 1$.*
- *If $p \nmid \Delta$ then:*
  - *$r = 5$ if $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$;*
  - *$r = \tau(p) + 1$ if $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$.*

Note that Theorem 1.2 follows easily from our Theorem 1.4, since in the case of Fibonacci numbers $b = 1$, $\Delta = 5$, $\nu_2(F_3) = 1$, $\nu_2(F_6) = 3$, and $\tau(p) = \alpha(p)$.

As a preliminary step in the proof of Theorem 1.3, we obtain some formulas for the k-adic valuation $\nu_k(u_n)$, which generalize a previous result of the second author. Precisely, Sanna [15] proved the following formulas for the p-adic valuation of u_n.

Theorem 1.5. *If p is a prime number such that $p \nmid b$, then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \varrho_p(n) & \text{if } \tau(p) \mid n, \\ 0 & \text{if } \tau(p) \nmid n, \end{cases}$$

for each positive integer n, where

$$\varrho_2(n) := \begin{cases} \nu_2(u_3) & \text{if } 2 \nmid \Delta,\ 2 \nmid n, \\ \nu_2(u_6) - 1 & \text{if } 2 \nmid \Delta,\ 2 \mid n, \\ \nu_2(u_2) - 1 & \text{if } 2 \mid \Delta, \end{cases}$$

and

$$\varrho_p(n) = \varrho_p := \begin{cases} \nu_p(u_{\tau(p)}) & \text{if } p \nmid \Delta, \\ \nu_3(u_3) - 1 & \text{if } p \mid \Delta,\ p = 3, \\ 0 & \text{if } p \mid \Delta,\ p \geq 5, \end{cases}$$

for $p \geq 3$.

Actually, Sanna's result [15, Theorem 1.5] is slightly different but it quickly turns out to be equivalent to Theorem 1.5 using [15, Lemma 2.1(v), Lemma 3.1, and Lemma 3.2]. Furthermore, in Sanna's paper it is assumed $\gcd(a,b)=1$, but the proof of [15, Theorem 1.5] works exactly in the same way also for $\gcd(a,b)\neq 1$.

From now on, let $k=p_1^{a_1}\cdots p_h^{a_h}$ be the prime factorization of k, where $p_1<\cdots<p_h$ are prime numbers and $a_1,\ldots,a_h$ are positive integers.

We prove the following generalization of Theorem 1.5.

Theorem 1.6. *If $k\geq 2$ is an integer relatively prime to b, then*

$$\nu_k(u_n)=\begin{cases}\nu_k(c_k(n)n) & \text{if } \tau(p_1\cdots p_h)\mid n,\\ 0 & \text{if } \tau(p_1\cdots p_h)\nmid n,\end{cases}$$

for any positive integer n, where

$$c_k(n):=\prod_{i=1}^{h} p_i^{\varrho_{p_i}(n)}.$$

Note that Theorem 1.6 is indeed a generalization of Theorem 1.5. In fact, if $k=p$ is a prime number then obviously

$$\nu_p(c_p(n)n)=\nu_p(p^{\varrho_p(n)}n)=\nu_p(n)+\varrho_p(n),$$

for each positive integer n.

2. Preliminaries

In this section we collect some preliminary facts needed to prove the results of this paper. We begin with some lemmas on k-regular sequences.

Lemma 2.1. *If $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are two k-regular sequences, then $(s(n)+t(n))_{n\geq 0}$ and $s(n)t(n)_{n\geq 0}$ are k-regular too. Precisely, if A is a finite set of generators of $\langle \ker_k(s(n)_{n\geq 0})\rangle$ and B is a finite set of generators of $\langle \ker_k(t(n)_{n\geq 0})\rangle$, then $A\cup B$ is a set of generators of $\langle \ker_k((s(n)+t(n))_{n\geq 0})\rangle$.*

Proof. See [1, Theorem 2.5]. □

Lemma 2.2. *If $s(n)_{n\geq 0}$ is a k-regular sequence, then for any integers $c\geq 1$ and $d\geq 0$ the subsequence $s(cn+d)_{n\geq 0}$ is k-regular.*

Proof. See [1, Theorem 2.6]. □

Lemma 2.3. *Any periodic sequence is k-regular.*

Proof. An ultimately periodic sequence is k-automatic for all $k\geq 2$, see [2, Theorem 5.4.2]. A k-automatic sequence is k-regular, see [1, Theorem 1.2]. □

The following lemma is essentially [1, Theorem 2.2(d) and remark (i) just below].

