Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Article à paraître, mis en ligne le 13 décembre 2017, 11 p.

Journal de Théorie des Nombres
de Bordeaux
Article mis en ligne le 13 décembre 2017

On the k-regularity of the k-adic valuation of Lucas sequences

par Nadir MURRU et Carlo SANNA

Résumé. Pour tous entiers $k \geq 2$ et $n \neq 0$, soit $\nu_k(n)$ le plus grand entier positif e tel que k^e divise n. De plus, soit $(u_n)_{n\geq 0}$ une suite de Lucas non dégénérée telle que $u_0 = 0$, $u_1 = 1$ et $u_{n+2} = au_{n+1} + bu_n$, pour certains entiers a et b. Shu et Yao ont montré que, pour tout nombre premier p, la suite $\nu_p(u_{n+1})_{n\geq 0}$ est p-régulière. Medina et Rowland ont déterminé le rang de $\nu_p(F_{n+1})_{n\geq 0}$, où F_n est le n-ième nombre de Fibonacci.

Nous montrons que si k et b sont premiers entre eux, alors $\nu_k(u_{n+1})_{n\geq 0}$ est une suite k-régulière. Si de plus k est un nombre premier, nous déterminons aussi le rang de cette suite. En outre, nous donnons des formules explicites pour $\nu_k(u_n)$, généralisant un théorème précédent de Sanna concernant les valuations p-adiques des suites de Lucas.

Abstract. For integers $k \geq 2$ and $n \neq 0$, let $\nu_k(n)$ denote the greatest nonnegative integer e such that k^e divides n. Moreover, let $(u_n)_{n\geq 0}$ be a nondegenerate Lucas sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$, for some integers a and b. Shu and Yao showed that for any prime number p the sequence $\nu_p(u_{n+1})_{n\geq 0}$ is p-regular, while Medina and Rowland found the rank of $\nu_p(F_{n+1})_{n\geq 0}$, where F_n is the n-th Fibonacci number.

We prove that if k and b are relatively prime then $\nu_k(u_{n+1})_{n\geq 0}$ is a k-regular sequence, and for k a prime number we also determine its rank. Furthermore, as an intermediate result, we give explicit formulas for $\nu_k(u_n)$, generalizing a previous theorem of Sanna concerning p-adic valuations of Lucas sequences.

1. Introduction

For integers $k \geq 2$ and $n \neq 0$, let $\nu_k(n)$ denote the greatest nonnegative integer e such that k^e divides n. In particular, if $k = p$ is a prime number then $\nu_p(\cdot)$ is the usual p-adic valuation. We shall refer to $\nu_k(\cdot)$ as the *k-adic valuation*, although, strictly speaking, for composite k this is not

Manuscrit reçu le 25 mai 2016, accepté le 13 septembre 2016.
2010 *Mathematics Subject Classification.* 11B37, 11B85, 11A99.
Mots-clefs. Lucas sequence, Fibonacci numbers, p-adic valuation, k-regular sequence, automatic sequence.

a "valuation" in the algebraic sense of the term, since it is not true that $\nu_k(mn) = \nu_k(m) + \nu_k(n)$ for all integers $m, n \neq 0$.

Valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [4, 6, 7, 8, 9, 10, 12, 14, 15, 18]). To this end, an important role is played by the family of k-regular sequences, which were first introduced and studied by Allouche and Shallit [1, 2, 3] with the aim of generalizing the concept of automatic sequences.

Given a sequence of integers $s(n)_{n\geq 0}$, its *k-kernel* is defined as the set of subsequences

$$\ker_k(s(n)_{n\geq 0}) := \{s(k^e n + i)_{n\geq 0} : e \geq 0,\ 0 \leq i < k^e\}.$$

Then $s(n)_{n\geq 0}$ is said to be *k-regular* if the $\mathbb{Z}$-module $\langle \ker_k(s(n)_{n\geq 0})\rangle$ generated by its k-kernel is finitely generated. In such a case, the *rank* of $s(n)_{n\geq 0}$ is the rank of this $\mathbb{Z}$-module.

Allouche and Shallit provided many examples of regular sequences. In particular, they showed that the sequence of p-adic valuations of factorials $\nu_p(n!)_{n\geq 0}$ is p-regular [1, Example 9], and that the sequence of 3-adic valuations of sums of central binomial coefficients

$$\nu_3\left(\sum_{i=0}^{n} \binom{2i}{i}\right)_{n\geq 0}$$

is 3-regular [1, Example 23]. Furthermore, for any polynomial $f(x) \in \mathbb{Q}[x]$ with no roots in the natural numbers, Bell [5] proved that the sequence $\nu_p(f(n))_{n\geq 0}$ is p-regular if and only if $f(x)$ factors as a product of linear polynomials in $\mathbb{Q}[x]$ times a polynomial with no root in the p-adic integers.

Fix two integers a and b, and let $(u_n)_{n\geq 0}$ be the Lucas sequence of characteristic polynomial $f(x) = x^2 - ax - b$, i.e., $(u_n)_{n\geq 0}$ is the integral sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$, for each integer $n \geq 0$. Assume also that $(u_n)_{n\geq 0}$ is nondegenerate, i.e., $b \neq 0$ and the ratio α/β of the two roots $\alpha, \beta \in \mathbb{C}$ of $f(x)$ is not a root of unity.

Using p-adic analysis, Shu and Yao [16, Corollary 1] proved the following result.

Theorem 1.1. *For each prime number p, the sequence $\nu_p(u_{n+1})_{n\geq 0}$ is p-regular.*

In the special case $a = b = 1$, i.e., when $(u_n)_{n\geq 0}$ is the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$, Medina and Rowland [11] gave an algebraic proof of Theorem 1.1 and also determined the rank of $\nu_p(F_{n+1})_{n\geq 0}$. Their result is the following.

Theorem 1.2. *For each prime number p the sequence $\nu_p(F_{n+1})_{n\geq 0}$ is p-regular. Precisely, for $p \neq 2, 5$ the rank of $\nu_p(F_{n+1})_{n\geq 0}$ is $\alpha(p) + 1$, where $\alpha(p)$ is the least positive integer such that $p \mid F_{\alpha(p)}$, while for $p = 2$ the rank is 5, and for $p = 5$ the rank is 2.*

In this paper, we extend Theorem 1.1 to k-adic valuations with k relatively prime to b; and we generalize Theorem 1.2 to nondegenerate Lucas sequences. Let $\Delta := a^2 + 4b$ be the discriminant of $f(x)$. Also, for each positive integer m relatively prime to b let $\tau(m)$ denote the *rank of apparition* of m in $(u_n)_{n\geq 0}$, i.e., the least positive integer n such that $m \mid u_n$ (which is well-defined, see, e.g., [13]).

Our first two results are the following.

Theorem 1.3. *If $k \geq 2$ is an integer relatively prime to b, then the sequence $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular.*

Theorem 1.4. *Let p be a prime number not dividing b, and let r be the rank of $\nu_p(u_{n+1})_{n\geq 0}$.*

- *If $p \mid \Delta$ then:*
  - *$r = 2$ if $p \in \{2,3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$;*
  - *$r = 3$ if $p \in \{2,3\}$ and $\nu_p(u_p) \neq 1$.*
- *If $p \nmid \Delta$ then:*
  - *$r = 5$ if $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$;*
  - *$r = \tau(p) + 1$ if $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$.*

Note that Theorem 1.2 follows easily from our Theorem 1.4, since in the case of Fibonacci numbers $b = 1$, $\Delta = 5$, $\nu_2(F_3) = 1$, $\nu_2(F_6) = 3$, and $\tau(p) = \alpha(p)$.