Lemma 2.4. *Let $s(n)_{n\geq 0}$ be a sequence of integers. If there exist some*

$$s_1 = s, s_2, \ldots, s_r \in \langle \ker_k(s(n)_{n\geq 0})\rangle \tag{2.1}$$

such that the sequences $s_j(kn+i)_{n\geq 0}$, with $0 \leq i < k$ and $1 \leq j \leq r$, are $\mathbb{Z}$-linear combinations of $s_1, \ldots, s_r$, then $s(n)_{n\geq 0}$ is k-regular and $\langle \ker_k(s(n)_{n\geq 0})\rangle$ is generated by $s_1, \ldots, s_r$.

Proof. It is sufficient to prove that $s(k^e n+i)_{n\geq 0} \in \langle s_1, \ldots, s_r\rangle$ for all integers $e \geq 0$ and $0 \leq i < k^e$. In fact, this claim implies that $\langle \ker_k(s(n)_{n\geq 0})\rangle \subseteq \langle s_1, \ldots, s_r\rangle$, while by (2.1) we have $\langle s_1, \ldots, s_r\rangle \subseteq \langle \ker_k(s(n)_{n\geq 0})\rangle$, hence $\langle \ker_k(s(n)_{n\geq 0})\rangle = \langle s_1, \ldots, s_r\rangle$ and so $s(n)_{n\geq 0}$ is k-regular. We proceed by induction on e. For $e = 0$ the claim is obvious since $s = s_1$. Suppose $e \geq 1$ and that the claim holds for $e-1$. We have $i = k^{e-1}j + i'$, for some integers $0 \leq j < k$ and $0 \leq i' < k^{e-1}$. Therefore, by the induction hypothesis,

$$\begin{aligned} s(k^e n+i)_{n\geq 0} = s(k^{e-1}(kn+j)+i')_{n\geq 0} & \\ &\in \langle s_1(kn+j)_{n\geq 0}, \ldots, s_r(kn+j)_{n\geq 0}\rangle \\ &\subseteq \langle s_1, \ldots, s_r\rangle, \end{aligned}$$

and the claim follows. □

The next lemma is well-known; we give the proof just for completeness.

Lemma 2.5. *The sequence $\nu_k(n+1)_{n\geq 0}$ is k-regular of rank 2. Indeed, $\langle \ker_k(\nu_k(n+1)_{n\geq 0})\rangle$ is generated by $\nu_k(n+1)_{n\geq 0}$ and the constant sequence $(1)_{n\geq 0}$.*

Proof. For all nonnegative integers n and $i < k$ we have

$$\nu_k(kn+i+1) = \begin{cases} 1+\nu_k(n+1) & \text{if } i = k-1, \\ 0 & \text{if } i < k-1. \end{cases}$$

Therefore, putting $s_1 = \nu_k(n+1)_{n\geq 0}$ and $s_2 = (1+\nu_k(n+1))_{n\geq 0}$ in Lemma 2.4, we obtain that $\langle \ker_k(\nu_k(n+1)_{n\geq 0})\rangle$ is generated by $\nu_k(n+1)_{n\geq 0}$ and $(1+\nu_k(n+1))_{n\geq 0}$, hence it is also generated by $\nu_k(n+1)_{n\geq 0}$ and $(1)_{n\geq 0}$, which are obviously linearly independent. Thus $\nu_k(n+1)_{n\geq 0}$ is k-regular of rank 2. □

Now we state a lemma that relates the k-adic valuation of an integer with its p_i-adic valuations. The proof is quite straightforward and we leave it to the reader.