As a preliminary step in the proof of Theorem 1.3, we obtain some formulas for the k-adic valuation $\nu_k(u_n)$, which generalize a previous result of the second author. Precisely, Sanna [15] proved the following formulas for the p-adic valuation of u_n.

Theorem 1.5. *If p is a prime number such that $p \nmid b$, then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \varrho_p(n) & \text{if } \tau(p) \mid n, \\ 0 & \text{if } \tau(p) \nmid n, \end{cases}$$

for each positive integer n, where

$$\varrho_2(n) := \begin{cases} \nu_2(u_3) & \text{if } 2 \nmid \Delta,\ 2 \nmid n, \\ \nu_2(u_6) - 1 & \text{if } 2 \nmid \Delta,\ 2 \mid n, \\ \nu_2(u_2) - 1 & \text{if } 2 \mid \Delta, \end{cases}$$

and

$$\varrho_p(n) = \varrho_p := \begin{cases} \nu_p(u_{\tau(p)}) & \text{if } p \nmid \Delta, \\ \nu_3(u_3) - 1 & \text{if } p \mid \Delta,\ p = 3, \\ 0 & \text{if } p \mid \Delta,\ p \geq 5, \end{cases}$$

for $p \geq 3$.

Actually, Sanna's result [15, Theorem 1.5] is slightly different but it quickly turns out to be equivalent to Theorem 1.5 using [15, Lemma 2.1(v), Lemma 3.1, and Lemma 3.2]. Furthermore, in Sanna's paper it is assumed $\gcd(a, b) = 1$, but the proof of [15, Theorem 1.5] works exactly in the same way also for $\gcd(a, b) \neq 1$.

From now on, let $k = p_1^{a_1} \cdots p_h^{a_h}$ be the prime factorization of k, where $p_1 < \cdots < p_h$ are prime numbers and $a_1, \ldots, a_h$ are positive integers.

We prove the following generalization of Theorem 1.5.

Theorem 1.6. *If $k \geq 2$ is an integer relatively prime to b, then*

$$\nu_k(u_n) = \begin{cases} \nu_k(c_k(n)n) & \text{if } \tau(p_1 \cdots p_h) \mid n, \\ 0 & \text{if } \tau(p_1 \cdots p_h) \nmid n, \end{cases}$$

for any positive integer n, where

$$c_k(n) := \prod_{i=1}^{h} p_i^{\varrho_{p_i}(n)}.$$

Note that Theorem 1.6 is indeed a generalization of Theorem 1.5. In fact, if $k = p$ is a prime number then obviously

$$\nu_p(c_p(n)n) = \nu_p(p^{\varrho_p(n)} n) = \nu_p(n) + \varrho_p(n),$$

for each positive integer n.

2. Preliminaries

In this section we collect some preliminary facts needed to prove the results of this paper. We begin with some lemmas on k-regular sequences.

Lemma 2.1. *If $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are two k-regular sequences, then $(s(n)+t(n))_{n \geq 0}$ and $s(n)t(n)_{n \geq 0}$ are k-regular too. Precisely, if A is a finite set of generators of $\langle \ker_k(s(n)_{n \geq 0}) \rangle$ and B is a finite set of generators of $\langle \ker_k(t(n)_{n \geq 0}) \rangle$, then $A \cup B$ is a set of generators of $\langle \ker_k((s(n)+t(n))_{n \geq 0}) \rangle$.*

Proof. See [1, Theorem 2.5]. □

Lemma 2.2. *If $s(n)_{n \geq 0}$ is a k-regular sequence, then for any integers $c \geq 1$ and $d \geq 0$ the subsequence $s(cn + d)_{n \geq 0}$ is k-regular.*

Proof. See [1, Theorem 2.6]. □

Lemma 2.3. *Any periodic sequence is k-regular.*

Proof. An ultimately periodic sequence is k-automatic for all $k \geq 2$, see [2, Theorem 5.4.2]. A k-automatic sequence is k-regular, see [1, Theorem 1.2]. □

The following lemma is essentially [1, Theorem 2.2(d) and remark (i) just below].