Lemma 2.6. *We have*

$$\nu_k(m) = \min_{i=1,\ldots,h} \left\lfloor \frac{\nu_{p_i}(m)}{a_i} \right\rfloor,$$

for any integer $m \geq 2$.

We conclude this section with two lemmas on the rank of apparition $\tau(n)$.

Lemma 2.7. *For each prime number p not dividing b,*

$$\tau(p) \mid p - (-1)^{p-1}\left(\frac{\Delta}{p}\right),$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. In particular, if $p \mid \Delta$ then $\tau(p) = p$.

Proof. The case $p = 2$ is easy. For $p > 2$ see [17, Lemma 1]. □

Lemma 2.8. *If m and n are two positive integers relatively prime to b, then*

$$\tau(\operatorname{lcm}(m, n)) = \operatorname{lcm}(\tau(m), \tau(n)).$$

Proof. See [13, Theorem 1(a)]. □

3. Proof of Theorem 1.6

Thanks to Lemma 2.6, we know that

$$\nu_k(u_n) = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(u_n)}{a_i} \right\rfloor. \tag{3.1}$$

Moreover, from Lemma 2.8 it follows that

$$\tau(p_1 \cdots p_h) = \operatorname{lcm}\{\tau(p_1), \dots, \tau(p_h)\}.$$

Therefore, on the one hand, if $\tau(p_1 \cdots p_h) \nmid n$ then $\tau(p_i) \nmid n$ for some $i \in \{1, \dots, h\}$, so that by Theorem 1.5 we have $\nu_{p_i}(u_n) = 0$, which together with (3.1) implies $\nu_k(u_n) = 0$, as claimed.

On the other hand, if $\tau(p_1 \cdots p_h) \mid n$ then $\tau(p_i) \mid n$ for $i = 1, \dots, h$. Hence, from (3.1), Theorem 1.5, and Lemma 2.6, we obtain

$$\nu_k(u_n) = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(n) + \varrho_{p_i}(n)}{a_i} \right\rfloor = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(c_k(n)n)}{a_i} \right\rfloor = \nu_k(c_k(n)n),$$

so that the proof is complete.

4. Proof of Theorem 1.3

Clearly, if Δ and k are fixed, then $c_k(n)$ depends only on the parity of n. Thus it follows easily from Theorem 1.6 that

$$\nu_k(u_{n+1}) = \nu_k(c_k(1)(n+1))\, s(n) + \nu_k(c_k(2)(n+1))\, t(n), \tag{4.1}$$

for each integer $n \geq 0$, where the sequences $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are defined by

$$s(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1,\ 2 \nmid n+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1,\ 2 \mid n+1, \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, by Lemma 2.5 and Lemma 2.2, we know that both $\nu_k(c_k(1)(n+1))_{n\geq 0}$ and $\nu_k(c_k(2)(n+1))_{n\geq 0}$ are k-regular sequences. On the other hand, by Lemma 2.3, also the sequences $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are k-regular, since obviously they are periodic.

In conclusion, using (4.1) and Lemma 2.1, we obtain that $\nu_k(u_{n+1})_{n\geq 0}$ is a k-regular sequence.

5. Proof of Theorem 1.4

We generalize Medina and Rowland's proof of Theorem 1.2. First, suppose that $p \mid \Delta$. By Lemma 2.7 we have $\tau(p) = p$. Moreover, it is clear that $\varrho_p(n) = \varrho_p$ does not depend on n. As a consequence, from Theorem 1.5 it follows easily that

$$\nu_p(u_{n+1}) = \nu_p(n+1) + s(n), \tag{5.1}$$

for any integer $n \geq 0$, where the sequence $s(n)_{n\geq 0}$ is defined by

$$s(n) := \begin{cases} \varrho_p & \text{if } n+1 \equiv 0 \bmod p, \\ 0 & \text{if } n+1 \not\equiv 0 \bmod p. \end{cases}$$

On the one hand, if $p \in \{2,3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$, then $\varrho_p = 0$. Thus $s(n)_{n\geq 0}$ is identically zero and it follows by (5.1) and Lemma 2.5 that $r = 2$. On the other hand, if $p \in \{2,3\}$ and $\nu_p(u_p) \neq 1$, then $\varrho_p \neq 0$. Moreover, for $i = 0, \ldots, p-1$ we have

$$s(pn+i) = \begin{cases} \varrho_p & \text{if } i = p-1, \\ 0 & \text{if } i \neq p-1, \end{cases}$$

hence from Lemma 2.4 it follows that $s(n)_{n\geq 0}$ is p-regular and that the module $\langle \ker_p(s(n)_{n\geq 0})\rangle$ is generated by $s(n)_{n\geq 0}$ and $(\varrho_p)_{n\geq 0}$. Therefore, by (5.1), Lemma 2.5, and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n\geq 0}$ is a p-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n\geq 0})\rangle$ is generated by $\nu_p(n+1)_{n\geq 0}$, $s(n)_{n\geq 0}$, and $(1)_{n\geq 0}$, which are clearly linearly independent, hence $r = 3$.

Now suppose $p \nmid \Delta$. By Lemma 2.7, we know that $p \equiv \varepsilon \bmod \tau(p)$, for some $\varepsilon \in \{-1, +1\}$. Furthermore, if $p = 2$ then it follows easily that $\tau(2) = 3$. As a consequence, from Theorem 1.5 we obtain that

$$\nu_p(u_{n+1}) = s(n) + t(n), \tag{5.2}$$

for any integer $n \geq 0$, where the sequences $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are defined by

$$s(n) := \begin{cases} \nu_p(n+1) + v & \text{if } n+1 \equiv 0 \bmod \tau(p) \\ 0 & \text{if } n+1 \not\equiv 0 \bmod \tau(p), \end{cases}$$

with $v := \nu_p(u_{\tau(p)})$, and

$$t(n) := \begin{cases} \nu_2(u_6) - \nu_2(u_3) - 1 & \text{if } p = 2,\ n+1 \equiv 0 \bmod 6, \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that $s(n)_{n\geq 0}$ is a p-regular sequence of rank $\tau(p)+1$. Let us define the sequences $s_j(n)_{n\geq 0}$, for $j = 0, \ldots, \tau(p)-1$, by

$$s_j(n) := \begin{cases} 1 & \text{if } n+j+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n+j+1 \not\equiv 0 \bmod \tau(p). \end{cases}$$

On the one hand, for $i = 0, \ldots, p-2$ we have

$$\begin{aligned} s(pn+i) &= \begin{cases} \nu_p(pn+i+1) + v & \text{if } pn+i+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } pn+i+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} v & \text{if } \varepsilon n+i+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } \varepsilon n+i+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} v & \text{if } n+(\varepsilon(i+1)-1)+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n+(\varepsilon(i+1)-1)+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= v \cdot s_{(\varepsilon(i+1)-1) \bmod \tau(p)}(n), \end{aligned}$$

since $p \nmid i+1$ and consequently $\nu_p(pn+i+1) = 0$.

On the other hand,

$$\begin{aligned} s(pn+p-1) &= \begin{cases} \nu_p(pn+p) + v & \text{if } p(n+1) \equiv 0 \bmod \tau(p), \\ 0 & \text{if } p(n+1) \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} \nu_p(n+1) + v + 1 & \text{if } n+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= s(n) + s_0(n), \end{aligned} \tag{5.3}$$

since $\nu_p(pn+p) = \nu_p(n+1) + 1$ and $\gcd(p, \tau(p)) = 1$.