Lemma 2.4. *Let $s(n)_{n\geq 0}$ be a sequence of integers. If there exist some*

$$s_1 = s, s_2, \dots, s_r \in \langle \ker_k(s(n)_{n\geq 0})\rangle \tag{2.1}$$

such that the sequences $s_j(kn+i)_{n\geq 0}$, with $0 \leq i < k$ and $1 \leq j \leq r$, are $\mathbb{Z}$-linear combinations of $s_1, \dots, s_r$, then $s(n)_{n\geq 0}$ is k-regular and $\langle \ker_k(s(n)_{n\geq 0})\rangle$ is generated by $s_1, \dots, s_r$.

Proof. It is sufficient to prove that $s(k^e n + i)_{n\geq 0} \in \langle s_1, \dots, s_r\rangle$ for all integers $e \geq 0$ and $0 \leq i < k^e$. In fact, this claim implies that $\langle \ker_k(s(n)_{n\geq 0})\rangle \subseteq \langle s_1, \dots, s_r\rangle$, while by (2.1) we have $\langle s_1, \dots, s_r\rangle \subseteq \langle \ker_k(s(n)_{n\geq 0})\rangle$, hence $\langle \ker_k(s(n)_{n\geq 0})\rangle = \langle s_1, \dots, s_r\rangle$ and so $s(n)_{n\geq 0}$ is k-regular. We proceed by induction on e. For $e = 0$ the claim is obvious since $s = s_1$. Suppose $e \geq 1$ and that the claim holds for $e-1$. We have $i = k^{e-1}j + i'$, for some integers $0 \leq j < k$ and $0 \leq i' < k^{e-1}$. Therefore, by the induction hypothesis,

$$\begin{aligned} s(k^e n + i)_{n\geq 0} &= s(k^{e-1}(kn+j) + i')_{n\geq 0} \\ &\in \langle s_1(kn+j)_{n\geq 0}, \dots, s_r(kn+j)_{n\geq 0}\rangle \\ &\subseteq \langle s_1, \dots, s_r\rangle, \end{aligned}$$

and the claim follows. □

The next lemma is well-known; we give the proof just for completeness.

Lemma 2.5. *The sequence $\nu_k(n+1)_{n\geq 0}$ is k-regular of rank 2. Indeed, $\langle \ker_k(\nu_k(n+1)_{n\geq 0})\rangle$ is generated by $\nu_k(n+1)_{n\geq 0}$ and the constant sequence $(1)_{n\geq 0}$.*

Proof. For all nonnegative integers n and $i < k$ we have

$$\nu_k(kn+i+1) = \begin{cases} 1 + \nu_k(n+1) & \text{if } i = k-1, \\ 0 & \text{if } i < k-1. \end{cases}$$

Therefore, putting $s_1 = \nu_k(n+1)_{n\geq 0}$ and $s_2 = (1+\nu_k(n+1))_{n\geq 0}$ in Lemma 2.4, we obtain that $\langle \ker_k(\nu_k(n+1)_{n\geq 0})\rangle$ is generated by $\nu_k(n+1)_{n\geq 0}$ and $(1+\nu_k(n+1))_{n\geq 0}$, hence it is also generated by $\nu_k(n+1)_{n\geq 0}$ and $(1)_{n\geq 0}$, which are obviously linearly independent. Thus $\nu_k(n+1)_{n\geq 0}$ is k-regular of rank 2. □

Now we state a lemma that relates the k-adic valuation of an integer with its p_i-adic valuations. The proof is quite straightforward and we leave it to the reader.

Lemma 2.6. *We have*

$$\nu_k(m) = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(m)}{a_i} \right\rfloor,$$

for any integer $m \geq 2$.

We conclude this section with two lemmas on the rank of apparition $\tau(n)$.

Lemma 2.7. *For each prime number p not dividing b,*

$$\tau(p) \mid p - (-1)^{p-1}\left(\frac{\Delta}{p}\right),$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. In particular, if $p \mid \Delta$ then $\tau(p) = p$.

Proof. The case $p = 2$ is easy. For $p > 2$ see [17, Lemma 1]. ☐

Lemma 2.8. *If m and n are two positive integers relatively prime to b, then*

$$\tau(\operatorname{lcm}(m, n)) = \operatorname{lcm}(\tau(m), \tau(n)).$$

Proof. See [13, Theorem 1(a)]. ☐

3. Proof of Theorem 1.6

Thanks to Lemma 2.6, we know that

$$\nu_k(u_n) = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(u_n)}{a_i} \right\rfloor. \tag{3.1}$$

Moreover, from Lemma 2.8 it follows that

$$\tau(p_1 \cdots p_h) = \operatorname{lcm}\{\tau(p_1), \dots, \tau(p_h)\}.$$

Therefore, on the one hand, if $\tau(p_1 \cdots p_h) \nmid n$ then $\tau(p_i) \nmid n$ for some $i \in \{1, \dots, h\}$, so that by Theorem 1.5 we have $\nu_{p_i}(u_n) = 0$, which together with (3.1) implies $\nu_k(u_n) = 0$, as claimed.

On the other hand, if $\tau(p_1 \cdots p_h) \mid n$ then $\tau(p_i) \mid n$ for $i = 1, \dots, h$. Hence, from (3.1), Theorem 1.5, and Lemma 2.6, we obtain

$$\nu_k(u_n) = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(n) + \varrho_{p_i}(n)}{a_i} \right\rfloor = \min_{i=1,\dots,h} \left\lfloor \frac{\nu_{p_i}(c_k(n)n)}{a_i} \right\rfloor = \nu_k(c_k(n)n),$$

so that the proof is complete.

4. Proof of Theorem 1.3

Clearly, if Δ and k are fixed, then $c_k(n)$ depends only on the parity of n. Thus it follows easily from Theorem 1.6 that

$$\nu_k(u_{n+1}) = \nu_k(c_k(1)(n+1))\, s(n) + \nu_k(c_k(2)(n+1))\, t(n), \tag{4.1}$$

for each integer $n \geq 0$, where the sequences $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are defined by

$$s(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1,\ 2 \nmid n+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1,\ 2 \mid n+1, \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, by Lemma 2.5 and Lemma 2.2, we know that both $\nu_k(c_k(1)(n+1))_{n\geq 0}$ and $\nu_k(c_k(2)(n+1))_{n\geq 0}$ are k-regular sequences. On the other hand, by Lemma 2.3, also the sequences $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are k-regular, since obviously they are periodic.

In conclusion, using (4.1) and Lemma 2.1, we obtain that $\nu_k(u_{n+1})_{n\geq 0}$ is a k-regular sequence.

5. Proof of Theorem 1.4

We generalize Medina and Rowland's proof of Theorem 1.2. First, suppose that $p \mid \Delta$. By Lemma 2.7 we have $\tau(p) = p$. Moreover, it is clear that $\varrho_p(n) = \varrho_p$ does not depend on n. As a consequence, from Theorem 1.5 it follows easily that

$$\nu_p(u_{n+1}) = \nu_p(n+1) + s(n), \tag{5.1}$$

for any integer $n \geq 0$, where the sequence $s(n)_{n\geq 0}$ is defined by

$$s(n) := \begin{cases} \varrho_p & \text{if } n+1 \equiv 0 \bmod p, \\ 0 & \text{if } n+1 \not\equiv 0 \bmod p. \end{cases}$$