Furthermore, for $i = 0, \ldots, p-1$ and $j = 0, \ldots, \tau(p)-1$,

$$\begin{aligned} s_j(pn+i) &= \begin{cases} 1 & \text{if } pn+i+j+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } pn+i+j+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} 1 & \text{if } n+(\varepsilon(i+j+1)-1)+1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n+(\varepsilon(i+j+1)-1)+1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= s_{(\varepsilon(i+j+1)-1) \bmod \tau(p)}(n). \end{aligned}$$

Summarizing, the sequences $s(pn+i)_{n\geq 0}$ and $s_j(pn+i)_{n\geq 0}$, for $0 \leq i < p$ and $0 \leq j < \tau(p)$, are $\mathbb{Z}$-linear combinations of $s(n)_{n\geq 0}$ and $s_j(n)_{n\geq 0}$.

Moreover, for $i = 0, \dots, p^2 - 1$ we have

$$
\begin{aligned}
s_0(p^2 n + i) &= \begin{cases} 1 & \text{if } p^2 n + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } p^2 n + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\
&= \begin{cases} 1 & \text{if } n + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\
&= s_{i \bmod \tau(p)}(n),
\end{aligned} \tag{5.4}
$$

hence, by (5.4) and (5.3), it follows that

$$
\begin{aligned}
s_{i \bmod \tau(p)}(n)_{n\geq 0} &= s_0(p^2 n + i)_{n\geq 0} \\
&= s(p^3 n + pi + p - 1)_{n\geq 0} - s(p^2 n + i)_{n\geq 0} \\
&\in \langle \ker_p(s(n)_{n\geq 0}) \rangle.
\end{aligned} \tag{5.5}
$$

Since $\tau(p) \mid p - \varepsilon$, we have

$$\tau(p) \leq p - \varepsilon \leq p + 1 < p^2,$$

hence by (5.5) we get that $s_j(n)_{n\geq 0} \in \langle \ker_p(s(n)_{n\geq 0}) \rangle$, for $0 \leq j < \tau(p)$.

Therefore, in light of Lemma 2.4, we obtain that $s(n)_{n\geq 0}$ is a p-regular sequence and that $\langle \ker_p(s(n)_{n\geq 0}) \rangle$ is generated by $s(n)_{n\geq 0}$ and $s_j(n)_{n\geq 0}$, with $j = 0, \dots, \tau(p) - 1$. It is straightforward to see that these last sequences are linearly independent, hence $s(n)_{n\geq 0}$ has rank $\tau(p) + 1$.

If $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$, then $t(n)_{n\geq 0}$ is identically zero, thus from (5.2) and the previous result on $s(n)$ we find that $r = \tau(p) + 1$.

So it remains only to consider the case $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$. Recall that in such a case $\tau(2) = 3$, and put $d := \nu_2(u_6) - \nu_2(u_3) - 1$. Obviously, the sequence $t(2n)_{n\geq 0}$ is identically zero, while

$$
\begin{aligned}
t(2n+1) &= \begin{cases} d & \text{if } \ 2n + 2 \equiv 0 \bmod 6, \\ 0 & \text{if } \ 2n + 2 \not\equiv 0 \bmod 6, \end{cases} \\
&= \begin{cases} d & \text{if } \ n + 1 \equiv 0 \bmod 3, \\ 0 & \text{if } \ n + 1 \not\equiv 0 \bmod 3, \end{cases} \\
&= d \cdot s_0(n).
\end{aligned}
$$

Thus, again from Lemma 2.4, we have that $t(n)$ is a 2-regular sequence and that $\langle \ker_p(t(n)_{n\geq 0}) \rangle$ is generated by $t(n)_{n\geq 0}$ and $d \cdot s_j(n)_{n\geq 0}$, for $j = 0, 1, 2$.

In conclusion, by (5.2) and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n\geq 0}$ is a 2-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n\geq 0}) \rangle$ is generated by $s(n)$, $t(n)$, and $s_j(n)$, for $j = 0, 1, 2$, which are linearly independent, hence $r = 5$. The proof is complete.

6. Concluding remarks

It might be interesting to understand if, actually, $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular for every integer $k \geq 2$, so that Theorem 1.3 holds even by dropping the assumption that k and b are relatively prime. A trivial observation is that if k and b have a common prime factor p such that $p \nmid a$, then $p \nmid u_n$ for all integers $n \geq 1$, and consequently $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular simple because it is identically zero. Thus the nontrivial case occurs when each of the prime factors of $\gcd(b, k)$ divides a.