On the one hand, if $p \in \{2,3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$, then $\varrho_p = 0$. Thus $s(n)_{n\geq 0}$ is identically zero and it follows by (5.1) and Lemma 2.5 that $r = 2$. On the other hand, if $p \in \{2,3\}$ and $\nu_p(u_p) \neq 1$, then $\varrho_p \neq 0$. Moreover, for $i = 0, \ldots, p-1$ we have

$$s(pn+i) = \begin{cases} \varrho_p & \text{if } i = p-1, \\ 0 & \text{if } i \neq p-1, \end{cases}$$

hence from Lemma 2.4 it follows that $s(n)_{n\geq 0}$ is p-regular and that the module $\langle \ker_p(s(n)_{n\geq 0}) \rangle$ is generated by $s(n)_{n\geq 0}$ and $(\varrho_p)_{n\geq 0}$. Therefore, by (5.1), Lemma 2.5, and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n\geq 0}$ is a p-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n\geq 0}) \rangle$ is generated by $\nu_p(n+1)_{n\geq 0}$, $s(n)_{n\geq 0}$, and $(1)_{n\geq 0}$, which are clearly linearly independent, hence $r = 3$.

Now suppose $p \nmid \Delta$. By Lemma 2.7, we know that $p \equiv \varepsilon \bmod \tau(p)$, for some $\varepsilon \in \{-1,+1\}$. Furthermore, if $p = 2$ then it follows easily that $\tau(2) = 3$. As a consequence, from Theorem 1.5 we obtain that

$$\nu_p(u_{n+1}) = s(n) + t(n), \tag{5.2}$$

for any integer $n \geq 0$, where the sequences $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ are defined by

$$s(n) := \begin{cases} \nu_p(n+1) + v & \text{if } n+1 \equiv 0 \bmod \tau(p) \\ 0 & \text{if } n+1 \not\equiv 0 \bmod \tau(p), \end{cases}$$

with $v := \nu_p(u_{\tau(p)})$, and

$$t(n) := \begin{cases} \nu_2(u_6) - \nu_2(u_3) - 1 & \text{if } p = 2,\ n + 1 \equiv 0 \bmod 6, \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that $s(n)_{n\geq 0}$ is a p-regular sequence of rank $\tau(p)+1$. Let us define the sequences $s_j(n)_{n\geq 0}$, for $j = 0, \dots, \tau(p) - 1$, by

$$s_j(n) := \begin{cases} 1 & \text{if } n + j + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + j + 1 \not\equiv 0 \bmod \tau(p). \end{cases}$$

On the one hand, for $i = 0, \dots, p - 2$ we have

$$\begin{aligned} s(pn + i) &= \begin{cases} \nu_p(pn + i + 1) + v & \text{if } pn + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } pn + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} v & \text{if } \varepsilon n + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } \varepsilon n + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} v & \text{if } n + (\varepsilon(i+1) - 1) + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + (\varepsilon(i+1) - 1) + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= v \cdot s_{(\varepsilon(i+1)-1) \bmod \tau(p)}(n), \end{aligned}$$

since $p \nmid i + 1$ and consequently $\nu_p(pn + i + 1) = 0$.

On the other hand,

$$\begin{aligned} s(pn + p - 1) &= \begin{cases} \nu_p(pn + p) + v & \text{if } p(n+1) \equiv 0 \bmod \tau(p), \\ 0 & \text{if } p(n+1) \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} \nu_p(n + 1) + v + 1 & \text{if } n + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= s(n) + s_0(n), \end{aligned} \tag{5.3}$$

since $\nu_p(pn + p) = \nu_p(n + 1) + 1$ and $\gcd(p, \tau(p)) = 1$.

Furthermore, for $i = 0, \dots, p - 1$ and $j = 0, \dots, \tau(p) - 1$,

$$\begin{aligned} s_j(pn + i) &= \begin{cases} 1 & \text{if } pn + i + j + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } pn + i + j + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= \begin{cases} 1 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\ &= s_{(\varepsilon(i+j+1)-1) \bmod \tau(p)}(n). \end{aligned}$$

Summarizing, the sequences $s(pn + i)_{n\geq 0}$ and $s_j(pn + i)_{n\geq 0}$, for $0 \leq i < p$ and $0 \leq j < \tau(p)$, are $\mathbb{Z}$-linear combinations of $s(n)_{n\geq 0}$ and $s_j(n)_{n\geq 0}$.

Moreover, for $i = 0, \dots, p^2 - 1$ we have

$$
\begin{aligned}
s_0(p^2 n + i) &= \begin{cases} 1 & \text{if } p^2 n + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } p^2 n + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\
&= \begin{cases} 1 & \text{if } n + i + 1 \equiv 0 \bmod \tau(p), \\ 0 & \text{if } n + i + 1 \not\equiv 0 \bmod \tau(p), \end{cases} \\
&= s_{i \bmod \tau(p)}(n),
\end{aligned} \tag{5.4}
$$

hence, by (5.4) and (5.3), it follows that

$$
\begin{aligned}
s_{i \bmod \tau(p)}(n)_{n\geq 0} &= s_0(p^2 n + i)_{n\geq 0} \\
&= s(p^3 n + pi + p - 1)_{n\geq 0} - s(p^2 n + i)_{n\geq 0} \\
&\in \langle \ker_p(s(n)_{n\geq 0}) \rangle.
\end{aligned} \tag{5.5}
$$

Since $\tau(p) \mid p - \varepsilon$, we have

$$\tau(p) \leq p - \varepsilon \leq p + 1 < p^2,$$

hence by (5.5) we get that $s_j(n)_{n\geq 0} \in \langle \ker_p(s(n)_{n\geq 0}) \rangle$, for $0 \leq j < \tau(p)$.

Therefore, in light of Lemma 2.4, we obtain that $s(n)_{n\geq 0}$ is a p-regular sequence and that $\langle \ker_p(s(n)_{n\geq 0}) \rangle$ is generated by $s(n)_{n\geq 0}$ and $s_j(n)_{n\geq 0}$, with $j = 0, \dots, \tau(p) - 1$. It is straightforward to see that these last sequences are linearly independent, hence $s(n)_{n\geq 0}$ has rank $\tau(p) + 1$.

If $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$, then $t(n)_{n\geq 0}$ is identically zero, thus from (5.2) and the previous result on $s(n)$ we find that $r = \tau(p) + 1$.

So it remains only to consider the case $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$. Recall that in such a case $\tau(2) = 3$, and put $d := \nu_2(u_6) - \nu_2(u_3) - 1$. Obviously, the sequence $t(2n)_{n\geq 0}$ is identically zero, while

$$
\begin{aligned}
t(2n+1) &= \begin{cases} d & \text{if } \ 2n + 2 \equiv 0 \bmod 6, \\ 0 & \text{if } \ 2n + 2 \not\equiv 0 \bmod 6, \end{cases} \\
&= \begin{cases} d & \text{if } \ n + 1 \equiv 0 \bmod 3, \\ 0 & \text{if } \ n + 1 \not\equiv 0 \bmod 3, \end{cases} \\
&= d \cdot s_0(n).
\end{aligned}
$$

Thus, again from Lemma 2.4, we have that $t(n)$ is a 2-regular sequence and that $\langle \ker_p(t(n)_{n\geq 0}) \rangle$ is generated by $t(n)_{n\geq 0}$ and $d \cdot s_j(n)_{n\geq 0}$, for $j = 0, 1, 2$.

In conclusion, by (5.2) and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n\geq 0}$ is a 2-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n\geq 0}) \rangle$ is generated by $s(n)$, $t(n)$, and $s_j(n)$, for $j = 0, 1, 2$, which are linearly independent, hence $r = 5$. The proof is complete.