Another natural question is if it is possible to generalize Theorem 1.4 in order to say something about the rank of $\nu_k(u_{n+1})_{n\geq 0}$ when k is composite. Probably, the easier cases are those when k is squarefree, or when k is a power of a prime number.

We leave these as open questions to the reader.

References

[1] J.-P. Allouche & J. Shallit, "The ring of k-regular sequences", *Theor. Comput. Sci.* **98** (1992), no. 2, p. 163-197.

[2] ———, *Automatic sequences: Theory, applications, generalizations*, Cambridge University Press, 2003, xvi+571 pages.

[3] ———, "The ring of k-regular sequences. II", *Theor. Comput. Sci.* **207** (2003), no. 1, p. 3-29.

[4] T. Amdeberhan, D. Manna & V. H. Moll, "The 2-adic valuation of Stirling numbers", *Exp. Math.* **17** (2008), no. 1, p. 69-82.

[5] J. P. Bell, "p-adic valuations and k-regular sequences", *Discrete Math.* **307** (2007), no. 23, p. 3070-3075.

[6] H. Cohn, "2-adic behavior of numbers of domino tilings", *Electron. J. Comb.* **6** (1999), no. 2, 7 pp. (electronic).

[7] S. Hong, J. Zhao & W. Zhao, "The 2-adic valuations of Stirling numbers of the second kind", *Int. J. Number Theory* **8** (2012), no. 4, p. 1057-1066.

[8] T. Lengyel, "The order of the Fibonacci and Lucas numbers", *Fibonacci Q.* **33** (1995), no. 3, p. 234-239.

[9] ———, "Exact p-adic orders for differences of Motzkin numbers", *Int. J. Number Theory* **10** (2014), no. 3, p. 653-667.

[10] D. Marques & T. Lengyel, "The 2-adic order of the Tribonacci numbers and the equation $T_n = m!$", *J. Integer Seq.* **17** (2014), no. 10, 8 pp. (electronic).

[11] L. A. Medina & E. Rowland, "p-regularity of the p-adic valuation of the Fibonacci sequence", *Fibonacci Q.* **53** (2015), no. 3, p. 265-271.

[12] A. Postnikov & B. E. Sagan, "What power of two divides a weighted Catalan number?", *J. Comb. Theory,* **114** (2007), no. 5, p. 970-977.

[13] M. Renault, "The period, rank, and order of the (a, b)-Fibonacci sequence mod m", *Math. Mag.* **86** (2013), no. 5, p. 372-380.

[14] C. Sanna, "On the p-adic valuation of harmonic numbers", *J. Number Theory* **166** (2016), p. 41-46.

[15] ———, "The p-adic valuation of Lucas sequences", *Fibonacci Q.* **54** (2016), p. 118-224.

[16] Z. Shu & J. Yao, "Analytic functions over $\mathbb{Z}_p$ and p-regular sequences", *C. R., Math., Acad. Sci. Paris* **349** (2011), no. 17-18, p. 947-952.

[17] L. Somer, "The divisibility properties of primary Lucas recurrences with respect to primes", *Fibonacci Q.* **18** (1980), p. 316-334.

[18] X. Sun & V. H. Moll, "The p-adic valuations of sequences counting alternating sign matrices", *J. Integer Seq.* **12** (2009), no. 3, 24 pp. (electronic).

Nadir Murru
Università degli Studi di Torino
Department of Mathematics
Via Carlo Alberto 10
10123 Torino, Italy
E-mail: nadir.murru@unito.it
URL: http://orcid.org/0000-0003-0509-6278

Carlo Sanna
Università degli Studi di Torino
Department of Mathematics
Via Carlo Alberto 10
10123 Torino, Italy
E-mail: carlo.sanna.dev@gmail.com
URL: http://orcid.org/0000-0002-2111-7596