6. Concluding remarks

It might be interesting to understand if, actually, $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular for every integer $k \geq 2$, so that Theorem 1.3 holds even by dropping the assumption that k and b are relatively prime. A trivial observation is that if k and b have a common prime factor p such that $p \nmid a$, then $p \nmid u_n$ for all integers $n \geq 1$, and consequently $\nu_k(u_{n+1})_{n\geq 0}$ is k-regular simple because it is identically zero. Thus the nontrivial case occurs when each of the prime factors of $\gcd(b, k)$ divides a.

Another natural question is if it is possible to generalize Theorem 1.4 in order to say something about the rank of $\nu_k(u_{n+1})_{n\geq 0}$ when k is composite. Probably, the easier cases are those when k is squarefree, or when k is a power of a prime number.

We leave these as open questions to the reader.

References

[1] J.-P. ALLOUCHE & J. SHALLIT, "The ring of k-regular sequences", *Theor. Comput. Sci.* **98** (1992), no. 2, p. 163-197.

[2] ———, *Automatic sequences: Theory, applications, generalizations*, Cambridge University Press, 2003, xvi+571 pages.

[3] ———, "The ring of k-regular sequences. II", *Theor. Comput. Sci.* **207** (2003), no. 1, p. 3-29.

[4] T. AMDEBERHAN, D. MANNA & V. H. MOLL, "The 2-adic valuation of Stirling numbers", *Exp. Math.* **17** (2008), no. 1, p. 69-82.

[5] J. P. BELL, "p-adic valuations and k-regular sequences", *Discrete Math.* **307** (2007), no. 23, p. 3070-3075.

[6] H. COHN, "2-adic behavior of numbers of domino tilings", *Electron. J. Comb.* **6** (1999), no. 2, 7 pp. (electronic).

[7] S. HONG, J. ZHAO & W. ZHAO, "The 2-adic valuations of Stirling numbers of the second kind", *Int. J. Number Theory* **8** (2012), no. 4, p. 1057-1066.

[8] T. LENGYEL, "The order of the Fibonacci and Lucas numbers", *Fibonacci Q.* **33** (1995), no. 3, p. 234-239.

[9] ———, "Exact p-adic orders for differences of Motzkin numbers", *Int. J. Number Theory* **10** (2014), no. 3, p. 653-667.

[10] D. MARQUES & T. LENGYEL, "The 2-adic order of the Tribonacci numbers and the equation $T_n = m!$", *J. Integer Seq.* **17** (2014), no. 10, 8 pp. (electronic).

[11] L. A. MEDINA & E. ROWLAND, "p-regularity of the p-adic valuation of the Fibonacci sequence", *Fibonacci Q.* **53** (2015), no. 3, p. 265-271.

[12] A. POSTNIKOV & B. E. SAGAN, "What power of two divides a weighted Catalan number?", *J. Comb. Theory,* **114** (2007), no. 5, p. 970-977.

[13] M. RENAULT, "The period, rank, and order of the (a, b)-Fibonacci sequence mod m", *Math. Mag.* **86** (2013), no. 5, p. 372-380.

[14] C. SANNA, "On the p-adic valuation of harmonic numbers", *J. Number Theory* **166** (2016), p. 41-46.

[15] ———, "The p-adic valuation of Lucas sequences", *Fibonacci Q.* **54** (2016), p. 118-224.

[16] Z. SHU & J. YAO, "Analytic functions over $\mathbb{Z}_p$ and p-regular sequences", *C. R., Math., Acad. Sci. Paris* **349** (2011), no. 17-18, p. 947-952.

[17] L. SOMER, "The divisibility properties of primary Lucas recurrences with respect to primes", *Fibonacci Q.* **18** (1980), p. 316-334.

[18] X. SUN & V. H. MOLL, "The p-adic valuations of sequences counting alternating sign matrices", *J. Integer Seq.* **12** (2009), no. 3, 24 pp. (electronic).

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